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AN UNCONDITIONALLY STABLE NUMERICAL SCHEME FOR A COMPETITION SYSTEM INVOLVING DIFFUSION TERMS

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Abstract. A system of difference equations is proposed to approximate the solution of a system of partial differential equations that is used to model competing species with diffusion. The approximation method is a new semi-implicit finite difference scheme that is shown to mimic the dynamical properties of the true solution. In addition, it is proven that the scheme is uniquely solvable and unconditionally stable. The asymptotic behavior of the difference scheme is studied by constructing upper and lower solutions for the difference scheme. The convergence rate of the numerical solution to the true solution of the system is also given.

Key words. Competing species, convergence, asymptotic behavior, implicit finite difference scheme.

1. Introduction

We consider the following system of nonlinear parabolic partial differential equations used to model dynamic population distribution or biomass of two species that are competing for resources while each undergoes diffusion:

$$p_t = m_1 \triangle p + a_1 p - b_1 p^2 - c_1 pq \qquad (t > 0,\ \mathbf{x} \in \Omega), \tag{1}$$

$$q_t = m_2 \triangle q + a_2 q - b_2 q^2 - c_2 pq \qquad (t > 0,\ \mathbf{x} \in \Omega), \tag{2}$$

$$\frac{\partial p}{\partial \eta}|_{\partial\Omega} = 0,\ \ \frac{\partial q}{\partial \eta}|_{\partial\Omega} = 0 \qquad (t > 0), \tag{3}$$

$$p(0, \mathbf{x}) = p_0(\mathbf{x}),\ q(0, \mathbf{x}) = q_0(\mathbf{x}) \qquad (\mathbf{x} \in \Omega). \tag{4}$$

Here, $p(t, \mathbf{x})$ and $q(t, \mathbf{x})$ denote the time-dependent populations of the two species, $\Omega \in \mathbb{R}^n$ is a bounded domain with outward normal η along the boundary. The Neumann boundary conditions suggest absence of migration. There is a substantial body of work about this system, where many properties of the solutions are extracted, including such considerations as coexistence and long-term population behaviors of the competing species; see, for example, [1], [2], [3], and [6] and references therein. If $c_1 = c_2 = 0$, each equation in the paired system has the form of a so-called Fisher's equation. Ways to approach the numerical solutions of these equations can be found in [4] and [5].

For a numerical approximation of (1)-(4), the author in [8] proposes a discretization that gives rise to a fully implicit finite difference scheme. For $\Omega \subset \mathbb{R}$, this takes on the form

$$\frac{p_i^{k+1} - p_i^k}{\triangle t} = m_1 \left[\frac{p_{i+1}^{k+1} - 2p_i^{k+1} + p_{i-1}^{k+1}}{(\triangle x)^2} \right] + p_i^{k+1}(a_1 - b_1 p_i^{k+1} - c_1 q_i^{k+1}) \tag{5}$$

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$$\frac{q_i^{k+1}-q_i^k}{\triangle t} = m_2\left[\frac{q_{i+1}^{k+1}-2q_i^{k+1}+q_{i-1}^{k+1}}{(\triangle x)^2}\right] + q_i^{k+1}(a_2 - b_2 q_i^{k+1} - c_2 p_i^{k+1}) \tag{6}$$

where $p_i^k = p(k\triangle t, i\triangle x)$, $q_i^k = q(k\triangle t, i\triangle x)$.

The author in [8] then used Picard iteration to construct sequences of decreasing upper solutions $\{\bar{p}_i^k\}^m$ and $\{\bar{q}_i^k\}^m$, and of increasing lower solutions $\{\underline{p}_i^k\}^m$ and $\{\underline{q}_i^k\}^m$, such that if the time mesh size $\triangle t$ is chosen sufficiently small,

$$\lim_{m\to\infty}\{\bar{p}_i^k\}^m = \lim_{m\to\infty}\{\underline{p}_i^k\}^m = p_i^k \quad \text{and}$$
$$\lim_{m\to\infty}\{\bar{q}_i^k\}^m = \lim_{m\to\infty}\{\underline{q}_i^k\}^m = q_i^k.$$

To study the asymptotic behavior of the numerical solution, the author in [8] studied the steady state solution of (5)-(6), or the solutions of the nonlinear algebraic system

$$m_1\left[\frac{p_{i+1}-2p_i+p_{i-1}}{(\triangle x)^2}\right] + a_1 p_i - b_1(p_i)^2 - c_1 p_i q_i = 0. \tag{7}$$

$$m_2\left[\frac{q_{i+1}-2q_i+q_{i-1}}{(\triangle x)^2}\right] + a_2 q_i - b_2(q_i)^2 - c_2 p_i q_i = 0. \tag{8}$$

In [9]-[12], the author studies the result under the conditions where the minimal solution $(\underline{p}_i^*, \underline{q}_i^*)$ is equal to the maximal solution $(\bar{p}_i^*, \bar{q}_i^*)$ of (7)-(8). If this is the case, the author shows that $(p_i^k, q_i^k) \to (\underline{p}_i^*, \underline{q}_i^*) = (\bar{p}_i^*, \bar{q}_i^*)$.

The fully implicit scheme proposed in [8] conserves the dynamic properties of the system (1)-(4). The author in [8] also applied this method in [13] for a coupled system of quasilinear elliptic equations. However, it takes a significant amount of time to approximate the numerical solution using Picard iteration. In addition, it is hard to estimate the convergence rate as $\triangle t$ and $\triangle \mathbf{x}$ approach zero. Finally, $\triangle t$ must be chosen sufficiently small to guarantee convergence of the fully implicit system to the theoretical solution.

In this paper, we develop a new method for numerical approximation of the true solution (p, q) to (1)-(4); call this numerical approximation (p^k, q^k) for the time being. We propose a nonstandard finite difference method for discretizing the system that ends up requiring that a semi-implicit system of difference equations be solved for (p^k, q^k) rather than a fully implicit system as in [8]. We find the numerical solution to the system of difference equations directly. Then, fully independent of the choice of $\triangle t$, we prove the nonnegativity of p^k and q^k, the stability of the difference scheme, and that (p^k, q^k) converges to the true solution (p, q) of the system. We also show its rate of convergence to be $\mathcal{O}(\Delta t + \Delta x^2)$. We construct an upper solution $(\bar{p}^k, \bar{q}^k)$ and a lower solution $(\underline{p}^k, \underline{q}^k)$ to the system of difference equations using a related system of ordinary differential equations. Having constructed these upper and lower solutions, we will then be able to give a sufficient condition for coexistence of solutions of the system of difference equations and to provide a complete analysis of the long-term behavior of the numerical solution to (1)-(4).

In Section 2, we will introduce the difference scheme used for the approximation of (1)-(4) for $\Omega \subset \mathbb{R}$. We prove existence of the numerical solution to the scheme, and that it is stable independent of the choice of Δt and $\Delta \mathbf{x}$. We finish by giving the convergence rate of the numerical scheme to the true solution. In Section 3, we give more properties of the asymptotic behavior of the numerical solutions to the system. In Section 4, we show how the same results extend naturally from a domain $\Omega \subset \mathbb{R}$ to one in $\mathbb{R}^2$. In Section 5, we present some results of numerical experiments that reflect the stability and convergence of the proposed difference scheme in one and two-dimensional spatial domains.

2. A Semi-Implicit Difference Scheme

We can reduce the quantity of parameters from eight to three by applying nondimensionalization to (1)-(4). This is done under the substitutions

$$T = a_1 t,\ \mathbf{X} = \mathbf{x} \cdot \sqrt{\frac{a_1}{m_1}},\ p(t,\mathbf{x}) = \frac{a_1}{b_1} \cdot u(T,\mathbf{X}), \quad \text{and} \quad q(t,\mathbf{x}) = \frac{a_2}{b_2} \cdot v(T,\mathbf{X}).$$

After applying these changes of parameters and renaming T and $\mathbf{X}$ back to t and $\mathbf{x}$, (1)-(4) becomes

$$u_t = \triangle u + u(1-u-av) \qquad (t>0,\ \mathbf{x} \in \Omega), \tag{9}$$

$$v_t = r_m \triangle v + \rho v(1-v-bu) \qquad (t>0,\ \mathbf{x} \in \Omega), \tag{10}$$

$$u_\eta|_{\partial\Omega} = 0,\ v_\eta|_{\partial\Omega} = 0 \qquad (t>0), \tag{11}$$

$$u(0,\mathbf{x}) = u_0(\mathbf{x}), v(0,\mathbf{x}) = v_0(\mathbf{x}) \qquad (\mathbf{x} \in \Omega), \tag{12}$$

where $a = \frac{a_2 c_1}{b_2}$, $b = \frac{a_1 c_2}{b_1}$, $r_m = \frac{m_2}{m_1}$, and $\rho = \frac{a_2}{a_1}$.

Remark 2.1. *For theoretical results about system (9)-(12), particularly regarding the relation between the parameters a, b, r_m, ρ and coexistence of two species represented by populations u and v, details may be found in Chapter 2 of* [6].

As stated in Section 1, to facilitate analysis of the difference scheme we study the finite difference approximation of system (9)-(12) for a one-dimensional domain $\Omega \subset \mathbb{R}$. We show how these results extend in a natural way to $\mathbb{R}^2$ in Section 4.

With this in mind, for $t > 0$ we introduce time step $t_k = k\triangle t$ for $k = 0, 1, 2, \ldots$, where $\triangle t$ is a fixed time step whose size will be given later; until then, the choice of time step is relatively unimportant. In $\mathbb{R}$, we choose $\Omega = (-L, L)$ and define the partition

$$\Omega_x = \{x_i |\, x_i = -L + i\triangle x,\ i = 1, 2, \ldots, N-1\}, \quad \text{where} \quad \triangle x = 2L/N.$$

We use u_i^k to represent the approximation to $u(t_k, x_i)$ and v_i^k for $v(t_k, x_i)$, where $(u(t_k, x_i), v(t_k, x_i))$ is the true solution of the system (9)-(12) at (t_k, x_i). Our choice of difference scheme for the system (9)-(12) is

$$\frac{u_i^{k+1} - u_i^k}{\triangle t} = \frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{(\triangle x)^2} + u_i^k - u_i^k u_i^{k+1} - a u_i^{k+1} v_i^k \tag{13}$$

$$\frac{v_i^{k+1} - v_i^k}{\triangle t} = r_m \left[\frac{v_{i+1}^{k+1} - 2v_i^{k+1} + v_{i-1}^{k+1}}{(\triangle x)^2} \right] + \rho v_i^k - \rho v_i^k v_i^{k+1} - \rho b u_i^k v_i^{k+1} \tag{14}$$

for $i = 0, 1, 2, ..., N$ and $k = 0, 1, 2, \ldots$. To account for Neumann boundary conditions (15) we define

$$u_{-1}^k = u_1^k,\ u_{N+1}^k = u_{N-1}^k, v_{-1}^k = v_1^k,\ v_{N+1}^k = v_{N-1}^k \tag{15}$$

for $k = 0, 1, 2, \ldots$, and for initial conditions (16) we let

$$u_i^0 = u_0(x_i),\ v_i^0 = v_0(x_i) \tag{16}$$

for $i = 0, 1, 2, \ldots, N$. Setting $r = \frac{\triangle t}{(\triangle x)^2}$, (13)-(14) is written as

$$-ru_{i+1}^{k+1} + [1 + 2r + \triangle t(u_i^k + av_i^k)]u_i^{k+1} - ru_{i-1}^{k+1} = (1 + \triangle t)u_i^k \tag{17}$$

$$-r_m r v_{i+1}^{k+1} + [1 + 2r_m r + \rho\triangle t(bu_i^k + v_i^k)]v_i^{k+1} - r_m r v_{i-1}^{k+1} = (1 + \rho\triangle t)v_i^k \tag{18}$$

for $k = 0, 1, 2, \ldots$ and for $i = 0, 1, 2, \ldots, N$.

For each time step $k = 0, 1, 2, \ldots$, obtaining the numerical approximation requires that we solve the linear systems

$$A_u^{(k)} U^{(k+1)} = B_u^{(k)} \quad \text{and} \quad A_v^{(k)} V^{(k+1)} = B_v^{(k)}, \tag{19}$$

where $A_u^{(k)}$ and $A_v^{(k)}$ are the $(N+1)\times(N+1)$ tridiagonal matrices given by

$$A_u^{(k)} = \begin{pmatrix} \alpha_0^{(k)} & -2r & 0 & \cdots & 0 \\ -r & \alpha_1^{(k)} & -r & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -r & \alpha_{N-1}^{(k)} & -r \\ 0 & \cdots & 0 & -2r & \alpha_N^{(k)} \end{pmatrix} \tag{20}$$

where $\alpha_i^{(k)} = 1 + 2r + \triangle t(u_i^k + av_i^k)$, $i = 0, 1, 2, \ldots, N$, and

$$A_v^{(k)} = \begin{pmatrix} \beta_0^{(k)} & -2r_m r & 0 & \cdots & 0 \\ -r_m r & \beta_1^{(k)} & -r_m r & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -r_m r & \beta_{N-1}^{(k)} & -r_m r \\ 0 & \cdots & 0 & -2r_m r & \beta_N^{(k)} \end{pmatrix}, \tag{21}$$

where $\beta_i^{(k)} = 1 + 2r_m r + \rho\triangle t(bu_i^k + v_i^k)$ for $i = 0, \cdots, N$. Matrices $A_u^{(k)}$ and $A_v^{(k)}$ in (20)-(21) reflect the Neumann boundary conditions (15) in their first and last rows.

Remark 2.2. *Choosing a scheme that is semi-implicit as in (13)-(14) guarantees that $A_u^{(k)}$ and $A_v^{(k)}$ are diagonally dominant matrices, an essential property on which the results of this paper heavily depend. Choosing a fully implicit scheme would not guarantee the required diagonal dominance of its corresponding matrices $A_u^{(k)}$ and $A_v^{(k)}$.*

Solution vectors $U^{(k+1)}$ and $V^{(k+1)}$ in (19) are defined as

$$U^{(k+1)} = \begin{pmatrix} u_0^{k+1} \\ u_1^{k+1} \\ \vdots \\ u_N^{k+1} \end{pmatrix}, \quad V^{(k+1)} = \begin{pmatrix} v_0^{k+1} \\ v_1^{k+1} \\ \vdots \\ v_N^{k+1} \end{pmatrix}, \tag{22}$$

and vectors $B_u^{(k)}$ and $B_v^{(k)}$ are given by

$$B_u^{(k)} = \begin{pmatrix} (1+\triangle t)u_0^k \\ (1+\triangle t)u_1^k \\ \vdots \\ (1+\triangle t)u_N^k \end{pmatrix}, \quad B_v^{(k)} = \begin{pmatrix} (1+\rho\triangle t)v_0^k \\ (1+\rho\triangle t)v_1^k \\ \vdots \\ (1+\rho\triangle t)v_N^k \end{pmatrix}. \tag{23}$$

If $u_0(x) > 0, v_0(x) > 0$, define for use in the theorem that follows the constants

$$u^0 = \max_{0\le i\le N}\{u_i^0\},\ v^0 = \max_{0\le i\le N}\{v_i^0\},\ K_u = \max\{1, u^0\},\ \text{and } K_v = \max\{1, v^0\}.$$

.

Theorem 2.3. *The system of difference equations (19) has a unique solution* (U^{k+1}, V^{k+1}) *for* $k = 0, 1, 2, \dots$. *Furthermore, for all* $i = 0, 1, 2, \dots, N$ *and* $k = 0, 1, 2, \dots,$

$$0 < u_i^{k+1} \le K_u \quad \text{and} \quad 0 < v_i^{k+1} \le K_v.$$

Remark 2.4. *The preceding theorem mentions no restriction on the sizes of* $\triangle t$ *and* $\triangle x$. *This omission is intentional since the scheme (13)-(14) unconditionally preserves the positivity of the solution and is unconditionally stable, independent of step size in time and space.*

Proof. We first consider existence and uniqueness. Since $u_0(x) > 0$ and $v_0(x) > 0$, we have $u_i^0 > 0$ and $v_i^0 > 0$ for $i = 0, 1, 2, \dots, N$. For $k = 0$ and for $i = 0, 1, 2, \dots, N$, mimicking the notation in (19)-(23) we have

$$A_u^{(0)} U^{(1)} = B_u^{(0)} \quad \text{and} \quad A_v^{(0)} V^{(1)} = B_v^{(0)}$$

where $\alpha_i^{(0)} = 1 + 2r + \triangle t(u_i^0 + a v_i^0)$ and $\beta_i^{(0)} = 1 + 2r_m r + \rho \triangle t(b u_i^0 + v_i^0)$. If we set $A_u^{(0)} = (a_{ij})$, then since $a_{ii} = \alpha_i^{(0)} = 1 + 2r + \triangle t(u_i^0 + a v_i^0) > 0$, since $a_{ij} \le 0$ for $i \ne j$, and since $\sum_{j=0}^N (a_{ij}) = 1 + \triangle t(u_i^0 + a v_i^0) > 0$ for $i = 0, 1, 2, \dots, N$, it follows that $A_u^{(0)}$ is an M-matrix (for the definition and properties of M-matrices used here, see Chapter 3.6 in [14]). Following Lemma 3.6.4 in [14], $(A_u^{(0)})^{-1}$ is a matrix with strictly positive entries. Therefore, $U^{(1)} = (A_u^{(0)})^{-1} B_u^{(0)}$ and $V^{(1)} = (A_v^{(0)})^{-1} B_v^{(0)}$ are both vectors with strictly positive entries: That is, $u_i^1 > 0$ for $i = 0, 1, \dots, N$, and $v_i^1 > 0$ for $i = 0, 1, 2, \dots, N$, respectively. Hence the system has a unique, positive solution for $k = 0$.

If we assume that the statement is true for k, then using the same argument we can show that it must be true also for $k+1$. By mathematical induction, existence, uniqueness, and positivity of solutions is guaranteed for all $k = 0, 1, \dots$.

To complete the proof, we must verify the upper bounds for solutions of (19). We apply induction again. For $k = 0$, assume by way of contradiction that $u_i^0 \le K_u$ and $v_i^0 \le K_v$ for $i = 0, 1, 2, \dots, N$, but that there exists i_1 or i_2, $0 \le i_1 \le N$ or $0 \le i_2 \le N$, such that $u_{i_1}^1 > K_u$ or $v_{i_2}^1 > K_v$.

Suppose without loss of generality that there is such an i_1 and set $M_u = \max\{u_0^1, u_1^1, \dots, u_N^1\}$, so that $M_u > K_u \ge 1$. Then there exists j_1, where $0 \le j_1 \le N$, such that $u_{j_1}^1 = M_u$. Setting $i = j_1$ and $k = 0$ in (17), we have

$$-r u_{j_1+1}^1 + [1 + 2r + \triangle t(u_{j_1}^0 + a v_{j_1}^0)] u_{j_1}^1 - r u_{j_1-1}^1 = (1 + \triangle t) u_{j_1}^0. \tag{24}$$

Since $u_{j_1}^1 \ge u_{j_1-1}^1$, $u_{j_1}^1 \ge u_{j_1+1}^1$, $a v_{j_1}^0 > 0$, $u_{j_1}^1 = M_u > K_u$, and $u_{j_1}^0 \le K_u$, we see that

$$\begin{aligned} &-r u_{j_1+1}^1 + [1 + 2r + \triangle t(u_{j_1}^0 + a v_{j_1}^0)] u_{j_1}^1 - r u_{j_1-1}^1 \\ &\qquad > r(2 - u_{j_1-1}^1 - u_{j_1+1}^1) + (1 + \triangle t u_{j_1}^0) u_{j_1}^1 > (1 + \triangle t) u_{j_1}^0. \end{aligned}$$

This contradicts (17), so it must be that $u_i^1 \le K_u$ for $i = 0, 1, \dots, N$. The same reasoning shows that $v_i^1 \le K_v$ for $i = 0, 1, 2, \dots, N$, establishing the claim for $k = 0$.

Under the assumption that the statement holds true for some k, we may apply the same argument to show that it holds for $k+1$. Hence by induction, the conclusion is true for all k, $k = 0, 1, 2, \ldots$. □

We are ready to turn to the question of convergence and convergence rate of solutions of the system of difference equations (13)-(16) to the true solution of (9)-(12).

Theorem 2.5. *If $u_i^0 > 0$ and $v_i^0 > 0$ for $i = 0, 1, 2, \cdots, N$, then for any $T > 0$, the solution of the difference scheme (13)-(16) converges to the solution of the system (9)-(12) on $[0, T] \times [-L, L]$ as $\triangle t, \triangle x \to 0$, with convergence rate $\mathcal{O}(\triangle t + \triangle x^2)$.*

Proof. Let $(u(t, x), v(t, x))$ be the solution of the system (9)-(12) and let $T > 0$ be given. We define

$$U_i^k = u(t_k, x_i) \quad \text{and} \quad V_i^k = v(t_k, x_i) \tag{25}$$

as the evaluation of paired solutions (u, v) to (9)-(12) at partition points (t_k, x_i) in $[0, T] \times \overline{\Omega} = [0, T] \times [-L, L]$ for $k = 0, 1, 2, \ldots, K_T$ and $i = 0, 1, 2, \ldots N$, where without loss of generality we may assume that $K_T \triangle t = T$. From (13)-(16), we have for $k = 0, 1, 2, \ldots, K_T$,

$$\begin{aligned} \frac{U_i^{k+1} - U_i^k}{\triangle t} &= \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{(\triangle x)^2} + U_i^{k+1} \\ &\quad - (U_i^{k+1})^2 - aU_i^{k+1}V_i^{k+1} + R_1(\triangle t, \triangle x), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{V_i^{k+1} - V_i^k}{\triangle t} &= r_m \left[\frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{(\triangle x)^2} \right] + \rho \, V_i^{k+1} \\ &\quad - \rho (V_i^{k+1})^2 - \rho \, b \, U_i^{k+1} V_i^{k+1} + R_2(\triangle t, \triangle x), \end{aligned} \tag{27}$$

where R_1 and R_2 are functions of $\triangle t$ and $\triangle x$ that are $\mathcal{O}(\triangle t + \triangle x^2)$. Additionally, the Neumann boundary conditions (15) imply that

$$\begin{aligned} U_1^k &= U_{-1}^k + \mathcal{O}(\triangle x^2),\ U_{N+1}^k = U_{N-1}^k + \mathcal{O}(\triangle x^2), \\ V_1^k &= V_{-1}^k + \mathcal{O}(\triangle x^2),\ V_{N+1}^k = V_{N-1}^k + \mathcal{O}(\triangle x^2) \end{aligned}$$

for $i = 0, 1, 2, \ldots, N$.

We now define error terms

$$X_i^k = U_i^k - u_i^k \quad \text{and} \quad Y_i^k = V_i^k - v_i^k,$$

for $k = 0, 1, 2, \ldots, K_T$ and $i = 0, 1, 2, \ldots, N$. From (16), we see that

$$X_i^0 = 0,\ Y_i^0 = 0 \ \text{ for } i = 0, \ldots, N.$$

For each $k \geq 0$, (13)-(14) and (26)-(27) imply that

$$\begin{aligned} \frac{X_i^{k+1} - X_i^k}{\triangle t} &= \frac{X_{i+1}^{k+1} - 2X_i^{k+1} + X_{i-1}^{k+1}}{\triangle x^2} + (U_i^{k+1} - u_i^k) \\ &\quad + [u_i^k u_i^{k+1} - (U_i^{k+1})^2] + a[u_i^{k+1} v_i^k - U_i^{k+1} V_i^{k+1}] + R_1(\triangle t, \triangle x), \end{aligned} \tag{28}$$

and

$$\begin{aligned}\frac{Y_i^{k+1}-Y_i^k}{\triangle t} &= r_m\left[\frac{Y_{i+1}^{k+1}-2Y_i^{k+1}+Y_{i-1}^{k-1}}{\triangle x^2}\right]+\rho(V_i^{k+1}-v_i^k)\\ &\quad+\rho[v_i^k v_i^{k+1}-(V_i^{k+1})^2]+\rho b(u_i^k v_i^{k+1}-U_i^{k+1}V_i^{k+1})+R_2(\triangle t,\triangle x).\end{aligned} \tag{29}$$

Theorem 2.3 implies that the numerical solution (u_i^k, v_i^k) is uniformly bounded. We also know the theoretical solutions u and v to (9)-(12) are bounded with their partial derivatives: the first partial with respect to time on $[0,T]$ and the second partial with respect to space on $\overline{\Omega}$. Hence u_t is uniformly bounded, guaranteeing $|U_i^{k+1}-U_i^k| = \mathcal{O}(\triangle t)$. This implies that there exists a positive constant C_1 independent of Δt and Δx such that the following error bounds hold:

$$|U_i^{k+1}-u_i^k| = |U_i^{k+1}-U_i^k+U_i^k-u_i^k| \le |X_i^k|+\mathcal{O}(\triangle t), \tag{30}$$

$$\begin{aligned}|u_i^k u_i^{k+1}-(U_i^{k+1})^2| &\le |u_i^k u_i^{k+1}-U_i^k u_i^{k+1}|+|U_i^k u_i^{k+1}-U_i^k U_i^{k+1}|\\ &\qquad+|U_i^k U_i^{k+1}-(U_i^{k+1})^2|\\ &\le C_1|X_i^k|+C_1|X_i^{k+1}|+\mathcal{O}(\triangle t),\end{aligned} \tag{31}$$

$$\begin{aligned}|u_i^{k+1}v_i^k-U_i^{k+1}V_i^{k+1}| &= |u_i^{k+1}v_i^k-u_i^{k+1}V_i^k+u_i^{k+1}V_i^k\\ &\qquad-U_i^{k+1}V_i^k+U_i^{k+1}V_i^k-U_i^{k+1}V_i^{k+1}|\\ &\le u_i^{k+1}|v_i^k-V_i^k|+V_i^k|u_i^{k+1}-U_i^{k+1}|+U_i^{k+1}|V_i^k-V_i^{k+1}|\\ &\le C_1|Y_i^k|+C_1|X_i^{k+1}|+\mathcal{O}(\triangle t),\end{aligned} \tag{32}$$

and likewise for this C_1,

$$|V_i^{k+1}-v_i^k| \le |Y_i^k|+\mathcal{O}(\triangle t), \tag{33}$$

$$|v_i^k v_i^{k+1}-(V_i^{k+1})^2| \le C_1|Y_i^k|+C_1|Y_i^{k+1}|+\mathcal{O}(\triangle t), \tag{34}$$

$$|u_i^k v_i^{k+1}-U_i^{k+1}V_i^{k+1}| \le C_1|X_i^k|+C_1|Y_i^{k+1}|+\mathcal{O}(\triangle t). \tag{35}$$

We now rewrite (28)-(29) as

$$\begin{aligned}&-rX_{i+1}^{k+1}+(1{+}2r)X_i^{k+1}-rX_{i-1}^{k+1}\\ &\qquad= X^k+\triangle t(U_i^{k+1}-u_i^k)+\triangle t[u_i^k u_i^{k+1}-(U_i^{k+1})^2]\\ &\qquad+a\Delta t(u_i^{k+1}v_i^k-U_i^{k+1}V_i^{k+1})+\triangle t\,R_1(\triangle t,\triangle x),\end{aligned} \tag{36}$$

$$\begin{aligned}&-r_m rY_{i+1}^{k+1}+(1{+}2r_m r)Y_i^{k+1}-r_m rY_{i-1}^{k+1}\\ &\qquad= Y^k+\rho\triangle t(V_i^{k+1}-v_i^k)+\rho\triangle t[v_i^k v_i^{k+1}-(V_i^{k+1})^2]\\ &\qquad+\rho\, b\triangle t(v_i^{k+1}u_i^k-U_i^{k+1}V_i^{k+1})+\triangle t\,R_2(\triangle t,\triangle x).\end{aligned} \tag{37}$$

System (36)-(37) leads to a linear system that involves the $(N+1) \times (N+1)$ dimensional tridiagonal matrices M_1 and M_2 given by

$$M_1 = \begin{pmatrix} 1+2r & -2r & 0 & \cdots & 0 \\ -r & 1+2r & -r & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -r & 1+2r & -r \\ 0 & \cdots & 0 & -2r & 1+2r \end{pmatrix}, \tag{38}$$

and

$$M_2 = \begin{pmatrix} 1+2r_m r & -2r_m r & 0 & \cdots & 0 \\ -r_m r & 1+2r_m r & -r_m r & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -r_m r & 1+2r_m r & -r_m r \\ 0 & \cdots & 0 & -2r_m r & 1+2r_m r \end{pmatrix}. \tag{39}$$

Since M_1 and M_2 are strictly diagonally dominant, following [15] we are assured that they are nonsingular matrices with $||M_i^{-1}|| \leq 1$ for $i = 1, 2$, where $|| \cdot ||$ can represent either the 1-norm or the ∞-norm of a matrix throughout, meaning its maximal absolute column sum or row sum, respectively. Since matrix norms may sometimes be defined without the submultiplicative property, we note that either norm represented by $|| \cdot ||$ is submultiplicative.

Writing (36)-(37) as a matrix system using (38)-(39) while remembering the Neumann boundary conditions (15) gives

$$M_1 X^{k+1} = X^k + E_1^{(k)} \quad \text{and} \quad M_2 Y^{k+1} = Y^k + E_2^{(k)}, \tag{40}$$

where $X^{k+1} = (X_0^{k+1}, X_1^{k+1}, \ldots, X_N^{k+1})^T$ with Y^{k+1} defined similarly, and where for $i = 0, 1, 2, \ldots, N$, if we set

$$\gamma_i^k = \triangle t(U_i^{k+1} - u_i^k) + \Delta t[u_i^k u_i^{k+1} - (U_i^{k+1})^2] + a\Delta t(u_i^{k+1} v_i^k - U_i^{k+1} V_i^{k+1})$$

and

$$\delta_i^k = \rho\triangle t(V_i^{k+1} - v_i^k) + \rho\Delta t[v_i^k v_i^{k+1} - (V_i^{k+1})^2] + \rho\, b\Delta t(v_i^{k+1} u_i^k - U_i^{k+1} V_i^{k+1}),$$

then, by resizing R_1 and R_2 as defined following (26)-(27) if necessary,

$$E_1^{(k)} = \begin{pmatrix} \gamma_0^k + \Delta t\, R_1(\Delta t, \Delta x) \\ \gamma_1^k + \Delta t\, R_1(\Delta t, \Delta x) \\ \vdots \\ \gamma_N^k + \Delta t\, R_1(\Delta t, \Delta x) \end{pmatrix} \quad \text{and} \quad E_2^{(k)} = \begin{pmatrix} \delta_0^k + \Delta t\, R_2(\Delta t, \Delta x) \\ \delta_1^k + \Delta t\, R_2(\Delta t, \Delta x) \\ \vdots \\ \delta_N^k + \Delta t\, R_2(\Delta t, \Delta x) \end{pmatrix}.$$

It follows from (40) that

$$X^{k+1} = M_1^{-1}\left(X^k + E_1^{(k)}\right) \quad \text{and} \quad Y^{k+1} = M_2^{-1}\left(Y^k + E_2^{(k)}\right), \tag{41}$$

which leads to

$$||X^{k+1}|| \leq ||M_1^{-1}|| \left(||X^k|| + ||E_1^{(k)}||\right) \leq ||X^k|| + ||E_1^{(k)}|| \tag{42}$$

and

$$||Y^{k+1}|| \leq ||M_2^{-1}||| \left(||Y^k|| + ||E_2^{(k)}||\right) \leq ||Y^k|| + ||E_2^{(k)}||. \tag{43}$$

Applying the estimates from (30)-(34) gives bounds

$$||E_1^{(k)}|| \leq \triangle t||X^k|| + C_1\triangle t||X^{k+1}|| + C_1\triangle t||Y^k|| + \triangle t||R_1(\triangle t, \triangle x)||, \tag{44}$$

$$||E_2^{(k)}|| \leq \triangle t||Y^k|| + C_1\triangle t||Y^{k+1}|| + C_1\triangle t||X^k|| + \triangle t||R_2(\triangle t, \triangle x)|| \tag{45}$$

for functions R_1 and R_2 that are $\mathcal{O}(\Delta t + \Delta x^2)$ as described.

Let us now choose $\triangle t$ such that $C_1\triangle t < \frac{1}{2}$, then by expanding $(1 - C_1\Delta t)^{-1}$ geometrically and substituting (44)-(45) into (42)-(43), we see that there exist constants C_2 and C_3 such that

$$\begin{aligned} ||X^{k+1}|| &\leq \frac{1+\triangle t}{1-C_1\triangle t}||X^k|| + \frac{C_1\triangle t}{1-C_1\triangle t}||Y^k|| \\ &\quad + \frac{C_1\triangle t}{1-C_1\triangle t}||R_1(\triangle t, \triangle x)|| \\ &\leq (1+C_2\triangle t)||X^k|| + C_3\triangle t||Y^k|| + C_3\triangle t||R_1(\triangle t, \triangle x)||, \end{aligned} \tag{46}$$

and such that for Y, we likewise obtain

$$||Y^{k+1}|| \leq (1+C_2\triangle t)||Y^k|| + C_3\triangle t||X^k|| + C_3\triangle t||R_2(\triangle t, \triangle x)||. \tag{47}$$

We now define $N^{(k)}$ by $N^{(k)} = ||X^k|| + ||Y^k||$, so that $N^{(0)} = 0$, and we define the function R_3 by $R_3(\Delta t, \Delta x) = ||R_1(\triangle t, \triangle x)|| + ||R_2(\triangle t, \triangle x)||$. Adding (46) and (47) and letting $C_2 + C_3 = C_4$ gives

$$\begin{aligned} N^{(k+1)} &\leq (1+C_4\triangle t)N^{(k)} + C_3\triangle t R_3(\triangle t, \triangle x) \\ &\leq (1+C_4\triangle t)^2 N^{(k-1)} + C_3\Delta t[1 + (1+C_4\triangle t)]R_3(\triangle t, \triangle x) \\ &\leq (1+C_4\triangle t)^3 N^{(k-2)} + C_3\Delta t[1 + (1+C_4\triangle t) + (1+C_4\Delta t)^2]R_3(\triangle t, \triangle x) \\ &\quad \vdots \\ &\leq (1+C_2\triangle t)^{k+1}N^{(0)} + C_3\Delta t\left[\frac{(1+C_4\triangle t)^{k+1}-1}{C_4\triangle t}\right]R_3(\triangle t, \triangle x) \\ &= 0 + C(T)R_3(\triangle t, \triangle x). \end{aligned}$$

where $C(T)$ is a constant that depends only on T.

□

3. Asymptotic Behavior of the Numerical Solution

We have considered results about the numerical solution to (17)-(18) and its approximation of the true solution of (9)-(12) on a bounded interval. In this section, our goal is to study the the asymptotic behavior of a numerical solution as it approximates the true solution. Similarly to the author in [8], we use the method of lower and upper solutions as the means of investigating this long-term behavior. We begin this study with a definition.

Definition 3.1. *We say that $(\underline{u}_i^k, \underline{v}_i^k)$ and $(\bar{u}_i^k, \bar{v}_i^k)$ are a lower solution and an upper solution, respectively, of system (17)-(18) if (a) $0 < \underline{u}_i^0 \leq u_i^0 \leq \bar{u}_i^0$ and $0 < \underline{v}_i^0 \leq v_i^0 \leq \bar{v}_i^0$ for all $i = 0, \ldots, N$, and if (b) for constants r and r_m defined* *in (17)-(18) for any $\triangle t > 0$, $(\underline{u}_i^k, \underline{v}_i^k)$ and $(\bar{u}_i^k, \bar{v}_i^k)$ satisfy the system of difference equations*

$$-r\bar{u}_{i+1}^{k+1} + (1 + 2r + \triangle t\bar{u}_i^k + a\triangle t\underline{v}_i^k)\bar{u}_i^{k+1} - r\bar{u}_{i-1}^{k+1} = (1 + \triangle t)\bar{u}_i^k \tag{48}$$

$$-r_m r\bar{v}_{i+1}^{k+1} + (1 + 2r_m r + \rho\triangle t\bar{v}_i^k + \rho\triangle tb\underline{u}_i^k)\bar{v}_i^{k+1} - r_m r\bar{v}_{i-1}^{k+1} = (1 + \rho\triangle t)\bar{v}_i^k \tag{49}$$

$$-r\underline{u}_{i+1}^{k+1} + (1 + 2r + \triangle t\underline{u}_i^k + a\triangle t\bar{v}_i^k)\underline{u}_i^{k+1} - r\underline{u}_{i-1}^{k+1} = (1 + \triangle t)\underline{u}_i^k \tag{50}$$

$$-r_m r\underline{v}_{i+1}^{k+1} + (1 + 2r_m r + \rho\triangle t\underline{v}_i^k + \rho\triangle tb\bar{u}_i^k)\underline{v}_i^{k+1} - r_m r\underline{v}_{i-1}^{k+1} = (1 + \rho\triangle t)\underline{v}_i^k \tag{51}$$

for all $k = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots, N$.

Remark 3.2. *We note that the Definition 3.1 is valid since, given initial conditions u_i^0 and v_i^0 for $i = 0, 1, 2, \ldots, N$, the system* (48) − (51) *is solvable for all k, $k = 0, 1, 2, \ldots$.*

We now turn to the following comparison theorem, which ensures that if initial conditions for (17)-(18) have desirable bounds as per Definition 3.1, then the corresponding inequalities are valid for all time steps k.

Theorem 3.3. *Let $(\underline{u}_i^k, \underline{v}_i^k)$ and $(\bar{u}_i^k, \bar{v}_i^k)$ be lower and upper solutions of (17)-(18), respectively, as in Definition 3.1. Then for all $i = 0, 1, 2, \ldots, N$ and all $k = 0, 1, 2, \ldots$,*

$$0 < \underline{u}_i^k \le u_i^k \le \bar{u}_i^k \quad \text{and} \quad 0 < \underline{v}_i^k \le v_i^k \le \bar{v}_i^k. \tag{52}$$

Proof. Using an argument similar to that used for Theorem 2.3, we are assured that all of $\underline{u}_i^k, u_i^k, \bar{u}_i^k, \underline{v}_i^k, v_i^k, \bar{v}_i^k > 0$ for $i = 0, 1, 2, \ldots, N$ and $k = 0, 1, 2, \ldots$.

We use mathematical induction to prove that $\underline{u}_i^k \le u_i^k \le \bar{u}_i^k$. By Definition 3.1, the conclusion holds for $k = 0$, for $i = 0, 1, 2, \ldots, N$. We assume that the conclusion is also true for some k, for $i = 0, 1, 2, \ldots, N$. Then for the $k + 1$ case, since the inequalities are assumed true for k we have that $\underline{v}_i^k \le v_i^k \le \bar{v}_i^k$. Then based on (48) and (50),

$$-r\bar{u}_{i+1}^{k+1} + (1 + 2r + \triangle t\bar{u}_i^k + a\triangle tv_i^k)\bar{u}_i^{k+1} - r\bar{u}_{i-1}^{k+1} \ge (1 + \triangle t)\bar{u}_i^k, \tag{53}$$

$$-r\underline{u}_{i+1}^{k+1} + (1 + 2r + \triangle t\underline{u}_i^k + a\triangle tv_i^k)\underline{u}_i^{k+1} - r\underline{u}_{i-1}^{k+1} \le (1 + \triangle t)\underline{u}_i^k. \tag{54}$$

Let us set $w_i^k = \bar{u}_i^k - u_i^k$ and $x_i^k = \underline{u}_i^k - u_i^k$. Then by assumption, $w_i^k \ge 0$ and $x_i^k \le 0$ for $i = 1, 2, \ldots, N$. If we subtract (17) from inequalities (53) and (54), we have

$$\begin{aligned} -rw_{i+1}^{k+1} + 2rw_i^{k+1} - rw_{i-1}^{k+1} &+ (1 + a\triangle tv_i^k)w_i^{k+1} \\ &+ \triangle t(\bar{u}_i^k\bar{u}_i^{k+1} - u_i^k u_i^{k+1}) \ge (1 + \triangle t)w_i^k, \end{aligned} \tag{55}$$

$$\begin{aligned} -rx_{i+1}^{k+1} + 2rx_i^{k+1} - rx_{i-1}^{k+1} &+ (1 + a\triangle tv_i^k)x_i^{k+1} + \\ &\triangle t(\underline{u}_i^k\underline{u}_i^{k+1} - u_i^k u_i^{k+1}) \le (1 + \triangle t)x_i^k. \end{aligned} \tag{56}$$

Since

$$\begin{aligned} \bar{u}_i^k\bar{u}_i^{k+1} - u_i^k u_i^{k+1} &= \bar{u}_i^k w_i^{k+1} + u_i^{k+1} w_i^k, \\ \underline{u}_i^k\underline{u}_i^{k+1} - u_i^k u_i^{k+1} &= \underline{u}_i^k x_i^{k+1} + u_i^{k+1} x_i^k, \end{aligned}$$

when we substitute these into (55) and (56), we obtain

$$\begin{aligned}-rw_{i+1}^{k+1} + 2rw_i^{k+1} - rw_{i-1}^{k+1} + (1 + a\triangle tv_i^k + \triangle t\bar{u}_i^k)w_i^{k+1} \\ \geq (1 + \triangle t - \triangle tu_i^{k+1})w_i^k,\end{aligned} \tag{57}$$

$$\begin{aligned}-rx_{i+1}^{k+1} + 2rx_i^{k+1} - rx_{i-1}^{k+1} + (1 + a\triangle tv_i^k + \triangle t\underline{u}_i^k)x_i^{k+1} \\ \leq (1 + \triangle t - \triangle tu_i^{k+1})x_i^k.\end{aligned} \tag{58}$$

From Theorem 2.3, we know that u_i^{k+1} is bounded by $K_u = \max\{1, \max u_0(x)\}$, so that if $\triangle t$ is small enough we are assured that $1 + \triangle t - \triangle tu_i^{k+1} > 0$. Let us pick such a Δt.

We must use the inductive hypotheses to show that $w_i^{k+1} \geq 0$ and $x_i^{k+1} \leq 0$ for $i = 0, 1, 2, \ldots, N$. Assuming this does not hold, there exist i_0 and j_0 such that $w_{i_0}^{k+1} < 0$ and $x_{j_0}^{k+1} > 0$. Without loss of generality, we may assume that

$$w_{i_0}^{k+1} = \min_{0\leq i\leq N} w_i^{k+1} \quad \text{and} \quad x_{j_0}^{k+1} = \max_{0\leq i\leq N} x_i^{k+1}.$$

If we set $i = i_0$ in (57) and $i = j_0$ in (58), then

$$\begin{aligned}-rw_{i_0+1}^{k+1} + 2rw_{i_0}^{k+1} - rw_{i_0-1}^{k+1} + (1 + a\triangle tv_{i_0}^k + \triangle t\bar{u}_{i_0}^k)w_{i_0}^{k+1} \\ \geq (1 + \triangle t - \triangle tu_{i_0}^{k+1})w_{i_0}^k,\end{aligned} \tag{59}$$

$$\begin{aligned}-rx_{j_0+1}^{k+1} + 2rx_{j_0}^{k+1} - rx_{j_0-1}^{k+1} + (1 + a\triangle tv_{j_0}^k + \triangle t\underline{u}_{j_0}^k)x_{j_0}^{k+1} \\ \leq (1 + \triangle t - \triangle tu_{j_0}^{k+1})x_{j_0}^k.\end{aligned} \tag{60}$$

Now since

$$-rw_{i_0+1}^{k+1} + 2rw_{i_0}^{k+1} - rw_{i_0-1}^{k+1} \leq 0 \quad \text{and} \quad (1 + a\triangle tv_{i_0}^k + \triangle t\bar{u}_{i_0}^k)w_{i_0}^{k+1} \leq 0,$$

and likewise

$$-rx_{j_0+1}^{k+1} + 2rx_{j_0}^{k+1} - rx_{j_0-1}^{k+1} \geq 0 \quad \text{and} \quad (1 + a\triangle tv_{j_0}^k + \triangle t\underline{u}_{j_0}^k)w_{j_0}^{k+1} \geq 0,$$

we see that

$$\begin{aligned}0 \geq -rw_{i_0+1}^{k+1} + 2rw_{i_0}^{k+1} - rw_{i_0-1}^{k+1} + (1 + a\triangle tv_{i_0}^k + \triangle t\bar{u}_{i_0}^k)w_{i_0}^{k+1} \\ \geq (1 + \triangle t - \triangle tu_{i_0}^{k+1})w_{i_0}^k,\end{aligned}$$

$$\begin{aligned}0 \leq -rx_{j_0+1}^{k+1} + 2rx_{j_0}^{k+1} - rx_{j_0-1}^{k+1} + (1 + a\triangle tv_{j_0}^k + \triangle t\underline{u}_{i_0}^k)x_{j_0}^{k+1} \\ \leq (1 + \triangle t - \triangle tu_{j_0}^{k+1})x_{j_0}^k.\end{aligned}$$

By the inductive assumption for k, as $w_{i_0}^k > 0$ and $x_{j_0}^k < 0$, we have that

$$(1 + \triangle t - \triangle tu_{i_0}^{k+1})w_{i_0}^k > 0 \quad \text{and} \quad (1 + \triangle t - \triangle tu_{j_0}^{k+1})x_{j_0}^k < 0.$$

This is a contradiction. Therefore, for all $k > 0$, $\underline{u}_i^{k+1} \leq u_i^{k+1} \leq \bar{u}_i^{k+1}$ for $i = 0, \ldots, N$, with a similar sequence of inequalities holding valid for v^{k+1}, $k = 1, 2, 3, \ldots$.

□

We now construct an upper solution and a lower solution of the system (17)-(18) in the following lemma.

Lemma 3.4. *Suppose that $0 < a, b < 1$, $\rho > 0$, that all of $\bar{u}^0, \underline{v}^0, \bar{v}^0, \underline{u}^0 > 0$, and that*

$$1 - \bar{u}^0 - a\underline{v}^0 < 0, \quad 1 - \underline{v}^0 - b\bar{u}^0 > 0,$$
$$1 - \bar{v}^0 - b\underline{u}^0 < 0, \quad 1 - \underline{u}^0 - a\bar{v}^0 > 0.$$

Then the system of difference equations

$$(1 + \triangle t\bar{u}^k + a\triangle t\underline{v}^k)\bar{u}^{k+1} = (1 + \triangle t)\bar{u}^k \tag{61}$$

$$(1 + \rho\triangle t\underline{v}^k + \rho b\triangle t\bar{u}^k)\underline{v}^{k+1} = (1 + \rho\triangle t)\underline{v}^k \tag{62}$$

$$(1 + \triangle t\underline{u}^k + a\triangle t\bar{v}^k)\underline{u}^{k+1} = (1 + \triangle t)\underline{u}^k \tag{63}$$

$$(1 + \rho\triangle t\bar{v}^k + \rho b\triangle t\underline{u}^k)\bar{v}^{k+1} = (1 + \rho\triangle t)\bar{v}^k \tag{64}$$

has a unique solution $(\bar{u}^{k+1}, \bar{v}^{k+1}, \underline{u}^{k+1}, \underline{v}^{k+1})$ for $k = 0, 1, 2, 3, \ldots$. Furthermore, for $k = 0, 1, 2, 3, \ldots$, we have

(a) $\bar{u}^k$, $\bar{v}^k$, $\underline{u}^k$, $\underline{v}^k > 0$,

(b) $1 - \bar{u}^k - a\underline{v}^k < 0$, $1 - \underline{v}^k - b\bar{u}^k > 0$, $1 - \bar{v}^k - b\underline{u}^k < 0$, *and* $1 - \underline{u}^k - a\bar{v}^k > 0$,

(c) $\bar{u}^{k+1} < \bar{u}^k$, $\underline{v}^{k+1} > \underline{v}^k$, $\bar{v}^{k+1} < \bar{v}^k$, *and* $\underline{u}^{k+1} > \underline{u}^k$, *and*

(d) $\lim_{k\to\infty} \bar{u}^k = \lim_{k\to\infty} \underline{u}^k = \dfrac{1-a}{1-ab}$ *and* $\lim_{k\to\infty} \bar{v}^k = \lim_{k\to\infty} \underline{v}^k = \dfrac{1-b}{1-ab}$.

Proof. We first establish (a) by mathematical induction. By assumption, the statement is true for $k = 0$.

Assume the statement is true for some $k > 0$. Then since $\bar{u}^k > 0$, $\bar{v}^k > 0$, $\underline{u}^k > 0$, $\underline{v}^k > 0$, and $\Delta t > 0$, we have

$$\bar{u}^{k+1} = \frac{(1 + \triangle t)\bar{u}^k}{1 + \triangle t\bar{u}^k + a\triangle t\underline{v}^k} > 0, \tag{65}$$

$$\underline{v}^{k+1} = \frac{(1 + \rho\triangle t)\underline{v}^k}{1 + \rho\triangle t\underline{v}^k + \rho b\triangle t\bar{u}^k} > 0, \tag{66}$$

$$\underline{u}^{k+1} = \frac{(1 + \triangle t)\underline{u}^k}{1 + \triangle t\underline{u}^k + a\triangle t\bar{v}^k} > 0, \tag{67}$$

$$\bar{v}^{k+1} = \frac{(1 + \rho\triangle t)\bar{v}^k}{1 + \rho\triangle t\bar{v}^k + \rho b\triangle t\underline{u}^k} > 0. \tag{68}$$

So (a) is true for $k + 1$ and, by mathematical induction, for all $k = 0, 1, 2, \ldots$; see Figure 1 as reference for the proof of this lemma.

We now turn to (b). Using (65) and (66), $1 - \bar{u}^{k+1} - a\underline{v}^{k+1} < 0$ is equivalent to

$$\begin{aligned}(1 - \bar{u}^k - a\underline{v}^k) &+ \triangle t[\rho\underline{v}^k + \rho b\bar{u}^k + a\underline{v}^k - \rho\bar{u}^k\underline{v}^k - \rho b(\bar{u}^k)^2 - a\rho\underline{v}^k \\ &- a\bar{u}^k\underline{v}^k - a^2(\underline{v}^k)^2] + (\triangle t)^2 a\rho\underline{v}^k(\underline{v}^k + b\bar{u}^k - \bar{u}^k - a\underline{v}^k) < 0.\end{aligned} \tag{69}$$

Since the statement we want to prove is true for k by the inductive hypothesis,

$$1 - \bar{u}^k - a\underline{v}^k < 0 \quad \text{and} \quad 1 - \underline{v}^k - b\bar{u}^k > 0,$$

which implies

$$\bar{u}^k > \frac{1-a}{1-ab} \quad \text{and} \quad \underline{v}^k < \frac{1-b}{1-ab}. \tag{70}$$

The inequalities in (70) imply that $(1-a)\underline{v}^k < (1-b)\bar{u}^k$. Then

$$(\triangle t)^2 a\rho\underline{v}^k(\underline{v}^k + b\bar{u}^k - \bar{u}^k - a\underline{v}^k) < 0. \tag{71}$$

Since $\bar{u}^k > 0$, $\underline{v}^k > 0$, $1-\bar{u}^k - a\underline{v}^k < 0$ and $1-\underline{v}^k - b\bar{u}^k > 0$,

$$\begin{aligned}
&\rho\underline{v}^k + \rho b\bar{u}^k + a\underline{v}^k - \rho\bar{u}^k\underline{v}^k - \rho b(\bar{u}^k)^2 - a\rho\underline{v}^k - a\bar{u}^k\underline{v}^k - a^2(\underline{v}^k)^2 \\
&= \rho b\bar{u}^k(1-\bar{u}^k - a\underline{v}_k) + a\underline{v}^k(1-\bar{u}^k - a\underline{v}_k) + a\rho\underline{v}^k(b\bar{u}^k - 1) + \rho\underline{v}^k(1-\bar{u}^k) \\
&< \rho b\bar{u}^k(1-\bar{u}^k - a\underline{v}_k) + a\underline{v}^k(1-\bar{u}^k - a\underline{v}_k) + a\rho\underline{v}^k(b\bar{u}^k - 1) + a\rho\underline{v}^k\underline{v}^k \\
&= \rho b\bar{u}^k(1-\bar{u}^k - a\underline{v}_k) + a\underline{v}^k(1-\bar{u}^k - a\underline{v}_k) + a\rho\underline{v}^k(b\bar{u}^k + v^k - 1) < 0.
\end{aligned} \tag{72}$$

Therefore, inequality (69) stands and we have established that

$$1 - \bar{u}^{k+1} - a\underline{v}^{k+1} < 0,$$

and the first inequality of (b) is established using induction.

Next we prove $1-\underline{v}^{k+1} - b\bar{u}^{k+1} > 0$. Let us substitute $\bar{u}^{k+1}$ and $\underline{v}^{k+1}$ from (65) and (66) into the inequality to obtain an equivalent form

$$1 - \frac{(1+\rho\triangle t)\underline{v}^k}{1+\rho\triangle t\underline{v}^k + \rho b\triangle t\bar{u}^k} - \frac{b(1+\triangle t)\bar{u}^k}{1+\triangle t\bar{u}^k + a\triangle t\underline{v}^k} > 0, \tag{73}$$

so that, as in the previous case,

$$\begin{aligned}
&(1-\underline{v}^k - b\bar{u}^k) + \triangle t[\rho b\bar{u}^k + \bar{u}^k + a\underline{v}^k - \bar{u}^k\underline{v}^k - a(\underline{v}^k)^2 - b\rho\bar{u}^k\underline{v}^k - \\
&\quad \rho b^2(\bar{u}^k)^2 - bu^k] + (\triangle t)^2 a\rho b\bar{u}^k(-\underline{v}^k - b\bar{u}^k + \bar{u}^k + a\underline{v}^k) > 0.
\end{aligned} \tag{74}$$

Since the statement is true for k, from (70), we have

$$1-\underline{v}^k - b\bar{u}^k > 0 \quad \text{and} \quad a\rho b\bar{u}^k(-\underline{v}^k - b\bar{u}^k + \bar{u}^k + a\underline{v}^k) > 0. \tag{75}$$

Therefore we consider the remaining portion

$$\begin{aligned}
&\rho b\bar{u}^k + \bar{u}^k + a\underline{v}^k - \bar{u}^k\underline{v}^k - a(\underline{v}^k)^2 - b\rho\bar{u}^k\underline{v}^k - \rho b^2(\bar{u}^k)^2 - bu^k \\
&= \rho b\bar{u}^k(1-\underline{v}^k - b\bar{u}^k) + \bar{u}^k(1-\underline{v}^k - b\bar{u}^k) + a\underline{v}^k(1-\underline{v}^k) + b\bar{u}^k(\bar{u}^k - 1).
\end{aligned} \tag{76}$$

Since $1-\underline{v}^k > b\bar{u}^k$, we have

$$a\underline{v}^k(1-\underline{v}^k) > ab\underline{v}^k\bar{u}^k,$$

which when substituted into (76) yields

$$\begin{aligned}
&\rho b\bar{u}^k + \bar{u}^k + a\underline{v}^k - \bar{u}^k\underline{v}^k - a(\underline{v}^k)^2 - b\rho\bar{u}^k\underline{v}^k - \rho b^2(\bar{u}^k)^2 - bu^k \\
&= \rho b\bar{u}^k(1-\underline{v}^k - b\bar{u}^k) + \bar{u}^k(1-\underline{v}^k - b\bar{u}^k) + a\underline{v}^k(1-\underline{v}^k) + b\bar{u}^k(\bar{u}^k - 1) \\
&> \rho b\bar{u}^k(1-\underline{v}^k - b\bar{u}^k) + \bar{u}^k(1-\underline{v}^k - b\bar{u}^k) + b\bar{u}^k(a\underline{v}^k + \bar{u}^k - 1).
\end{aligned} \tag{77}$$

Exploiting the inductive hypotheses $1-\bar{u}^k - a\underline{v}^k < 0$ and $1-\underline{v}^k - b\bar{u}^k > 0$ again, we have

$$\rho b\bar{u}^k + \bar{u}^k + a\underline{v}^k - \bar{u}^k\underline{v}^k - a(\underline{v}^k)^2 - b\rho\bar{u}^k\underline{v}^k - \rho b^2(\bar{u}^k)^2 - bu^k > 0, \tag{78}$$

and invoking mathematical induction we have shown the second inequality in (b) holds for all $k = 0, 1, 2, \ldots$.

As the proof of the last two inequalities in (b) is similar to that as for the first two parts, we omit it here.

To establish part (c) of Lemma 3.4, we observe that

$$\bar{u}^{k+1} - \bar{u}^k = \frac{(1+\triangle t)\bar{u}^k}{1+\triangle t\bar{u}^k + a\triangle t\underline{v}^k} - \bar{u}^k = \frac{\triangle t\bar{u}^k(1-\bar{u}^k - a\underline{v}^k)}{1+\triangle t\bar{u}^k + a\triangle t\underline{v}^k} \tag{79}$$

and

$$\underline{v}^{k+1} - \underline{v}^k = \frac{(1+\rho\triangle t)\underline{v}^k}{1+\rho\triangle t\underline{v}^k + \rho b\triangle t\bar{u}^k} - \underline{v}^k = \frac{\rho\triangle t\underline{v}^k(1-\underline{v}^k - b\bar{u}^k)}{1+\rho\triangle t\underline{v}^k + \rho b\triangle t\bar{u}^k}, \tag{80}$$

so that the result follows immediately from part (b). Likewise, $\bar{v}^{k+1} < \bar{v}^k$ and $\underline{u}^{k+1} > \underline{u}^k$.

Finally, for part (d), we call on part (b) of Lemma 3.4, which ensures that

$$\bar{u}^k > \frac{1-a}{1-ab},\ \underline{v}^k < \frac{1-b}{1-ab},\ \bar{v}^k > \frac{1-b}{1-ab},\ \text{ and }\ \underline{u}^k < \frac{1-a}{1-ab}. \tag{81}$$

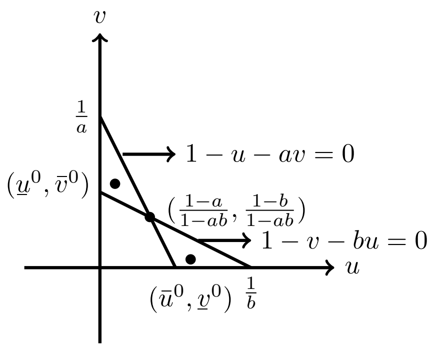


Figure 1: Bounded regions as described in Lemma 3.4
with initial conditions $(\underline{u}^0, \bar{v}^0)$ and $(\bar{u}^0, \underline{v}^0)$.

Since $\{\bar{u}^k\}$ and $\{\bar{v}^k\}$ are decreasing and bounded below, and likewise $\{\underline{u}^k\}$ and $\{\underline{v}^k\}$ are increasing and bounded above, their limits exist. Then we have

$$\lim_{k\to\infty} \bar{u}^k = L_1 \geq \frac{1-a}{1-ab} > 0, \lim_{k\to\infty} \underline{v}^k = L_2 \geq \underline{v}^0 > 0, \tag{82}$$

and

$$\lim_{k\to\infty} \bar{v}^k = L_3 \geq \frac{1-b}{1-ab} > 0, \lim_{k\to\infty} \underline{u}^k = L_4 > \underline{u}^0 > 0. \tag{83}$$

Taking the limit in (61)-(64), we have

$$(1+\triangle tL_1 + a\triangle tL_2)L_1 = (1+\triangle t)L_1, \tag{84}$$

$$(1+\rho\triangle tL_2 + \rho b\triangle tL_1)L_2 = (1+\rho\triangle t)L_2, \tag{85}$$

$$(1+\triangle tL_4 + a\triangle tL_3)L_4 = (1+\triangle t)L_4, \tag{86}$$

$$(1+\rho\triangle tL_3 + \rho b\triangle L_4)L_3 = (1+\rho\triangle t)L_3. \tag{87}$$

Solving system (84)-(85) and system (86)-(87), and since $L_1, L_2, L_3, L_4 > 0$, we have

$$L_1 = L_4 = \frac{1-a}{1-ab} \quad \text{and} \quad L_2 = L_3 = \frac{1-b}{1-ab}. \tag{88}$$

□

Theorem 3.5. *Define K_1, K_2, K_3 and K_4 by*

$$K_1 = \min u_0(x), \;\; K_2 = \max u_0(x), \;\; K_3 = \min v_0(x), \;\; \text{and } K_4 = \max v_0(x).$$

Suppose that $0 < a,\ b < 1$, and that $0 < K_1 \le K_2 \le 1/b$ and $0 < K_3 \le K_4 \le 1/a$. Then the solution (u_i^k, v_i^k) to system (17)-(18) satisfies

$$\lim_{k\to\infty} u_i^k = \frac{1-a}{1-ab} \quad \text{and} \quad \lim_{k\to\infty} v_i^k = \frac{1-b}{1-ab} \tag{89}$$

for $i = 0, 1, \ldots, N$.

Proof. Since $0 < K_1 \le K_2 \le 1/b$ and $0 < K_3 \le K_4 \le 1/a$, we may choose a point $(\underline{u}^0, \bar{v}^0)$ in Region I and another point $(\bar{u}^0, \underline{v}^0)$ in Region II, both as shown in Figure 2, such that

$$\underline{u}^0 \le K_1, \;\; \bar{v}^0 \ge K_4, \;\; \underline{v}^0 \le K_3, \;\; \text{and } \bar{u}^0 \ge K_1.$$

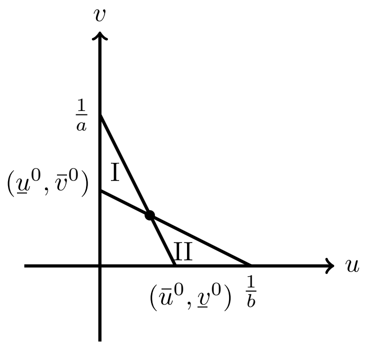


Figure 2: Regions I and II with initial conditions as described in Theorem 3.5.

Using Lemma 3.4, we have

$$\lim_{k\to\infty} \bar{u}^k = \lim_{k\to\infty} \underline{u}^k = \frac{1-a}{1-ab} \quad \text{and} \quad \lim_{k\to\infty} \bar{v}^k = \lim_{k\to\infty} \underline{v}^k = \frac{1-b}{1-ab} \tag{90}$$

for $i = 0, 1, \ldots, N$.

Since $\bar{u}^k, \underline{u}^k, \bar{v}^k$, and $\underline{v}^k$ also satisfy (48)-(51), $(\bar{u}^k, \bar{v}^k)$ and $(\underline{u}^k, \underline{v}^k)$ are an upper solution and a lower solution, respectively, of system (13)-(14). By Theorem 3.3, we have

$$\underline{u}^k \le u_i^k \le \bar{u}^k \quad \text{and} \quad \underline{v}^k \le v_i^k \le \bar{v}^k \tag{91}$$

for $i = 0, \ldots, N$ and $k = 0, 1, 2, \ldots$.

Letting $k \to \infty$ and using the squeezing theorem, we have

$$\lim_{k\to\infty} u_i^k = \frac{1-a}{1-ab} \quad \text{and} \quad \lim_{k\to\infty} v_i^k = \frac{1-b}{1-ab}$$

for $i = 0, 1, \ldots, N$.

□

In the next lemma, we test what happens when the interaction rate of one species exceeds unity. We confirm that the other species is driven to extinction.

Lemma 3.6. *Suppose that* $0 < b < 1$, $a > 1$, $\rho > 0$, *and that* $\bar{u}^0 > 0$ *and* $\underline{v}^0 > 0$ *satisfy*

$$1 - \bar{u}^0 - a\underline{v}^0 < 0 \quad \textit{and} \quad 1 - \underline{v}^0 - b\bar{u}^0 > 0.$$

Suppose further that $(\bar{u}^k, \underline{v}^k)$ *satisfies the system of difference equations*

$$\begin{aligned} (1 + \triangle t\bar{u}^k + a\triangle t\underline{v}^k)\bar{u}^{k+1} &= (1 + \triangle t)\bar{u}^k \\ (1 + \rho\triangle t\underline{v}^k + \rho b\triangle t\bar{u}^k)\underline{v}^{k+1} &= (1 + \rho\triangle t)\underline{v}^k \end{aligned} \tag{92}$$

for $k = 0, 1, 2, \ldots$. *Then the following are satisfied:*

(a) $\bar{u}^k$, $\underline{v}^k > 0$, $1 - \bar{u}^k - a\underline{v}^k < 0$, *and* $1 - \underline{v}^k - b\bar{u}^k > 0$*;*

(b) $\bar{u}^{k+1} < \bar{u}^k$ *and* $\underline{v}^{k+1} > \underline{v}^k$*; and*

(c) $\lim_{k\to\infty} \bar{u}^k = 0$ *and* $\lim_{k\to\infty} \underline{v}^k = 1$.

Proof. As before, we proceed to prove (a) by induction; note Figure 3 throughout the proof of this lemma. By assumption, the statement is true for $k = 0$.

Assume the statement is true for $k > 0$. Then for $k + 1$, using (65) and (66) as before,

$$1 - \bar{u}^{k+1} - a\underline{v}^{k+1} < 0$$

is equivalent to

$$\begin{aligned} (1-\bar{u}^k - a\underline{v}^k) + \triangle t[\rho\underline{v}^k + b\rho\bar{u}^k + a\underline{v}^k - \rho\bar{u}^k\underline{v}^k - b\rho(\bar{u}^k)^2 - \\ a\rho\underline{v}^k - a\bar{u}^k\underline{v}^k - a^2(\underline{v}^k)^2] + (\triangle t)^2 a\rho\underline{v}^k(\underline{v}^k + b\bar{u}^k - \bar{u}^k - a\underline{v}^k) < 0. \end{aligned} \tag{93}$$

Then since

$$1 - \bar{u}^k - a\underline{v}^k < 0 \quad \text{and} \quad 1 - \underline{v}^k - b\bar{u}^k > 0$$

imply that

$$\underline{v}^k < 1 \quad \text{and} \quad (1 - a)\underline{v}^k < (1 - b)\bar{u}^k. \tag{94}$$

Hence

$$a\rho\underline{v}^k(\underline{v}^k + b\bar{u}^k - \bar{u}^k - a\underline{v}^k) < 0, \tag{95}$$

and we have

$$\begin{aligned} &\rho\underline{v}^k + \rho b\bar{u}^k + a\underline{v}^k - \rho\bar{u}^k\underline{v}^k - \rho b(\bar{u}^k)^2 - a\rho\underline{v}^k - a\bar{u}^k\underline{v}^k - a^2(\underline{v}^k)^2 \\ &= \rho b\bar{u}^k(1 - \bar{u}^k - a\underline{v}_k) + a\underline{v}^k(1 - \bar{u}^k - a\underline{v}_k) + a\rho\underline{v}^k(b\bar{u}^k - 1) + \rho\underline{v}^k(1 - \bar{u}^k) \\ &< \rho b\bar{u}^k(1 - \bar{u}^k - a\underline{v}_k) + a\underline{v}^k(1 - \bar{u}^k - a\underline{v}_k) + a\rho\underline{v}^k(b\bar{u}^k - 1) + a\rho\underline{v}^k\underline{v}^k \\ &= \rho b\bar{u}^k(1 - \bar{u}^k - a\underline{v}_k) + a\underline{v}^k(1 - \bar{u}^k - a\underline{v}_k) + a\rho\underline{v}^k(b\bar{u}^k + v^k - 1) < 0. \end{aligned} \tag{96}$$

This implies that

$$1 - \bar{u}^{k+1} - a\underline{v}^{k+1} < 0.$$

Since similar reasoning we may establish that $1 - \underline{v}^{k+1} - b\bar{u}^{k+1} > 0$, mathematical induction implies that part (a) is true for all k, $k = 0, 1, 2, \ldots$.

To prove part (b), we use the system of difference equations

$$\begin{aligned} \bar{u}^{k+1} - \bar{u}^k &= \frac{\triangle \bar{u}^k(1 - \bar{u}^k - a\underline{v}^k)}{1 + \triangle t\bar{u}^k + a\triangle t\underline{v}^k} < 0 \\ \underline{v}^{k+1} - \underline{v}^k &= \frac{\rho\triangle t\underline{v}^k(1 - \underline{v}^k - b\bar{u}^k)}{1 + \rho\triangle t\underline{v}^k + \rho b\triangle t\bar{u}^k} > 0 \end{aligned} \tag{97}$$

from part (a). This leads directly to $\bar{u}^{k+1} < \bar{u}^k$ and $\underline{v}^{k+1} > \underline{v}^k$, completing the proof of (b).

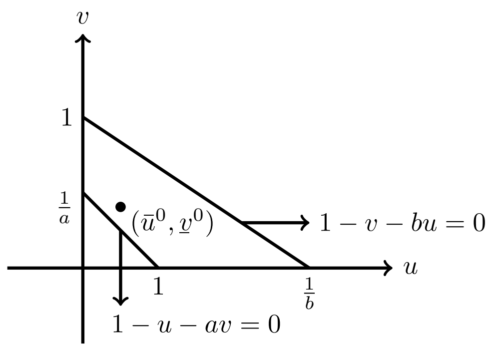


Figure 3: Initial conditions $(\bar{u}^0, \underline{v}^0)$ and the region of interest in Lemma 3.6.

We now turn to part (c) to finish the proof of Lemma 3.6. From part (a),

$$0 < \bar{u}^k < 1/b \quad \text{and} \quad 0 < \underline{v}^k < 1. \tag{98}$$

Then the sequence $\{\bar{u}^k\}$ is decreasing and bounded below, while $\{\underline{v}^k\}$ is increasing and bounded above; hence their limits exist so that there exist constants L_1 and L_2 such that

$$0 \le \lim_{k\to\infty} \bar{u}^k = L_1 \le 1/b < 1 \quad \text{and} \quad \underline{v}^0 \le \lim_{k\to\infty} \underline{v}^k = L_2 \le 1. \tag{99}$$

Taking the limit as $k \to \infty$ in (92), we have

$$\begin{aligned} (1 + \triangle tL_1 + a\triangle tL_2)L_1 &= (1 + \triangle t)L_1 \\ (1 + \rho\triangle tL_2 + \rho b\triangle tL_1)L_2 &= (1 + \rho\triangle t)L_2. \end{aligned} \tag{100}$$

When this limiting system is solved, using $L_1 \ge 0$, $L_2 > 0$, $a > 1$ and $b < 1$, we arrive at

$$L_1 = 0 \quad \text{and} \quad L_2 = 1.$$

This establishes (c).

□

Theorem 3.7. *Define M_1, M_2, M_3 and M_4 by*

$$M_1 = \min u_0(x),\ M_2 = \max u_0(x),\ M_3 = \min v_0(x),\ \textit{and}\ M_4 = \max v_0(x).$$

Assume that $0 < b < 1$, $a > 1$, and that $0 < M_1 \le M_2 \le 1/b$ with $0 < M_3 \le M_4 \le 1$. Then the solution (u_i^k, v_i^k) to system (17)-(18) satisfies

$$\lim_{k\to\infty} u_i^k = 0 \quad \textit{and} \quad \lim_{k\to\infty} v_i^k = 1$$

for all $i = 0, 1, 2, \ldots, N$.

Proof. Since $0 < M_1 \leq M_2 \leq 1/b$ and $0 < M_3 \leq M_4 \leq 1$, we can choose two points $(\underline{u}^0, \bar{v}^0)$ and $(\bar{u}^0, \underline{v}^0)$ in Region I as shown such that

$$\underline{u}^0 = 0,\ \bar{v}^0 = 1,\ \underline{v}^0 \leq M_3,\ \text{and } \bar{u}^0 \geq M_2.$$

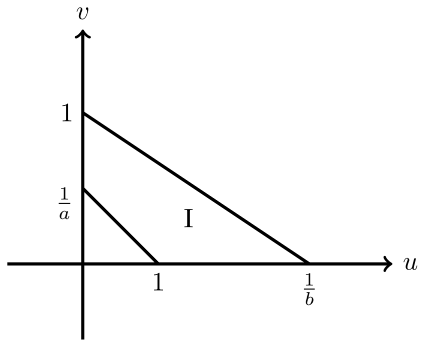


Figure 4: Initial conditions $(\bar{u}^0, \underline{v}^0)$ and the region described in Theorem 3.7.

Applying Lemma 3.6 yields

$$\lim_{k\to\infty} \bar{u}^k = 0 \ \text{ and } \ \lim_{k\to\infty} \underline{v}^k = 1. \tag{101}$$

for $i = 0, 1, 2, \ldots, N$.

Since $\underline{u}^0 = 0$ and $\bar{v}^0 = 1$, from (63)-(64) we have that $\underline{u}^k = 0$ and $\bar{v}^k = 1$. Since all of $\bar{u}^k$, $\underline{u}^k$, $\bar{v}^k$, and $\underline{v}^k$ also satisfy (48)-(51), $(\bar{u}^k, \bar{v}^k)$ and $(\underline{u}^k, \underline{v}^k)$ are an upper solution and a lower solution of system (13)-(14). Drawing on Theorem 3.3, we have the inequalities

$$\underline{u}^k \leq u_i^k \leq \bar{u}^k \ \text{ and } \ \underline{v}^k \leq v_i^k \leq \bar{v}^k \tag{102}$$

for $i = 0, \ldots, N$ and $k = 0, 1, 2, \ldots$. This implies that

$$\lim_{k\to\infty} u_i^k = 0,\ \lim_{k\to\infty} v_i^k = 1.$$

for $i = 0, 1, 2, \ldots, N$.

□

Theorem 3.8. *If $0 < a < 1$, $b > 1$, $\max u_0(x) \leq 1$ and $\max v_0(x) \leq 1/a$, then the solution (u_i^k, v_i^k) to system (17)-(18) satisfies*

$$\lim_{k\to\infty} u_i^k = 1 \ \text{ and } \ \lim_{k\to\infty} v_i^k = 0 \tag{103}$$

for $i = 0, 1, 2, \ldots, N$.

Proof. As the proof is similar to that for Theorem 3.7, we omit here. □

4. The Extension to $\mathbb{R}^2$

It is valuable to discuss the case where $\Omega \subset \mathbb{R}^2$ here to demonstrate why the results carry over in a natural way to higher dimensions. This became necessary to develop the numerical approximation in Section 5 in the case of a two-dimensional domain. To accomplish this, we modify the discretization of (13)-(16) by renaming any u_i and v_i to populations in two dimensions u_{ij} and v_{ij} in (17)-(18).

For a rectangular domain $(-L, L) \times (-M, M) \subset \mathbb{R}^2$, we have

$$\begin{aligned}\Omega_{x,\,y} = \{(x_i, y_j)|\, x_i = -L + i\triangle x,\, y_j = -M + j\triangle y,\\ 1 \le i \le N_x - 1,\, 1 \le j \le N_y - 1\},\end{aligned}$$

where $\triangle x = 2L/N_x$ and $\triangle y = 2M/N_y$.

Then for the discretization of diffusion terms in expanding (17) and (18), and given that w_{ij}^k now means $w(t_k, x_i, y_j)$, we include on the right-hand sides

$$\frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{(\triangle y)^2} \quad \text{and} \quad \frac{v_{i,j+1}^{k+1} - 2v_{i,j}^{k+1} + v_{i,j-1}^{k+1}}{(\triangle y)^2}, \tag{104}$$

respectively. We also replace r by $r_x = \frac{\Delta t}{(\Delta x)^2}$ and $r_y = \frac{\Delta t}{(\Delta y)^2}$ and leave r_m unchanged. We must include Neumann boundary conditions (15) extending to u and v along all four boundaries in the natural way. Then taking all terms with time step $k+1$ to one side gives rise to the replacement of (19) by the system of equations

$$G_u^{(k)}\mathbf{U}^{(k+1)} = \mathbf{H}_u^{(k)} \quad \text{and} \quad G_v^{(k)}\mathbf{V}^{(k+1)} = \mathbf{H}_v^{(k)} \tag{105}$$

for $i = 0, 1, 2, \ldots, N_x$ and $j = 0, 1, 2, \ldots, N_y$, to be solved for $k = 0, 1, 2, \ldots$.

For $j = 0, 1, 2, \ldots, N_y$, we let $\Lambda_j(k)$ and $\Lambda_j'(k)$ be the tridiagonal, $(N_x+1) \times (N_x+1)$ matrices given by

$$\Lambda_j(k) = \begin{pmatrix} \alpha_{0j}^{(k)} & -2r_x & 0 & \cdots & 0 \\ -r_x & \alpha_{1j}^{(k)} & -r_x & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & -r_x & \alpha_{N_x-1,j}^{(k)} & -r_x \\ 0 & \cdots & 0 & -2r_x & \alpha_{N_x,j}^{(k)} \end{pmatrix} \tag{106}$$

where $\alpha_{ij}^{(k)} = 1 + 2r_x + 2r_y + \triangle t(u_{ij}^k + av_{ij}^k)$ for $i = 0, 1, 2, \ldots, N_x$, and

$$\Lambda_j'(k) = \begin{pmatrix} \beta_{0j}^{(k)} & -2r_m r_x & 0 & \cdots & 0 \\ -r_m r_x & \beta_{1j}^{(k)} & -r_m r_x & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & -r_m r_x & \beta_{N_x-1,j}^{(k)} & -r_m r_x \\ 0 & \cdots & 0 & -2r_m r_x & \beta_{N_x,j}^{(k)} \end{pmatrix}, \tag{107}$$

where $\beta_{ij}^{(k)} = 1 + 2r_m r_x + 2r_m r_y + \rho\triangle t(bu_{ij}^k + v_{ij}^k)$ for $i = 0, 1, 2, \ldots, N_x$.

We then define R_y to be the $(N_x+1) \times (N_x+1)$ diagonal matrix given by $R_y = r_y I_{N_x+1}$, where I_n is the $n \times n$ identity matrix, and we let Θ mean the $(N_x+1) \times (N_x+1)$ zero matrix. Finally, for $j = 0, 1, 2, \ldots, N_y$, and for $W = U$ or $W = V$, we define the $1 \times N_x$ row vectors

$$W_j^{(k')} = (w_{0j}^{k'}, w_{1j}^{k'}, w_{2j}^{k'}, \ldots, w_{N_x,j}^{k'}). \tag{108}$$

With these definitions in hand and setting $M = (N_x + 1) \cdot (N_y + 1)$, in (105) the matrices $G_u^{(k)}$ and $G_v^{(k)}$ are the $M \times M$ block matrices defined by

$$G_u^{(k)} = \begin{pmatrix} \Lambda_0(k) & -2R_y & \Theta & \cdots & \Theta \\ -R_y & \Lambda_1(k) & -R_y & & \Theta \\ & \ddots & \ddots & \ddots & \\ \Theta & & -R_y & \Lambda_{N_y-1}(k) & -R_y \\ \Theta & \cdots & \Theta & -2R_y & \Lambda_{N_y}(k) \end{pmatrix} \quad \text{and} \tag{109}$$

$$G_v^{(k)} = \begin{pmatrix} \Lambda_0'(k) & -2r_m R_y & \Theta & \cdots & \Theta \\ -2r_m R_y & \Lambda_1'(k) & -r_m R_y & & \Theta \\ & \ddots & \ddots & \ddots & \\ \Theta & & -r_m R_y & \Lambda_{N_y-1}'(k) & -r_m R_y \\ \Theta & \cdots & \Theta & -2r_m R_y & \Lambda_{N_y}'(k) \end{pmatrix}. \tag{110}$$

Drawing on $U_j^{(k+1)}$ and $V_j^{(k+1)}$ as the row vectors described in (108), we define the two block $\mathbb{R}^M$ vectors $\mathbf{U}^{(k+1)}$ and $\mathbf{V}^{(k+1)}$ in (105) by

$$\mathbf{U}^{(k+1)} = \left(U_0^{(k+1)} \,|\, U_1^{(k+1)} \,|\, \ldots \,|\, U_{N_y}^{(k+1)}\right)^T \quad \text{and}$$
$$\mathbf{V}^{(k+1)} = \left(V_0^{(k+1)} \,|\, V_1^{(k+1)} \,|\, \ldots \,|\, V_{N_y}^{(k+1)}\right)^T.$$

We can now finish definitions of the individual parts of (105) by defining

$$\mathbf{H}_u^{(k)} = (1 + \triangle t) \cdot \mathbf{U}^{(k)} \quad \text{and} \quad \mathbf{H}_v^{(k)} = (1 + \rho \triangle t) \cdot \mathbf{V}^{(k)}.$$

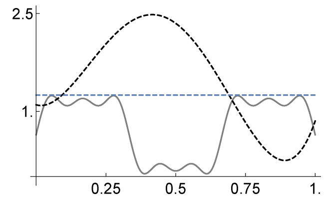
We observe that $\Lambda_j(0)$ and $\Lambda'(0)$ in (106)-(107) are both positive definite for each $j = 0, 1, 2, \ldots, N_y$ so that block diagonal elements of $G_u(0)$ and $G_v(0)$ in (109)-(110) are positive definite. In addition, the off-diagonal blocks of $G_u(0)$ and $G_v(0)$ are negative matrices and the row sums are positive, so that the higher-dimensional analog of the result of Theorem 2.3 holds for $\Omega = (-L, L) \times (-M, M) \subset \mathbb{R}^2$ as well as for $\Omega = (-L, L) \subset \mathbb{R}$. That is, mathematical induction can be used to (19) in order to arrive at the result for (105) as that for (19): For the two dimensional case, existence of a solution to the system of difference equations, as well as its uniqueness, positivity, and uniform boundedness of u_{ij}^k and v_{ij}^k, for all $i = 0, 1, 2, \ldots, N_x$ and $j = 0, 1, 2, \ldots, N_y$ follows for $k = 0, 1, 2, \ldots$.

5. Numerical results

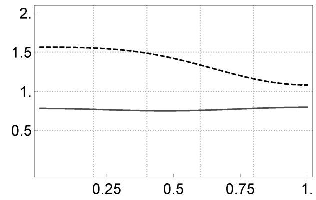
Now that we have verified the stability and convergence of the numerical scheme and studied the long term behavior of the numerical solution, and now that we have shown that the numerical solution preserves the properties of the theoretical solution, in this section we examine the results from some computational experiments.

For all of the following results, we have chosen $\rho = 0.7$ and $r_m = 0.8$ in (9)-(12). While we have varied Δt as a convergence test in Table 1, there is no restriction on the size of Δt for any mesh. For one-dimensional graphs, we compare over a mesh of 50 spatial subintervals.

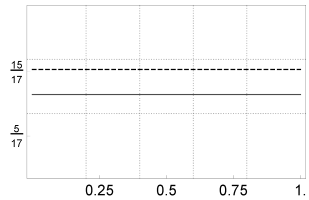
We consider the case of coexistent, competing species over a one-dimensional domain $0 \le x \le 1$. We have chosen $u(0, x) = \frac{77}{1800}(15 + 15 \sin 3\pi x + 5 \sin 9\pi x + 3 \sin 15\pi x)$ and $v(0, x) = \frac{1}{20}(980x^4 - 1730x^3 + 770x^2 - 25x + 22)$ and $a = 2/5$ and



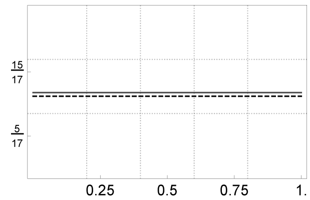
(a) Initial conditions $u(0, x)$ and $v(0, x)$



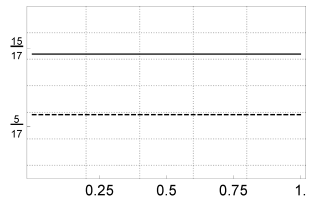
(b) Approximation to $u(0.1, x)$ and $v(0.1, x)$



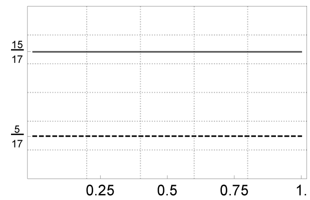
(c) Approximation to $u(1.0, x)$ and $v(1.0, x)$



(d) Approximation to $u(2.5, x)$ and $v(2.5, x)$



(e) Approximation to $u(10, x)$ and $v(10, x)$



(f) Approximation to $u(40, x)$ and $v(40, x)$

Figure 5: The limits for species coexistence predicted by Theorem 3.5 for $\Delta t = 0.1$, $a = 2/5$, $b = 4/5$; u is solid and v is dashed.

$b = 4/5$, since if $0 < u_0(x) \le 5/4$ and $0 < v_0(x) \le 5/2$ for $0 \le x \le 1$, then by Theorem 3.5 we have that

$$\lim_{k\to\infty} u_i^k = \frac{1-b}{1-ab} = \frac{5}{17} \quad \text{and} \quad \lim_{k\to\infty} v_i^k = \frac{1-a}{1-ab} = \frac{15}{17}$$

for $i = 0, 1, 2, \ldots, N$.

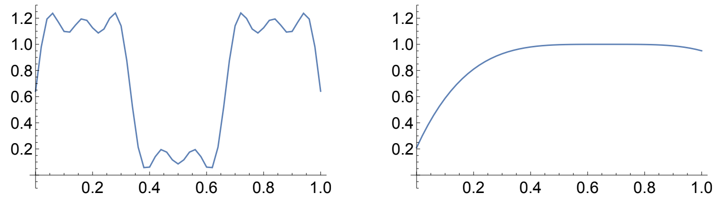
We compare the numerical solutions for various choices of $\triangle t$ and different $\triangle x$ in Cases 1 and 2; the results are summarized in Tables 1 and 2.

Case 1: Fix $\triangle x = 0.02$ and vary $\triangle t$.

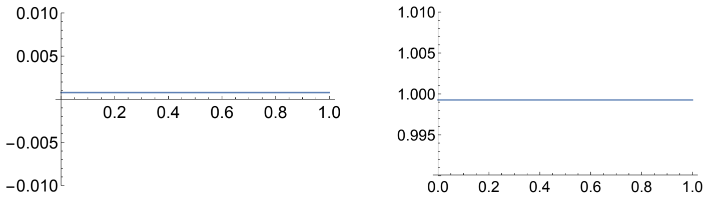
Let $(u(t, x), v(t, x))$ represent the numerical solution corresponding to $\triangle t$ and let $(u_1(t, x), v_1(t, x))$ represent the numerical solution corresponding to $\triangle t_1$. The the differences between $u(t, x)$ and $u_1(t, x)$ and between $v(t, x)$ and $v_1(t, x)$ are shown at $t = 20$.

Table 1: Solution differences for varied $\triangle t$.

$\triangle t$	$\triangle t_1$	$\max\lvert u(20,x) - u_1(20,x)\rvert$	$\max\lvert v(20,x) - v_1(20,x)\rvert$
0.1	0.05	0.0116	0.0242
0.01	0.005	0.0099	0.0207



(a) Initial condition $u(0,x)$ (b) Initial condition $v(0,x)$



(c) Approximation to $u(60,x)$ (d) Approximation to $v(60,x)$

Figure 6: The limits of 0 and 1 being approached as $t \to \infty$ as predicted in Theorem 3.7 for $\Delta t = 0.1$, $a = 10/9$, $b = 4/5$.

CASE 2: Fix $\triangle t = 0.01$ and vary $\triangle x$.

Let $(u(t,x), v(t,x))$ represent the numerical solution corresponding to $\triangle x$ and let $(u_1(t,x), v_1(t,x))$ represent the numerical solution corresponding to $\triangle x_1$. The differences between $u(t,x)$ and $u_1(t,x)$ and between $v(t,x)$ and $v_1(t,x)$ are shown at $t = 20$.

Table 2: Solution differences for varied $\triangle x$.

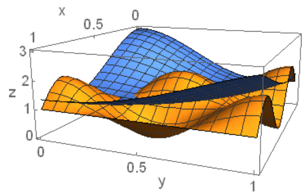
$\triangle x$	$\triangle x_1$	$\max\lvert u(20,x) - u_1(20,x)\rvert$	$\max\lvert v(20,x) - v_1(20,x)\rvert$
0.1	0.05	1.5471e-04	3.1988e-04
0.05	0.025	2.9465e-05	6.0942e-05

The order of convergence in Tables 1-2 follows the convergence results of Theorem 2.5. Graphs provided in Figure 5 - Figure 7 confirm limit results in Theorems 3.5, 3.7, and 3.8.

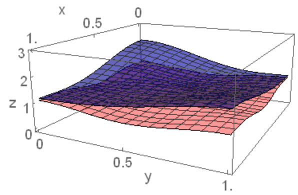
In Figure 6, we consider the case of one competitor dominating the other over the one-dimensional domain of existence $0 \le x \le 1$. We have chosen $u(0,x) = \frac{77}{1800}(15 + 15\sin 3\pi x + 5\sin 9\pi x + 3\sin 15\pi x)$ again for one initial condition that shows the species with population $u(t,x)$ is concentrated closer to the boundary of the domain, which function is used due to being a truncated Fourier series approximation to a step function. Now, however, we choose $v(0,x) = 1 - 4(x - \frac{2}{3})^4 < 1$ so that $a = 10/9$ and $b = 4/5$ since if $0 < u_0(x) \le 5/4$ and $0 < v_0(x) \le 1$ for $0 \le x \le 1$. Then, by Theorem 3.7, we have that

$$\lim_{k\to\infty} u_i^k = 0 \quad \text{and} \quad \lim_{k\to\infty} v_i^k = 1$$

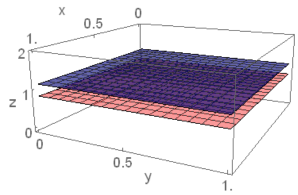
for $i = 0, 1, 2, \ldots, N$. Once again, the theorem predicts the results that are demonstrated in Figure 6: the extinction of the species whose population is given by $u(t,x)$ and uniform distribution of the competing species across the domain.



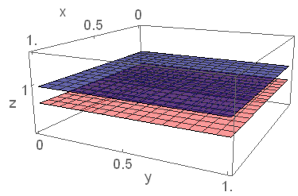
(a) Initial conditions $u(0, x, y)$ and $v(0, x, y)$



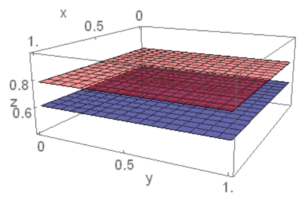
(b) Approximations of $u(0.05, x, y)$ and $v(0.05, x, y)$



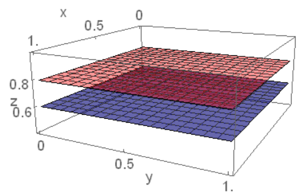
(c) Approximation of $u(0.5, x, y)$ and $v(0.5, x, y)$



(d) Approximations of $u(1.0, x, y)$ and $v(1.0, x, y)$



(e) Approximation of $u(10.0, x, y)$ and $v(10.0, x, y)$



(f) Approximations of $u(15.0, x, y)$ and $v(15.0, x, y)$

Figure 7: The case of coexistence for $\Omega \in \mathbb{R}^2$ in Theorem 3.5 over $(x, y) \in [0, 1] \times [0, 1]$ for $\Delta t = 0.01$, $a = 1/3$, $b = 1/2$.

Finally, in Figure 7 we consider the case of competing species over a two-dimensional square domain $[0,1]\times[0,1]$. We have chosen $u(0,x,y) = 1+0.9\sin 3\pi x \cos 2\pi y$ and $v(0,x,y) = 2+\cos(4\sqrt{x^2+y^2})$ so that if we choose $a = 1/3$ and $b = 1/2$ then following Theorem 3.7, we see that $0 < u(0,x,y) \leq 2$ and $0 < v(0,x,y) \leq 3$. Hence we have that

$$\lim_{k\to\infty} u_i^k = \frac{1-b}{1-ab} = \frac{4}{5} \quad \text{and} \quad \lim_{k\to\infty} v_i^k = \frac{1-a}{1-ab} = \frac{3}{5}$$

for $i = 0,1,2,\ldots,N$. In this case, due to heavier computational cost of solving an $(N_x+1)\times(N_y+1)$ matrix system at each time step, we have used $\Delta t = \Delta x = 0.05$. Despite being a much bigger system than for the for the one-dimensional case, in *Mathematica* we were able to generate the data shown in Figure 7 in less than ten seconds. Once again, Theorem 3.5 predicts what is demonstrated in the numerical approximation as $k \to \infty$.

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