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A FINITE DIFFERENCE METHOD FOR STOCHASTIC NONLINEAR SECOND-ORDER BOUNDARY-VALUE PROBLEMS DRIVEN BY ADDITIVE NOISES

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Abstract. In this paper, we present a finite difference method for stochastic nonlinear second-order boundary-value problems (BVPs) driven by additive noises. We first approximate the white noise process with its piecewise constant approximation to obtain an approximate stochastic BVP. The solution to the new BVP is shown to converge to the solution of the original BVP at $\mathcal{O}(h)$ in the mean-square sense. The approximate BVP is shown to have certain regularity properties which are not true in general for the solution to the original stochastic BVP. The standard finite difference method for deterministic BVPs is then applied to approximate the solution of the new stochastic BVP. Convergence analysis is presented for the numerical solution based on the standard finite difference method. We prove that the finite difference solution converges to the solution to the original stochastic BVP at $\mathcal{O}(h)$ in the mean-square sense. Finally, we perform several numerical examples to validate the theoretical results.

Key words. Stochastic nonlinear boundary-value problems, finite difference method, additive white noise, mean-square convergence, order of convergence.

1. Introduction

In this paper, we investigate the convergence properties of a finite difference method applied to scalar stochastic nonlinear second-order boundary-value problems (BVPs) driven by additive white noises. More specifically, we are interested in the stochastic BVP (SBVP)

$$u'' = f(x,u) + g(x)\dot{W}(x), \quad x \in (a,b), \quad u(a) = \alpha, \quad u(b) = \beta, \tag{1}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are given functions. Here, α and β are deterministic real constants and $\dot{W}$ is the white noise. The white noise is a generalized function or a distribution and it can be written informally as $\dot{W}(x) = \frac{dW(x)}{dx}$ in the sense of distribution. Here, $W(x)$ is the one-dimensional standard Brownian motion (or Wiener process) which is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_x\}_{a \le x \le b}$ satisfying the usual conditions (*i.e.*, the filtration is right-continuous and contains all P-null sets in $\mathcal{F}$) and carrying a standard one-dimensional Brownian motion W. We note that the stochastic process $W = W(x),\ x \in [a,b]$ has the following important properties:

1. $W(a) = 0$ with probability one.
2. The trajectories (or sample paths) $x \to W(x)$ are continuous for $x \in [a,b]$.
3. For every $a \le x < y \le b$, the increment $W(y) - W(x)$ is normally distributed with mean 0 and variance $y - x$. Symbolically, we write $W(y) - W(x) \sim \mathcal{N}(0, y-x)$.
4. $W(x)$ has independent increments *i.e.*, for every partition $a = x_0 \le x_1 < \cdots < x_N = b$, the increments $\Delta W_i = W(x_i) - W(x_{i-1}),\ i = 1, 2, \ldots, N$, are independent.

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Although Brownian paths are not differentiable pointwise, we may interpret their derivative in a distributional sense to get a generalized stochastic process called white noise $\dot{W} = \frac{dW}{dx}$. The term "white noise" arises from the spectral theory of stationary random processes, according to which white noise has a "flat" power spectrum that is uniformly distributed over all frequencies (like white light). This can be observed from the Fourier representation of Brownian motion.

In our analysis, we assume that the SBVP (1) has a unique solution. The existence and uniqueness of the solution to SBVPs were established by Nualart and Pardoux in [29, 30]. We further assume that the function g is continuous on $[a, b]$ and satisfies the uniform Lipschitz condition with Lipschitz constant L_g:

$$|g(x) - g(y)| \leq L_g |x - y| . \tag{2}$$

Finally, we assume that the nonlinear function $f(x, u)$ satisfies the following conditions

(1) $f(x, u)$ and $f_u(x, u)$ are continuous functions on the set $D = \{(x, u) \mid x \in [a, b],\ u \in \mathbb{R}\}$,

(2) there exist constants K_1 and K_2 such that

$$0 < K_1 \leq f_u(x, u) \leq K_2, \quad \text{for all } (x, u) \in D. \tag{3}$$

Using the Mean-Value Theorem, it follows that f satisfies the following uniform Lipschitz condition on D in the variable u with uniform Lipschitz constant $L_f = K_2$

$$|f(x, u) - f(x, v)| \leq L_f |u - v| , \quad \text{for all } (x, u),\ (x, v) \in D = [a, b] \times \mathbb{R}. \tag{4}$$

We remark that (1) is a formal notation due to poor regularity of the white noise. A solution to the SBVP (1) is defined in terms of integral equations. To define the solution u, we first introduce a new variable $v = u'$. Then we convert (1) into the system

$$u' = v, \quad v' = f(x, u) + g(x)\dot{W}(x), \quad x \in (a, b), \quad u(a) = \alpha, \quad u(b) = \beta. \tag{5}$$

The stochastic process $(u, v) \in \mathbb{R}^2$ is a solution to (5) if (u, v) satisfies the integral equations

$$u(x) = u(a) + \int_a^x v(y)dy, \quad x \in (a, b), \tag{6a}$$

$$v(x) = v(a) + \int_a^x f\left(y, u(y)\right) dy + \int_a^x g(y)dW(y), \quad x \in (a, b), \tag{6b}$$

with the boundary conditions $u(a) = \alpha$ and $u(b) = \beta$. The integral in (6a) and the first integral in (6b) are pathwise Riemann integrals. However, the last integral in (6b) is an Itô stochastic integral. Since the Brownian paths are of unbounded variation on $[a, x]$ for every $x > a$, the latter integral cannot be defined as a Riemann-Stieltjes integral.

Stochastic differential equations (SDEs) are used to describe more realistic models. They provide suitable mathematical tools to model real-world problems with uncertainties that may be originated from various sources such as side (initial and boundary) conditions, geometry representation of the domain, and input parameters. Many areas of applications use SDEs including physics, biology, finance, economics, insurance, signal processing and filtering, population dynamics, and genetics; see for examples [17, 25, 31, 32, 33, 34, 38].

Unlike deterministic BVPs, there are very few SDEs with exact analytical solutions. Therefore, numerical methods are usually necessary to approximate their solutions.

Numerical methods for stochastic initial-value problems (SIVPs) were investigated by several authors; see for instance [1, 2, 7, 8, 9, 22, 23, 25, 26, 27, 28, 35, 36, 37, 39] just to mention a few. However, numerical approximations of SBVPs have received much less attention [2, 3, 4, 5, 6, 11, 13, 12, 14, 15, 16, 19, 41, 42, 40]. In particular, Arciniega and Allen [3, 4, 5] proposed a stochastic shooting method procedure to approximate solutions to linear and nonlinear SBVPs. The stochastic shooting method is similar to the standard shooting method for deterministic boundary-value problems. It consists of transforming SBVPs to a family of SIVPs, which can be solved using standard numerical methods for SIVPs such as the Euler-Maruyama method or Milstein's method. The shooting method requires solutions to a family of SIVPs and nonlinear equations. Allen *et al.* [2] presented finite difference and finite element methods to approximate solutions to linear parabolic and elliptic stochastic partial differential equations (SPDEs) driven by additive white noise. They approximated the white noise process $\dot{W}$ with a piecewise constant random process and obtained an approximate solution that converges to the solution of the original problem. They further showed certain regularity conditions are met by the solution to the new approximate problem. The regularity conditions allowed them to use the standard analysis techniques in the finite difference and finite element methods. Some homotopy continuation methods for stochastic two-point BVPs driven by additive noises were proposed in [12]. However, to the best of our knowledge, there has been no work in the literature which studies the finite difference method for the stochastic nonlinear BVPs.

In [13], Cao *et al.* studied the well-posedness and optimal error estimates of spectral finite element approximations for the boundary-value problems of stochastic semi-linear elliptic SPDEs driven by white or colored Gaussian noises. They established a covariance operator dependent condition for the well-posedness of SPDE through the convergence analysis for a sequence of solutions of SPDEs with the noise term in the SPDE replaced by its spectral projections. To obtain numerical solutions, they applied the finite element method to the SPDEs whose noises are the spectral projections of the original noise. More recently, Cao *et al.* [14] investigated the well posedness and the finite element approximations for the stochastic boundary-value problem (1) driven by a fractional Brownian motion with Hurst index $H \leq 1/2$. They obtained the existence of a unique solution for the SBVP by analyzing the convergence of a sequence of approximate solutions of the stochastic equation with the fractional noise replaced by its piecewise constant approximations. They also derived an error estimate between the exact solution of the original problem and its approximations that are used in the well-posedness analysis. Moreover, they applied the Galerkin finite element method to the approximate noise-driven equation, and obtained the overall error estimate of the finite element solution through an finite element error estimate for the approximate stochastic problem. More precisely, they showed that, with continuous piecewise linear finite elements, the mean square convergence rate of the finite element approximations is $\mathcal{O}(h^{H+1/2})$, which is consistent with the existing result for white noise $H = 1/2$; see, *e.g.* [2].

In this paper, we propose a new first-order finite difference method for nonlinear stochastic two-point boundary-value problems of the form (1). As in [2], we first construct a new approximate stochastic BVP by replacing the white noise $\dot{W}$ with a piecewise constant function. We prove that the solution to the new approximate SBVP converges to the solution of (1) at $\mathcal{O}(h)$ in the mean-square sense. We then use the standard finite difference method for deterministic BVPs to discritize the new SBVP. We prove that the proposed finite difference scheme is convergent in mean-square with order $\mathcal{O}(h)$, when the second-order accurate three-point difference formulas are used to approximate the second derivative. Our theoretical results are supported by several numerical results.

The rest of paper is organized as follows: In section 2, we present the stochastic finite difference method for solving (1). We also introduce some preliminary results which will be needed in our error analysis. In section 3, we provide the convergence properties of the proposed scheme. In section 4, we present several numerical examples to validate our theoretical results. We conclude and discuss our results in section 5.

2. The stochastic finite difference method

For simplicity, we assume homogeneous boundary conditions. We note that this assumption can always be fulfilled by letting $v = u - w$, where $w(x) = \frac{\alpha(b-x)+\beta(x-a)}{b-a}$, which satisfies the boundary conditions $w(a) = \alpha$ and $w(b) = \beta$. It can be shown that v satisfies the homogeneous SBVP

$$v'' = \bar{f}(x, v) + g(x)\dot{W}(x), \quad x \in (a, b), \quad v(a) = v(b) = 0,$$

where $\bar{f}(x, v) = f(x, v + w)$. Once we obtain a numerical approximation, say v_h, to the above homogeneous SBVP, then an approximate solution to the original SBVP (1) with inhomogeneous Dirichlet boundary conditions can be found by adding the function linear function $w(x)$ to v_h.

Remark 2.1. *Here, we only consider the case of Dirichlet boundary conditions. We note that the proposed approach can be easily extended to solve SBVPs subject to other boundary conditions such as the mixed Neumann-Dirichlet boundary conditions of the form* $u'(a) = \alpha,\ u(b) = \beta$ *and the periodic boundary conditions* $u(a) = u(b),\ u'(a) = u'(b)$. *Details are omitted to save space.*

The finite difference method applied to the deterministic BVP, $u'' = f(x, u, u'),\ x \in (a, b),\ u(a) = \alpha, \quad u(b) = \beta$, provides high-order accuracy when using higher-order approximations of the derivatives u' and u'' at the mesh points. However, the order of accuracy depends on the regularity of the exact solution u. Thus, high-order accuracy cannot be achieved if the exact solution has poor regularity. For instance, when the finite difference method for deterministic BVP $u'' = f(x, u, u')$ is based on the three-point formulas to approximate $u'(x_i)$ and $u''(x_i)$, the exact solution u is required to belong to $C^4(a, b)$. Similarly for the stochastic BVP (1), convergence proofs of the finite difference method require certain regularity conditions on the stochastic process u. Unfortunately, the required conditions are not satisfied for the SBVP (1) as the Wiener process W is nowhere differentiable almost surely. Consequently, the stochastic process $u''(x)$ in (1) is nowhere differentiable as well. It is well-known that if the exact solution to a given differential equation has poor regularity, then the order of error estimates does not improve when applying high order numerical methods. We refer the reader to [2, 18, 20, 19] for some discussions.

To overcome the above difficulty, we regularize the SBVP (1) by replacing the noise $\dot{W}$ with its piecewise constant approximation. It turns out that the solution to the new SBVP has better regularity which allows us to apply the standard error estimates for the finite difference method.

2.1. Approximation of the process W and its regularity. We follow the approach presented in [6]. We divide the computational domain $[a, b]$ into N subintervals $I_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, N$, where $a = x_0 < x_1 < \cdots < x_N = b$. Let $h_i = x_i - x_{i-1}$ be the length of I_i. Let $h = \max_{1\le i\le N} h_i$ and $h_{\min} = \min_{1\le i\le N} h_i$ to be the lengths of the largest and smallest subintervals, respectively. In our analysis, we assume that the mesh is quasi-uniform in the sense that there exists a constant $K \ge 1$ (independent of the mesh size h) such that

$$h \le K h_{\min}. \tag{7}$$

Next, we approximate the Wiener process $W(x)$, $x \in I_i$ by the linear polynomial $W_i(x)$ that interpolates $W(x)$ at the endpoints of I_i *i.e.*,

$$\begin{aligned} W_i(x) =& W(x_{i-1}) + \frac{W(x_i) - W(x_{i-1})}{x_i - x_{i-1}}(x - x_{i-1}) \\ =& W(x_{i-1}) + \frac{\Delta W_i}{h_i}(x - x_{i-1}),\ x \in I_i, \quad i = 1, 2, \ldots, N, \end{aligned}$$

where $\Delta W_i = W(x_i) - W(x_{i-1})$ and $h_i = x_i - x_{i-1}$. Thus, an approximation of $W(x)$, $x \in [a, b]$ is given by the linear spline interpolating function $\hat{W}$

$$\hat{W}(x) = \sum_{i=1}^{N} W_i(x)\chi_i(x), \quad x \in [a, b], \tag{8}$$

where $\chi_i(x)$ is the indicator function *i.e.*, $\chi_i(x) = 1$ if $x \in I_i$ and $\chi_i(x) = 0$ otherwise.

We remark that the new approximate process $\hat{W}(x)$ is continuous over $[a, b]$ and of bounded variation. Furthermore, unlike the original process $W(x)$, the approximate process $\hat{W}(x)$ has piecewise constant derivatives given by

$$\frac{d\hat{W}(x)}{dx} = \sum_{i=1}^{N} \frac{\Delta W_i}{h_i}\chi_i(x) = \sum_{i=1}^{N} \frac{\eta_i}{\sqrt{h_i}}\chi_i(x), \quad x \in [a, b], \tag{9}$$

where $\eta_i = \frac{\Delta W_i}{\sqrt{h_i}} \sim \mathcal{N}(0, 1)$ is the standard Gaussian random variable with mean zero and variance one.

Before we discuss the main properties of the approximate process $\hat{W}$, we introduce some notation. Let $\mathbb{E}[v]$ be the expected value of the random variable v defined on a probability space $(\Omega, \mathcal{F}, P)$. It is defined by as the Lebesgue integral $\mathbb{E}[v] = \int_\Omega v dP$, provided that the integral exists.

Next, we introduce some norms. The L^2-norm of a function $u(x)$ over the intervals I_i and $I = [a, b]$ are, respectively, defined by $\|u\|_{0,I_i} = \left(\int_{I_i} u^2(x)dx\right)^{1/2}$ and $\|u\| = \left(\sum_{i=1}^{N} \|u\|_{0,I_i}^2\right)^{1/2}$. The standard L^∞-norm of $u(x)$ on I_i and on I are, respectively, defined by

$$\|u\|_{\infty,I_i} = \sup_{x\in I_i} |u(x)|, \quad \|u\|_\infty = \max_{1\le i\le N} \|u\|_{\infty,I_i}.$$

Let $H^s(I_i)$, $s = 1, 2, \ldots$ be the standard Sobolev space $H^s(I_i) = \left\{u : \int_{I_i} |u^{(k)}(x)|^2 dx < \infty,\ 0 \le k \le s\right\}$. The $H^s(I_i)$-norm is defined as $\|u\|_{s,I_i} = \left(\sum_{k=0}^{s} \left\|u^{(k)}\right\|_{0,I_i}^2\right)^{1/2}$.

The norm on the whole computational domain I is defined as $\|u\|_s = (\sum_{i=1}^N \|u\|_{s,I_i}^2)^{\frac{1}{2}}$. We remark that if $u \in H^s(I)$, then $\|u\|_s$ is the standard Sobolev norm $(\sum_{k=0}^{s} \|u^{(k)}\|^2)^{\frac{1}{2}}$.

Throughout this paper, the notation C (with or without a subscript) will be used to denote a generic deterministic positive constant which might not be the same in each appearance. It might depend on the functions f and g, but it is always independent of the mesh size h.

Next, we recall the regularity of the approximate process $\hat{W}(x)$. The following lemma, from [6], provides some important properties of $\hat{W}(x)$. These results will be used to derive an estimate of $\mathbb{E}[\|\hat{u}\|_s]$, where $\hat{u}$ is the exact solution of the new approximate SBVP obtained by replacing $W(x)$ with $\hat{W}(x)$.

Lemma 2.1. *The approximate process $\hat{W}(x)$ satisfies the following properties: The paths of $\frac{d\hat{W}}{dx}$ belong to $L^2[a,b]$ and*

$$\mathbb{E}\left[\left\|\frac{d\hat{W}}{dx}\right\|^2\right] = N \le Ch^{-1}. \tag{10}$$

Furthermore, if the non-random function $\phi(x)$ satisfies a Lipschitz condition on $[a,b]$ with Lipschitz constant $L > 0$, i.e., $|\phi(x) - \phi(y)| \le L|x - y|$, $\forall\ x,\ y \in [a,b]$, then

$$\mathbb{E}\left[\left(\int_a^b \phi(y)dW(y) - \int_a^b \phi(y)d\hat{W}(y)\right)^2\right] \le (b-a)L^2h^2. \tag{11}$$

Proof. The proof of this lemma is given in [6], more precisely in its Lemma 2.1. ☐

2.2. New approximate SBVP. Replacing $\dot{W}(x)$ with $\frac{d\hat{W}(x)}{dx}$, we obtain the following approximate SBVP

$$\hat{u}'' = f(x,\hat{u}) + g(x)\frac{d\hat{W}(x)}{dx}, \quad \hat{u}(a) = \hat{u}(b) = 0, \tag{12a}$$

where

$$\frac{d\hat{W}(x)}{dx} = \sum_{i=1}^{N} \frac{\eta_i}{\sqrt{h_i}}\chi_i(x), \quad x \in [a,b]. \tag{12b}$$

We will show that the sequence of random variables $\hat{u}(x)$ converges in the mean-square sense to the solution $u(x)$ of the SBVP (1). In the following theorem we state this convergence result along with its order of convergence.

Theorem 1. *Let u and $\hat{u}$ be the exact solutions to (1) and (12), respectively. Suppose that f and g satisfy the conditions (2)-(4). If $L_f < \frac{2\sqrt{2}}{(b-a)^2}$, then there exists a constant C independent of h such that*

$$\max_{x\in[a,b]} \mathbb{E}\left[|u(x) - \hat{u}(x)|^2\right] \le Ch^2. \tag{13}$$

$$\max_{x\in[a,b]} \left|\mathbb{E}[u(x)] - \mathbb{E}[\hat{u}(x)]\right|^2 \le Ch^2. \tag{14}$$

In particular, we have the following convergence results at the mesh points

$$\max_{i=0,1,\dots,N} \mathbb{E}\left[|u(x_i)-\hat{u}(x_i)|^2\right] \leq Ch^2. \tag{15}$$

$$\max_{i=0,1,\dots,N} \left|\mathbb{E}\left[u(x_i)\right]-\mathbb{E}\left[\hat{u}(x_i)\right]\right|^2 \leq Ch^2. \tag{16}$$

Proof. Let $G(x,y)$ be the Green's function associated with the BVP $v''=\phi(x)$ subject to $v(a)=v(b)=0$ so that $v(x)=\int_a^b G(x,y)\phi(y)dy$. The function $G(x,y)$ is explicitly given by

$$G(x,y)=\begin{cases} G_1(x,y)=\frac{(x-b)(y-a)}{b-a}, & a\leq y\leq x\leq b,\\ G_2(x,y)=\frac{(x-a)(y-b)}{b-a}, & a\leq x\leq y\leq b.\end{cases} \tag{17}$$

Then the integral forms of (1) with $\alpha=\beta=0$ and (12) are given by

$$u(x)=\int_a^b G(x,y)f(y,u(y))dy+\int_a^b G(x,y)g(y)dW(y), \tag{18}$$

$$\hat{u}(x)=\int_a^b G(x,y)f(y,\hat{u}(y))dy+\int_a^b G(x,y)g(y)d\hat{W}(y). \tag{19}$$

Subtracting (19) from (18) gives

$$\begin{aligned} u(x)-\hat{u}(x)=&\int_a^b G(x,y)\left(f(y,u(y))-f(y,\hat{u}(y))\right)dy\\ &+\int_a^b G(x,y)g(y)dW(y)-\int_a^b G(x,y)g(y)d\hat{W}(y). \end{aligned} \tag{20}$$

We observe that

$$\max_{a\leq x,\ y\leq b}|G(x,y)|=\frac{b-a}{4}. \tag{21}$$

Consequently, we have

$$|u(x)-\hat{u}(x)|\leq\frac{b-a}{4}\int_a^b|f(y,u(y))-f(y,\hat{u}(y))|\,dy+T(x),$$

where

$$T(x)=\left|\int_a^b G(x,y)g(y)dW(y)-\int_a^b G(x,y)g(y)d\hat{W}(y)\right|.$$

Squaring both sides, using the inequality $(x+y)^2\leq 2x^2+2y^2$, and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |u(x)-\hat{u}(x)|^2\leq&\frac{(b-a)^2}{8}\left(\int_a^b|f(y,u(y))-f(y,\hat{u}(y))|\,dy\right)^2+2T^2(x)\\ \leq&\frac{(b-a)^3}{8}\int_a^b|f(y,u(y))-f(y,\hat{u}(y))|^2\,dy+2T^2(x). \end{aligned}$$

Now, using the Lipschitz condition (4), we get

$$|u(x)-\hat{u}(x)|^2\leq\frac{(b-a)^3L_f^2}{8}\int_a^b|u(y)-\hat{u}(y)|^2\,dy+2T^2(x).$$

Taking the expectation of both sides, we obtain

$$\begin{aligned}\mathbb{E}\left[|u(x)-\hat{u}(x)|^2\right] \leq & \frac{(b-a)^3 L_f^2}{8}\int_a^b \mathbb{E}\left[|u(y)-\hat{u}(y)|^2\right]dy + \mathbb{E}\left[T^2(x)\right] \\ \leq & \frac{(b-a)^4 L_f^2}{8}\max_{x\in[a,b]}\mathbb{E}\left[|u(x)-\hat{u}(x)|^2\right] + 2\max_{x\in[a,b]}\mathbb{E}\left[T^2(x)\right].\end{aligned}$$

Taking the maximum of both sides yields

$$\max_{x\in[a,b]}\mathbb{E}\left[|u(x)-\hat{u}(x)|^2\right] \leq \frac{(b-a)^4 L_f^2}{8}\max_{x\in[a,b]}\mathbb{E}\left[|u(x)-\hat{u}(x)|^2\right] + 2\max_{x\in[a,b]}\mathbb{E}\left[T^2(x)\right].$$

Therefore, for $\frac{(b-a)^4 L_f^2}{8} < 1$, we have

$$\max_{x\in[a,b]}\mathbb{E}\left[|u(x)-\hat{u}(x)|^2\right] \leq \frac{2}{1-\frac{(b-a)^4 L_f^2}{8}}\max_{x\in[a,b]}\mathbb{E}\left[T^2(x)\right]. \tag{22}$$

Next, we will estimate $\max_{x\in[a,b]}\mathbb{E}\left[T^2(x)\right]$. We note that

$$\mathbb{E}\left[T^2(x)\right] = \mathbb{E}\left[\left(\int_a^b G(x,y)g(y)dW(y) - \int_a^b G(x,y)g(y)d\hat{W}(y)\right)^2\right]. \tag{23}$$

We remark that both components of the Green's function $G_1(x,y)$ and $G_2(x,y)$ are Lipschitz continuous with respect to the variable y with Lipschitz constants $\max_{y\in[a,x]}\left|\frac{\partial G_1(x,y)}{\partial y}\right| = \max_{y\in[a,x]}\left|\frac{x-b}{b-a}\right| = 1$ and $\max_{y\in[x,b]}\left|\frac{\partial G_2(x,y)}{\partial y}\right| = \max_{y\in[x,b]}\left|\frac{x-a}{b-a}\right| = 1$, respectively. Hence, for all $y,\ z\in[a,b]$, we have

$$\left|G(x,y)-G(x,z)\right| \leq 2\left|y-z\right|.$$

Furthermore, since the functions $g(y)$ and $G(x,y)$ are bounded and Lipschitz functions, then for all $y,\ z\in[a,b]$, we have

$$\begin{aligned}& |G(x,y)g(y)-G(x,z)g(z)| \\ = & |G(x,y)g(y)-G(x,z)g(y)+G(x,z)g(y)-G(x,z)g(z)| \\ \leq & |g(y)|\,|G(x,y)g(y)-G(x,z)| + |G(x,z)|\,|g(y)-g(z)| \\ \leq & \max_{y\in[a,b]}|g(y)|\,|G(x,y)-G(x,z)| + \max_{y,z\in[a,b]}|G(x,z)|\,|g(y)-g(z)| \\ \leq & 2M_g\,|y-z| + \frac{b-a}{4}L_g\,|y-z| = \hat{L}\,|y-z|,\end{aligned}$$

where $M_g = \max_{y\in[a,b]}|g(y)|$, L_g is a Lipschitz constant for g on $[a,b]$, and $\hat{L} = 2M_g + \frac{b-a}{4}L_g$. Thus, $G(x,y)g(y)$ is also a Lipschitz function with respect to y with Lipschitz constant $\hat{L}$. Applying (11) with $\phi(y) = G(x,y)g(y)$, (23) yields, for all $x\in[a,b]$,

$$\mathbb{E}\left[T^2(x)\right] \leq (b-a)\hat{L}^2h^2.$$

Taking the maximum on both sides, we get

$$\max_{x\in[a,b]}\mathbb{E}\left[T^2(x)\right] \leq (b-a)\hat{L}^2h^2. \tag{24}$$

Now, combining (22) and (24), we obtain, for all $\epsilon > 0$,

$$\max_{x\in[a,b]}\mathbb{E}\left[|u(x)-\hat{u}(x)|^2\right] \leq \frac{2}{1-\frac{(b-a)^4 L_f^2}{8}}(b-a)\hat{L}^2h^2 = Ch^2, \tag{25}$$

which establishes (13) with $C = \frac{2}{1-\frac{(b-a)^4 L_f^2}{8}}(b-a)\hat{L}^2$. We note that (14) follows from (15) and Jensen's Inequality $\phi(E[X]) \le E[\phi(X)]$ with $\phi(X) = X^2$ and $X = u(x) - \hat{u}(x)$. Since $\max_{x\in[a,b]} |v(x)| \le \max_{i=0,1,\dots,N} |v(x_i)|$, we deduce the estimates (15) (16). $\square$

2.3. Regularity of the solution $\hat{u}$. We show that the solution $\hat{u}(x)$ to the approximate SBVP (12) has better regularity than the solution u of the original problem. In the next theorem, we state and prove the following important regularity results of the approximate solution $\hat{u}$.

Theorem 2. *Suppose that $\hat{u}$ is the solution to (12). Assume that f and g satisfy the conditions (2)-(4). We further assume that $g \in C^2[a,b]$ and $f \in C^2([a,b]\times\mathbb{R})$. Then $\hat{u} \in H^4[a,b]$. Furthermore, there exists a deterministic constant C independent of mesh size such that*

$$\mathbb{E}\left[\left\|\hat{u}^{(s)}\right\|^2\right] \le Ch^{-1}, \quad s = 0,1,2. \tag{26}$$

Finally, we have the following pointwise estimates

$$\mathbb{E}\left[\left|\hat{u}^{(s)}(x)\right|^2\right] \le Ch^{-1}, \quad s = 0,1,2,3,4, \quad x \in I_i, \quad i = 1,2,\dots,N. \tag{27a}$$

$$\mathbb{E}\left[\left|\hat{u}'(x)\right|^4\right] \le Ch^{-1}, \quad x \in I_i, \quad i = 1,2,\dots,N. \tag{27b}$$

Proof. If $f \in C^2([a,b]\times\mathbb{R})$ and $g \in C^2[a,b]$ then the right-hand side, $f(x,\hat{u}) + g(x)\frac{d\hat{W}(x)}{dx}$, of (12a) is in $H^2[a,b]$ since $\frac{d\hat{W}(x)}{dx}$ is piecewise constant on $[a,b]$. Thus, $\hat{u} \in H^4[a,b]$.

Next, we will prove (26). Multiplying (12a) by $\hat{u}$ and integrating over $[a,b]$, we get

$$\int_a^b \hat{u}''\hat{u}dx = \int_a^b \hat{u}f(x,\hat{u})dx + \int_a^b g(x)\hat{u}\frac{d\hat{W}(x)}{dx}dx.$$

Integrating by parts and using $\hat{u}(a) = \hat{u}(b) = 0$ yields

$$\|\hat{u}'\|^2 = -\int_a^b \hat{u}f(x,\hat{u})dx - \int_a^b g(x)\hat{u}\frac{d\hat{W}(x)}{dx}dx.$$

If $f \in C^2([a,b]\times\mathbb{R})$ then the function $\phi(x) = f(x,\hat{u}(x))$ is continuous on $[a,b]$. Consequently, it is bounded over $[a,b]$. Hence, we have

$$\begin{aligned}\|\hat{u}'\|^2 \le& \int_a^b |\hat{u}|\,|f(x,\hat{u})|\,dx + \int_a^b |g(x)|\,|\hat{u}|\left|\frac{d\hat{W}(x)}{dx}\right|dx\\ \le& M_f\int_a^b |\hat{u}|\,dx + M_g\int_a^b |\hat{u}|\left|\frac{d\hat{W}(x)}{dx}\right|dx,\end{aligned}$$

where $M_f = \max\limits_{x\in[a,b]} |f(x,u(x))|$ and $M_g = \max\limits_{x\in[a,b]} |g(x)|$.

Applying the Cauchy-Schwartz inequality, we get

$$\|\hat{u}'\|^2 \le M_f\sqrt{b-a}\,\|\hat{u}\| + M_g\,\|\hat{u}\|\left\|\frac{d\hat{W}(x)}{dx}\right\|. \tag{28}$$

On the other hand, since $\hat{u}(a)=0$, the Fundamental Theorem of Calculus gives

$$\begin{aligned}\left|\hat{u}(x)\right| =&\left|\hat{u}(a)+\int_a^x \hat{u}'(s)ds\right| = \left|\int_a^x \hat{u}'(s)ds\right|\\ \leq& \int_a^x \left|\hat{u}'(s)\right|ds \leq \int_a^b \left|\hat{u}'(s)\right|ds, \quad \forall\ x\in[a,b].\end{aligned}$$

Squaring both sides and using the Cauchy-Schwartz inequality, we get

$$\left|\hat{u}(x)\right|^2 \leq (b-a)\int_a^b \left|\hat{u}'(s)\right|^2 ds = (b-a)\left\|\hat{u}'\right\|^2.$$

Integrating over the interval $[a,b]$, we obtain

$$\|\hat{u}\| \leq (b-a)\left\|\hat{u}'\right\|. \tag{29}$$

Combining (28) and (29), we arrive at

$$\left\|\hat{u}'\right\|^2 \leq M_f(b-a)^{3/2}\left\|\hat{u}'\right\| + M_g(b-a)\left\|\hat{u}'\right\|\left\|\frac{d\hat{W}}{dx}\right\|,$$

which gives

$$\left\|\hat{u}'\right\| \leq M_f(b-a)^{3/2} + M_g(b-a)\left\|\frac{d\hat{W}}{dx}\right\|.$$

Applying the inequality $(A+B)^2 \leq 2A^2+2B^2$, we get

$$\left\|\hat{u}'\right\|^2 \leq 2M_f^2(b-a)^3 + 2M_g^2(b-a)^2\left\|\frac{d\hat{W}}{dx}\right\|^2. \tag{30}$$

Taking the expectation of both sides and using (10), we obtain

$$\mathbb{E}\left[\left\|\hat{u}'\right\|^2\right] \leq C_1 + C_2h^{-1} \leq Ch^{-1}. \tag{31}$$

Next, we estimate $\mathbb{E}\left[\|\hat{u}\|^2\right]$. From (29), we have

$$\|\hat{u}\|^2 \leq (b-a)^2\left\|\hat{u}'\right\|^2. \tag{32}$$

Taking the expectation and applying (31), we obtain

$$\mathbb{E}\left[\|\hat{u}\|^2\right] \leq (b-a)^2\mathbb{E}\left[\left\|\hat{u}'\right\|^2\right] \leq (b-a)^2C_1h^{-1} \leq Ch^{-1}. \tag{33}$$

Now we are ready to prove (26) for $s=2$. Multiplying (12a) by $\hat{u}''$ and integrating over $[a,b]$ yields

$$\begin{aligned}\left\|\hat{u}''\right\|^2 =& \int_a^b \hat{u}'' f(x,\hat{u})dx + \int_a^b g(x)\hat{u}''\frac{d\hat{W}(x)}{dx}dx\\ \leq& M_f\int_a^b \left|\hat{u}''\right| dx + M_g\int_a^b \left|\hat{u}''\right|\left|\frac{d\hat{W}(x)}{dx}\right| dx.\end{aligned}$$

A simple application of the Cauchy-Schwarz inequality gives

$$\left\|\hat{u}''\right\|^2 \leq M_f\sqrt{b-a}\left\|\hat{u}''\right\| + M_g\left\|\hat{u}''\right\|\left\|\frac{d\hat{W}}{dx}\right\|.$$

Consequently, we have

$$\left\|\hat{u}''\right\| \leq M_f\sqrt{b-a} + M_g\left\|\frac{d\hat{W}}{dx}\right\| \leq C_1 + C_2\left\|\frac{d\hat{W}}{dx}\right\|. \tag{34}$$

Squaring both sides, applying the inequality $(A+B)^2 \leq 2A^2+2B^2$, taking the expectation of both sides, and invoking the estimates (10), we obtain

$$\mathbb{E}\left[\|\hat{u}''\|^2\right] \leq C_3 + C_4 h^{-1} \leq C h^{-1},$$

which completes the proof of (26).

Finally, we will prove the estimate (27). Since $\hat{u}(a)=0$, we have by the Fundamental Theorem of Calculus, $\forall\, x \in [a,b]$,

$$\begin{aligned}\left|\hat{u}(x)\right| = \left|\hat{u}(a) + \int_a^x \hat{u}'(s)ds\right| = \left|\int_a^x \hat{u}'(s)ds\right| \leq \int_a^x \left|\hat{u}'(s)\right|ds \leq \int_a^b \left|\hat{u}'(s)\right|ds \\ \leq (b-a)^{1/2}\left\|\hat{u}'\right\|.\end{aligned} \tag{35}$$

Squaring both sides, taking the expectation, and applying the estimate (26) with $s=1$, we get

$$\mathbb{E}\left[\left|\hat{u}(x)\right|^2\right] \leq (b-a)\mathbb{E}\left[\|\hat{u}'\|^2\right] \leq (b-a)C_1 h^{-1} = C h^{-1},$$

which completes the proof of (27a) for $s=0$.

Next, we prove (27a) for $s=1$. Since $\hat{u}(a)=\hat{u}(b)=0$, Rolle's Theorem states that there exists $\xi \in (a,b)$ such that $\hat{u}'(\xi)=0$. Applying the Fundamental Theorem of Calculus, we write

$$\begin{aligned}\left|\hat{u}'(x)\right| =& \left|\hat{u}'(\xi) + \int_\xi^x \hat{u}''(s)ds\right| = \left|\int_\xi^x \hat{u}''(s)ds\right| \\ \leq& \int_a^b \left|\hat{u}''(s)\right|ds \leq (b-a)^{1/2}\left\|\hat{u}''\right\|, \quad \forall\, x \in [a,b].\end{aligned}$$

Taking the expectation and applying the estimate (26) with $s=2$, we obtain

$$\mathbb{E}\left[\left|\hat{u}'(x)\right|^2\right] \leq (b-a)\mathbb{E}\left[\|\hat{u}''\|^2\right] \leq (b-a)C_2 h^{-1} = C h^{-1},$$

which completes the proof of (27a) for $s=1$.

The proof of the estimate (27b) is similar to the proof of (27a) with $s=1$. Indeed, from (35), we have

$$\mathbb{E}\left[\left|\hat{u}'(x)\right|^4\right] \leq (b-a)^2\mathbb{E}\left[\|\hat{u}''\|^4\right]. \tag{36}$$

Next, we will estimate $\mathbb{E}\left[\|\hat{u}''\|^4\right]$. From (34), we have

$$\mathbb{E}\left[\|\hat{u}''\|^4\right] \leq C_3 + C_4\mathbb{E}\left[\left\|\frac{d\hat{W}}{dx}\right\|^4\right]. \tag{37}$$

Using $E[|\Delta W_i|^4] = 3h_i^2$, we have

$$\begin{aligned}\mathbb{E}\left[\left\|\frac{d\hat{W}}{dx}\right\|^4\right] =&\mathbb{E}\left[\sum_{i=1}^{N}\left(\int_{I_i}\frac{(\Delta W_i)^2}{h_i^2}dx\right)^2\right]\\ =&\mathbb{E}\left[\sum_{i=1}^{N}\frac{(\Delta W_i)^4}{h_i^2}\right] = \sum_{i=1}^{N}\frac{\mathbb{E}\left[(\Delta W_i)^4\right]}{h_i^2}\\ =&\sum_{i=1}^{N}\frac{3h_i^2}{h_i^2} = 3N \le Ch^{-1},\end{aligned}$$

since, from (7), $N \le \frac{b-a}{h_{min}} \le \frac{(b-a)K}{h} = Ch^{-1}$, where $C = (b-a)K$ is a constant independent of h.

Combining this estimate and (37) we establish (27b).

Next, we prove (27a) for $s = 2$. Equation (12a) gives $\hat{u}''(x) = f(x, \hat{u}(x)) + g(x)\frac{d\hat{W}(x)}{dx}$. Consequently, we have

$$|\hat{u}''(x)| \le M_f + M_g\frac{|\Delta W_i|}{h_i} \le C_1\left(1 + \frac{|\Delta W_i|}{h_i}\right), \quad x \in I_i, \quad i = 1, 2, \ldots, N.$$

Squaring both sides and using the inequality $(A + B)^2 \le 2A^2 + 2B^2$ yields

$$|\hat{u}''(x)|^2 \le C_2\left(1 + \frac{(\Delta W_i)^2}{h_i^2}\right), \quad x \in I_i, \quad i = 1, 2, \ldots, N.$$

Taking the expectation of both sides and using $E[(\Delta W_i)^2] = h_i$, we get

$$\mathbb{E}\left[|\hat{u}''(x)|^2\right] \le C\left(1 + \frac{h_i}{h_i^2}\right) \le Ch^{-1}, \quad x \in I_i, \quad i = 1, 2, \ldots, N.$$

Next, we will prove (27a) for $s = 3$. Differentiating (12a) with respect to x, we get

$$\text{(38)} \quad \hat{u}''' = f_x(x, \hat{u}) + \hat{u}' f_u(x, \hat{u}) + g'(x)\frac{\Delta W_i}{h_i}, \quad x \in I_i, \quad i = 1, 2, \ldots, N,$$

since $\frac{\Delta W_i}{h_i}$ is piecewise constant. Since $g \in C^2[a, b]$ and $f \in C^2([a, b] \times \mathbb{R})$, we have

$$|\hat{u}'''| \le C_1 + C_2\,|\hat{u}'| + C_3\frac{|\Delta W_i|}{h_i}.$$

Squaring both sides and using the inequality $\left(\sum_{j=1}^{n} c_i\right)^2 \le n\sum_{j=1}^{n} c_i^2$ with $n = 3$, we get

$$|\hat{u}'''|^2 \le 3C_1^2 + 3C_2^2\,|\hat{u}'|^2 + 3C_3^2\frac{|\Delta W_i|^2}{h_i^2}, \quad x \in I_i, \quad i = 1, 2, \ldots, N.$$

Taking the expectation of both sides, using (27a) with $s = 1$, and applying $E[(\Delta W_i)^2] = h_i$, we obtain

$$\mathbb{E}\left[|\hat{u}'''|^2\right] \le 3C_1^2 + 3C_2^2\mathbb{E}\left[|\hat{u}'|^2\right] + 3C_3^2\frac{\mathbb{E}\left[|\Delta W_i|^2\right]}{h_i^2} \le Ch^{-1}, \quad x \in I_i, \quad i = 1, 2, \ldots, N.$$

Finally, we will prove (27a) for $s = 4$. Differentiating (38) with respect to x, we get

$$\hat{u}^{(4)} = f_{xx}(x, \hat{u}) + 2\hat{u}' f_{xu}(x, \hat{u}) + \hat{u}'' f_u(x, \hat{u}) + (\hat{u}')^2 f_{uu}(x, \hat{u}) + g''(x)\frac{\Delta W_i}{h_i}, \quad x \in I_i, \quad i = 1, 2, \ldots, N.$$

Since $g \in C^2[a, b]$ and $f \in C^2([a, b] \times \mathbb{R})$, we have

$$\left|\hat{u}^{(4)}\right| \leq C_4 + C_5 |\hat{u}'| + C_6 |\hat{u}''| + C_7 |\hat{u}'|^2 + C_8 \frac{|\Delta W_i|}{h_i}.$$

Squaring both sides and using the inequality $\left(\sum_{j=1}^{n} c_i\right)^2 \leq n \sum_{j=1}^{n} c_i^2$ with $n = 5$ gives

$$\left|\hat{u}^{(4)}\right|^2 \leq 5C_4^2 + 5C_5^2 |\hat{u}'|^2 + 5C_6^2 |\hat{u}''|^2 + 5C_7^2 |\hat{u}'|^4 + 5C_8^2 \frac{|\Delta W_i|^2}{h_i^2}, \quad x \in I_i, \quad i = 1, 2, \ldots, N.$$

Taking the expectation of both sides, using (27a) with $s = 1, 2$, (27b), and applying $E[(\Delta W_i)^2] = h_i$, we obtain, for all $x \in I_i$, $i = 1, 2, \ldots, N$,

$$\mathbb{E}\left[\left|\hat{u}^{(4)}\right|^2\right] \leq 5C_4^2 + 5C_5^2 \mathbb{E}\left[|\hat{u}'|^2\right] + 5C_6^2 \mathbb{E}\left[|\hat{u}''|^2\right] + 5C_7^2 \mathbb{E}\left[|\hat{u}'|^4\right] + 5C_8^2 \frac{\mathbb{E}\left[|\Delta W_i|^2\right]}{h_i^2} \leq C h^{-1},$$

which completes the proof of the theorem. ☐

Remark 2.2. *We remark that the exact solution u of the original SBVP (1) has poor regularity since W is nowhere differentiable. However, the solution $\hat{u}(x)$ of the approximate SBVP (12) has better regularity ($\hat{u} \in H^4(a, b)$). Thus, standard numerical methods such as the finite difference method can be applied to approximate its approximate solution. Furthermore, the standard analysis techniques in the finite difference method can be applied.*

2.4. The proposed stochastic finite difference (SFD) scheme. We remark that (12) becomes a two-point BVP with random inhomogeneous term. Hence, we can apply the standard finite difference method for deterministic two-point BVPs [10, 21]. The stochastic finite difference method described below is analogous to the standard finite difference method for numerical solution of deterministic BVPs.

As before, we divide the interval $[a, b]$ into N intervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, N$. The points need not to be equally spaced. For simplicity, we assume the points are uniformly distributed

$$x_i = a + hi, \quad i = 0, 1, \ldots, N, \quad h = \frac{b - a}{N}.$$

We use the central difference formula to approximate $\hat{u}''(x_i)$

(39)

$$\hat{u}''(x_i) = \frac{\hat{u}(x_i + h) - 2\hat{u}(x_i) + \hat{u}(x_i - h)}{h^2} - \frac{h^2}{12}\hat{u}^{(4)}(\xi_i), \quad \xi_i \in (x_i - h, x_i + h).$$

Substituting the centered-difference formulas (39) into (12), we get, for $i = 1, 2, \dots,$ $N-1$,

$$\frac{\hat{u}(x_i+h) - 2\hat{u}(x_i) + \hat{u}(x_i-h)}{h^2} - \frac{h^2}{12}\hat{u}^{(4)}(\xi_i) = f\left(x_i, \hat{u}(x_i)\right) + g(x_i)\frac{\eta_i}{\sqrt{h}}. \tag{40}$$

Neglecting the term involving ξ_i and using u_i to denote the approximate value of $\hat{u}(x_i)$, we obtain the following discrete finite-difference method

$$\begin{aligned} u_0 &= 0, \\ \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} &= f\left(x_i, u_i\right) + g(x_i)\frac{\eta_i}{\sqrt{h}}, \quad i = 1, 2, \dots, N-1, \\ u_N &= 0. \end{aligned} \tag{41}$$

Multiplying throughout by h^2 and letting $g_i = g(x_i)$, the nonlinear system (41) becomes

$$\begin{aligned} u_0 &= 0, \\ u_{i+1} - 2u_i - h^2 f\left(x_i, u_i\right) + u_{i-1} - h^{3/2} g_i \eta_i &= 0, \quad i = 1, 2, \dots, N-1. \\ u_N &= 0. \end{aligned} \tag{42}$$

The nonlinear system (42) can be written as

$$\mathbf{F}(\mathbf{u}) = \mathbf{0}, \tag{43a}$$

where $\mathbf{u} = [u_1, u_2, \dots, u_{N-1}]^t$, $\mathbf{F} = [f_1, f_2, \dots, f_{N-1}]^t$ with

$$\begin{aligned} f_1 &= u_2 - 2u_1 - h^2 f\left(x_1, u_1\right) - h^{3/2} g_1 \eta_1, \\ f_2 &= u_3 - 2u_2 - h^2 f\left(x_2, u_2\right) + u_1 - h^{3/2} g_2 \eta_2, \\ &\vdots \\ f_{N-2} &= u_{N-1} - 2u_{N-2} - h^2 f\left(x_{N-2}, u_{N-2}\right) + u_{N-3} - h^{3/2} g_{N-2} \eta_{N-2}, \\ f_{N-1} &= -2u_{N-1} - h^2 f\left(x_{N-1}, u_{N-1}\right) + u_{N-2} - h^{3/2} g_{N-1} \eta_{N-1}. \end{aligned} \tag{43b}$$

Theorem 3. *Suppose that the assumption (3) is satisfied. Then the system (43) has a unique solution.*

Proof. The proof is elementary and can be found in [24]. □

2.5. Implementation. The nonlinear system (43) can be solved using Newton's method for nonlinear systems to approximate the solution to this system

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} - J^{-1}(\mathbf{u}^{(k)})\mathbf{F}(\mathbf{u}^{(k)}), \quad k = 0, 1, ...$$

A sequence of iterates $\left\{\mathbf{u}^{(k)} = [u_1^{(k)}, u_2^{(k)}, \dots, u_{N-1}^{(k)}]^t\right\}$ is generated that converges to the solution of system (43), provided that the initial approximation $\mathbf{u}^{(0)}$ is sufficiently close to the solution $\mathbf{u}$ and that the Jacobian matrix for the system is nonsingular. For system (43), the Jacobian matrix $J(\mathbf{u})$ is tridiagonal with ij-th entry

$$J_{i,j}(\mathbf{u}) = \begin{cases} 1, \quad i = j-1, \ j = 2, 3, \cdots, N-1, \\ -2 - h^2 f_u\left(x_i, u_i\right), \quad i = j, \ j = 1, 2, \cdots, N-1, \\ 1, \quad i = j+1, \ j = 2, 3, \cdots, N-2. \end{cases} \tag{44}$$

Newton's method for nonlinear systems requires that at each iteration, the $(N-1) \times (N-1)$ linear system

$$J(\mathbf{u}^{(k)})\mathbf{y}^{(k)} = \mathbf{F}(\mathbf{u}^{(k)}),$$

be solved for $\mathbf{y}^{(k)}$ since

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} - \mathbf{y}^{(k)}, \quad k = 0, 1, \cdots$$

Because $J(\mathbf{u})$ is tridiagonal, the tridiagonal linear system can be solved efficiently using Crout factorization for tridiagonal linear systems. Note that ΔW_i can be simulated using $\sqrt{h_i}\eta_i$, where $\eta_i \sim \mathcal{N}(0,1)$ is normally distributed with mean zero and variance one.

Remark 2.3. *Other boundary conditions can also be handled. For instance, if the mixed Dirichlet-Neumann boundary conditions of the form $\hat{u}(a) = \alpha$, $\hat{u}'(b) = \beta$ are used then the resulting system is*

$$\begin{aligned} u_0 &= \alpha, \\ u_{i+1} - 2u_i - h^2 f(x_i, u_i) + u_{i-1} - h^{3/2} g_i \eta_i &= 0, \quad i = 1, 2, \ldots, N, \\ \frac{u_{N+1} - u_{N-1}}{2h} &= \beta. \end{aligned} \tag{45}$$

3. Convergence analysis

In this section, we prove that the finite difference solution u_i, $i = 1, 2, \ldots, N$ converges to the exact solution $u(x_i)$ of the original problem (1). First, we prove that the SFD solution provide accurate approximations to the solution $\hat{u}$ of (12). Then we deduce that the SFD solution converges to the exact solution u of (1).

Remark 3.1. *The proofs in this paper require that the function f is smooth and f_u is bounded and positive on the set D. These assumptions are usually the hypotheses of the existence and uniqueness theorem for the deterministic BVP $u'' = f(x,u)$, $x \in (a,b)$, $u(a) = \alpha$, $\quad u(b) = \beta$.*

Unlike the deterministic case, the required regularity conditions are not satisfied for (1) to carry out the standard error estimates of the proposed finite difference scheme. However, the Wiener process W is approximated by piecewise linear random processes $\hat{W}$ to facilitate the convergence proof for the SFD method. In other words, the approximation of the Wiener process is used to improve the regularity of the solution to the approximated SBVP (12), so that standard analysis techniques in the finite difference method can be applied. More precisely, Theorem 2 indicates that the solution $\hat{u}(x)$ of the approximated SBVP (12) has certain regularity which allows us to obtain an error estimate.

It follows from Theorem 2 that the SBVP (12) possesses a unique solution $\hat{u}(x) \in H^4(a,b)$. Consequently, we can apply the estimate (27). In the next theorem, we prove that $\mathbb{E}\left[\left|\hat{u}(x_i) - u_i\right|^2\right] = \mathcal{O}(h^3)$. Thus, the SFD solution u_i converges to $\hat{u}$ as $h \to 0$ in the mean-square sense with order of convergence 3/2.

Theorem 4. *Let $\hat{u}$ be the exact solution of (12). Suppose that the assumptions of Theorem 2 are satisfied. Let u_i be the SFD solution defined in (42), then there exists a positive constant C independent of h such that*

$$\max_{i=0,1,\ldots,N} \mathbb{E}\left[\left|\hat{u}(x_i) - u_i\right|^2\right] \leq Ch^3. \tag{46}$$

$$\max_{i=0,1,\ldots,N} \left|\mathbb{E}\left[\hat{u}(x_i)\right] - \mathbb{E}\left[u_i\right]\right|^2 \leq Ch^3. \tag{47}$$

Proof. Subtracting (41) from (40) and letting $e_i = \hat{u}(x_i) - u_i$, we obtain

$$\frac{e_{i+1} - 2e_i + e_{i-1}}{h^2} = f(x_i, \hat{u}(x_i)) - f(x_i, u_i) + h^2 \gamma_i, \quad \text{where } \gamma_i = \frac{1}{12}\hat{u}^{(4)}(\xi_i).$$

Using the the Mean Value Theorem, we get

$$\frac{e_{i+1}-2e_i+e_{i-1}}{h^2}=e_i f_u\left(x_i,\hat{v}_i\right)+h^2\gamma_i, \tag{48}$$

where $\hat{v}_i$ lies between $\hat{u}(x_i)$ and u_i.

Collecting terms, (48) can be written as

$$\left(2+h^2 f_u\left(x_i,\hat{v}_i\right)\right)e_i=e_{i-1}+e_{i+1}-h^4\gamma_i. \tag{49}$$

Let $e=\max(|e_1|,\ldots,|e_{N-1}|)$ and pick the index i such that $|e_i|=e$. Then from (49), we get

$$\left|2+h^2 f_u\left(x_i,\hat{v}_i\right)\right||e_i|\le|e_{i-1}|+|e_{i+1}|+h^4|\gamma_i|.$$

Since $0<K_1\le f_u(x,u)$, we have

$$\left(2+K_1h^2\right)\left||e_i|\le|e_{i-1}|+|e_{i+1}|+h^4|\gamma_i|.\right.$$

Since $|e_i|=e$, we immediately have $\left(2+K_1h^2\right)e\le e+e+h^4|\gamma_i|$, which gives $K_1h^2e\le h^4|\gamma_i|$. Consequently, we get

$$e\le\frac{|\gamma_i|}{K_1}h^2. \tag{50}$$

Squaring both sides, taking the expectation, and applying the estimate (27a), we obtain

$$\mathbb{E}\left[e^2\right]\le\frac{\mathbb{E}\left[|\gamma_i|^2\right]}{K_1^2}h^4=\frac{\mathbb{E}\left[\left|\hat{u}^{(4)}(\xi_i)\right|^2\right]}{144K_1^2}h^4\le\frac{C_1}{144K_1^2}h^3.$$

Thus, for all $i=0,1,\ldots,N$, we have

$$\mathbb{E}\left[\left|\hat{u}(x_i)-u_i\right|^2\right]\le Ch^3.$$

Taking the maximum yields (46). Applying Jensen's Inequality $\phi(E[X])\le E[\phi(X)]$ with $\phi(X)=X^2$ and $X=\hat{u}(x_i)-u_i$ and invoking (46) yields (47). □

Since u_i converges to $\hat{u}(x_i)$ and $\hat{u}(x_i)$ converges to $u(x_i)$ in the mean-square sense, the SFD solution u_i converges to the exact solution $u(x_i)$ of the original BVP (1) in the mean-square sense. We state this result in the following corollary.

Corollary 3.1. *Suppose that the assumptions of Theorem 4 are satisfied. Let u be the exact solution of (1). Let u_i be the SFD solution defined in (42), then there exists a positive constant C independent of h such that*

$$\max_{i=0,1,\ldots,N}\mathbb{E}\left[\left|u(x_i)-u_i\right|^2\right]\quad\le\quad Ch^2. \tag{51}$$

$$\max_{i=0,1,\ldots,N}\left|\mathbb{E}\left[u(x_i)\right]-\mathbb{E}\left[u_i\right]\right|^2\quad\le\quad Ch^2. \tag{52}$$

Proof. Using the inequality $(a+b)^2\le2(a^2+b^2)$, we have

$$\begin{aligned}\left|u(x_i)-u_i\right|^2=&\left|(u(x_i)-\hat{u}(x_i))+(\hat{u}(x_i)-u_i)\right|^2\\\le&2\left|u(x_i)-\hat{u}(x_i)\right|^2+2\left|\hat{u}(x_i)-u_i\right|^2.\end{aligned}$$

Taking the expectation of both sides and using the estimates (15) and (46), we get, for all $i = 0, 1, \ldots, N$,

$$\begin{aligned} &\max_{i=0,1,\ldots,N} \mathbb{E}\left[|u(x_i) - u_i|^2\right] \\ \leq &2 \max_{i=0,1,\ldots,N} \mathbb{E}\left[|u(x_i) - \hat{u}(x_i)|^2\right] + 2 \max_{i=0,1,\ldots,N} \mathbb{E}\left[|\hat{u}(x_i) - u_i|^2\right] \\ \leq &2C_1 h^2 + 2C_2 h^3 \leq Ch^2, \end{aligned}$$

which completes the proof of (51). Similarly, we have

$$\begin{aligned} \left|\mathbb{E}\left[u(x_i)\right] - \mathbb{E}\left[u_i\right]\right|^2 = &\left|\mathbb{E}\left[u(x_i)\right] - \mathbb{E}\left[\hat{u}(x_i)\right] + \mathbb{E}\left[\hat{u}(x_i)\right] - \mathbb{E}\left[u_i\right]\right|^2 \\ \leq &2\left|\mathbb{E}\left[u(x_i)\right] - \mathbb{E}\left[\hat{u}(x_i)\right]\right|^2 + 2\left|\mathbb{E}\left[\hat{u}(x_i)\right] - \mathbb{E}\left[u_i\right]\right|^2. \end{aligned}$$

Taking the maximum of both sides and using the estimates (16) and (47) we establish (52). □

4. Numerical experiments

In this section, we present several numerical examples to validate the convergence of the proposed SFD method for solving stochastic two-point boundary-value problems. For simplicity, we consider uniform meshes obtained by subdividing the computational domain $[a, b]$ into N subintervals, where $N = 2^n$, $n = 2, 3, \ldots, 10$. For each step size $h = \frac{b-a}{N}$, we perform M runs with different sample paths of the Brownian motion W. In our experiments, we observed similar convergence results when using $M = 1,000$, $M = 10,000$, $M = 100,000$ and $M = 1,000,000$. To save space, we only include results using $M = 10,000$. In our numerical experiments, we used a random number generator to produce independent pseudo-random numbers from the distribution $\mathcal{N}(0, 1)$. We performed several runs of code with different random seeds so that we have results from many sample paths of the W from which to collect the desired results. All expected values are approximated by computing averages over $M = 10,000$ trials.

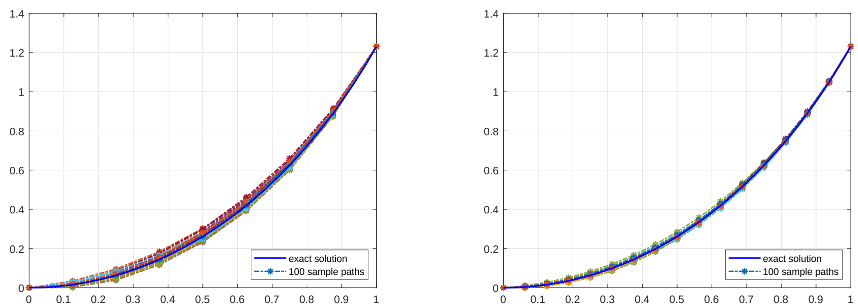

Figure 1: The mean solution $E[u(x_i)]$ and 100 sample paths obtained using the SFD method for Example 4.1 using $N = 8$ (left) and $N = 16$ (right).

Example 4.1. Consider the second-order nonlinear Bratu problem

(53) $u'' = 2e^u + \dot{W}(x), \quad x \in (0, 1), \quad u(0) = 0, \quad u(1) = -2\ln(\cos(1)).$

The exact expected solution is $\mathbb{E}[u(x)] = -2\ln(\cos(x))$. We apply the proposed SFD method (42) to solve (53) on a uniform mesh having $N = 2^n,\ n = 2, 3, \ldots, 10$ elements. In Figure 1 we present the exact mean solution $\mathbb{E}[u(x)]$ and 100 sample paths using $N = 8, 16$. The exact mean value $\mathbb{E}[u(x)]$ and the mean of 10,000 sample paths using $N = 8$ and $N = 16$ are shown in Figure 2. In Figure 3, we present the errors $\mathbb{E}[u(x_i)] - \mathbb{E}[u_i]$, where $\mathbb{E}[u_i]$ is the average of 10,000 simulations. In Table 1, we present the maximum errors $\max_{i=0,1,\ldots,N} \left|\mathrm{E}[u(x_i)] - \mathbb{E}[u_i]\right|^2$ and their orders of convergence. These results suggest $\mathcal{O}(h^2)$ convergence rate in the mean-square sense. This is in full agreement with the theoretical result.

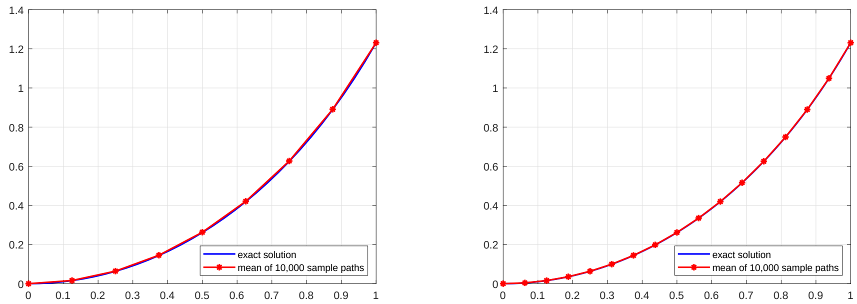

Figure 2: Mean solution and the mean of 10,000 sample paths for Example 4.1 using $N = 8$ (left) and $N = 16$ (right).

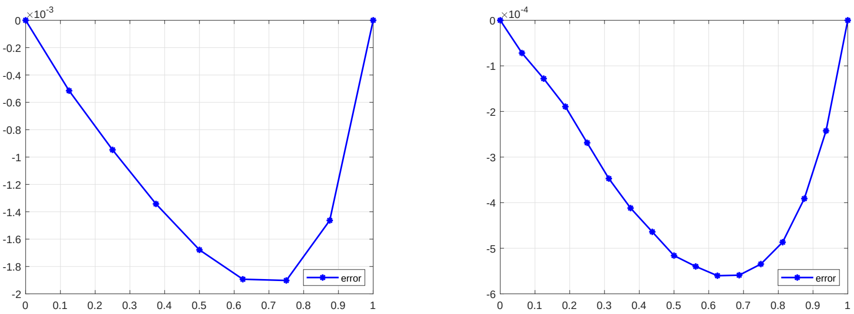

Figure 3: The error $E[u(x_i)] - E[u_i]$ for Example 4.1 using $N = 8$ (left) and $N = 16$ (right). $E[u_i]$ is obtaining by averaging the solution of 10,000 simulations.

Example 4.2. Consider the stochastic two-point BVP

$$\begin{cases} u'' = -\sin(u') - \sqrt{1+u^2} - 4\sin(2x) + \sin(2\cos(2x)) \\ \qquad + \sqrt{1+\sin^2(2x)} + \dot{W}(x), \\ u(0) = u(\pi) = 0. \end{cases} \tag{54}$$

It is easy to verify that the exact mean solution is given by

$$\mathbb{E}[u(x)] = \sin(2x), \quad x \in [0, \pi].$$

Table 1. The errors $\max_{i=0,\dots,N} \left|\mathrm{E}\left[u(x_i)\right] - \mathbb{E}\left[u_i\right]\right|^2$ and their orders of convergence for Example 4.1 on uniform meshes having $N = 2^n,\ n = 2, \dots, 10$ elements using $M = 10,000$ simulations.

N	$\max_{i=0,\dots,N} \left|\mathrm{E}\left[u(x_i)\right] - \mathbb{E}\left[u_i\right]\right|^2$	order
4	1.0988e-03	NA
8	2.6744e-04	2.0386
16	6.5413e-05	2.0316
32	1.6277e-05	2.0067
64	4.0663e-06	2.0011
128	1.0178e-06	1.9983
256	2.5452e-07	1.9996
512	6.3639e-08	1.9998
1024	1.5911e-08	1.9999

We solve (54) using the proposed SFD method (42) on a uniform mesh having $N = 2^n$, $n = 2, 3, \dots, 10$ elements. In Figure 4 we present the exact mean solution $\mathbb{E}\left[u(x)\right]$ and 100 sample paths using $N = 8$ and $N = 16$. The exact mean value $\mathbb{E}\left[u(x)\right]$ and the mean of 10,000 sample paths using $N = 8$ and $N = 16$ are shown in Figure 5. In Figure 6, we use 10,000 simulations to show the errors $\mathbb{E}\left[u(x_i)\right] - \mathbb{E}\left[u_i\right]$ using $N = 8$ and $N = 16$. In Table 2, we present the maximum errors $\max_{i=0,1,\dots,N} \left|\mathrm{E}\left[u(x_i)\right] - \mathbb{E}\left[u_i\right]\right|^2$ as well as their orders of convergence. Again, these results suggest $\mathcal{O}(h^2)$ convergence rate in the mean-square sense as the theory predicts. This is in full agreement with the theoretical result. We conclude that our proposed SFD scheme is convergent with order one in the mean-square sense.

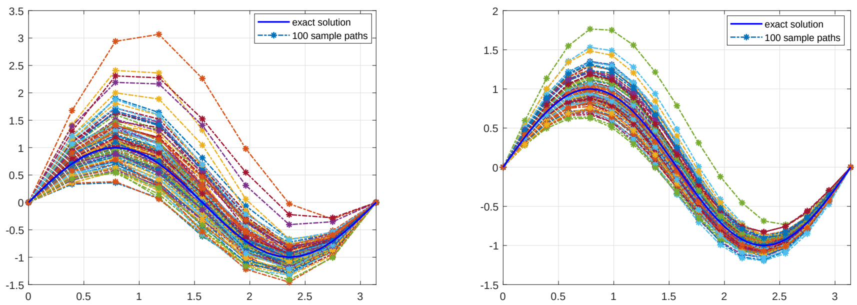

Figure 4: The mean solution $E[x]$ and 100 sample paths obtained using the SFD method for Example 4.2 using $N = 8$ (left) and $N = 16$ (right).

5. Concluding remarks

In this paper, we presented a stochastic finite difference (SFD) method for numerically solving nonlinear stochastic boundary-value problems (SBVPs) driven by additive white noises. We first constructed an approximate SBVP by replacing the white noise process with its piecewise constant approximation. The solution to the

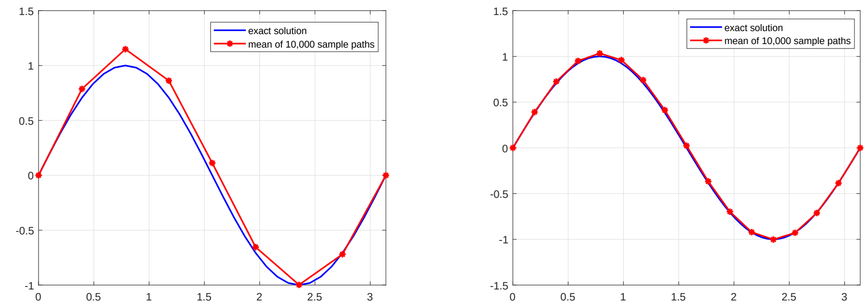

Figure 5: Mean solution and the mean of 10,000 sample paths for Example 4.2 using $N = 8$ (left) and $N = 16$ (right).

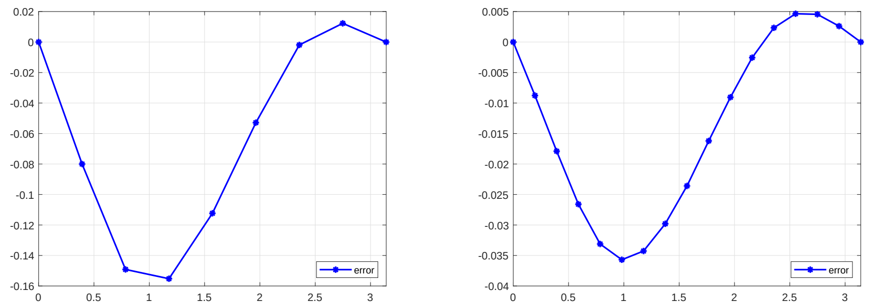

Figure 6: The error $E[u] - E[u_i]$ for Example 4.2 using $N = 8$ (left) and $N = 16$ (right). $E[u_i]$ is obtaining by averaging the solution of 10,000 simulations.

TABLE 2. The errors $\max\limits_{i=0,\ldots,N} \left|\mathrm{E}\left[u(x_i)\right] - \mathbb{E}\left[u_i\right]\right|^2$ and their orders of convergence for Example 4.2 on uniform meshes having $N = 2^n,\ n = 2, \ldots, 10$ elements using $M = 10,000$ simulations.

N	$\max\limits_{i=0,\ldots,N} \left\|\mathrm{E}\left[u(x_i)\right] - \mathbb{E}\left[u_i\right]\right\|^2$	order
4	5.3574e+00	NA
8	3.1915e-01	4.0692
16	6.8026e-02	2.2301
32	1.5299e-02	2.1527
64	3.7495e-03	2.0286
128	9.1528e-04	2.0344
256	2.2722e-04	2.0101
512	5.6379e-05	2.0109
1024	1.4062e-05	2.0034

new approximate SBVP is shown to converge to the solution of the original problem at $\mathcal{O}(h)$ in the mean-square sense. Furthermore, we proved that the solution to the approximate SBVP has better regularity conditions. The regularity conditions allowed us to apply the standard analysis techniques in the finite difference method. We proved pointwise convergence at all nodes. The order of convergence is proved to be $\mathcal{O}(h)$ in the mean-square sense, when the second-order accurate three-point formula to approximate the second derivative is used. Our numerical examples suggested that the proposed SFD scheme has optimal convergence order. We are currently investigating SFD methods for stochastic partial differential equations.

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