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ANALYSIS OF A GALERKIN FINITE ELEMENT METHOD APPLIED TO A SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEM IN THREE DIMENSIONS

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Abstract. We consider a linear singularly perturbed reaction-diffusion problem in three dimensions and its numerical solution by a Galerkin finite element method with trilinear elements. The problem is discretised on a Shishkin mesh with N intervals in each coordinate direction. Derivation of an error estimate for such a method is usually based on the (Shishkin) decomposition of the solution into distinct layer components. Our contribution is to provide a careful and detailed analysis of the trilinear interpolants of these components. From this analysis it is shown that, in the usual energy norm the errors converge at a rate of $\mathcal{O}(N^{-2}+\varepsilon^{1/2}N^{-1}\ln N)$. This is validated by numerical results.

Key words. Reaction-diffusion, finite element, Shishkin mesh, three-dimensional.

1. Introduction

Consider the following three-dimensional singularly perturbed reaction-diffusion problem posed on the unit cube

$$
\begin{aligned}
Lu &:= -\varepsilon^2\Delta u + bu = f \quad \text{in } \Omega := (0,1)^3,\\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where the reaction coefficient $b(x,y,z) \geq 2\beta^2$, and β is a positive constant. In the case of interest, the diffusion parameter, ε, can be arbitrarily small, i.e. $0 < \varepsilon \ll 1$, and so the problem (1) is singularly perturbed. As such, its solution typically exhibits layers of width $\mathcal{O}(\varepsilon)$ along the boundary of the domain, $\partial\Omega$.

It is well known that computing numerical solutions to singularly perturbed problems presents many difficulties. Solutions to these problems tend to be anisotropic in nature on regions along the boundary or interior of the domain. Over the course of the past five decades there have been many advances in devising specialised numerical schemes to deal with this phenomenon. Many of these schemes fall into the category of "*fitted mesh*" methods, where a standard discretisation is applied on a specially designed non-uniform mesh.

The idea of using non-uniform meshes to solve singularly perturbed problems was first introduced in 1969 by Bakhvalov [4]. However, it wasn't until the early 1990s, with the introduction of the piecewise uniform mesh of Shishkin [19, 20], that fitted mesh methods gained major prominence in the literature. In particular, the application of finite difference methods to the one- and two-dimensional analogues of (1) is well understood, see, e.g., [7, 17].

Finite element methods (FEMs) for one- and two-dimensional singularly perturbed reaction-diffusion problems, discretised on Shishkin meshes, are also well documented in the literature. In the one-dimensional setting, Sun and Stynes [22]

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prove almost optimal convergence using piecewise polynomial elements, see also [10, Thm. 6.6]. In two dimensions, Li and Navon [8] provide the first numerical analysis for a Galerkin FEM applied to a singularly perturbed reaction-diffusion problem. They prove almost second-order convergence in the L^2-norm. Using piecewise polynomial elements of order k, and quantifying the error in the energy norm, Apel [3] proves convergence of $\mathcal{O}(N^{-k-1} + \varepsilon^{1/2} N^{-k} (\ln N)^{k+1})$. Using piecewise bilinear elements, Liu et al. [11] prove convergence in the energy norm of $\mathcal{O}(N^{-2} + \varepsilon^{1/2} N^{-1} \ln N)$. In that same paper, the authors provide the first full numerical analysis of a sparse grid FEM applied to a singularly perturbed problem. For a two-scale sparse grid method applied to a reaction-diffusion problem they prove that the method converges at the same rate as the standard Galerkin FEM. Madden and Russell [15]. extend the results of [11] and prove convergence of a multiscale sparse grid FEM applied to the same problem.

The analysis of three-dimensional singularly perturbed reaction-diffusion problems has received, comparatively, little attention. The work of most relevance to this paper is that of Shishkin and Shishkina [21], which provides a valuable solution decomposition in three dimensions. Crucial bounds on the derivatives of each of the solution decomposition components are also given. Chadha and Kopteva [5] provide maximum norm *a posteriori* error estimates for a finite difference method applied to a semi-linear reaction-diffusion problem in three dimensions. We also make note of the work of Lopez and co-authors [12, 13, 14].

The numerical solution of three dimensional problems is computationally demanding, which motivates the use of sparse grid techniques, such as those referred to above. In particular, the authors have devised and analysed a two-scale sparse grid method for (1) [18]. However, that analysis establishes the difference between the sparse grid and standard Galerkin solutions, so, for completeness, a full analysis of the latter method is required; this paper provides that. It does so by employing a Shishkin decomposition of the solution into distinct layer components, established by Shishkin and Shishkina [21]. We then provide a careful and detailed analysis of the trilinear interpolants of these components. From this, the numerical analysis of the method follows. It should be noted that the interpolation results given here also form the basis for the analysis of other techniques, such as the balanced norm method of Lin and Stynes [9] and the FOSLS-type method of Adler et al. [1].

This paper is organised as follows. In Section 2.1 we present a canonical example of a solution to (1), which motivates a three-dimensional piecewise uniform mesh. The Shishkin decomposition is presented in Section 2.2, and standard bounds on the components are reported. The heart of this paper is in Section 3, where we give a detailed analysis of the decomposition components in the energy norm. The FEM is given in Section 4, and the error estimate is readily deduced from the preceding interpolation analysis. Numerical results, presented in Section 5 verify that the theoretical results are sharp.

Notation. We use the following standard notation for function spaces and norms (see, e.g., [6]):

- let $\mathcal{C}^p(\Omega)$ be the space of all real-valued functions, v, defined on Ω, such that v, and all its partial derivatives up to order p, are continuous on Ω;
- $\mathcal{C}^{m,\alpha}(\Omega) = \{v \in \mathcal{C}^m(\bar{\Omega}); \forall \beta, |\beta| = m, \exists C_\beta, \forall x, y \in \Omega, |\partial^\beta v(x) - \partial^\beta v(y)| \leq C_\beta \|x - y\|^\alpha\}$;

- the space $L^p(\Omega) := \{v : \int_\Omega |v|^p < \infty\}$;
- the Sobolev space $W^{m,p}(\Omega) := \{v \in L^p(\Omega); \forall\alpha, |\alpha| \le m, \partial^\alpha v \in L^p(\Omega)\}$;
- let T be a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that
  - $Tu = u|_{\partial\Omega}, \quad u \in W^{1,p}(\Omega) \cap \mathcal{C}(\bar{\Omega})$,
  - $\|Tu\|_{L^p(\partial\Omega)} \le C(p,\Omega)\|u\|_{W^{1,p}(\Omega)}, \quad u \in W^{1,p}(\Omega)$,

  where Tu is the *trace* of u.
- the spaces $H^1(\Omega) := \{v \in L^2(\Omega); \partial v \in L^2(\Omega)\}$, and $H_0^1(\Omega) := \{v \in H^1(\Omega); Tu = 0\}$;
- the norms $\|v\|_{0,\Omega} := \left(\int_\Omega |v|^2 dx\right)^{1/2}, \quad \|v\|_{4,\Omega} := \left(\sum_{|\alpha|\le 4} \int_\Omega |\partial^\alpha v|^2 dx\right)^{1/2}$,
  $\|v\|_{0,\infty,\Omega} := \operatorname{ess.sup}_{x\in\Omega} |v(x)|$, and $\|v\|_{1,\infty,\Omega} := \max_{|\alpha|\le 1}\{\operatorname{ess.sup}_{x\in\Omega} |\partial^\alpha v(x)|\}$.

We use $B(\cdot,\cdot)$ to denote the bilinear form associated with the reaction-diffusion problem (1), that is

$$B(u,v) := \varepsilon^2(\nabla u, \nabla v) + (bu, v).$$

The corresponding energy norm is

$$\|u\|_\varepsilon := \{\varepsilon^2\|\nabla u\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2\}^{1/2}. \tag{2}$$

Throughout this paper we shall use the letter C, with or without subscript, to denote a generic positive constant that is independent of the diffusion parameter ε and the discretisation parameter N. It may stand for different values in different places.

2. Shishkin mesh and decomposition

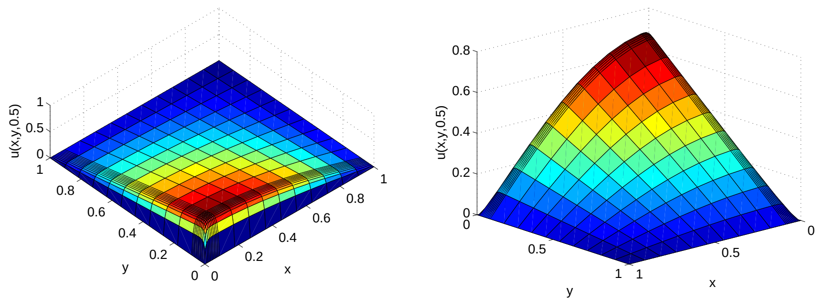
2.1. A three-dimensional piecewise uniform mesh. Solutions to three-dimen -sional problems are, by their nature, more difficult to visualise than their one- and two-dimensional counterparts. Furthermore, since the problem is singularly perturbed its solution possesses layers along the boundary of the domain. In an attempt to provide some insight into the nature of the layers, we consider the following sample problem, taken from Chadha and Kopteva [5].

Example 1. *In* (1)*, set* $b \equiv 1$ *and* f *such that*

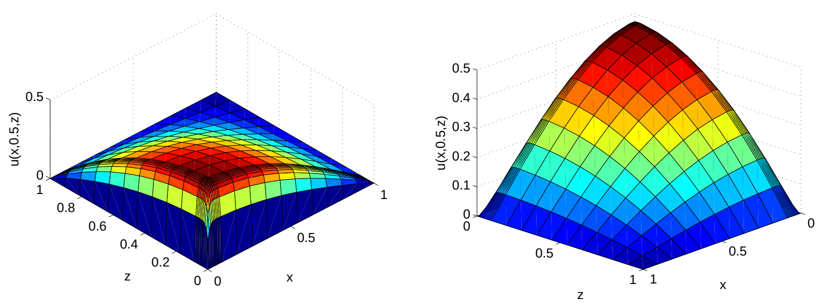
$$\begin{aligned} u = &\left(\cos\left(\frac{\pi x}{2}\right) - \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right)\left(1 - y - \frac{e^{-y/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right) \\ &\cdot \left(1 - z^2 - \frac{e^{-z/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right). \end{aligned} \tag{3}$$

This problem exhibits $1D$ exponential layers near the faces of the domain, $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$, as well as 2D layers near the edges, $(0, 0, z)$, $(0, y, 0)$ and $(x, 0, 0)$, and a $3D$ layer at the origin $(0, 0, 0)$. We visualise u in Figure 1 by taking cross-sections at $z = 1/2$, $y = 1/2$ and $x = 1/2$ in sub-plots (a), (b) and (c), respectively.

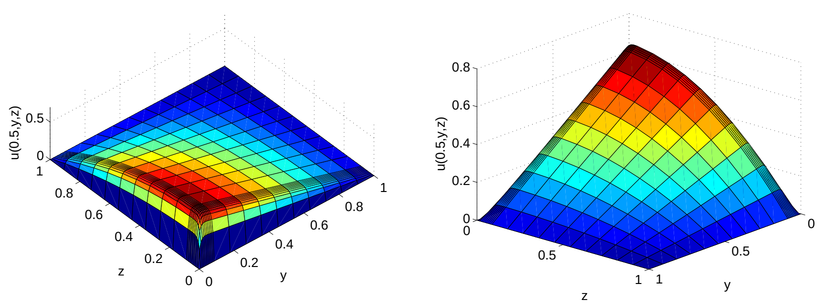
Remark 2. *The problem that has* (3) *as its solution is artificially simplified. In general, solutions to* (1) *feature six* $1D$*, twelve* $2D$ *and eight* $3D$ *layers. Therefore, when the interior of the domain is included, there are 27 distinct regions to be analysed. However, Example 1 captures the essence of the problem: it features* $1D$*,* $2D$ *and* $3D$ *layers, and so possesses all the mathematical complexity of the most general version of* (1)*, but the amount of notation and repetitive analysis is greatly*



(A) Solution to Test Problem (3), with the z-axis fixed at $z = 1/2$.



(B) Solution to Test Problem (3), with the y-axis fixed at $y = 1/2$.



(C) Solution to Test Problem (3), with the x-axis fixed at $x = 1/2$.

FIGURE 1. Solution to Test Problem (3).

reduced. Extending the mesh to resolve all 26 possible layers in the solution to (1) *is trivial computationally, but would greatly increase the amount of technical mathematical arguments, without providing any new insights.*

Motivated by Example 1 we now define and label the sub-regions of the domain, Ω. To start, faces of the boundary, $\partial\Omega$, are denoted by

(4)

$$\Gamma_1 := \{(x,0,z)|0 \le x \le 1, \quad 0 \le z \le 1\}, \quad \Gamma_2 := \{(0,y,z)|0 \le y \le 1, \quad 0 \le z \le 1\},$$
$$\Gamma_3 := \{(x,y,0)|0 \le x \le 1, \quad 0 \le y \le 1\}, \quad \Gamma_4 := \{(x,1,z)|0 \le x \le 1, \quad 0 \le z \le 1\},$$
$$\Gamma_5 := \{(1,y,z)|0 \le y \le 1, \quad 0 \le z \le 1\}, \quad \Gamma_6 := \{(x,y,1)|0 \le x \le 1, \quad 0 \le y \le 1\}.$$

Let $E_1 := \Gamma_1 \cap \Gamma_3$, $E_2 := \Gamma_2 \cap \Gamma_3$, $E_3 := \Gamma_1 \cap \Gamma_2$ be the three edges that intersect at the point $(0, 0, 0)$, and let $c := \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ denote the corner at the point $(0, 0, 0)$. See Figure 2.

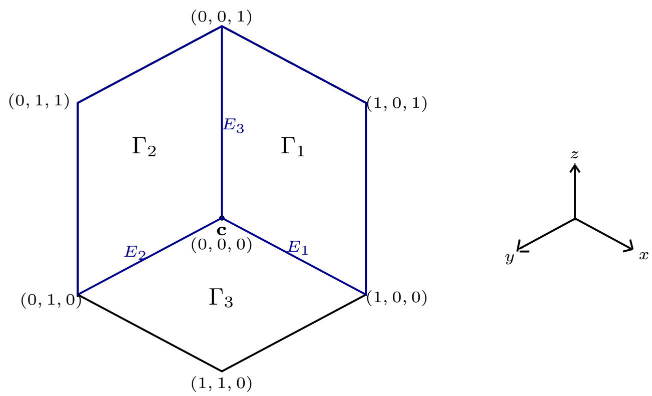


FIGURE 2. Labelling of faces, edges and corner of interest for Test Problem (3).

Let N be a positive integer. Let τ be the transition parameter that specifies the point where the mesh transitions between coarse and fine, defined as

$$\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon \ln N}{\beta}\right\}.$$

Assumption 3. *We make the assumption that $\varepsilon \le CN^{-1}$, as otherwise the analysis can be carried out on a uniform mesh using standard arguments. In particular, the subsequent analysis assumes that*

$$\tau = 2\varepsilon\beta^{-1} \ln N.$$

We also make the assumption that $b, f \in \mathcal{C}^{4,\alpha}(\bar{\Omega})$, $\alpha \in (0, 1)$. Along with this we assume that the corner compatibility conditions are such that $u \in \mathcal{C}^{6,\alpha}(\Omega) \cap \mathcal{C}^{3,\alpha}(\bar{\Omega})$, (see [21, 23, 24] *for more details).*

Divide the one-dimensional x-interval $[0, 1]$ into two subintervals: $[0, \tau]$ and $[\tau, 1]$. These subintervals are then both partitioned into $N/2$ mesh intervals. This gives a one-dimensional piecewise uniform mesh on the x-interval that we denote as $T_N[0, 1]$. The corresponding meshes in the y- and z-directions are constructed in the same way. The three-dimensional Shishkin mesh, $T_{N,N,N}(\bar{\Omega})$, is constructed by taking the tensor product of these three one-dimensional meshes, as visualised in Figure 3.

We now present notation used to denote the various subdomains of Ω. We use the subscripts B, I, and U to represent, respectively, the intervals $[0, \tau]$ (which contains the boundary layer), $[\tau, 1]$ (the interior region), and $[0, 1]$ (the unit interval). These can be combined so that, for example, Ω_{BBB} is the corner layer region; see Figure 4. Also, for example, Ω_{UBI} is the region $[0, 1] \times [0, \tau] \times [\tau, 1] = \Omega_{BBI} \cup \Omega_{IBI}$.

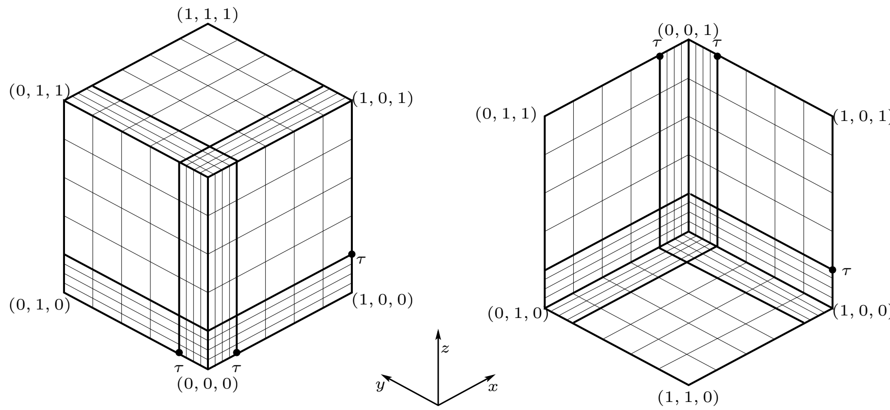


FIGURE 3. Two perspectives of a Shishkin mesh on the unit cube.

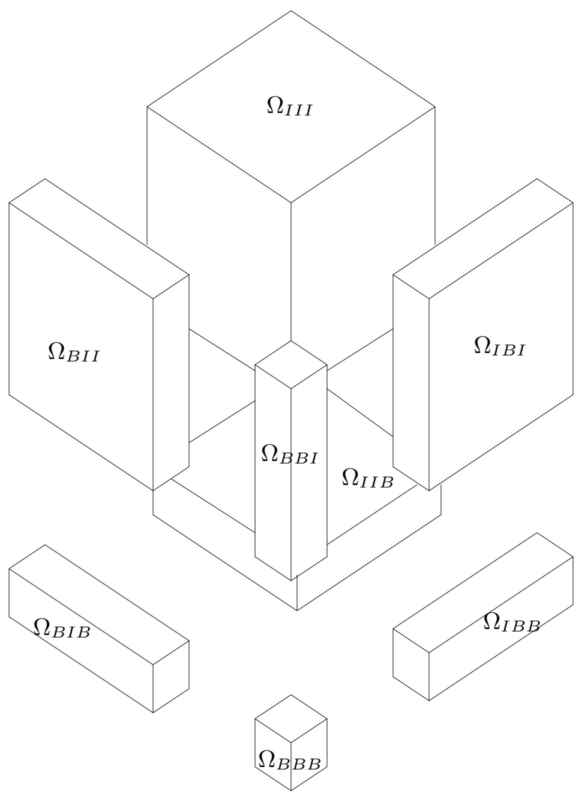


FIGURE 4. Decomposition of the mesh into layer regions.

2.2. Solution decomposition. In order to present a thorough analysis of a numerical method on a Shishkin mesh, a solution decomposition is required. We use a variant of the solution decomposition proposed by Shishkin and Shishkina [21, Section 3.2]. This decomposition represents u as the sum of

- a regular component v,
- components r_1, r_2 and r_3, corresponding to the $1D$ layers associated with Γ_i, $i = 1, 2, 3$,
- components s_1, s_2 and s_3, corresponding to the $2D$ layers associated with E_i, $i = 1, 2, 3$, and
- a component t, corresponding to the $3D$ layer associated with the corner c.

Lemma 4 ([21, Theorem 3.2.2])**.** *Let $b, f \in \mathcal{C}^{4,\alpha}(\bar{\Omega})$, $\alpha \in (0, 1)$. Then the solution, u, to* (1) *can be decomposed as*

$$u = v + \sum_{i=1}^{3} r_i + \sum_{i=1}^{3} s_i + t, \tag{5}$$

where for $l, m, n \geq 0$ there exists a constant C, such that

(6a)
$$\left| \frac{\partial^{l+m+n} v}{\partial x^l \partial y^m \partial z^n}(x, y, z) \right| \leq C \left(1 + \varepsilon^{2-l-m-n}\right), \qquad \textit{for} \quad 0 \leq l + m + n \leq 4,$$
(6b)
$$\left| \frac{\partial^{l+m+n} r_1}{\partial x^l \partial y^m \partial z^n}(x, y, z) \right| \leq C(1 + \varepsilon^{2-l-n})\varepsilon^{-m} e^{-\beta y/\varepsilon}, \qquad \textit{for} \quad 0 \leq l + m + n \leq 3,$$
(6c)
$$\left| \frac{\partial^{l+m+n} r_2}{\partial x^l \partial y^m \partial z^n}(x, y, z) \right| \leq C(1 + \varepsilon^{2-m-n})\varepsilon^{-l} e^{-\beta x/\varepsilon}, \qquad \textit{for} \quad 0 \leq l + m + n \leq 3,$$
(6d)
$$\left| \frac{\partial^{l+m+n} r_3}{\partial x^l \partial y^m \partial z^n}(x, y, z) \right| \leq C(1 + \varepsilon^{2-l-m})\varepsilon^{-n} e^{-\beta z/\varepsilon}, \qquad \textit{for} \quad 0 \leq l + m + n \leq 3,$$
(6e)
$$\left| \frac{\partial^{l+m+n} s_1}{\partial x^l \partial y^m \partial z^n}(x, y, z) \right| \leq C(1 + \varepsilon^{2-l})\varepsilon^{-m-n} e^{-\beta(y+z)/\varepsilon}, \qquad \textit{for} \quad 0 \leq l + m + n \leq 3,$$
(6f)
$$\left| \frac{\partial^{l+m+n} s_2}{\partial x^l \partial y^m \partial z^n}(x, y, z) \right| \leq C(1 + \varepsilon^{2-m})\varepsilon^{-l-n} e^{-\beta(x+z)/\varepsilon}, \qquad \textit{for} \quad 0 \leq l + m + n \leq 3,$$
(6g)
$$\left| \frac{\partial^{l+m+n} s_3}{\partial x^l \partial y^m \partial z^n}(x, y, z) \right| \leq C(1 + \varepsilon^{2-n})\varepsilon^{-l-m} e^{-\beta(x+y)/\varepsilon}, \qquad \textit{for} \quad 0 \leq l + m + n \leq 3,$$
(6h)
$$\left| \frac{\partial^{l+m+n} t}{\partial x^l \partial y^m \partial z^n}(x, y, z) \right| \leq C\varepsilon^{-l-m-n} e^{-\beta(x+y+z)/\varepsilon}, \qquad \textit{for} \quad 0 \leq l + m + n \leq 3.$$

The following lemma provides bounds on derivatives in the L^2-norm required for the analysis of the interpolation error.

Lemma 5. *For $0 \le l+m+n \le 3$, where the solution, u, to* (1) *is once again decomposed as in* (5)*, there exists a constant, C, such that*

$$\begin{aligned}
\left\| \frac{\partial^{l+m+n} r_1}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega_{UBU}} &\le C(1+\varepsilon^{2-l-n})\varepsilon^{1/2-m}, \\
\left\| \frac{\partial^{l+m+n} r_1}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega\setminus\Omega_{UBU}} &\le C(1+\varepsilon^{2-l-n})\varepsilon^{1/2-m} N^{-2}, \\
\left\| \frac{\partial^{l+m+n} r_2}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega_{BUU}} &\le C(1+\varepsilon^{2-m-n})\varepsilon^{1/2-l}, \\
\left\| \frac{\partial^{l+m+n} r_2}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega\setminus\Omega_{BUU}} &\le C(1+\varepsilon^{2-m-n})\varepsilon^{1/2-l} N^{-2}, \\
\left\| \frac{\partial^{l+m+n} s_1}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega_{UBB}} &\le C(1+\varepsilon^{2-l})\varepsilon^{1-m-n}, \\
\left\| \frac{\partial^{l+m+n} s_1}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega\setminus\Omega_{UBB}} &\le C(1+\varepsilon^{2-l})\varepsilon^{1-m-n} N^{-2}, \\
\left\| \frac{\partial^{l+m+n} s_2}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega_{BUB}} &\le C(1+\varepsilon^{2-m})\varepsilon^{1-l-n}, \\
\left\| \frac{\partial^{l+m+n} s_2}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega\setminus\Omega_{BUB}} &\le C(1+\varepsilon^{2-m})\varepsilon^{1-l-n} N^{-2}, \\
\left\| \frac{\partial^{l+m+n} t}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega_{BBB}} &\le C\varepsilon^{3/2-l-m-n}, \\
\left\| \frac{\partial^{l+m+n} t}{\partial x^l \partial y^m \partial z^n} \right\|_{0,\Omega\setminus\Omega_{BBB}} &\le C\varepsilon^{3/2-l-m-n} N^{-2}.
\end{aligned}$$

Analogous bounds for r_3 and s_3, on the relevant subdomains, also hold.

Proof. The results follow directly from Lemma 4 and the definition of the L^2-norm. □

3. Trilinear interpolation

Recall the one-dimensional piecewise uniform mesh, $T_N[0,1]$, and the associated three-dimensional Shishkin mesh $T_{N,N,N}(\bar{\Omega})$ defined in Section 2.1. Let $V_N[0,1] \subset H^1[0,1]$ be the space of piecewise linear functions on $T_N[0,1]$. The space of piecewise trilinear functions is $V_{N,N,N}(\Omega) = V_N[0,1] \times V_N[0,1] \times V_N[0,1]$. We now define $I_{N,N,N} : \mathcal{C}(\bar{\Omega}) \to V_{N,N,N}(\Omega)$ to be the nodal piecewise trilinear interpolation operator on $T_{N,N,N}(\Omega)$.

Since we are analysing a finite element method on a layer-adapted mesh, specialised anisotropic interpolation estimates are needed. However, as highlighted by Roos [16, §3.1], the approach that is usually taken for two-dimensional problems is not useful in three dimensions.

To obtain useful interpolation error estimates on the layer regions of the domain we use the following anisotropic interpolation analysis results of Apel and Dobrowolski [2] and Apel [3].

Lemma 6. [2, Theorem 3] *Let Υ be any mesh brick of size $h_x \times h_y \times h_z$. Let $\phi \in H^2(\Upsilon) \cap \mathcal{C}(\bar{\Omega})$. Then its piecewise trilinear nodal interpolant $I_{N,N,N}\phi$ satisfies the bounds*

$$\begin{aligned}\|\phi - I_{N,N,N}\phi\|_{0,\Upsilon} \leq & C(h_x^2\|\phi_{xx}\|_{0,\Upsilon} + h_y^2\|\phi_{yy}\|_{0,\Upsilon} + h_z^2\|\phi_{zz}\|_{0,\Upsilon} \\ & + h_xh_y\|\phi_{xy}\|_{0,\Upsilon} + h_xh_z\|\phi_{xz}\|_{0,\Upsilon} + h_yh_z\|\phi_{yz}\|_{0,\Upsilon}).\end{aligned} \tag{7}$$

The result (7) is the natural three-dimensional extension of its two-dimensional counterpart for bilinear interpolants, see, e.g., [11, Lemma 2.2]. Providing that $\phi \in H^2(K)$, the derivative of the interpolation error may be bounded as

$$\|(\phi - I_{N,N}\phi)_x\|_{0,K} \leq C(h_x\|\phi_{xx}\|_{0,K} + k_y\|\phi_{xy}\|_{0,K}), \tag{8}$$

where K is a mesh rectangle of size $h_x \times h_y$ and $I_{N,N}\phi$ is the standard bilinear interpolant of ϕ; see [11, Lemma 2]. For $\phi \in H^2$ the estimate (8) does not have a "natural" analogue in three dimensions for trilinear interpolants in the L^2-norm. However, by assuming additional regularity on ϕ, one has the following bounds.

Lemma 7. [2, Theorem 4] *Let Υ be any mesh brick of size $h_x \times h_y \times h_z$. Let $\phi \in W^{3,2}(\Upsilon)\cap\mathcal{C}(\bar{\Omega})$. Then its piecewise trilinear nodal interpolant $I_{N,N,N}\phi$ satisfies the bounds*

$$\begin{aligned}&\|(\phi - I_{N,N,N}\phi)_x\|_{0,\Upsilon} \\ &\quad \leq C(h_x\|\phi_{xx}\|_{0,\Upsilon} + h_xh_y\|\phi_{xxy}\|_{0,\Upsilon} + h_xh_z\|\phi_{xxz}\|_{0,\Upsilon} \\ &\quad + h_yh_z\|\phi_{xyz}\|_{0,\Upsilon} + h_x^2\|\phi_{xxx}\|_{0,\Upsilon} + h_y^2\|\phi_{xyy}\|_{0,\Upsilon} + h_z^2\|\phi_{xzz}\|_{0,\Upsilon}),\end{aligned} \tag{9a}$$

$$\begin{aligned}&\|(\phi - I_{N,N,N}\phi)_y\|_{0,\Upsilon} \\ &\quad \leq C(h_y\|\phi_{yy}\|_{0,\Upsilon} + h_xh_y\|\phi_{xyy}\|_{0,\Upsilon} + h_xh_z\|\phi_{xyz}\|_{0,\Upsilon} \\ &\quad + h_yh_z\|\phi_{yyz}\|_{0,\Upsilon} + h_x^2\|\phi_{xxy}\|_{0,\Upsilon} + h_y^2\|\phi_{yyy}\|_{0,\Upsilon} + h_z^2\|\phi_{yzz}\|_{0,\Upsilon}),\end{aligned} \tag{9b}$$

$$\begin{aligned}&\|(\phi - I_{N,N,N}\phi)_z\|_{0,\Upsilon} \\ &\quad \leq C(h_z\|\phi_{zz}\|_{0,\Upsilon} + h_xh_y\|\phi_{xyz}\|_{0,\Upsilon} + h_xh_z\|\phi_{xzz}\|_{0,\Upsilon} \\ &\quad + h_yh_z\|\phi_{yzz}\|_{0,\Upsilon} + h_x^2\|\phi_{xxz}\|_{0,\Upsilon} + h_y^2\|\phi_{yyz}\|_{0,\Upsilon} + h_z^2\|\phi_{zzz}\|_{0,\Upsilon}).\end{aligned} \tag{9c}$$

Using these results, we now present an analysis for the interpolation error in the L^2-norm.

Remark 8. *Generally, one only assumes that the solution to the variational form of (1) is found in $H_0^1(\Omega)$. However, in addition, u solves (1), so, in light of Assumption 3, which is required for the solution decomposition to exist, the premise of Lemma 7 is entirely reasonable.*

Lemma 9. *Suppose $\Omega = (0,1)^3$. Let $u \in H_0^1(\Omega)$ be the solution to (1), and $I_{N,N,N}u$ be its piecewise trilinear nodal interpolant. Then there exists a constant C, independent of ε and N such that*

$$\|u - I_{N,N,N}u\|_{0,\Omega} \leq CN^{-2}.$$

Proof. Recall the solution decomposition (5): $u = v + \sum_{i=1}^3 r_i + \sum_{i=1}^3 s_i + t$. We analyse each component of u separately, frequently making use of Assumption 3.

Analysing the v component first, we have from (6a) and (7) that

$$\begin{aligned}
&\|v - I_{N,N,N}v\|_{0,\Omega} \\
\le& CN^{-2}\Bigg(\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{0,\Omega} + \left\|\frac{\partial^2 v}{\partial y^2}\right\|_{0,\Omega} + \left\|\frac{\partial^2 v}{\partial z^2}\right\|_{0,\Omega} + \left\|\frac{\partial^2 v}{\partial x\partial y}\right\|_{0,\Omega} \\
&+ \left\|\frac{\partial^2 v}{\partial x\partial z}\right\|_{0,\Omega} + \left\|\frac{\partial^2 v}{\partial y\partial z}\right\|_{0,\Omega}\Bigg) \\
\le& CN^{-2}.
\end{aligned}$$

For the r_1 term on the region $\Omega_{UBU} := \Omega_{BBB} \cup \Omega_{IBB} \cup \Omega_{IBI} \cup \Omega_{BBI}$, we have, by Lemmas 5 and 6, that

$$\begin{aligned}
&\|r_1 - I_{N,N,N}r_1\|_{0,\Omega_{UBU}} \\
\le& C\Bigg(N^{-2}\left\|\frac{\partial^2 r_1}{\partial x^2}\right\|_{0,\Omega_{UBU}} + (\varepsilon N^{-1}\ln N)^2\left\|\frac{\partial^2 r_1}{\partial y^2}\right\|_{0,\Omega_{UBU}} \\
&+ N^{-2}\left\|\frac{\partial^2 r_1}{\partial z^2}\right\|_{0,\Omega_{UBU}} + N^{-1}(\varepsilon N^{-1}\ln N)\left\|\frac{\partial^2 r_1}{\partial x\partial y}\right\|_{0,\Omega_{UBU}} \\
&+ N^{-2}\left\|\frac{\partial^2 r_1}{\partial x\partial z}\right\|_{0,\Omega_{UBU}} + (\varepsilon N^{-1}\ln N)N^{-1}\left\|\frac{\partial^2 r_1}{\partial y\partial z}\right\|_{0,\Omega_{UBU}}\Bigg) \\
\le& C\Big(\varepsilon^{1/2}N^{-2} + \varepsilon^{1/2}N^{-2}\ln^2 N + \varepsilon^{1/2}N^{-2} + \varepsilon^{1/2}N^{-2}\ln N \\
&+\varepsilon^{1/2}N^{-2} + \varepsilon^{1/2}N^{-2}\ln N\Big) \\
\le& C\varepsilon^{1/2}N^{-2}\ln^2 N.
\end{aligned}$$

On the region $\Omega\backslash\Omega_{UBU}$, applying a triangle inequality and (6b) yields

$$\begin{aligned}
&\|r_1 - I_{N,N,N}r_1\|_{0,\Omega\backslash\Omega_{UBU}} \\
\le& \|r_1\|_{0,\Omega\backslash\Omega_{UBU}} + \|I_{N,N,N}r_1\|_{0,\Omega\backslash\Omega_{UBU}} \\
\le& C\varepsilon^{1/2}N^{-2} + C\sqrt{\mathrm{meas}(\Omega\backslash\Omega_{UBU})}\|r_1\|_{0,\infty,\Omega\backslash\Omega_{UBU}} \\
\le& CN^{-2}.
\end{aligned}$$

Combining these last two inequalities we can see that $\|r_1 - I_{N,N,N}r_1\|_{0,\Omega} \le CN^{-2}$. Analogous arguments can be applied to show that the same bound holds for the r_2 and r_3 components.

Next we analyse the s_1 component, which is associated with the 2D layer near $x=0$. By using Lemmas 5 and 6 on the region $\Omega_{UBB} := \Omega_{BBB} \cup \Omega_{IBB}$ one has

$$\begin{aligned}
&\|s_1 - I_{N,N,N}s_1\|_{0,\Omega_{UBB}}\\
\le& C\biggl(N^{-2}\left\|\frac{\partial^2 s_1}{\partial x^2}\right\|_{0,\Omega_{UBB}} + (\varepsilon N^{-1}\ln N)^2\left\|\frac{\partial^2 s_1}{\partial y^2}\right\|_{0,\Omega_{UBB}}\\
&+ (\varepsilon N^{-1}\ln N)^2\left\|\frac{\partial^2 s_1}{\partial z^2}\right\|_{0,\Omega_{UBB}} + N^{-1}(\varepsilon N^{-1}\ln N)\left\|\frac{\partial^2 s_1}{\partial x\partial y}\right\|_{0,\Omega_{UBB}}\\
&+ N^{-1}(\varepsilon N^{-1}\ln N)\left\|\frac{\partial^2 s_1}{\partial x\partial z}\right\|_{0,\Omega_{UBB}} + (\varepsilon N^{-1}\ln N)^2\left\|\frac{\partial^2 s_1}{\partial y\partial z}\right\|_{0,\Omega_{UBB}}\biggr)\\
\le& C\bigl(\varepsilon N^{-2} + \varepsilon N^{-2}\ln^2 N + \varepsilon N^{-2}\ln^2 N + \varepsilon N^{-2}\ln N\\
&+\varepsilon N^{-2}\ln N + \varepsilon N^{-2}\ln^2 N\bigr)\\
\le& C\varepsilon N^{-2}\ln^2 N.
\end{aligned}$$

Using (6e) on the region $\Omega\backslash\Omega_{UBB}$ gives

$$\begin{aligned}
&\|s_1 - I_{N,N,N}s_1\|_{0,\Omega\backslash\Omega_{UBB}}\\
\le&\|s_1\|_{0,\Omega\backslash\Omega_{UBB}} + \|I_{N,N,N}s_1\|_{0,\Omega\backslash\Omega_{UBB}}\\
\le& C\varepsilon N^{-2} + C\sqrt{\text{meas}(\Omega\backslash\Omega_{UBB})}\|s_1\|_{0,\infty,\Omega\backslash\Omega_{UBB}} \le CN^{-2}.
\end{aligned}$$

Similar results are valid for s_2 and s_3.

Finally we look at the corner layer function, t, first over the region Ω_{BBB}. By applying Lemma 6 and bounds from Lemma 5 one has

$$\begin{aligned}
&\|t - I_{N,N,N}t\|_{0,\Omega_{BBB}}\\
\le& C(\varepsilon N^{-1}\ln N)^2\biggl(\left\|\frac{\partial^2 t}{\partial x^2}\right\|_{0,\Omega_{BBB}} + \left\|\frac{\partial^2 t}{\partial y^2}\right\|_{0,\Omega_{BBB}} + \left\|\frac{\partial^2 t}{\partial z^2}\right\|_{0,\Omega_{BBB}}\\
&+ \left\|\frac{\partial^2 t}{\partial x\partial y}\right\|_{0,\Omega_{BBB}} + \left\|\frac{\partial^2 t}{\partial x\partial z}\right\|_{0,\Omega_{BBB}} + \left\|\frac{\partial^2 t}{\partial y\partial z}\right\|_{0,\Omega_{BBB}}\biggr)\\
\le& C\varepsilon^{3/2}N^{-2}\ln^2 N.
\end{aligned}$$

Using the result (6h) on the region $\Omega\backslash\Omega_{BBB}$ yields

$$\|t - I_{N,N,N}t\|_{0,\Omega\backslash\Omega_{BBB}} \le \|t\|_{0,\infty,\Omega\backslash\Omega_{BBB}} \le CN^{-2}.$$

Collecting these results and invoking Assumption 3 yields the desired result. □

Lemma 10. *Let u and $I_{N,N,N}u$ be as defined in Lemma 9. Then there exists a constant C, independent of ε and N such that*

$$\varepsilon\|\nabla(u - I_{N,N,N}u)\|_{0,\Omega} \le C\varepsilon^{1/2}N^{-1}\ln N.$$

Proof. We decompose u as in the solution decomposition (5), and analyse each term in turn. By Lemma 7 we have

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(v - I_{N,N,N}v)\right\|_{0,\Omega} \\
\leq & C\varepsilon N^{-1}\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{0,\Omega} + C\varepsilon N^{-2}\Bigg(\left\|\frac{\partial^3 v}{\partial x^2\partial y}\right\|_{0,\Omega} + \left\|\frac{\partial^3 v}{\partial x^2\partial z}\right\|_{0,\Omega} \\
& + \left\|\frac{\partial^3 v}{\partial x\partial y\partial z}\right\|_{0,\Omega} + \left\|\frac{\partial^3 v}{\partial x^3}\right\|_{0,\Omega} + \left\|\frac{\partial^3 v}{\partial x\partial y^2}\right\|_{0,\Omega} + \left\|\frac{\partial^3 v}{\partial x\partial z^2}\right\|_{0,\Omega}\Bigg) \\
\leq & C(\varepsilon N^{-1} + N^{-2}) \leq CN^{-2}.
\end{aligned}$$

For the r_1 term on the region $\Omega_{UBU} := \Omega_{BBB} \cup \Omega_{IBB} \cup \Omega_{IBI} \cup \Omega_{BBI}$, Lemmas 5 and 7 give that

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(r_1 - I_{N,N,N}r_1)\right\|_{0,\Omega_{UBU}} \\
\leq & C\varepsilon\Bigg(N^{-1}\left\|\frac{\partial^2 r_1}{\partial x^2}\right\|_{0,\Omega_{UBU}} + \varepsilon N^{-2}\ln N\left\|\frac{\partial^3 r_1}{\partial x^2\partial y}\right\|_{0,\Omega_{UBU}} \\
& + N^{-2}\left\|\frac{\partial^3 r_1}{\partial x^2\partial z}\right\|_{0,\Omega_{UBU}} + \varepsilon N^{-2}\ln N\left\|\frac{\partial^3 r_1}{\partial x\partial y\partial z}\right\|_{0,\Omega_{UBU}} + N^{-2}\left\|\frac{\partial^3 r_1}{\partial x^3}\right\|_{0,\Omega_{UBU}} \\
& + \varepsilon^2 N^{-2}\ln^2 N\left\|\frac{\partial^3 r_1}{\partial x\partial y^2}\right\|_{0,\Omega_{UBU}} + N^{-2}\left\|\frac{\partial^3 r_1}{\partial x\partial z^2}\right\|_{0,\Omega_{UBU}}\Bigg) \\
\leq & C\varepsilon(\varepsilon^{3/2}N^{-1} + \varepsilon^{3/2}N^{-2}\ln N + N^{-2} + \varepsilon^{3/2}N^{-2}\ln N \\
& + N^{-2} + \varepsilon^{3/2}N^{-2}\ln^2 N + N^{-2}) \\
\leq & C(N^{-2} + \varepsilon^{3/2}N^{-1}),
\end{aligned} \tag{10}$$

while on the region $\Omega\backslash\Omega_{UBU}$ the results of Lemmas 4 and 5, together with an inverse inequality, yield

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(r_1 - I_{N,N,N}r_1)\right\|_{0,\Omega\backslash\Omega_{UBU}} \\
\leq & \varepsilon\left(\left\|\frac{\partial r_1}{\partial x}\right\|_{0,\Omega\backslash\Omega_{UBU}} + \left\|\frac{\partial}{\partial x}I_{N,N,N}r_1\right\|_{0,\Omega\backslash\Omega_{UBU}}\right) \\
\leq & C\varepsilon\left(\max_{(x,y,z)\in\Omega\backslash\Omega_{UBU}} e^{-\beta y/\varepsilon} + N\left\|I_{N,N,N}r_1\right\|_{0,\Omega\backslash\Omega_{UBU}}\right) \\
\leq & C\varepsilon\left(N^{-2} + N\left\|r_1\right\|_{0,\infty,\Omega\backslash\Omega_{UBU}}\right) \\
\leq & CN^{-2}.
\end{aligned}$$

Notice that the pointwise bound on r_1 does not depend on x. Similarly, the pointwise bound on r_3 does not depend on x, so analogous arguments can be used to bound the r_3 term. The pointwise bound on r_2 *does* depend on x, so different arguments are used. We proceed as follows. By applying Lemmas 5 and 7 on the region $\Omega_{BUU} := \Omega_{BBB} \cup \Omega_{BIB} \cup \Omega_{BII} \cup \Omega_{BBI}$, one has

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(r_2 - I_{N,N,N}r_2)\right\|_{0,\Omega_{BUU}} \\
\le& C\varepsilon\Bigg(\varepsilon N^{-1}\ln N\left\|\frac{\partial^2 r_2}{\partial x^2}\right\|_{0,\Omega_{BUU}} + \varepsilon N^{-2}\ln N\left\|\frac{\partial^3 r_2}{\partial x^2\partial y}\right\|_{0,\Omega_{BUU}} \\
&+ \varepsilon N^{-2}\ln N\left\|\frac{\partial^3 r_2}{\partial^2 x\partial z}\right\|_{0,\Omega_{BUU}} + N^{-2}\left\|\frac{\partial^3 r_2}{\partial x\partial y\partial z}\right\|_{0,\Omega_{BUU}} \\
&+ \varepsilon^2 N^{-2}\ln^2 N\left\|\frac{\partial^3 r_2}{\partial^3 x}\right\|_{0,\Omega_{BUU}} + N^{-2}\left\|\frac{\partial^3 r_2}{\partial x\partial^2 y}\right\|_{0,\Omega_{BUU}} + N^{-2}\left\|\frac{\partial^3 r_2}{\partial x\partial^2 z}\right\|_{0,\Omega_{BUU}}\Bigg) \\
\le& C\Big(\varepsilon^{1/2}N^{-1}\ln N + \varepsilon^{1/2}N^{-2}\ln N + \varepsilon^{1/2}N^{-2}\ln N \\
&+ \varepsilon^{1/2}N^{-2} + \varepsilon^{1/2}N^{-2}\ln^2 N + \varepsilon^{1/2}N^{-2} + \varepsilon^{1/2}N^{-2}\Big) \\
\le& C\varepsilon^{1/2}N^{-1}\ln N. \qquad (11)
\end{aligned}$$

On the region $\Omega\backslash\Omega_{BUU}$, Lemmas 4 and 5, along with an inverse estimate, yield

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(r_2 - I_{N,N,N}r_2)\right\|_{0,\Omega\backslash\Omega_{BUU}} \\
\le& C\varepsilon\left(\left\|\frac{\partial r_2}{\partial x}\right\|_{0,\Omega\backslash\Omega_{BUU}} + \left\|\frac{\partial}{\partial x}I_{N,N,N}r_2\right\|_{0,\Omega\backslash\Omega_{BUU}}\right) \\
\le& C\varepsilon\left(\varepsilon^{-1/2}N^{-2} + N\|I_{N,N,N}r_2\|_{0,\Omega\backslash\Omega_{BUU}}\right) \\
\le& C\varepsilon\left(\varepsilon^{-1/2}N^{-2} + N\|r_2\|_{0,\infty,\Omega\backslash\Omega_{BUU}}\right) \\
\le& CN^{-2}.
\end{aligned}$$

Moving on to the $2D$ layer, using Lemmas 5 and 7, firstly on the region $\Omega_{UBB} := \Omega_{BBB}\cup\Omega_{IBB}$, and then Lemma 4 together with an inverse inequality on the region $\Omega\backslash\Omega_{UBB}$ one has

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(s_1 - I_{N,N,N}s_1)\right\|_{0,\Omega_{UBB}} \\
\le& C\varepsilon\Bigg(N^{-1}\left\|\frac{\partial^2 s_1}{\partial x^2}\right\|_{0,\Omega_{UBB}} + \varepsilon N^{-2}\ln N\left\|\frac{\partial^3 s_1}{\partial^2 x\partial y}\right\|_{0,\Omega_{UBB}} + \varepsilon N^{-2}\ln N\left\|\frac{\partial^3 s_1}{\partial^2 x\partial z}\right\|_{0,\Omega_{UBB}} \\
&+ \varepsilon^2 N^{-2}\ln^2 N\left\|\frac{\partial^3 s_1}{\partial x\partial y\partial z}\right\|_{0,\Omega_{UBB}} + N^{-2}\left\|\frac{\partial^3 s_1}{\partial^3 x}\right\|_{0,\Omega_{UBB}} \\
&+ \varepsilon^2 N^{-2}\ln^2 N\left\|\frac{\partial^3 s_1}{\partial x\partial^2 y}\right\|_{0,\Omega_{UBB}} + \varepsilon^2 N^{-2}\ln^2 N\left\|\frac{\partial^3 s_1}{\partial x\partial^2 z}\right\|_{0,\Omega_{UBB}}\Bigg) \\
\le& C\big(\varepsilon^2 N^{-1} + \varepsilon^2 N^{-2}\ln N + \varepsilon^2 N^{-2}\ln N \\
&+ \varepsilon^2 N^{-2}\ln^2 N + N^{-2} + \varepsilon^2 N^{-2}\ln^2 N + \varepsilon^2 N^{-2}\ln^2 N\big) \\
\le& C\left(N^{-2} + \varepsilon^2 N^{-1} + \varepsilon^2 N^{-2}\ln^2 N\right),
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon\left\|\frac{\partial}{\partial x}(s_1 - I_{N,N,N}s_1)\right\|_{0,\Omega\setminus\Omega_{UBB}} &\le \varepsilon\left(\left\|\frac{\partial s_1}{\partial x}\right\|_{0,\Omega\setminus\Omega_{UBB}} + \left\|\frac{\partial}{\partial x}I_{N,N,N}s_1\right\|_{0,\Omega\setminus\Omega_{UBB}}\right)\\
&\le C\varepsilon\left(N^{-2} + N\left\|I_{N,N,N}s_1\right\|_{0,\Omega\setminus\Omega_{UBB}}\right)\\
&\le C\varepsilon\left(N^{-2} + N\left\|s_1\right\|_{0,\infty,\Omega\setminus\Omega_{UBB}}\right)\\
&\le CN^{-2}.
\end{aligned}$$

For the s_2 component, by applying Lemmas 5 and 7 on the region $\Omega_{BUB} := \Omega_{BBB}\cup \Omega_{BIB}$ one obtains

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(s_2 - I_{N,N,N}s_2)\right\|_{0,\Omega_{BUB}}\\
\le& C\varepsilon\Bigg(\varepsilon N^{-1}\ln N\left\|\frac{\partial^2 s_2}{\partial x^2}\right\|_{0,\Omega_{BUB}} + \varepsilon N^{-2}\ln N\left\|\frac{\partial^3 s_2}{\partial^2 x\partial y}\right\|_{0,\Omega_{BUB}}\\
&+ \varepsilon^2 N^{-2}\ln^2 N\left\|\frac{\partial^3 s_2}{\partial^2 x\partial z}\right\|_{0,\Omega_{BUB}} + \varepsilon N^{-2}\ln N\left\|\frac{\partial^3 s_2}{\partial x\partial y\partial z}\right\|_{0,\Omega_{BUB}}\\
&+ \varepsilon^2 N^{-2}\ln^2 N\left\|\frac{\partial^3 s_2}{\partial^3 x}\right\|_{0,\Omega_{BUB}} + N^{-2}\left\|\frac{\partial^3 s_2}{\partial x\partial^2 y}\right\|_{0,\Omega_{BUB}}\\
&+ \varepsilon^2 N^{-2}\ln^2 N\left\|\frac{\partial^3 s_2}{\partial x\partial^2 z}\right\|_{0,\Omega_{BUB}}\Bigg)\\
\le& C\big(\varepsilon N^{-1}\ln N + \varepsilon N^{-2}\ln N + \varepsilon N^{-2}\ln^2 N\\
&+ \varepsilon N^{-2}\ln N + \varepsilon N^{-2}\ln N + \varepsilon N^{-2}\ln^2 N + \varepsilon N^{-2} + \varepsilon N^{-2}\ln^2 N\big)\\
\le& C\varepsilon N^{-1}\ln N,
\end{aligned}$$

while on $\Omega\setminus\Omega_{BUB}$, Lemmas 4 and 5, along with an inverse inequality, yield

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(s_2 - I_{N,N,N}s_2)\right\|_{0,\Omega\setminus\Omega_{BUB}}\\
\le& \varepsilon\left(\left\|\frac{\partial s_2}{\partial x}\right\|_{0,\Omega\setminus\Omega_{BUB}} + \left\|\frac{\partial}{\partial x}I_{N,N,N}s_2\right\|_{0,\Omega\setminus\Omega_{BUB}}\right)\\
\le& C\varepsilon\left(N^{-2} + N\left\|I_{N,N,N}s_2\right\|_{0,\Omega\setminus\Omega_{BUB}}\right)\\
\le& C\varepsilon\left(N^{-2} + N\left\|s_2\right\|_{0,\infty,\Omega\setminus\Omega_{BUB}}\right)\\
\le& CN^{-2}.
\end{aligned}$$

A bound for s_3 is found using a similar argument to that of s_2.

Finally, we deal with the corner layer component t. On the region Ω_{BBB}, Lemmas 5 and 7 give

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial}{\partial x}(t-I_{N,N,N}t)\right\|_{0,\Omega_{BBB}}\\
\le& C\varepsilon^2N^{-1}\ln N\left\|\frac{\partial^2 t}{\partial x^2}\right\|_{0,\Omega_{BBB}}+C\varepsilon^3N^{-2}\ln^2 N\Bigg(\left\|\frac{\partial^3 t}{\partial^2 x\partial y}\right\|_{0,\Omega_{BBB}}\\
&+\left\|\frac{\partial^3 t}{\partial^2 x\partial z}\right\|_{0,\Omega_{BBB}}+\left\|\frac{\partial^3 t}{\partial x\partial y\partial z}\right\|_{0,\Omega_{BBB}}+\left\|\frac{\partial^3 t}{\partial^3 x}\right\|_{0,\Omega_{BBB}}\\
&+\left\|\frac{\partial^3 t}{\partial x\partial^2 y}\right\|_{0,\Omega_{BBB}}+\left\|\frac{\partial^3 t}{\partial x\partial^2 z}\right\|_{0,\Omega_{BBB}}\Bigg)\\
\le& C\left(\varepsilon^{3/2}N^{-1}\ln N+\varepsilon^{3/2}N^{-2}\ln^2 N\right)\\
\le& C\varepsilon^{3/2}N^{-1}\ln N,
\end{aligned}$$

while on the region $\Omega\backslash\Omega_{BBB}$ one has, by Lemmas 4 and 5 and an inverse inequality, that

$$\begin{aligned}
\varepsilon\left\|\frac{\partial}{\partial x}(t-I_{N,N,N}t)\right\|_{0,\Omega\backslash\Omega_{BBB}} &\le \varepsilon\left(\left\|\frac{\partial t}{\partial x}\right\|_{0,\Omega\backslash\Omega_{BBB}}+\left\|\frac{\partial}{\partial x}I_{N,N,N}t\right\|_{0,\Omega\backslash\Omega_{BBB}}\right)\\
&\le \varepsilon\left(\varepsilon^{1/2}N^{-2}+N\left\|I_{N,N,N}t\right\|_{0,\Omega\backslash\Omega_{BBB}}\right)\\
&\le \varepsilon\left(\varepsilon^{1/2}N^{-2}+N\left\|t\right\|_{0,\infty,\Omega\backslash\Omega_{BBB}}\right)\\
&\le CN^{-2}.
\end{aligned}$$

Using Assumption 3 it is clear that the bounds from (10) and (11) dominate all the others. Combining them gives the desired result for $\varepsilon\|\partial/\partial x(u-I_{N,N,N}u)\|_{0,\Omega}$. Using the same approach, corresponding bounds for $\varepsilon\|\partial/\partial y(u-I_{N,N,N}u)\|_{0,\Omega}$ and $\varepsilon\|\partial/\partial z(u-I_{N,N,N}u)\|_{0,\Omega}$ can be obtained, completing the proof. □

We now present the main result of this section.

Theorem 11. *Let u and $I_{N,N,N}u$ be defined as in Lemma 9. Then there exists a constant C, independent of ε and N, such that*

$$\|u-I_{N,N,N}u\|_\varepsilon \le C\left(N^{-2}+\varepsilon^{1/2}N^{-1}\ln N\right).$$

Proof. The desired result is a direct consequence of the definition (2) of the energy norm and Lemmas 9 and 10. □

4. A Galerkin finite element method

The weak form of (1) is : *find $u\in H_0^1(\Omega)$ such that*

$$B(u,v):=\varepsilon^2(\nabla u,\nabla v)+(bu,v)=(f,v)\quad \forall v\in H_0^1(\Omega). \tag{12}$$

Restricting u and v to a suitable finite dimensional subspace of $H_0^1(\Omega)$ gives a Galerkin finite element method for the problem. A standard choice is the space of piecewise trilinear functions defined on the tensor product Shishkin mesh described in Section 2. One first forms the one-dimensional piecewise linear basis functions $\psi_i^N(x)$ on $T_{N_x}[0,1]$:

$$\psi_i^N(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \le x < x_i, \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x_i \le x < x_{i+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Define the basis functions $\psi_j^N(y)$ on $T_{N_y}[0,1]$, and $\psi_k^N(z)$ on $T_{N_z}[0,1]$ in a similar manner. Now define the space of piecewise trilinear functions of a given mesh $V_{N,N,N}(\Omega) \subset H_0^1(\Omega)$ as

$$\begin{aligned} V_{N,N,N}(\Omega) = \operatorname{span}\left\{\psi_i^N(x)\psi_j^N(y)\psi_k^N(z)\right\}, \\ \text{for } i = 1,2,\ldots,N-1, \quad j = 1,2,\ldots,N-1, \quad k = 1,2,\ldots,N-1. \end{aligned} \tag{14}$$

The Galerkin finite element method for (1) is then defined as: *find* $u_{N,N,N} \in V_{N,N,N}(\Omega)$ *such that*

$$B(u_{N,N,N}, v_{N,N,N}) = (f, v_{N,N,N}) \quad \text{for all } v_{N,N,N} \in V_{N,N,N}(\Omega). \tag{15}$$

Theorem 12. *Let u be the solution to* (1)*, and $u_{N,N,N}$ the solution to* (15)*. Then there exists a constant C, independent of ε and N, such that*

$$\|u - u_{N,N,N}\|_\varepsilon \le C(N^{-2} + \varepsilon^{1/2} N^{-1} \ln N).$$

Proof. The trilinear form defined in (12) is continuous and coercive, so (15) possesses a unique solution. Thus, one has the following quasioptimal bound:

$$\|u - u_{N,N,N}\|_\varepsilon \le C \inf_{\psi \in V_{N,N,N}(\Omega)} \|u - \psi\|_\varepsilon.$$

Since $I_{N,N,N}u \in V_{N,N,N}(\Omega)$, the result follows as an immediate consequence of Theorem 11. $\square$

5. Numerical results

To verify the analytical results of the previous section we use the test problem (3), taken from [5],

$$-\varepsilon^2 \Delta u + u = f \quad \text{on } (0,1)^3, \tag{16}$$

where $u = 0$ on the boundary, and f is chosen such that

$$\begin{aligned} u = \left(\cos\left(\frac{\pi x}{2}\right) - \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right)\left(1 - y - \frac{e^{-y/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right) \\ \cdot \left(1 - z^2 - \frac{e^{-z/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right). \end{aligned}$$

This problem exhibits exponential boundary layers near $x = 0$, $y = 0$ and $z = 0$, as well as at the point $(0,0,0)$. See Figure 1.

In Table 1 we present, in the energy norm, the errors for the standard Galerkin FEM for the test problem (16), for a range of values of N and ε.

The results in Table 1 show that Theorem 12 is sharp. When $\varepsilon = 1$, it is obvious that the error is proportional to N^{-1}. For small N and ε, the $\mathcal{O}(N^{-2})$ term in Theorem 12 appears to dominate. For larger N, it is clear that the $\mathcal{O}(\varepsilon^{1/2} N^{-1} \ln N)$

TABLE 1. Errors, $\|u - u_{N,N,N}\|_\varepsilon$, for the standard Galerkin FEM with trilinear elements for various N and ε.

ε^2	$\|u - u_{N,N,N}\|_\varepsilon \times \rho$	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
1	$\rho = 1$	8.455e-3	4.201e-3	2.097e-3	1.048e-3	5.241e-4	2.620e-4	1.310e-4
	$\rho = N$	0.034	0.034	0.034	0.034	0.034	0.034	0.034
	$\rho = N^2$	0.135	0.269	0.537	1.073	2.147	4.293	8.585
	$\rho = N/(\varepsilon^{1/2} \ln N)$	2.440e-2	1.616e-2	1.210e-2	9.676e-3	8.065e-3	6.912e-3	6.048e-3
10^{-2}	$\rho = 1$	6.458e-2	4.264e-2	2.196e-2	1.107e-2	5.547e-3	2.775e-3	1.388e-3
	$\rho = N$	0.258	0.341	0.351	0.354	0.355	0.355	0.355
	$\rho = N^2$	1.033	2.729	5.622	11.336	22.721	45.466	90.964
	$\rho = N/(\varepsilon^{1/2} \ln N)$	0.589	0.519	0.400	0.323	0.270	0.232	0.203
10^{-4}	$\rho = 1$	2.815e-2	2.141e-2	1.477e-2	9.417e-3	5.700e-3	3.336e-3	1.909e-3
	$\rho = N$	0.113	0.171	0.236	0.301	0.365	0.427	0.489
	$\rho = N^2$	0.450	1.370	3.781	9.643	23.347	54.657	125.110
	$\rho = N/(\varepsilon^{1/2} \ln N)$	0.812	0.824	0.852	0.870	0.877	0.880	0.881
10^{-6}	$\rho = 1$	1.121e-2	7.060e-3	4.815e-3	3.069e-3	1.858e-3	1.087e-3	6.221e-4
	$\rho = N$	0.045	0.056	0.077	0.098	0.119	0.139	0.159
	$\rho = N^2$	0.179	0.452	1.233	3.143	7.610	17.809	40.770
	$\rho = N/(\varepsilon^{1/2} \ln N)$	1.023	0.859	0.879	0.896	0.904	0.907	0.908
10^{-8}	$\rho = 1$	7.413e-3	2.503e-3	1.540e-3	9.740e-4	5.893e-4	3.449e-04	1.973e-4
	$\rho = N$	0.030	0.020	0.025	0.031	0.038	0.044	0.051
	$\rho = N^2$	0.119	0.160	0.394	0.997	2.414	5.651	12.930
	$\rho = N/(\varepsilon^{1/2} \ln N)$	2.139	0.963	0.889	0.899	0.907	0.910	0.911
10^{-10}	$\rho = 1$	6.916e-3	1.370e-3	5.245e-4	3.100e-4	1.865e-4	1.091e-4	6.242e-5
	$\rho = N$	0.028	0.011	0.008	0.010	0.012	0.014	0.016
	$\rho = N^2$	0.111	0.088	0.134	0.317	0.764	1.788	4.091
	$\rho = N/(\varepsilon^{1/2} \ln N)$	6.310	1.667	0.957	0.905	0.908	0.910	0.911
10^{-12}	$\rho = 1$	6.864e-3	1.200e-3	2.557e-4	1.038e-4	5.928e-5	3.452e-5	1.974e-5
	$\rho = N$	0.027	0.010	0.004	0.003	0.004	0.004	0.005
	$\rho = N^2$	0.110	0.077	0.065	0.106	0.243	0.566	1.294
	$\rho = N/(\varepsilon^{1/2} \ln N)$	1.981	4.617	1.476	0.958	0.912	0.911	0.911

term is dominating. These results can be further visualised in the log-log plot of Figure 5.

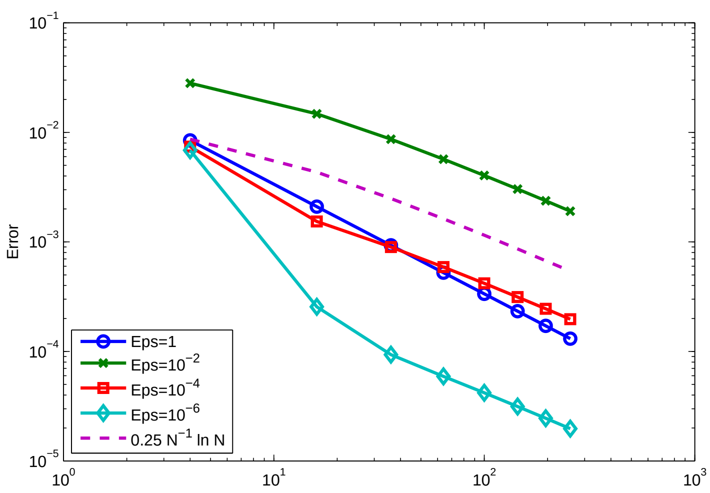


FIGURE 5. A log-log plot of the errors for the standard Galerkin FEM in three dimensions for various N and ε.

6. Concluding remarks and observations

It has been shown by Liu et al., that for a Galerkin finite element method applied to a singularly perturbed reaction-diffusion problem in two dimensions, one has the following result

$$\|u - u_{N,N}\|_\varepsilon \leq C(N^{-2} + \varepsilon^{1/2} N^{-1} \ln N), \tag{17}$$

where $u_{N,N}$ is the numerical solution computed on a Shishkin mesh [11]. Theorem 12 proves that the same result applies to the analogous problem in three dimensions, suggesting that the error estimate of this method is independent of the dimension in which the problem is posed. However, although this result is not surprising, until now, a proof is not available in the literature.

This theorem has other applications. For example, Liu et al. not only prove (17), but also establish a similar bound for a two-scale sparse grid method applied to a two-dimensional problem [11]. Since the development of sparse grid methods for the three-dimensional problem is even more pertinent, in a companion paper we have done just that [18]. However, the analysis hinges on the theory presented here.

Finally, we note that the estimate given above is not "balanced", in the sense in which that term is introduced by Lin and Stynes [9]. That is, the presence of $\varepsilon^{1/2}$ term in the error estimate means this component is under-weighted (and the results presented in Table 1 verify that this under-weighting is observed in practice, and is not just an artefact of the analysis). However, to extend the FOSLS-type approach of Adler et al. [1], for example, requires the results of Theorem 11 (as well as addition interpolation results for higher derivatives). Therefore, this theorem has applications beyond its direct use to prove Theorem 12.

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