INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 2, Pages 172–194

ON THE LINEAR THERMOELASTICITY WITH TWO POROSITIES: NUMERICAL ASPECTS

NOELIA BAZARRA, JOSÉ R. FERNÁNDEZ, MARI CARME LESEDUARTE, ANTONIO MAGAÑA, AND RAMÓN QUINTANILLA

Abstract. In this work we analyze, from the numerical point of view, a dynamic problem involving a thermoelastic rod. Two porosities are considered: the first one is the macro-porosity, connected with the pores of the material, and the other one is the micro-porosity, linked with the fissures of the skeleton. The mechanical problem is written as a set of hyperbolic and parabolic partial differential equations. An existence and uniqueness result and an energy decay property are stated. Then, a fully discrete approximation is introduced using the finite element method and the backward Euler scheme. A discrete stability property and a priori error estimates are proved, from which the linear convergence of the algorithm is derived under suitable additional regularity conditions. Finally, some numerical simulations are presented to show the behaviour of the approximation.

Key words. Thermoelasticity with two porosities, finite elements, a priori error estimates, numerical simulations.

1. Introduction

The study of thermoelastic problems with double porosity has become a topic of increasing interest during the last twenty years. Some possible applications of this kind of models have been found, for instance, in geophysics or in biomechanics (bones) [1, 2, 3, 4, 5]. The main idea of this model is to consider two porosities: the first one is the macro-porosity, which is connected with the pores of the material, meanwhile the second one is related to the fissures in the skeleton. Straughan [4] pointed out that "a good example of this may be seen in the pictures in [6] where they show a pile of rocks, but the rocks themselves are full of fissures (or cracks), and the macro porosity degrades over a period of ten years leaving a pile of finer material characteristic of the micro porous structure". It is usual to find relations of this theory with the law of Darcy, and the presentation of the theory involves displacement, pressure associated with the pores and pressure associated with the fissures [4, 7, 8].

Since the first works of Barenblatt et al. [1, 9], a large number of papers have been published dealing with mathematical issues as the existence and uniqueness of solutions or the energy decay (see, for instance, [2, 4, 7, 8, 10, 11, 12, 13, 14, 15]).

To describe the behaviour of porous solids materials some proposals have been stated. Nowadays, the theory proposed by Nunziato and Cowin [16] is commonly accepted as one of the non-classical elasticity theories. Grosso modo, it is supposed that in the materials there is a skeleton or material matrix that is elastic, and the interstices are voids in the material. A lot of contributions can be found dealing with this theory, even with applications to geological materials such as rocks and soils or to manufactured materials such as ceramics and pressed powders [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

Received by the editors October 31, 2018.
2000 *Mathematics Subject Classification.* 65M60, 65M15, 65M12, 74K10, 74F05.

Based on the theory of Nunziato and Cowin, and following a rational process (not just intuitive), Ieşan and Quintanilla [28] set a model where it is supposed the existence of two porous structures, one associated with the material pores and the other with the microporosity. The material skeleton supporting both structures and the interactions between them is described by the constitutive equations. Both structures have influences on the elastic deformations of the material matrix and also over the heat conduction through the material. That means the porous structures and the heat conduction are strongly coupled. This alternative approach is currently under research and several qualitative results have been obtained [17, 29, 30, 31]. Moreover, we can see our theory as a particular sub-case of the theory proposed in [32]. Notice that the theory we consider here coincides with the classical thermoelasticity theory if no porous structures are considered.

We want to highlight two issues. The first one is the novelty of the model, and the second one is that our approach is mainly theoretical. From our point of view, we believe that any theory needs a mathematical and physical analysis that allows to decide its applications to the real-world situations. Our paper is addressed in this line. We also want to remark the similarities, from a mathematical perspective, between the equations for elastic materials with double porosity and those for microstretch materials. That means that the equations that we study in this paper can also be viewed as the equations used to describe a mixture of microstretch materials if their macroscopic structures coincide.

Ieşan and Quintanilla [28] introduced only the thermal dissipation in their model. We consider dissipation also in the porous structures. To be precise, we will consider one dissipative mechanism on each porous structure.

We restrict our attention to the one-dimensional problem. This work is parallel to [33], where the existence of a unique solution and an energy decay property were proved. Here, we provide the numerical analysis of the corresponding variational problem, obtaining a discrete stability property, proving some a priori error estimates and performing some numerical simulations which show the behaviour of the solution.

The paper is outlined as follows. The mathematical model is briefly described in Section 2 following the parallel contribution [33], deriving its variational formulation. An existence and uniqueness result, and an energy decay property, are also stated. Then, in Section 3 a fully discrete approximation is introduced, based on the finite element method and the backward Euler scheme. A discrete stability property is proved, a priori error estimates are obtained for the approximative solutions and, under suitable regularity assumptions, the linear convergence of the algorithm is derived. Finally, some numerical simulations are presented in Section 4

2. The model and its variational formulation

In this section, following [33] we describe briefly the model, derive its variational formulation and state the main results (see [33] for further details).

Let us denote by $[0,\ell]$, $\ell>0$, and $[0,T]$, $T>0$, the one-dimensional rod of length ℓ and the time interval of interest, respectively. Moreover, let $x\in[0,\ell]$ and $t\in[0,T]$ be the spatial and time variables. In order to simplify the writing, we do not indicate the dependence of the functions on x and t, the time derivatives are denoted by one (first-order) or two (second-order) dots over a variable and the subscript x under a variable represents its spatial derivative.

Therefore, the mechanical problem of a one-dimensional thermoelastic rod with double porosity is written as follows (see [33]).

Problem P. *Find the longitudinal displacement field* $u : [0,\ell]\times[0,T]\to\mathbb{R}$*, the macroscopic porosity field* $\phi : [0,\ell]\times[0,T]\to\mathbb{R}$*, the microscopic porosity field* $\psi : [0,\ell]\times[0,T]\to\mathbb{R}$ *and the temperature field* $\theta : [0,\ell]\times[0,T]\to\mathbb{R}$ *such that*

$$(1)\quad \rho\ddot{u} = \mu u_{xx} + b\phi_x + d\psi_x - \beta\theta_x \quad in \quad (0,\ell)\times(0,T),$$

$$(2)\quad \kappa_1\ddot{\phi} = \alpha\phi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\phi - \alpha_3\psi + \gamma_1\theta - \varepsilon_1\dot{\phi} - \varepsilon_2\dot{\psi} \quad in \quad (0,\ell)\times(0,T),$$

$$(3)\quad \kappa_2\ddot{\psi} = b_1\phi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\phi - \alpha_2\psi + \gamma_2\theta - \varepsilon_3\dot{\phi} - \varepsilon_4\dot{\psi} \quad in \quad (0,\ell)\times(0,T),$$

$$(4)\quad c\dot{\theta} = \kappa\theta_{xx} - \beta\dot{u}_x - \gamma_1\dot{\phi} - \gamma_2\dot{\psi} \quad in \quad (0,\ell)\times(0,T),$$

$$(5)\quad u(0,t) = u(\ell,t) = 0, \quad \phi(0,t) = \phi(\ell,t) = 0 \quad for\ a.e. \quad t\in(0,T),$$

$$(6)\quad \psi(0,t) = \psi(\ell,t) = 0, \quad \theta(0,t) = \theta(\ell,t) = 0 \quad for\ a.e. \quad t\in(0,T),$$

$$(7)\quad u(x,0) = u_0(x), \quad \phi(x,0) = \phi_0(x), \quad \psi(x,0) = \psi_0(x) \quad for\ a.e.\ x\in(0,\ell),$$

$$(8)\quad \dot{u}(x,0) = v_0(x), \quad \dot{\phi}(x,0) = e_0(x), \quad \dot{\psi}(x,0) = \xi_0(x) \quad for\ a.e.\ x\in(0,\ell),$$

$$(9)\quad \theta(x,0) = \theta_0(x) \quad for\ a.e.\ x\in(0,\ell).$$

Here, ρ denotes the mass density, κ is the thermal conductivity, c denotes the thermal capacity, κ_1 and κ_2 represent the coefficients of inertia for each porous structure and the remaining constants are constitutive constants of the material. In particular ε_i, $i = 1,2,3,4$, correspond to the porous dissipation.

In order to obtain the variational formulation of Problem P, let $Y = L^2(0,\ell)$ and $E = H^1(0,\ell)$ and denote by $(\cdot,\cdot)$ the scalar product in the space Y, with corresponding norm $\|\cdot\|$. Moreover, let us define the variational space V as follows,

$$V = \{v\in H^1(0,\ell)\,;\, v(0) = v(\ell) = 0\},$$

with scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$.

By using the integration by parts and the above Dirichlet boundary conditions at $x = 0,\ \ell$, we write the variational formulation of the corresponding thermomechanical problem, based on Problem P, in terms of the velocity $v = \dot{u}$, the macroscopic porosity speed $e = \dot{\phi}$, the microscopic porosity speed $\xi = \dot{\psi}$ and the temperature θ.

Problem VP. *Find the velocity field* $v : [0,T]\to V$*, the macroscopic porosity speed field* $e : [0,T]\to V$*, the microscopic porosity speed field* $\xi : [0,T]\to V$ *and the temperature field* $\theta : [0,T]\to V$ *such that* $v(0) = v_0$*,* $e(0) = e_0$*,* $\xi(0) = \xi_0$*,* $\theta(0) = \theta_0$*, and, for a.e.* $t\in(0,T)$ *and for all* $z,\ r,\ l,\ m\in V$*,*

$$(10)\quad \rho(\dot{v}(t),z) + \mu(u_x(t),z_x) = b(\phi_x(t),z) + d(\psi_x(t),z) - \beta(\theta_x(t),z),$$

$$(11)\quad \begin{aligned}&\kappa_1(\dot{e}(t),r) + \alpha(\phi_x(t),r_x) + b_1(\psi_x(t),r_x) + \varepsilon_1(e(t),r) + \alpha_1(\phi(t),r) + \alpha_3(\psi(t),r)\\ &\quad +\varepsilon_2(\xi(t),r) = -b(u_x(t),r) + \gamma_1(\theta(t),r),\end{aligned}$$

$$(12)\quad \begin{aligned}&\kappa_2(\dot{\xi}(t),l) + b_1(\phi_x(t),l_x) + \gamma(\psi_x(t),l_x) + \alpha_3(\phi(t),l) + \alpha_2(\psi(t),l) + \varepsilon_4(\xi(t),l)\\ &\quad +\varepsilon_3(e(t),l) = -d(u_x(t),l) + \gamma_2(\theta(t),l),\end{aligned}$$

$$(13)\quad c(\theta_t(t),m) + \kappa(\theta_x(t),m_x) = -\gamma_1(e(t),m) - \gamma_2(\xi(t),m) - \beta(v_x(t),m),$$

where the displacement, macroscopic porosity and microscopic porosity fields are then recovered from the relations

$$(14) \quad u(t) = \int_0^t v(s)\,ds + u_0, \quad \phi(t) = \int_0^t e(s)\,ds + \phi_0, \quad \psi(t) = \int_0^t \xi(s)\,ds + \psi_0.$$

We assume the following conditions on the constitutive constants to guarantee that the internal mechanical energy W defined as

$$W = \mu|u_x|^2 + 2bu_x\phi + 2du_x\psi + \alpha|\phi_x|^2 + \gamma|\psi_x|^2 + 2b_1\phi_x\psi_x + \alpha_1|\phi|^2 + \alpha_{\mathbf{2}}|\psi|^{\mathbf{2}} + 2\alpha_3\phi\psi$$

is positive:

$$(15) \quad \mu > 0, \quad \mu\alpha_1 > b^2, \quad \alpha\gamma > b_1^2, \quad \mu\alpha_1\alpha_2 + 2bd\alpha_3 - d^2\alpha_1 - b^2\alpha_2 - \alpha_3^2\mu > 0.$$

We note that from these relations we also obtain that

$$(16) \qquad \mu\alpha_2 > d^2, \quad \alpha_1\alpha_2 > \alpha_3^2.$$

The mechanical dissipation of the system is given by

$$(17) \qquad D_1 = \varepsilon_1|\dot{\varphi}|^2 + \varepsilon_4|\dot{\psi}|^2 + 2(\varepsilon_2 + \varepsilon_3)\dot{\varphi}\dot{\psi}.$$

In our mathematical study it also plays a relevant role in the expression

$$(18) \qquad D_2 = \kappa|\theta_x|^2$$

and, in general, $D^* = D_1 + D_2$. If we want that D^* to be positive, we need to impose that

$$(19) \qquad \kappa > 0, \ \varepsilon_1\varepsilon_4 > \frac{1}{4}(\varepsilon_2 + \varepsilon_3)^2, \ \varepsilon_1 > 0.$$

The following result is proved in [33].

Theorem 2.1. *Let the assumptions (15)-(19) hold and assume that* ρ, c, κ_1, $\kappa_2 > 0$ *and* u_0, v_0, ϕ_0, e_0, ψ_0, ξ_0, $\theta_0 \in E$. *Therefore, Problem VP has a unique solution with the following regularity:*

$$u \in C^2([0,T];Y) \cap C^1([0,T];E), \quad \theta \in C^1([0,T];Y) \cap C([0,T];E),$$
$$\varphi \in C^2([0,T];Y) \cap C^1([0,T];E), \quad \psi \in C^2([0,T];Y) \cap C^1([0,T];E).$$

Moreover, this solution is asymptotically stable.

3. Fully discrete approximations: a priori error estimates

In this section, we will provide the numerical analysis of Problem VP, introducing a fully discrete approximation. This is done in two steps. First, we assume that the interval $[0,\ell]$ is divided into M subintervals $a_0 = 0 < a_1 < \ldots < a_M = \ell$ of length $h = a_{i+1} - a_i = \ell/M$ and so, we construct the finite dimensional space $V^h \subset V$, approximating the variational space V, given by

$$(20) \qquad \begin{aligned} V^h = \{w^h \in C([0,\ell]) \ ; \ w^h_{|_{[a_i,a_{i+1}]}} \in P_1([a_i,a_{i+1}]) \quad i = 0,\ldots,M-1, \\ w^h(0) = w^h(\ell) = 0\}, \end{aligned}$$

where $P_1([a_i,a_{i+1}])$ represents the space of polynomials of degree less or equal to one in the subinterval $[a_i,a_{i+1}]$; i.e. the finite element space V^h is composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by u_0^h, v_0^h, ϕ_0^h, e_0^h, ψ_0^h, ξ_0^h and θ_0^h, are given by

$$(21) \qquad \begin{aligned} u_0^h = \mathcal{P}^h u_0, \quad v_0^h = \mathcal{P}^h v_0, \quad \phi_0^h = \mathcal{P}^h \phi_0, \quad e_0^h = \mathcal{P}^h e_0, \\ \psi_0^h = \mathcal{P}^h \psi_0, \quad \xi_0^h = \mathcal{P}^h \xi_0, \quad \theta_0^h = \mathcal{P}^h \theta_0, \end{aligned}$$

where $\mathcal{P}^h$ is the $L^2(0,\ell)$-projection operator over V^h (see [34]).

Secondly, we consider a partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0,1,\dots,N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. *Find the discrete velocity field* $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$, *the discrete macroscopic porosity speed field* $e^{hk} = \{e_n^{hk}\}_{n=0}^N \subset V^h$, *the discrete microscopic porosity speed field* $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset V^h$ *and the discrete temperature field* $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset V^h$ *such that* $v_0^{hk} = v_0^h$, $e_0^{hk} = e_0^h$, $\xi_0^{hk} = \xi_0^h$, $\theta_0^{hk} = \theta_0^h$ *and, for* $n = 1,\dots,N$ *and for all* z^h, r^h, l^h, $m^h \in V^h$,

$$\rho(\delta v_n^{hk}, z^h) + \mu((u_n^{hk})_x, z_x^h) = b((\phi_n^{hk})_x, z^h) + d((\psi_n^{hk})_x, z^h) - \beta((\theta_n^{hk})_x, z^h), \tag{22}$$

$$\begin{aligned}&\kappa_1(\delta e_n^{hk}, r^h) + \alpha((\phi_n^{hk})_x, r_x^h) + b_1((\psi_n^{hk})_x, r_x^h) + \varepsilon_1(e_n^{hk}, r^h) + \alpha_1(\phi_n^{hk}, r^h)\\ &\quad+\varepsilon_2(\xi_n^{hk}, r^h) + \alpha_3(\psi_n^{hk}, r^h) = -b((u_n^{hk})_x, r^h) + \gamma_1(\theta_n^{hk}, r^h),\end{aligned} \tag{23}$$

$$\begin{aligned}&\kappa_2(\delta \xi_n^{hk}, l^h) + b_1((\phi_n^{hk})_x, l_x^h) + \gamma((\psi_n^{hk})_x, l_x^h) + \alpha_3(\phi_n^{hk}, l^h) + \alpha_2(\psi_n^{hk}, l^h)\\ &\quad+\varepsilon_4(\xi_n^{hk}, l^h) + \varepsilon_3(e_n^{hk}, l^h) = -d((u_n^{hk})_x, l^h) + \gamma_2(\theta_n^{hk}, l^h),\end{aligned} \tag{24}$$

$$\begin{aligned}&c(\delta\theta_n^{hk}, m^h) + \kappa((\theta_n^{hk})_x, m_x^h) = -\gamma_1(e_n^{hk}, m^h) - \gamma_2(\xi_n^{hk}, m^h)\\ &\quad-\beta((v_n^{hk})_x, m^h),\end{aligned} \tag{25}$$

where the discrete displacement, macroscopic porosity and microscopic porosity fields, u_n^{hk}, ϕ_n^{hk} and ψ_n^{hk} respectively, are now recovered from the relations

$$u_n^{hk} = k\sum_{j=1}^n v_j^{hk} + u_0^h, \quad \phi_n^{hk} = k\sum_{j=1}^n e_j^{hk} + \phi_0^h, \quad \psi_n^{hk} = k\sum_{j=1}^n \xi_j^{hk} + \psi_0^h. \tag{26}$$

Under the assumptions on the coefficients (see (15)-(19)), using the well-known Lax-Milgram lemma, we can prove that there exists a unique discrete solution to Problem VP^{hk}.

Remark 3.1. We note that we have used the backward Euler scheme for the sake of simplicity in the calculations and the writing. However, we point out that other (conservative) schemes (see, e.g., [35]) could be used but it would complicate the proofs and the stability and energy decay results could not be obtained.

The following stability result is proved doing some algebraic manipulations.

Theorem 3.2. *Under the assumptions of Theorem 2.1, it follows that the sequences* $\{u^{hk}, v^{hk}, \phi^{hk}, e^{hk}, \theta^{hk}, \psi^{hk}\}$ *generated by Problem* VP^{hk} *satisfy the stability estimate:*

$$\begin{aligned}&\|v_n^{hk}\|^2 + \|(u_n^{hk})_x\|^2 + \|e_n^{hk}\|^2 + \|(\phi_n^{hk})_x\|^2 + \|\phi_n^{hk}\|^2 + \|\xi_n^{hk}\|^2 + \|(\psi_n^{hk})_x\|^2\\ &\quad+\|\psi_n^{hk}\|^2 + \|\theta_n^{hk}\|^2 \le C,\end{aligned}$$

where C is a positive constant which is independent of the discretization parameters h and k.

Proof. For the sake of clarity in the writing of this proof, we remove the superscripts h and k in all the variables.

Taking $z^h = v_n$ as a test function in discrete variational equation (22) it follows that

$$\rho(\delta v_n, v_n) + \mu((u_n)_x, (v_n)_x) = b((\phi_n)_x, v_n) + d((\psi_n)_x, v_n) - \beta((\theta_n)_x, v_n).$$

Thus, using the estimates

$$(\delta v_n, v_n) \geq \frac{1}{2k}\left\{\|v_n\|^2 - \|v_{n-1}\|^2\right\},$$
$$((u_n)_x, (v_n)_x) \geq \frac{1}{2k}\left\{\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2\right\},$$

and Cauchy's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \ a, b, \epsilon \in \mathbb{R}, \ \epsilon > 0, \tag{27}$$

we find that

$$\frac{\rho}{2k}\left\{\|v_n\|^2 - \|v_{n-1}\|^2\right\} + \frac{\mu}{2k}\left\{\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2\right\}$$
$$\leq C\Big(\|(\phi_n)_x\|^2 + \|v_n\|^2 + \|(\psi_n)_x\|^2\Big) - \beta((\theta_n)_x, v_n). \tag{28}$$

Taking $m^h = \theta_n$ as a test function in discrete variational equation (25) we obtain

$$c(\delta\theta_n, \theta_n) + \kappa((\theta_n)_x, (\theta_n)_x) = -\gamma_1(e_n, \theta_n) - \gamma_2(\xi_n, \theta_n) - \beta((v_n)_x, \theta_n),$$

and then, using the estimates

$$(\delta\theta_n, \theta_n) \geq \frac{1}{2k}\left\{\|\theta_n\|^2 - \|\theta_{n-1}\|^2\right\},$$
$$-\beta((v_n)_x, \theta_n) = \beta(v_n, (\theta_n)_x),$$

and Cauchy's inequality (27) it follows that

$$\frac{c}{2k}\left\{\|\theta_n\|^2 - \|\theta_{n-1}\|^2\right\} \leq \beta(v_n, (\theta_n)_x) + C\Big(\|\theta_n\|^2 + \|e_n\|^2 + \|\xi_n\|^2\Big). \tag{29}$$

Now, taking $r^h = e_n$ as a test function in discrete variational equation (23) we have

$$\kappa_1(\delta e_n, e_n) + \alpha((\phi_n)_x, e_n) + b_1((\psi_n)_x, (e_n)_x) + \varepsilon_1(e_n, e_n) + \alpha_1(\phi_n, e_n)$$
$$+\varepsilon_2(\xi_n, e_n) + \alpha_3(\psi_n, e_n) = -b((u_n)_x, e_n) + \gamma_1(\theta_n, e_n),$$

and therefore, using the estimates

$$(\delta e_n, e_n) \geq \frac{1}{2k}\left\{\|e_n\|^2 - \|e_{n-1}\|^2\right\},$$
$$((\phi_n)_x, (e_n)_x) \geq \frac{1}{2k}\left\{\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2 + \|(\phi_n - \phi_{n-1})_x\|^2\right\},$$
$$(\phi_n, e_n) = \frac{1}{2k}\left\{\|\phi_n\|^2 - \|\phi_{n-1}\|^2 + \|\phi_n - \phi_{n-1}\|^2\right\},$$

and inequality (27), we find that

$$\frac{\kappa_1}{2k}\left\{\|e_n\|^2 - \|e_{n-1}\|^2\right\} + \frac{\alpha}{2k}\left\{\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2 + \|(\phi_n - \phi_{n-1})_x\|^2\right\}$$
$$+\frac{\alpha_1}{2k}\left\{\|\phi_n\|^2 - \|\phi_{n-1}\|^2 + \|\phi_n - \phi_{n-1}\|^2\right\} + b_1((\psi_n)_x, (e_n)_x)$$
$$+\alpha_3(\psi_n, e_n) \leq C(\|(u_n)_x\|^2 + \|e_n\|^2 + \|\xi_n\|^2) + \gamma_1(\theta_n, e_n). \tag{30}$$

Proceeding in a similar form, we obtain the following estimates for the microscopic porosity field:

$$\frac{\kappa_2}{2k}\left\{\|\xi_n\|^2 - \|\xi_{n-1}\|^2\right\} + \frac{\gamma}{2k}\left\{\|(\psi_n)_x\|^2 - \|(\psi_{n-1})_x\|^2 + \|(\psi_n - \psi_{n-1})_x\|^2\right\}$$
$$+\frac{\alpha_2}{2k}\left\{\|\psi_n\|^2 - \|\psi_{n-1}\|^2 + \|\psi_n - \psi_{n-1}\|^2\right\} + \alpha_3(\phi_n, \xi_n)$$
$$+b_1((\phi_n)_x, (\xi_n)_x) \leq C(\|(u_n)_x\|^2 + \|\xi_n\|^2 + \|e_n\|^2) + \gamma_2(\theta_n, \xi_n). \tag{31}$$

Observing that

$$\begin{aligned}
&b_1((\psi_n)_x,(e_n)_x)+b_1((\phi_n)_x,(\xi_n)_x)\\
&\quad=b_1((\psi_n)_x,(\delta\phi_n)_x)+b_1((\phi_n)_x,(\delta\psi_n)_x)\\
&\quad=\frac{b_1}{k}\big[((\psi_n)_x,(\phi_n)_x)-((\psi_{n-1})_x,(\phi_{n-1})_x)+((\psi_n-\psi_{n-1})_x,(\phi_n-\phi_{n-1})_x)\big],\\
&\alpha_3(\psi_n,e_n)+\alpha_3(\phi_n,\xi_n)=\alpha_3(\psi_n,\delta\phi_n)+\alpha_3(\phi_n,\delta\psi_n)\\
&\quad=\frac{\alpha_3}{k}\big[(\psi_n,\phi_n)-((\psi_{n-1})_x,\phi_{n-1})+((\psi_n-\psi_{n-1})_x,\phi_n-\phi_{n-1})\big],
\end{aligned}$$

and that, using the required conditions (see (15)-19)),

$$\begin{aligned}
&\frac{\alpha}{2k}\|(\phi_n-\phi_{n-1})_x\|^2+\frac{\gamma}{2k}\|(\psi_n-\psi_{n-1})_x\|^2+\frac{b_1}{k}((\psi_n-\psi_{n-1})_x,(\phi_n-\phi_{n-1})_x)\geq 0,\\
&\frac{\alpha_1}{2k}\|(\phi_n-\phi_{n-1})_x\|^2+\frac{\alpha_2}{2k}\|\psi_n-\psi_{n-1}\|^2+\frac{\alpha_3}{k}((\phi_n-\phi_{n-1})_x,\psi_n-\psi_{n-1})\geq 0,
\end{aligned}$$

combining estimates (28), (29), (30) and (31), we find that

$$\begin{aligned}
&\frac{\rho}{2k}\left\{\|v_n\|^2-\|v_{n-1}\|^2\right\}+\frac{\mu}{2k}\left\{\|(u_n)_x\|^2-\|(u_{n-1})_x\|^2\right\}\\
&\quad+\frac{\kappa_1}{2k}\left\{\|e_n\|^2-\|e_{n-1}\|^2\right\}+\frac{\alpha}{2k}\left\{\|(\phi_n)_x\|^2-\|(\phi_{n-1})_x\|^2\right\}\\
&\quad+\frac{\alpha_1}{2k}\left\{\|\phi_n\|^2-\|\phi_{n-1}\|^2\right\}+\frac{b_1}{k}\left\{((\psi_n)_x,(\phi_n)_x)-((\psi_{n-1})_x,(\phi_{n-1})_x)\right\}\\
&\quad+\frac{\kappa_2}{2k}\left\{\|\xi_n\|^2-\|\xi_{n-1}\|^2\right\}+\frac{\gamma}{2k}\left\{\|(\psi_n)_x\|^2-\|(\psi_{n-1})_x\|^2\right\}\\
&\quad+\frac{\alpha_2}{2k}\left\{\|\psi_n\|^2-\|\psi_{n-1}\|^2\right\}+\frac{\alpha_3}{k}\left\{(\psi_n,\phi_n)-(\psi_{n-1},\phi_{n-1})\right\}\\
&\quad+\frac{c}{2k}\left\{\|\theta_n\|^2-\|\theta_{n-1}\|^2\right\}\leq C\Big(\|e_n\|^2+\|v_n\|^2+\|(\psi_n)_x\|^2+\|\xi_n\|^2\Big).
\end{aligned}$$

Summing up to n the previous estimates, it follows that

$$\begin{aligned}
&\rho\|v_n\|^2+\mu\|(u_n)_x\|^2+\kappa_1\|e_n\|^2+\alpha\|(\phi_n)_x\|^2+\alpha_1\|\phi_n\|^2+\kappa_2\|\xi_n\|^2+\gamma\|(\psi_n)_x\|^2\\
&\qquad+\alpha_2\|\psi_n\|^2+2b_1((\psi_n)_x,(\phi_n)_x)+2\alpha_3(\psi_n,\phi_n)\\
&\quad\leq Ck\sum_{j=1}^{n}\Big(\|e_j\|^2+\|v_j\|^2+\|(\psi_j)_x\|^2+\|\xi_j\|^2\Big)+C\big(\|v_0\|^2+\|(u_0)_x\|^2+\|e_0\|^2\\
&\qquad+\|(\phi_0)_x\|^2+\|\phi_0\|^2+\|\theta_0\|^2+\|(\psi_0)_x\|^2+\|\psi_0\|^2\big).
\end{aligned}$$

Now, using again required conditions (16), we can choose $\zeta_1,\ \zeta_2>0$ such that

$$b_1/\gamma<\zeta_1<\alpha/b_1,\quad \alpha_3/\alpha_2<\zeta_2<\alpha_1/\alpha_3,$$

which imply that

$$\begin{aligned}
&\alpha\|(\phi_n)_x\|^2+\gamma\|(\psi_n)_x\|^2+2b_1((\psi_n)_x,(\phi_n)_x)\\
&\qquad\geq(\alpha-b_1\zeta_1)\,\|(\phi_n)_x\|^2+\left(\gamma-\frac{b_1}{\zeta_1}\right)\|(\psi_n)_x\|^2,\\
&\alpha_1\|\phi_n\|^2+\alpha_3\|\psi_n\|^2+2\alpha_3(\phi_n,\psi_n)\geq(\alpha_1-\alpha_3\zeta_2)\,\|\phi_n\|^2+\left(\alpha_2-\frac{\alpha_3}{\zeta_2}\right)\|\psi_n\|^2,
\end{aligned}$$

and applying a discrete version of Gronwall's inequality (see, e.g., [36]) we obtain the desired stability property. □

The following energy decay property is derived from the previous stability theorem.

Corollary 3.3. *If we define the discrete energy at time $t = t_n$ by E_n^{hk} as follows:*

$$\begin{aligned} E_n^{hk} &= \rho\|v_n^{hk}\|^2 + \mu\|(u_n^{hk})_x\|^2 + \kappa_1\|e_n^{hk}\|^2 + \alpha\|(\phi_n^{hk})_x\|^2 + \alpha_1\|\phi_n^{hk}\|^2 \\ &+\kappa_2\|\xi_n^{hk}\|^2 + \gamma\|(\psi_n^{hk})_x\|^2 + \alpha_2\|\psi_n^{hk}\|^2 + c\|\theta_n^{hk}\|^2 + 2b_1((\phi_n^{hk})_x, (\psi_n^{hk})_x) \\ &+2\alpha_3(\phi_n^{hk}, \psi_n^{hk}) + 2b((u_n^{hk})_x, \phi_n^{hk}) + 2d((u_n^{hk})_x, \psi_n^{hk}), \end{aligned} \tag{32}$$

then we have

$$\frac{E_n^{hk} - E_{n-1}^{hk}}{k} \le 0.$$

In the rest of this section, we will derive some a priori error estimates for the numerical errors $u_n - u_n^{hk}$, $v_n - v_n^{hk}$, $\phi_n - \phi_n^{hk}$, $e_n - e_n^{hk}$, $\psi_n - \psi_n^{hk}$, $\xi_n - \xi_n^{hk}$ and $\theta_n - \theta_n^{hk}$. We have the following.

Theorem 3.4. *Under the assumptions of Theorem 3.2, if we denote by (v, e, ξ, θ) the solution to Problem VP and by $(v^{hk}, e^{hk}, \xi^{hk}, \theta^{hk})$ the solution to Problem VP^{hk}, then we have the following a priori error estimates, for all $z^h = \{z_j^h\}_{j=0}^N \subset V^h$, $r^h = \{r_j^h\}_{j=0}^N \subset V^h$, $l^h = \{l_j^h\}_{j=0}^N \subset V^h$ and $m^h = \{m_j^h\}_{j=0}^N \subset V^h$,*

$$\begin{aligned} &\max_{0\le n\le N} \Big\{\|\theta_n - \theta_n^{hk}\|^2 + \|(\psi_n - \psi_n^{hk})_x\|^2 + \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 \\ &\quad +\|e_n - e_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|\phi_n - \phi_n^{hk}\|^2 + \|\psi_n - \psi_n^{hk}\|^2\Big\} \\ &\le C\,k\sum_{j=1}^N \Big(\|\dot\theta_j - \delta\theta_j\|^2 + \|\theta_j - m_j^h\|^2 + \|(\theta_j - m_j^h)_x\|^2 \\ &\quad +\|\dot v_j - \delta v_j\|^2 + \|v_j - z_j^h\|^2 + \|(v_j - z_j^h)_x\|^2 + \|(\dot u_j - \delta u_j)_x\|^2 \\ &\quad +\|\dot e_j - \delta e_j\|^2 + \|\dot\phi_j - \delta\phi_j\|^2 + \|(\dot\phi_j - \delta\phi_j)_x\|^2 + \|e_j - r_j^h\|^2 + \|(e_j - r_j^h)_x\|^2 \\ &\quad +\|\dot\xi_j - \delta\xi_j\|^2 + \|\dot\psi_j - \delta\psi_j\|^2 + \|(\dot\psi_j - \delta\psi_j)_x\|^2 + \|\xi_j - l_j^h\|^2 + \|(\xi_j - l_j^h)_x\|^2\Big) \\ &\quad + \max_{0\le n\le N}\Big\{\|v_n - z_n^h\|^2 + \|e_n - r_n^h\|^2 + \|\xi_n - l_n^h\|^2 + \|\theta_n - m_n^h\|^2\Big\} \\ &\quad +\frac{C}{k}\sum_{j=1}^{N-1}\|v_j - z_j^h - (v_{j+1} - z_{j+1}^h)\|^2 + \frac{C}{k}\sum_{j=1}^{N-1}\|e_j - r_j^h - (e_{j+1} - r_{j+1}^h)\|^2 \\ &\quad +\frac{C}{k}\sum_{j=1}^{N-1}\|\xi_j - l_j^h - (\xi_{j+1} - l_{j+1}^h)\|^2 + \frac{C}{k}\sum_{j=1}^{N-1}\|\theta_j - m_j^h - (\theta_{j+1} - m_{j+1}^h)\|^2 \\ &\quad +C\Big(\|\theta_0 - \theta_0^h\|^2 + \|(\psi_0 - \psi_0^h)_x\|^2 + \|v_0 - v_0^h\|^2 + \|(u_0 - u_0^h)_x\|^2 \\ &\quad +\|e_0 - e_0^h\|^2 + \|(\phi_0 - \phi_0^h)_x\|^2 + \|\phi_0 - \phi_0^h\|^2 + \|\xi_0 - \xi_0^h\|^2 + \|(\psi_0 - \psi_0^h)_x\|^2 \\ &\quad +\|\psi_0 - \psi_0^h\|^2\Big), \end{aligned} \tag{33}$$

where $C > 0$ is a positive constant which is independent of the discretization parameters h and k, but depending on the continuous solution, and $\delta\theta_j = (\theta_j - \theta_{j-1})/k$, $\delta\psi_j = (\psi_j - \psi_{j-1})/k$, $\delta\xi_j = (\xi_j - \xi_{j-1})/k$, $\delta v_j = (v_j - v_{j-1})/k$, $\delta u_j = (u_j - u_{j-1})/k$, $\delta\phi_j = (\phi_j - \phi_{j-1})/k$ and $\delta e_j = (e_j - e_{j-1})/k$.

Proof. First, we obtain some estimates for the temperature field. Then, we subtract variational equation (13) at time $t = t_n$ for a test function $m = m^h \in V^h \subset V$ and discrete variational equation (25) to obtain, for all $m^h \in V^h$,

$$\begin{aligned} &c(\dot\theta_n - \delta\theta_n^{hk}, m^h) + \kappa((\theta_n - \theta_n^{hk})_x, m_x^h) = -\gamma_1(e_n - e_n^{hk}, m^h) - \gamma_2(\xi_n - \xi_n^{hk}, m^h) \\ &\quad -\beta((v_n - v_n^{hk})_x, m^h) \end{aligned}$$

and so, we have, for all $m^h \in V^h$,

$$\begin{aligned} & c(\dot\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk})+\kappa((\theta_n-\theta_n^{hk})_x,(\theta_n-\theta_n^{hk})_x)+\gamma_1(e_n-e_n^{hk},\theta_n-\theta_n^{hk}) \\ & \quad +\gamma_2(\xi_n-\xi_n^{hk},\theta_n-\theta_n^{hk})+\beta((v_n-v_n^{hk})_x,\theta_n-\theta_n^{hk}) \\ & \quad = c(\dot\theta_n-\delta\theta_n^{hk},\theta_n-m^h)+\kappa((\theta_n-\theta_n^{hk})_x,(\theta_n-m^h)_x)+\gamma_1(e_n-e_n^{hk},\theta_n-m^h) \\ & \quad +\gamma_2(\xi_n-\xi_n^{hk},\theta_n-m^h)+\beta((v_n-v_n^{hk})_x,\theta_n-m^h). \end{aligned}$$

Taking into account that

$$\begin{aligned} & (\dot\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk})\geq(\dot\theta_n-\delta\theta_n,\theta_n-\theta_n^{hk}) \\ & \quad +\frac{1}{2k}\left\{\|\theta_n-\theta_n^{hk}\|^2-\|\theta_{n-1}-\theta_{n-1}^{hk}\|^2\right\}, \\ & ((v_n-v_n^{hk})_x,\theta_n-m^h)=-(v_n-v_n^{hk},(\theta_n-m^h)_x), \\ & ((v_n-v_n^{hk})_x,\theta_n-\theta_n^{hk})=-(v_n-v_n^{hk},(\theta_n-\theta_n^{hk})_x), \end{aligned}$$

where $\delta\theta_n=(\theta_n-\theta_{n-1})/k$, using Cauchy-Schwarz inequality and inequality (27) we have

$$\begin{aligned} & \frac{c}{2k}\left[\|\theta_n-\theta_n^{hk}\|^2-\|\theta_{n-1}-\theta_{n-1}^{hk}\|^2\right]\leq C\Big(\|\dot\theta_n-\delta\theta_n\|^2+\|v_n-v_n^{hk}\|^2+\|\theta_n-m^h\|^2 \\ & \quad +\|(\theta_n-m^h)_x\|^2+\|e_n-e_n^{hk}\|^2+\|\xi_n-\xi_n^{hk}\|^2+\|\theta_n-\theta_n^{hk}\|^2\Big) \\ & \quad +(\delta\theta_n-\delta\theta_n^{hk},\theta_n-m^h)+\beta(v_n-v_n^{hk},(\theta_n-\theta_n^{hk})_x)\quad\forall m^h\in V^h. \end{aligned} \tag{34}$$

Secondly, we get the estimates for the velocity field. Thus, subtracting variational equation (10) at time $t=t_n$ for a test function $z=z^h\in V^h\subset V$ and discrete variational equation (22) we obtain, for all $z^h\in V^h$,

$$\begin{aligned} & \rho(\dot v_n-\delta v_n^{hk},z^h)+\mu((u_n-u_n^{hk})_x,z_x^h)-b((\phi_n-\phi_n^{hk})_x,z^h)-d((\psi_n-\psi_n^{hk})_x,z^h) \\ & \quad +\beta((\theta_n-\theta_n^{hk})_x,z^h)=0, \end{aligned}$$

and so we find that, for all $z^h\in V^h$,

$$\begin{aligned} & \rho(\dot v_n-\delta v_n^{hk},v_n-v_n^{hk})+\mu((u_n-u_n^{hk})_x,(v_n-v_n^{hk})_x)-b((\phi_n-\phi_n^{hk})_x,v_n-v_n^{hk}) \\ & \quad -d((\psi_n-\psi_n^{hk})_x,v_n-v_n^{hk})+\beta((\theta_n-\theta_n^{hk})_x,v_n-v_n^{hk}) \\ & \quad =\rho(\dot v_n-\delta v_n^{hk},v_n-z^h)+\mu((u_n-u_n^{hk})_x,(v_n-z^h)_x)-b((\phi_n-\phi_n^{hk})_x,v_n-z^h) \\ & \quad -d((\psi_n-\psi_n^{hk})_x,v_n-z^h)+\beta((\theta_n-\theta_n^{hk})_x,v_n-z^h). \end{aligned}$$

Taking into account that

$$\begin{aligned} & (\dot v_n-\delta v_n^{hk},v_n-v_n^{hk})\geq(\dot v_n-\delta v_n,v_n-v_n^{hk}) \\ & \quad +\frac{1}{2k}\left[\|v_n-v_n^{hk}\|^2-\|v_{n-1}-v_{n-1}^{hk}\|^2\right], \\ & ((u_n-u_n^{hk})_x,(v_n-v_n^{hk})_x)=((u_n-u_n^{hk})_x,(\dot u_n-\delta u_n)_x) \\ & \quad +\frac{1}{2k}\big[\|(u_n-u_n^{hk})_x\|^2-\|(u_{n-1}-u_{n-1}^{hk})_x\|^2 \\ & \quad +\|(u_n-u_n^{hk})_x-((u_{n-1}-u_{n-1}^{hk}))_x\|^2\big], \\ & ((\theta_n-\theta_n^{hk})_x,v_n-z^h)=-(\theta_n-\theta_n^{hk},(v_n-z^h)_x) \end{aligned}$$

where $\delta u_n=(u_n-u_{n-1})/k$, $\delta v_n=(v_n-v_{n-1})/k$ and we recall that $v_n^{hk}=\delta u_n^{hk}=(u_n^{hk}-u_{n-1}^{hk})/k$, using again Cauchy-Schwarz inequality and inequality (27) we have, for all $z^h \in V^h$,

$$\frac{\rho}{2k}\left[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\right] + \frac{\mu}{2k}\left[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\right]$$
$$\leq C\Big(\|v_n - v_n^{hk}\|^2 + \|\dot{v}_n - \delta v_n\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2$$
$$+\|(\psi_n - \psi_n^{hk})_x\|^2 + \|v_n - z^h\|^2 + \|(v_n - z^h)_x\|^2 + \|\theta_n - \theta_n^{hk}\|^2$$
$$+\|(\dot{u}_n - \delta u_n)_x\|^2\Big) - \beta((\theta_n - \theta_n^{hk})_x, v_n - v_n^{hk})$$
$$+(\delta v_n - \delta v_n^{hk}, v_n - w^h). \tag{35}$$

Next, we obtain the estimates for the macroscropic porosity speed field. Then, we subtract variational equation (11) at time $t = t_n$ for a test function $r = r^h \in V^h \subset V$ and discrete variational equation (23) to get

$$\kappa_1(\dot{e}_n - \delta e_n^{hk}, r^h) + \alpha((\phi_n - \phi_n^{hk})_x, r_x^h) + b_1((\psi_n - \psi_n^{hk})_x, r_x^h) + \varepsilon_1(e_n - e_n^{hk}, r^h)$$
$$+\alpha_1(\phi_n - \phi_n^{hk}, r^h) + \alpha_3(\psi_n - \psi_n^{hk}, r^h) + b((u_n - u_n^{hk})_x, r^h)$$
$$-\gamma_1(\theta_n - \theta_n^{hk}, r^h) + \varepsilon_2(\xi_n - \xi_n^{hk}, r^h) = 0,$$

and so we find that, for all $r^h \in V^h$,

$$\kappa_1(\dot{e}_n - \delta e_n^{hk}, e_n - e_n^{hk}) + \alpha((\phi_n - \phi_n^{hk})_x, (e_n - e_n^{hk})_x)$$
$$+b_1((\psi_n - \psi_n^{hk})_x, (e_n - e_n^{hk})_x) + \varepsilon_1(e_n - e_n^{hk}, e_n - e_n^{hk})$$
$$+\alpha_1(\phi_n - \phi_n^{hk}, e_n - e_n^{hk}) + \alpha_3(\psi_n - \psi_n^{hk}, e_n - e_n^{hk})$$
$$+b((u_n - u_n^{hk})_x, e_n - e_n^{hk}) - \gamma_1(\theta_n - \theta_n^{hk}, e_n - e_n^{hk}) + \varepsilon_2(\xi_n - \xi_n^{hk}, e_n - e_n^{hk})$$
$$= \kappa_1(\dot{e}_n - \delta e_n^{hk}, e_n - r^h) + \alpha((\phi_n - \phi_n^{hk})_x, (e_n - r^h)_x)$$
$$+b_1((\psi_n - \psi_n^{hk})_x, (e_n - r^h)_x) + \varepsilon_1(e_n - e_n^{hk}, e_n - r^h)$$
$$+\alpha_1(\phi_n - \phi_n^{hk}, e_n - r^h) + \alpha_3(\psi_n - \psi_n^{hk}, e_n - r^h)$$
$$+b((u_n - u_n^{hk})_x, e_n - r^h) - \gamma_1(\theta_n - \theta_n^{hk}, e_n - r^h) + \varepsilon_2(\xi_n - \xi_n^{hk}, e_n - r^h).$$

Taking into account that

$$(\dot{e}_n - \delta e_n^{hk}, e_n - e_n^{hk}) \geq (\dot{e}_n - \delta e_n, e_n - e_n^{hk})$$
$$+\frac{1}{2k}\left[\|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2\right],$$
$$((\phi_n - \phi_n^{hk})_x, (e_n - e_n^{hk})_x) = ((\phi_n - \phi_n^{hk})_x, ((\phi_t)_n - \delta\phi_n)_x)$$
$$+\frac{1}{2k}\left[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\right.$$
$$\left.+\|(\phi_n - \phi_n^{hk})_x - (\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\right],$$
$$(\phi_n - \phi_n^{hk}, e_n - e_n^{hk}) = (\phi_n - \phi_n^{hk}, \dot{\phi}_n - \delta\phi_n)$$
$$+\frac{1}{2k}\left[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2 + \|\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})\|^2\right],$$

where $\delta\phi_n = (\phi_n - \phi_{n-1})/k$, $\delta e_n = (e_n - e_{n-1})/k$ and we recall that $e_n^{hk} = \delta\phi_n^{hk} = (\phi_n^{hk} - \phi_{n-1}^{hk})/k$, using again Cauchy-Schwarz inequality and inequality (27) we have, for all $r^h \in V^h$,

$$
\begin{aligned}
&\frac{\kappa_1}{2k}\left[\|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2\right] \\
&\quad + \frac{\alpha}{2k}\big[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x \\
&\qquad + \|(\phi_n - \phi_n^{hk})_x - (\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\big] \\
&\quad + \frac{\alpha_1}{2k}\big[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2 \\
&\qquad + \|\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})\|^2\big] \\
&\quad + \alpha_3(\psi_n - \psi_n^{hk}, e_n - e_n^{hk}) + b_1((\psi_n - \psi_n^{hk})_x, (e_n - e_n^{hk})_x) \\
&\leq C\Big(\|e_n - e_n^{hk}\|^2 + \|\dot{e}_n - \delta e_n\|^2 + \|\phi_n - \phi_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 \\
&\quad + \|\xi_n - \xi_n^{hk}\|^2 + \|\dot{\phi}_n - \delta\phi_n\|^2 + \|\theta_n - \theta_n^{hk}\|^2 \\
&\quad + \|e_n - r^h\|^2 + \|(e_n - r^h)_x\|^2 + \|(\dot{\phi}_n - \delta\phi_n)_x\|^2 + \|(\psi_n - \psi_n^{hk})_x\|^2 \\
&\quad + \|(u_n - u_n^{hk})_x\|^2 + (\delta e_n - \delta e_n^{hk}, e_n - r^h)\Big).
\end{aligned}
\tag{36}
$$

Finally, proceeding in a similar form we obtain the estimates for the microscopic porosity speed field. We omit the details for the sake of clarity in the writing. Then, we have the following estimates, for all $l^h \in V^h$,

$$
\begin{aligned}
&\frac{\kappa_2}{2k}\left[\|\xi_n - \xi_n^{hk}\|^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|^2\right] + \frac{\gamma}{2k}\big[\|(\psi_n - \psi_n^{hk})_x\|^2 - \|(\psi_{n-1} - \psi_{n-1}^{hk})_x \\
&\quad + \|(\psi_n - \psi_n^{hk})_x - (\psi_{n-1} - \psi_{n-1}^{hk})_x\|^2\big] + \frac{\alpha_2}{2k}\big[\|\psi_n - \psi_n^{hk}\|^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|^2 \\
&\quad + \|\psi_n - \psi_n^{hk} - (\psi_{n-1} - \psi_{n-1}^{hk})\|^2\big] \\
&\leq C\Big(\|\xi_n - \xi_n^{hk}\|^2 + \|\dot{\xi}_n - \delta\xi_n\|^2 + \|\psi_n - \psi_n^{hk}\|^2 + \|(\psi_n - \psi_n^{hk})_x\|^2 + \|\phi_n - \phi_n^{hk}\|^2 \\
&\quad + \alpha_3(\phi_n - \phi_n^{hk}, \xi_n - \xi_n^{hk}) + b_1((\phi_n - \phi_n^{hk})_x, (\xi_n - \xi_n^{hk})_x) + \|e_n - e_n^{hk}\|^2 \\
&\quad + \|\dot{\psi}_n - \delta\psi_n\|^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|\xi_n - l^h\|^2 + \|(\xi_n - l^h)_x\|^2 + \|(\dot{\psi}_n - \delta\psi_n)_x\|^2 \\
&\quad + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|(u_n - u_n^{hk})_x\|^2 + (\delta\xi_n - \delta\xi_n^{hk}, \xi_n - l^h)\Big),
\end{aligned}
\tag{37}
$$

where $\delta\psi_n = (\psi_n - \psi_{n-1})/k$ and $\delta\xi_n = (\xi_n - \xi_{n-1})/k$.

Now, combining estimates (34), (35), (36) and (37) we find that, for all z^h, r^h, l^h, $m^h \in V^h$,

$$
\begin{aligned}
&\frac{c}{2k}\left[\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\right] + \frac{\gamma}{2k}\big[\|(\psi_n - \psi_n^{hk})_x\|^2 - \|(\psi_{n-1} - \psi_{n-1}^{hk})_x\|^2 \\
&\qquad + \|(\psi_n - \psi_n^{hk})_x - (\psi_{n-1} - \psi_{n-1}^{hk})_x\|^2\big] + b_1((\phi_n - \phi_n^{hk})_x, (\xi_n - \xi_n^{hk})_x) \\
&\qquad + \frac{\rho}{2k}\left[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\right] + \alpha_3(\phi_n - \phi_n^{hk}, \xi_n - \xi_n^{hk}) \\
&\qquad + \frac{\mu}{2k}\big[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2 \\
&\qquad\quad + \|(u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x\|^2\big] \\
&\qquad + \frac{\kappa_1}{2k}\left[\|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha}{2k}\big[\|(\phi_n-\phi_n^{hk})_x\|^2-\|(\phi_{n-1}-\phi_{n-1}^{hk})_x\\
&\qquad+\|(\phi_n-\phi_n^{hk})_x-(\phi_{n-1}-\phi_{n-1}^{hk})_x\|^2\big]\\
&+\frac{\alpha_1}{2k}\big[\|\phi_n-\phi_n^{hk}\|^2-\|\phi_{n-1}-\phi_{n-1}^{hk}\|^2\\
&\qquad+\|\phi_n-\phi_n^{hk}-(\phi_{n-1}-\phi_{n-1}^{hk})\|^2\big]\\
&+\frac{\kappa_2}{2k}\big[\|\xi_n-\xi_n^{hk}\|^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|^2\big]\\
&+\frac{\alpha_2}{2k}\big[\|\psi_n-\psi_n^{hk}\|^2-\|\psi_{n-1}-\psi_{n-1}^{hk}\|^2\\
&\qquad+\|\psi_n-\psi_n^{hk}-(\psi_{n-1}-\psi_{n-1}^{hk})\|^2\big]\\
\le C\Big(&\|\dot\theta_n-\delta\theta_n\|^2+\|v_n-v_n^{hk}\|^2+\|\theta_n-m^h\|^2+\|(\theta_n-m^h)_x\|^2\\
&+\|e_n-e_n^{hk}\|^2+\|\xi_n-\xi_n^{hk}\|^2+\|\theta_n-\theta_n^{hk}\|^2+(\delta\theta_n-\delta\theta_n^{hk},\theta_n-z^h)\\
&+\|\dot v_n-\delta v_n\|^2+\|(u_n-u_n^{hk})_x\|^2+\|(\phi_n-\phi_n^{hk})_x\|^2+\|(\psi_n-\psi_n^{hk})_x\|^2\\
&+\|v_n-z^h\|^2+\|(v_n-z^h)_x\|^2+\|(\dot u_n-\delta u_n)_x\|^2+(\delta v_n-\delta v_n^{hk},v_n-z^h)\\
&+\|\dot e_n-\delta e_n\|^2+\|\phi_n-\phi_n^{hk}\|^2+\|\dot\phi_n-\delta\phi_n\|^2+\|(\dot\phi_n-\delta\phi_n)_x\|^2\\
&+\|e_n-r^h\|^2+\|(e_n-r^h)_x\|^2+(\delta e_n-\delta e_n^{hk},e_n-r^h)\\
&+\|\dot\xi_n-\delta\xi_n\|^2+\|\dot\psi_n-\delta\psi_n\|^2+\|(\dot\psi_n-\delta\psi_n)_x\|^2+\|\xi_n-l^h\|^2\\
&+\|(\xi_n-l^h)_x\|^2+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-l^h)\Big).
\end{aligned}
$$

From conditions (15)-(16) it follows that

$$
\begin{aligned}
&\frac{\alpha}{2k}\|(\phi_n-\phi_n^{hk})_x-(\phi_{n-1}-\phi_{n-1}^{hk})_x\|^2+\frac{\gamma}{2k}\|\psi_n-\psi_n^{hk}-(\psi_{n-1}-\psi_{n-1}^{hk})\|^2\\
&\quad+\frac{b_1}{k}((\phi_n-\phi_n^{hk})_x-(\phi_{n-1}-\phi_{n-1}^{hk})_x,\psi_n-\psi_n^{hk}-(\psi_{n-1}-\psi_{n-1}^{hk}))\ge 0,\\
&\frac{\alpha_1}{2k}\|\phi_n-\phi_n^{hk}-(\phi_{n-1}-\phi_{n-1}^{hk})\|^2+\frac{\alpha_2}{2k}\|\psi_n-\psi_n^{hk}-(\psi_{n-1}-\psi_{n-1}^{hk})\|^2\\
&\quad+\frac{\alpha_3}{k}(\phi_n-\phi_n^{hk}-(\phi_{n-1}-\phi_{n-1}^{hk}),\psi_n-\psi_n^{hk}-(\psi_{n-1}-\psi_{n-1}^{hk}))\ge 0.
\end{aligned}
$$

Therefore, keeping in mind that

$$
\begin{aligned}
&b_1((\psi_n-\psi_n^{hk})_x,(e_n-e_n^{hk})_x)+b_1((\phi_n-\phi_n^{hk})_x,(\xi_n-\xi_n^{hk})_x)\\
&\quad=b_1((\psi_n-\psi_n^{hk})_x,(\dot\phi_n-\delta\phi_n)_x)+b_1((\phi_n-\phi_n^{hk})_x,(\dot\psi_n-\delta\psi_n)_x)\\
&\qquad+\frac{b_1}{k}\big[((\phi_n-\phi_n^{hk})_x,(\psi_n-\psi_n^{hk})_x)-(\phi_{n-1}-\phi_{n-1}^{hk},(\psi_{n-1}-\psi_{n-1}^{hk})_x)\\
&\qquad+((\phi_n-\phi_n^{hk})_x-(\phi_{n-1}-\phi_{n-1}^{hk})_x,(\psi_n-\psi_n^{hk})_x-(\psi_{n-1}-\psi_{n-1}^{hk})_x)\big],\\
&\alpha_3(\psi_n-\psi_n^{hk},e_n-e_n^{hk})+\alpha_3(\phi_n-\phi_n^{hk},\xi_n-\xi_n^{hk})\\
&\quad=\alpha_3(\psi_n-\psi_n^{hk},\dot\phi_n-\delta\phi_n^{hk})+\alpha_3(\dot\psi_n-\delta\psi_n,\phi_n-\phi_n^{hk})\\
&\qquad+\frac{\alpha_3}{k}\big[(\phi_n-\phi_n^{hk},\psi_n-\psi_n^{hk})-(\phi_{n-1}-\phi_{n-1}^{hk},\psi_{n-1}-\psi_{n-1}^{hk})\\
&\qquad+(\phi_n-\phi_n^{hk}-(\phi_{n-1}-\phi_{n-1}^{hk},\psi_n-\psi_n^{hk}-(\psi_{n-1}-\psi_{n-1}^{hk}))\big],
\end{aligned}
$$

multiplying the previous estimates by $2k$ and summing up to n, we obtain, for all $z^h=\{z_j^h\}_{j=0}^n\subset V^h$, $r^h=\{r_j^h\}_{j=0}^n\subset V^h$, $l^h=\{l_j^h\}_{j=0}^n\subset V^h$ and $m^h=\{m_j^h\}_{j=0}^n\subset V^h$,

$$
\begin{aligned}
&c\|\theta_n-\theta_n^{hk}\|^2+\gamma\|(\psi_n-\psi_n^{hk})_x\|^2+2b_1((\phi_n-\phi_n^{hk})_x,(\psi_n-\psi_n^{hk})_x)\\
&+\rho\|v_n-v_n^{hk}\|^2\qquad+2\alpha_3(\phi_n-\phi_n^{hk},\psi_n-\psi_n^{hk})+\mu\|(u_n-u_n^{hk})_x\|^2\\
&+\kappa_1\|e_n-e_n^{hk}\|^2+\alpha\|(\phi_n-\phi_n^{hk})_x\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_1\|\phi_n-\phi_n^{hk}\|^2+\kappa_2\|\xi_n-\xi_n^{hk}\|^2+\alpha_2\|\psi_n-\psi_n^{hk}\|^2\\
\leq{}& Ck\sum_{j=1}^n\Big(\|\dot\theta_j-\delta\theta_j\|^2+\|v_j-v_j^{hk}\|^2+\|\theta_j-m_j^h\|^2+\|(\theta_j-m_j^h)_x\|^2\\
&+\|e_j-e_j^{hk}\|^2+\|\xi_j-\xi_j^{hk}\|^2+\|\theta_j-\theta_j^{hk}\|^2+(\delta\theta_j-\delta\theta_j^{hk},\theta_j-z_j^h)\\
&+\|\dot v_j-\delta v_j\|^2+\|(u_j-u_j^{hk})_x\|^2+\|(\phi_j-\phi_j^{hk})_x\|^2+\|(\psi_j-\psi_j^{hk})_x\|^2\\
&+\|v_j-z_j^h\|^2+\|(v_j-z_j^h)_x\|^2+\|(\dot u_j-\delta u_j)_x\|^2+(\delta v_j-\delta v_j^{hk},v_j-z_j^h)\\
&+\|\dot e_j-\delta e_j\|^2+\|\phi_j-\phi_j^{hk}\|^2+\|\dot\phi_j-\delta\phi_j\|^2+\|(\dot\phi_j-\delta\phi_j)_x\|^2\\
&+\|e_j-r_j^h\|^2+\|(e_j-r_j^h)_x\|^2+(\delta e_j-\delta e_j^{hk},e_j-r_j^h)\\
&+\|\dot\xi_j-\delta\xi_j\|^2+\|\dot\psi_j-\delta\psi_j\|^2+\|(\dot\psi_j-\delta\psi_j)_x\|^2+\|\xi_j-l_j^h\|^2\\
&+\|(\xi_j-l_j^h)_x\|^2+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-l_j^h)\Big)+C\Big(\|\theta_0-\theta_0^h\|^2+\|(\psi_0-\psi_0^h)_x\|^2\\
&+\|v_0-v_0^h\|^2+\|(u_0-u_0^h)_x\|^2+\|e_0-e_0^h\|^2+\|(\phi_0-\phi_0^h)_x\|^2\\
&+\|\phi_0-\phi_0^h\|^2+\|\xi_0-\xi_0^h\|^2+\|(\psi_0-\psi_0^h)_x\|^2+\|\psi_0-\psi_0^h\|^2\Big).
\end{aligned}
$$

Using again assumptions (15) and (16), as in the proof of Theorem 3.2, we can choose ζ_1, $\zeta_2>0$ such that

$$
b_1/\gamma<\zeta_1<\alpha/b_1,\quad \alpha_3/\alpha_2<\zeta_2<\alpha_1/\alpha_3,
$$

and therefore,

$$
\begin{aligned}
&\alpha\|(\phi_n-\phi_n^{hk})_x\|^2+\gamma\|(\psi_n-\psi_n^{hk})_x\|^2+2b_1((\psi_n-\psi_n^{hk})_x,(\phi_n-\phi_n^{hk})_x)\\
&\qquad\geq(\alpha-b_1\zeta_1)\|(\phi_n-\phi_n^{hk})_x\|^2+\left(\gamma-\frac{b_1}{\zeta_1}\right)\|(\psi_n-\psi_n^{hk})_x\|^2,\\
&\alpha_1\|\phi_n-\phi_n^{hk}\|^2+\alpha_2\|\psi_n-\psi_n^{hk}\|^2+2\alpha_3(\phi_n-\phi_n^{hk},\psi_n-\psi_n^{hk})\\
&\qquad\geq(\alpha_1-\alpha_3\zeta_2)\|\phi_n-\phi_n^{hk}\|^2+\left(\alpha_2-\frac{\alpha_3}{\zeta_2}\right)\|\phi_n-\phi_n^{hk}\|^2.
\end{aligned}
$$

Thus, we have, for all $z^h=\{z_j^h\}_{j=0}^n\subset V^h$, $r^h=\{r_j^h\}_{j=0}^n\subset V^h$, $l^h=\{l_j^h\}_{j=0}^n\subset V^h$ and $m^h=\{m_j^h\}_{j=0}^n\subset V^h$,

$$
\begin{aligned}
&\|\theta_n-\theta_n^{hk}\|^2+\|(\psi_n-\psi_n^{hk})_x\|^2+\|v_n-v_n^{hk}\|^2+\|(u_n-u_n^{hk})_x\|^2+\|e_n-e_n^{hk}\|^2\\
&+\|(\phi_n-\phi_n^{hk})_x\|^2+\|\phi_n-\phi_n^{hk}\|^2+\|\xi_n-\xi_n^{hk}\|^2+\|\psi_n-\psi_n^{hk}\|^2\\
\leq{}& Ck\sum_{j=1}^n\Big(\|\dot\theta_j-\delta\theta_j\|^2+\|v_j-v_j^{hk}\|^2+\|\theta_j-m_j^h\|^2+\|(\theta_j-m_j^h)_x\|^2\\
&+\|e_j-e_j^{hk}\|^2+\|\xi_j-\xi_j^{hk}\|^2+\|\theta_j-\theta_j^{hk}\|^2+(\delta\theta_j-\delta\theta_j^{hk},\theta_j-z_j^h)\\
&+\|\dot v_j-\delta v_j\|^2+\|(u_j-u_j^{hk})_x\|^2+\|(\phi_j-\phi_j^{hk})_x\|^2+\|(\psi_j-\psi_j^{hk})_x\|^2\\
&+\|v_j-z_j^h\|^2+\|(v_j-z_j^h)_x\|^2+\|(\dot u_j-\delta u_j)_x\|^2+(\delta v_j-\delta v_j^{hk},v_j-z_j^h)\\
&+\|\dot e_j-\delta e_j\|^2+\|\phi_j-\phi_j^{hk}\|^2+\|\dot\phi_j-\delta\phi_j\|^2+\|(\dot\phi_j-\delta\phi_j)_x\|^2\\
&+\|e_j-r_j^h\|^2+\|(e_j-r_j^h)_x\|^2+(\delta e_j-\delta e_j^{hk},e_j-r_j^h)\\
&+\|\dot\xi_j-\delta\xi_j\|^2+\|\dot\psi_j-\delta\psi_j\|^2+\|(\dot\psi_j-\delta\psi_j)_x\|^2+\|\xi_j-l_j^h\|^2\\
&+\|(\xi_j-l_j^h)_x\|^2+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-l_j^h)\Big)+C\Big(\|\theta_0-\theta_0^h\|^2+\|(\psi_0-\psi_0^h)_x\|^2\\
&+\|v_0-v_0^h\|^2+\|(u_0-u_0^h)_x\|^2+\|e_0-e_0^h\|^2+\|(\phi_0-\phi_0^h)_x\|^2\\
&+\|\phi_0-\phi_0^h\|^2+\|\xi_0-\xi_0^h\|^2+\|(\psi_0-\psi_0^h)_x\|^2+\|\psi_0-\psi_0^h\|^2\Big).
\end{aligned}
$$

Keeping in mind that

$$\begin{aligned}
k\sum_{j=1}^{n}(\delta v_j-\delta v_j^{hk},v_j-z_j^h)&=\sum_{j=1}^{n}(v_j-v_j^{hk}-(v_{j-1}-v_{j-1}^{hk}),v_j-z_j^h)\\
&=(v_n-v_n^{hk},v_n-z_n^h)+(v_0^h-v_0,v_1-z_1^h)\\
&\quad+\sum_{j=1}^{n-1}(v_j-v_j^{hk},v_j-z_j^h-(v_{j+1}-z_{j+1}^h)),\\
k\sum_{j=1}^{n}(\delta e_j-\delta e_j^{hk},e_j-r_j^h)&=\sum_{j=1}^{n}(e_j-e_j^{hk}-(e_{j-1}-e_{j-1}^{hk}),e_j-r_j^h)\\
&=(e_n-e_n^{hk},e_n-r_n^h)+(e_0^h-e_0,e_1-r_1^h)\\
&\quad+\sum_{j=1}^{n-1}(e_j-e_j^{hk},e_j-r_j^h-(e_{j+1}-r_{j+1}^h)),\\
k\sum_{j=1}^{n}(\delta \xi_j-\delta \xi_j^{hk},\xi_j-l_j^h)&=\sum_{j=1}^{n}(\xi_j-\xi_j^{hk}-(\xi_{j-1}-\xi_{j-1}^{hk}),\xi_j-l_j^h)\\
&=(\xi_n-\xi_n^{hk},\xi_n-l_n^h)+(\xi_0^h-\xi_0,\xi_1-l_1^h)\\
&\quad+\sum_{j=1}^{n-1}(\xi_j-\xi_j^{hk},\xi_j-l_j^h-(\xi_{j+1}-l_{j+1}^h)),
\end{aligned}$$

$$\begin{aligned}
k\sum_{j=1}^{n}(\delta \theta_j-\delta \theta_j^{hk},\theta_j-m_j^h)&=\sum_{j=1}^{n}(\theta_j-\theta_j^{hk}-(\theta_{j-1}-\theta_{j-1}^{hk}),\theta_j-m_j^h)\\
&=(\theta_n-\theta_n^{hk},\theta_n-m_n^h)+(\theta_0^h-\theta_0,\theta_1-m_1^h)\\
&\quad+\sum_{j=1}^{n-1}(\theta_j-\theta_j^{hk},\theta_j-m_j^h-(\theta_{j+1}-m_{j+1}^h)),
\end{aligned}$$

using a discrete version of Gronwall's inequality (see, for instance, [36]), it follows a priori error estimates (33). □

Now, we point out that estimates (33) can be used to obtain the convergence order under some additional regularity conditions. Hence, we have the following result which states the linear convergence of the algorithm.

Corollary 3.5. *Let the assumptions of Theorem 3.4 still hold. If we assume that the solution to Problem VP has the additional regularity:*

$$\begin{aligned}
&u,\,\phi,\,\psi\in H^2(0,T;H^1(0,\ell))\cap H^3(0,T;Y)\cap C^1([0,T];H^2(0,\ell)),\\
&\theta\in H^2(0,T;Y)\cap C([0,T];H^2(0,\ell))\cap H^1(0,T;V),
\end{aligned}\tag{38}$$

and we use the finite element space V^h defined in (20) and the discrete initial conditions u_0^h, v_0^h, ϕ_0^h, e_0^h, ψ_0^h, ξ_0^h and θ_0^h given in (21), the linear convergence of the algorithm is deduced; i.e. there exists a positive constant $C>0$, independent of the discretization parameters h and k, such that

$$\begin{aligned}
\max_{0\le n\le N}&\Big\{\|\theta_n-\theta_n^{hk}\|+\|\xi_n-\xi_n^{hk}\|+\|(\psi_n-\psi_n^{hk})_x\|+\|\psi_n-\psi_n^{hk}\|+\|v_n-v_n^{hk}\|\\
&+\|(u_n-u_n^{hk})_x\|+\|e_n-e_n^{hk}\|+\|(\phi_n-\phi_n^{hk})_x\|+\|\phi_n-\phi_n^{hk}\|\Big\}\\
&\le C(h+k).
\end{aligned}$$

The proof of the linear convergence is done proceeding in a classical way. First, we have the following approximation result by finite elements (see [34] for details):

$$
\begin{aligned}
& k\sum_{j=1}^{N}\Big(\inf_{z_j^h\in V^h}\|v_j-z_j^h\|^2+\inf_{z_j^h\in V^h}\|(v_j-z_j^h)_x\|^2+\inf_{r_j^h\in V^h}\|e_j-r_j^h\|^2\\
&+\inf_{r_j^h\in V^h}\|(e_j-r_j^h)_x\|^2+\inf_{m_j^h\in V^h}\|\theta_j-m_j^h\|^2+\inf_{m_j^h\in V^h}\|(\theta_j-m_j^h)_x\|^2\\
&+\inf_{l_j^h\in V^h}\|\xi_j-l_j^h\|^2+\inf_{l_j^h\in V^h}\|(\xi_j-l_j^h)_x\|^2\Big)\\
&+\max_{0\le n\le N}\inf_{z_n^h\in V^h}\|v_n-z_n^h\|^2+\max_{0\le n\le N}\inf_{r_n^h\in V^h}\|e_n-r_n^h\|^2\\
&+\max_{0\le n\le N}\inf_{m_n^h\in V^h}\|\theta_n-m_n^h\|^2+\max_{0\le n\le N}\inf_{l_n^h\in V^h}\|\xi_n-l_n^h\|^2\\
\le\ & Ch^2\Big(\|u\|^2_{C^1([0,T];H^2(0,\ell))}+\|\phi\|^2_{C^1([0,T];H^2(0,\ell))}+\|\psi\|^2_{C^1([0,T];H^2(0,\ell))}\\
&+\|\theta\|^2_{C([0,T];H^2(0,\ell))}\Big).
\end{aligned}
$$

From the additional regularity (38), we find that

$$
\begin{aligned}
&\|v_0-v_0^h\|^2+\|(u_0-u_0^h)_x\|^2+\|e_0-e_0^h\|^2+\|\phi_0-\phi_0^h\|^2+\|(\phi_0-\phi_0^h)_x\|^2\\
&\quad+\|\theta_0-\theta_0^h\|^2+\|(\psi_0-\psi_0^h)_x\|^2+\|\psi_0-\psi_0^h\|^2+\|\xi_0-\xi_0^h\|^2\\
&\le Ch^2\big(\|u_0\|^2_{H^2(0,\ell)}+\|v_0\|^2_{H^2(0,\ell)}+\|\phi_0\|^2_{H^2(0,\ell)}+\|e_0\|^2_{H^2(0,\ell)}\\
&\qquad+\|\theta_0\|^2_{H^2(0,\ell)}+\|\psi_0\|^2_{H^2(0,\ell)}+\|\xi_0\|^2_{H^2(0,\ell)}\big),\\
&k\sum_{j=1}^{N}\Big[\|(\dot u_j-\delta u_j)_x\|^2+\|\dot v_j-\delta v_j\|^2+\|(\dot\phi_j-\delta\phi_j)_x\|^2+\|\dot\psi_j-\delta\psi_j\|^2+\|\dot\xi_j-\delta\xi_j\|^2\\
&\qquad+\|\dot e_j-\delta e_j\|^2+\|\dot\theta_j-\delta\theta_j\|^2+\|(\dot\psi_j-\delta\psi_j)_x\|^2+\|\dot\phi_j-\delta\phi_j\|^2\Big]\\
&\le Ck^2\Big(\|u\|^2_{H^2(0,T;H^1(0,\ell))}+\|u\|^2_{H^3(0,T;Y)}+\|\phi\|^2_{H^2(0,T;H^1(0,\ell))}\\
&\qquad+\|\phi\|^2_{H^3(0,T;Y)}+\|\psi\|^2_{H^3(0,T;Y)}+\|\psi\|^2_{H^2(0,T;H^1(0,\ell))}+\|\theta\|^2_{H^2(0,T;Y)}\Big).
\end{aligned}
$$

Finally, the remaining terms in estimates (33) can be bounded as follows (see [36, 37] for details),

$$
\begin{aligned}
&\frac{1}{k}\sum_{j=1}^{N-1}\|v_j-z_j^h-(v_{j+1}-z_{j+1}^h)\|^2+\frac{1}{k}\sum_{j=1}^{N-1}\|e_j-r_j^h-(e_{j+1}-r_{j+1}^h)\|^2\\
&\quad+\frac{1}{k}\sum_{j=1}^{N-1}\|\theta_j-m_j^h-(\theta_{j+1}-m_{j+1}^h)\|^2+\frac{1}{k}\sum_{j=1}^{N-1}\|\xi_j-l_j^h-(\xi_{j+1}-l_{j+1}^h)\|^2\\
&\le Ch^2\Big(\|u\|^2_{H^2(0,T;H^1(0,\ell))}+\|\phi\|^2_{H^2(0,T;H^1(0,\ell))}+\|\psi\|^2_{H^2(0,T;H^1(0,\ell))}\\
&\qquad+\|\theta\|^2_{H^1(0,T;H^1(0,\ell))}\Big).
\end{aligned}
$$

Thus, keeping in mind the previous estimates and using a priori error estimates (33), we derive the linear convergence of the algorithm.

4. Numerical results

In this final section, we describe the numerical scheme implemented in MATLAB for solving Problem VP^{hk}, and show some numerical examples to demonstrate the accuracy of the approximation and the behaviour of the solution.

Let the finite element space be defined in (20), for $n=1,2,\dots,N$ and given u_{n-1}^{hk}, v_{n-1}^{hk}, w_{n-1}^{hk}, e_{n-1}^{hk}, ϕ_{n-1}^{hk}, ξ_{n-1}^{hk}, $\theta_{n-1}^{hk}\in V^h$, the discrete velocity v_n^{hk}, the discrete macroscopic porosity speed e_n^{hk}, the discrete microscopic porosity speed ξ_n^{hk}

and the discrete temperature θ_n^{hk}, at time $t = t_n$, are then obtained from equations (22), (23), (24) and (25), respectively. That is, we solve the following linear problem, for all z^h, r^h, l^h, $m^h \in V^h$,

$$\begin{aligned}\rho(v_n^{hk}, z^h) + \mu k^2((v_n^{hk})_x, z_x^h) &= \rho(v_{n-1}^{hk}, z^h) - \mu k((u_{n-1}^{hk})_x, z_x^h)\\ &\quad +bk((\phi_n^{hk})_x, z^h) + dk((\psi_n^{hk})_x, z^h) - \beta k((\theta_n^{hk})_x, z^h),\end{aligned}$$

$$\begin{aligned}&\kappa_1(e_n^{hk}, r^h) + \alpha k^2((e_n^{hk})_x, r_x^h) + \varepsilon_1 k(e_n^{hk}, r^h) + \alpha_1 k^2(e_n^{hk}, r^h)\\ &\quad = \kappa_1(e_{n-1}^{hk}, r^h) + \alpha k((\phi_{n-1}^{hk})_x, r_x^h) - b_1 k((\psi_n^{hk})_x, r_x^h) - \varepsilon_2 k(\xi_n^{hk}, r^h)\\ &\qquad -bk((u_n^{hk})_x, r^h) - \alpha_1 k(\phi_{n-1}^{hk}, r^h) - \alpha_3 k(\psi_n^{hk}, r^h) + \gamma_1 k(\theta_n^{hk}, r^h),\end{aligned}$$

$$\begin{aligned}&\kappa_2(\xi_n^{hk}, l^h) + \gamma k^2((\xi_n^{hk})_x, l_x^h) + \alpha_2 k^2(\xi_n^{hk}, l^h) + \varepsilon_4 k(\xi_n^{hk}, l^h)\\ &\quad = \kappa_2(\xi_{n-1}^{hk}, l^h) - \gamma k((\psi_{n-1}^{hk})_x, l_x^h) - b_1 k((\phi_n^{hk})_x, l_x^h) - \varepsilon_3 k(e_n^{hk}, l^h)\\ &\qquad -\alpha_3 k(\phi_n^{hk}, l^h) - dk((u_n^{hk})_x, l^h) - \alpha_2 k(\xi_{n-1}^{hk}, l^h) + \gamma_2 k(\theta_n^{hk}, l^h),\end{aligned}$$

$$\begin{aligned}c(\theta_n^{hk}, m^h) + \kappa k((\theta_n^{hk})_x, m_x^h) &= c(\theta_{n-1}^{hk}, m^h) - \gamma_1 k(e_n^{hk}, m^h) - \gamma_2 k(\xi_n^{hk}, m^h)\\ &\quad -\beta k((v_n^{hk})_x, m^h),\end{aligned}$$

where we recall that the discrete displacement, macroscopic porosity and microscopic porosity fields, u_n^{hk}, ϕ_n^{hk} and ψ_n^{hk}, are now recovered from the relations

$$u_n^{hk} = u_{n-1}^{hk} + kv_n^{hk}, \quad \phi_n^{hk} = \phi_{n-1}^{hk} + ke_n^{hk}, \quad \psi_n^{hk} = \psi_{n-1}^{hk} + k\xi_n^{hk}.$$

This numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical run ($h = k = 0.01$) took about 3.7 seconds of CPU time.

4.1. First example: numerical convergence. As an academical example, in order to show the accuracy of the approximations the following simpler problem is considered.

Problem $\mathbf{P}^{ex}$. *Find the displacement field $u : [0,1]\times[0,1] \to \mathbb{R}$, the macroscopic porosity field $\phi : [0,1]\times[0,1] \to \mathbb{R}$, the microscopic porosity field $\psi : [0,1]\times[0,1] \to \mathbb{R}$ and the temperature field $\theta : [0,1] \times [0,1] \to \mathbb{R}$ such that*

(39) $\ddot{u} = 2u_{xx} + \phi_x + \psi_x - \theta_x + F_1 \quad in \quad (0,1) \times (0,1),$

(40) $\ddot{\phi} = \phi_{xx} + \psi_{xx} - u_x - 2\phi - \psi + \theta - 2\dot{\phi} - \dot{\psi} + F_2 \quad in \quad (0,1) \times (0,1),$

(41) $\ddot{\psi} = \phi_{xx} + \psi_{xx} - u_x - \phi - \psi + \theta - 2\dot{\psi} - \dot{\phi} + F_3 \quad in \quad (0,1) \times (0,1),$

(42) $\dot{\theta} = \theta_{xx} - \dot{v}_x - \dot{\phi} - \dot{\psi} + F_4 \quad in \quad (0,1) \times (0,1),$

(43) $u(0,t) = u(1,t) = 0, \quad \phi(0,t) = \phi(1,t) = 0 \quad for\ a.e. \quad t \in (0,1),$

(44) $\psi(0,t) = \psi(1,t) = 0, \quad \theta(0,t) = \theta(1,t) = 0 \quad for\ a.e. \quad t \in (0,1),$

(45) $u(x,0) = \phi(x,0) = \psi(x,0) = x(x-1) \quad for\ a.e.\ x \in (0,1),$

(46) $\dot{u}(x,0) = \dot{\phi}(x,0) = \dot{\psi}(x,0) = x(x-1) \quad for\ a.e.\ x \in (0,1),$

(47) $\theta(x,0) = x(x-1) \quad for\ a.e.\ x \in (0,1),$

where F_1, F_2, F_3, F_4 are artificial volume functions defined as, for all $(x,t) \in (0,1) \times (0,1)$,

$$\begin{aligned}F_1(x,t) &= e^t(3x(x-1) - 3 - 2x),\\ F_2(x,t) &= e^t(6x(x-1) - 5 + 2x),\\ F_3(x,t) &= e^t(6x(x-1) - 5 + 2x),\\ F_4(x,t) &= e^t(3x(x-1) - 3 - 2x).\end{aligned}$$

We note that Problem P^{ex} corresponds to Problem P with the following data:

$$\ell = 1, \quad T = 1, \quad \rho = 1, \quad \mu = 2, \quad b = 1, \quad d = 1, \quad \beta = 1, \quad \kappa = 1,$$
$$\kappa_1 = 1, \quad \kappa_2 = 1, \quad b_1 = 1, \quad \alpha = 1, \quad \alpha_1 = 2, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \gamma = 1,$$
$$\gamma_1 = 1, \quad \gamma_2 = 1, \quad c = 1, \quad \varepsilon_1 = 2, \quad \varepsilon_2 = 1, \quad \varepsilon_3 = 1, \quad \varepsilon_4 = 2,$$

and the initial conditions:

$$u_0 = v_0 = \phi_0 = e_0 = \psi_0 = \xi_0 = \theta_0 = x(x-1).$$

Although we added volume forces in all the equations, we note that the analysis presented in the previous section can be extended straightforwardly.

The exact solution to Problem P^{ex} can be calculated and it has the following form for all $(x,t) \in (0,1) \times (0,1)$:

$$u(x,t) = \phi(x,t) = \psi(x,t) = \theta(x,t) = e^t\, x(x-1).$$

Thus, the approximation errors estimated by

$$E_n^{hk} = \max_{0 \le n \le N} \Big\{ \|\theta_n - \theta_n^{hk}\| + \|(\psi_n - \psi_n^{hk})_x\| + \|\psi_n - \psi_n^{hk}\| + \|(u_n - u_n^{hk})_x\| \\ + \|v_n - v_n^{hk}\| + \|e_n - e_n^{hk}\| + \|(\phi_n - \phi_n^{hk})_x\| + \|\phi_n - \phi_n^{hk}\| + \|\xi_n - \xi_n^{hk}\| \Big\}$$

are presented in TABLE 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in FIGURE 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 3.5, is achieved.

TABLE 1. Example 1: Numerical errors for some h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.932221	0.929693	0.928971	0.928868	0.928843	0.928837	0.928836
$1/2^4$	0.471044	0.465977	0.464502	0.464289	0.464236	0.464221	0.464219
$1/2^5$	0.245245	0.235583	0.232657	0.232227	0.232120	0.232092	0.232086
$1/2^6$	0.139334	0.122758	0.117178	0.116326	0.116110	0.116051	0.116041
$1/2^7$	0.093661	0.069877	0.060236	0.058591	0.058162	0.058042	0.058024
$1/2^8$	0.075704	0.047102	0.033053	0.030119	0.029293	0.029054	0.029020
$1/2^9$	0.068924	0.038156	0.020987	0.016528	0.015057	0.014594	0.014526
$1/2^{10}$	0.066315	0.034779	0.016121	0.010492	0.008259	0.007428	0.007295
$1/2^{11}$	0.065327	0.033480	0.014271	0.008059	0.005238	0.003955	0.003711
$1/2^{12}$	0.064997	0.032992	0.013573	0.007136	0.004018	0.002365	0.001971
$1/2^{13}$	0.064903	0.032827	0.013304	0.006784	0.003555	0.001701	0.001172

If we assume now that there are not volume forces, and we use the final time $T = 10$, the following data

$$\ell = 1, \quad T = 10, \quad \rho = 1, \quad \mu = 4, \quad b = 2, \quad d = 3, \quad \beta = 2, \quad \kappa = 4,$$
$$\kappa_1 = 1, \quad \kappa_2 = 1, \quad b_1 = 1, \quad \alpha = 2, \quad \alpha_1 = 2, \quad \alpha_2 = 3, \quad \alpha_3 = 1, \quad \gamma = 2,$$
$$\gamma_1 = 1, \quad \gamma_2 = 1, \quad c = 5, \quad \varepsilon_1 = 2, \quad \varepsilon_2 = 1, \quad \varepsilon_3 = 1, \quad \varepsilon_4 = 2,$$

and the initial conditions

$$u_0 = v_0 = \phi_0 = e_0 = \psi_0 = \xi_0 = 0, \quad \theta_0 = 10x(x-1),$$

taking the discretization parameters $h = 10^{-3}$ and $k = 10^{-3}$, the evolution in time of the discrete energy E_n^{hk}, defined in (32), is plotted in FIGURE 2 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

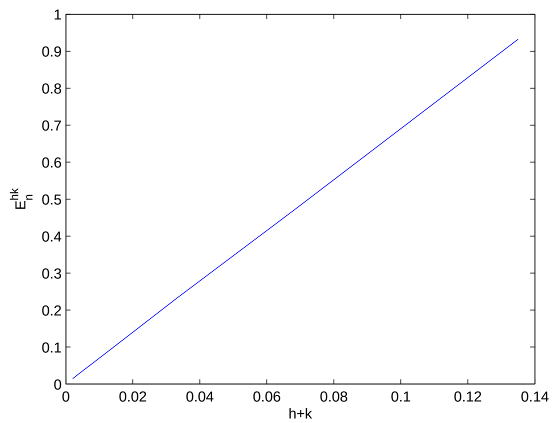

FIGURE 1. Example 1: Asymptotic constant error.

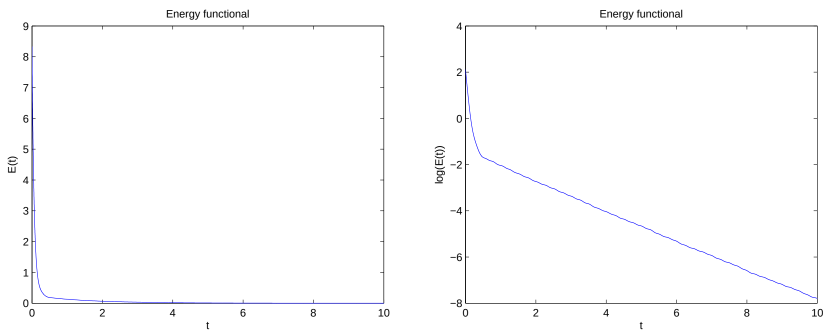

FIGURE 2. Example 1: Evolution in time of the discrete energy (natural and semi-log scales).

4.2. Second example: dependence on the parameter b_1. As a second example, we will analyze the dependence of the solution with respect to parameter b_1. Then, we will use the following data:

$$\ell = 1, \quad T = 1, \quad \rho = 1, \quad \mu = 2, \quad b = 1, \quad d = 1, \quad \beta = 1, \quad \kappa = 1,$$
$$\kappa_1 = 1, \quad \kappa_2 = 1, \quad \alpha = 10, \quad \alpha_1 = 2, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \gamma = 10,$$
$$\gamma_1 = 1, \quad \gamma_2 = 1, \quad c = 1, \quad \varepsilon_1 = 2, \quad \varepsilon_2 = 1, \quad \varepsilon_3 = 1, \quad \varepsilon_4 = 2,$$

and the initial conditions

$$u_0 = v_0 = \phi_0 = e_0 = \psi_0 = \xi_0 = 0, \quad \theta_0 = 10x(x-1).$$

Taking the discretization parameters $k = h = 10^{-3}$, we will assume that parameter b_1 takes values 10, 1, 0.1, 0.01. Therefore, in FIGURES 3 and 4 we plot the obtained results at final time. We can see that displacements have a similar shape for all the values of the parameter. However, concerning the macroscopic porosity we find that, for the greater value of the parameter, the quadratic behaviour changes completely. The same issue is also obtained for the microscopic porosity. Finally, the temperature has an oscillating shape, with big differences for high values of the parameter.

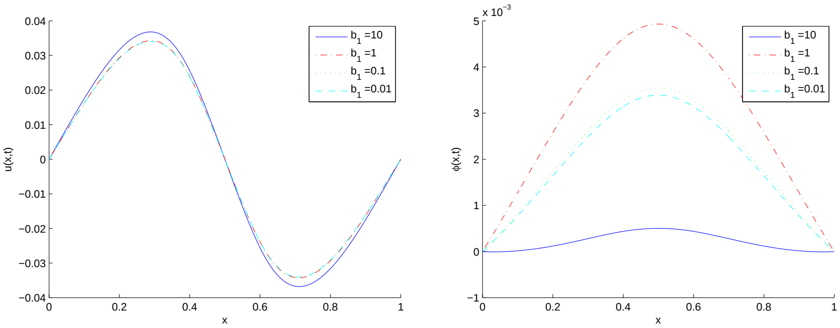

FIGURE 3. Example 2: Displacements and macroscopic porosity fields at final time for some values of b_1.

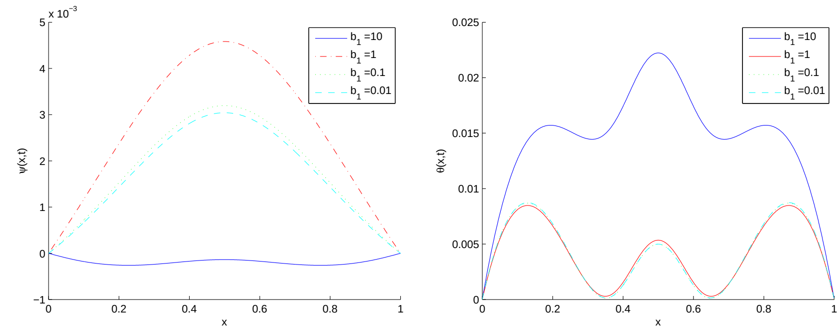

FIGURE 4. Example 2: Microscopic porosity and temperature fields at final time for some values of b_1.

4.3. Third example: dependence on the porous dissipation parameters ε_1, ε_2, ε_3 and ε_4. As a third example, we will analyze the dependence of the solution with respect to porous dissipation parameters ε_1, ε_2, ε_3 and ε_4. Then, we will use the following data:

$$\ell = 1, \quad T = 1, \quad \rho = 1, \quad \mu = 2, \quad b = 1, \quad d = 1, \quad \beta = 1, \quad \kappa = 1,$$
$$\kappa_1 = 1, \quad \kappa_2 = 1, \quad b_1 = 1, \quad \alpha = 5, \quad \alpha_1 = 2, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \gamma = 2,$$
$$\gamma_1 = 1, \quad \gamma_2 = 1, \quad c = 1,$$

and the initial conditions:

$$u_0 = v_0 = \phi_0 = e_0 = \psi_0 = \xi_0 = 0, \quad \theta_0 = 10x(x-1).$$

We will simulate the following cases: (i) $\varepsilon_1 = 2$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$ and $\varepsilon_4 = 2$, (ii) $\varepsilon_1 = 20$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$ and $\varepsilon_4 = 20$, (iii) $\varepsilon_1 = 1$, $\varepsilon_2 = 10$, $\varepsilon_3 = 10$ and $\varepsilon_4 = 1$, and (iv) $\varepsilon_1 = 10$, $\varepsilon_2 = 0$, $\varepsilon_3 = 0$ and $\varepsilon_4 = 0$.

Taking the discretization parameters $h = 10^{-3}$ and $k = 10^{-3}$, in FIGURE 5 the evolution in time of the discrete energy in both natural and semi-log scales is plotted for cases (i)-(iv). We can see that an exponential decay is almost achieved unless for case (iii), which does not satisfy the required conditions to obtain the theoretical results. In fact, we obtain that the dissipation in that case is not positive.

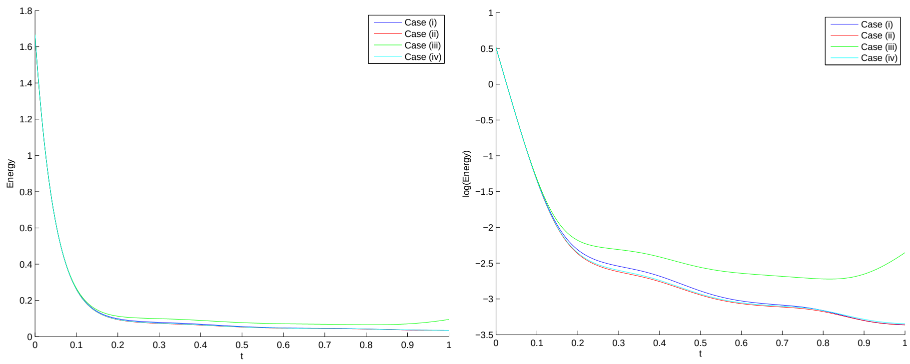

FIGURE 5. Example 3: Evolution in time of the discrete energy (natural and semi-log scales) for some values of the viscosity parameters.

4.4. Fourth example: modification of the dissipation mechanism. As a final example, we will consider a modification of the dissipation mechanism analysed in Problem P. Therefore, we will study the following problem:

Problem P^1. *Find the displacement field* $u : [0, \ell] \times [0, T] \to \mathbb{R}$*, the macroscopic porosity field* $\phi : [0, \ell] \times [0, T] \to \mathbb{R}$*, the microscopic porosity field* $\psi : [0, \ell] \times [0, T] \to \mathbb{R}$ *and the temperature field* $\theta : [0, \ell] \times [0, T] \to \mathbb{R}$ *such that*

$$\begin{aligned}
&\rho\ddot{u} = \mu u_{xx} + b\phi_x + d\psi_x - \beta\theta_x \quad \textit{in} \quad (0, \ell) \times (0, T),\\
&\kappa_1\ddot{\phi} = \alpha\phi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\phi - \alpha_3\psi + \gamma_1\theta - \alpha^*\dot{\phi}_{xx} + \omega_1\theta_{xx} \quad \textit{in} \quad (0, \ell) \times (0, T),\\
&\kappa_2\ddot{\psi} = b_1\phi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\phi - \alpha_2\psi + \gamma_2\theta \quad \textit{in} \quad (0, \ell) \times (0, T),\\
&c\dot{\theta} = \kappa\theta_{xx} - \beta\dot{u}_x - \gamma_1\dot{\phi} - \gamma_2\dot{\psi} + \omega_2\dot{\phi}_{xx} \quad \textit{in} \quad (0, \ell) \times (0, T),\\
&u(0, t) = u(\ell, t) = 0, \quad \phi(0, t) = \phi(\ell, t) = 0 \quad \textit{for a.e.} \quad t \in (0, T),\\
&\psi(0, t) = \psi(\ell, t) = 0, \quad \theta(0, t) = \theta(\ell, t) = 0 \quad \textit{for a.e.} \quad t \in (0, T),\\
&u(x, 0) = u_0(x), \quad \phi(x, 0) = \phi_0(x), \quad \psi(x, 0) = \psi_0(x) \quad \textit{for a.e. } x \in (0, \ell),\\
&\dot{u}(x, 0) = v_0(x), \quad \dot{\phi}(x, 0) = e_0(x), \quad \dot{\psi}(x, 0) = \xi_0(x) \quad \textit{for a.e. } x \in (0, \ell),\\
&\theta(x, 0) = \theta_0(x) \quad \textit{for a.e. } x \in (0, \ell).
\end{aligned}$$

We note that Problem P^1 can be numerically studied in a similar form as for Problem P. So, our aim now is to show if the energy also dissipates. We used the following data:

$$\begin{aligned}
&\ell = 1, \quad T = 10, \quad \rho = 1, \quad \mu = 4, \quad b = 2, \quad d = 3, \quad \beta = 2, \quad \kappa = 4,\\
&\kappa_1 = 1, \quad \kappa_2 = 1, \quad b_1 = 1, \quad \alpha = 2, \quad \alpha_1 = 2, \quad \alpha_2 = 3, \quad \alpha_3 = 1, \quad \gamma = 1,\\
&\gamma_1 = 1, \quad \gamma_2 = 1, \quad c = 5, \quad \omega_1 = 1, \quad \omega_2 = 1,
\end{aligned}$$

and the initial conditions

$$u_0 = v_0 = \phi_0 = e_0 = \psi_0 = \xi_0 = 0, \quad \theta_0 = 10x(x-1).$$

Taking the discretization parameters $h = 10^{-3}$ and $k = 10^{-3}$, in FIGURE 6 the evolution of the energy is plotted in both natural and semi-log scales. As can be seen, the exponential decay seems to be achieved, although some oscillations are found.

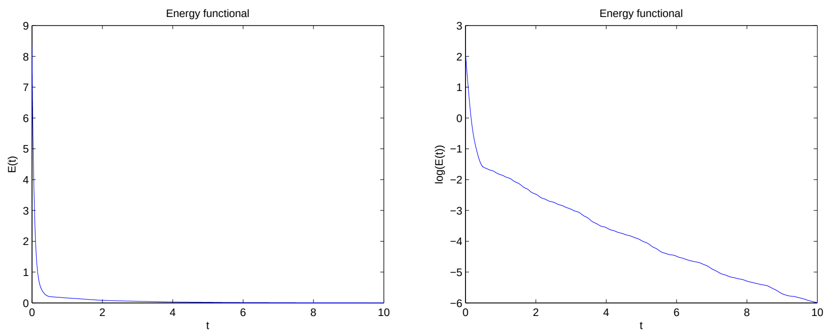

FIGURE 6. Example 4: Evolution in time of the discrete energy (natural and semi-log scales).

5. Conclusions

In this paper we analyzed, from the numerical point of view, a dynamic problem involving a thermoelastic rod. Two porosities, related to pores of the material (macroporosity) and to fissures of the skeleton (microporosity), were also included into the model. The variational formulation was written as a parabolic system of coupled linear variational equations in terms of the velocity, the speeds of both porosities and the temperature. Then, we introduced a fully discrete scheme using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. We proved a discrete stability result and obtained some a priori error estimates. Finally, we presented some numerical simulations to show the convergence of the numerical scheme and the decay of the discrete energy (Example 1), the dependence on the coupling parameter b_1 (Example 2), the dependence on the porous dissipation parameters ε_1, ε_2, ε_3, ε_4 (Example 3) and the influence of another dissipation mechanism (Example 4).

Acknowledgments

The authors gratefully acknowledge the anonymous reviewers for the suggestions. The fruitful discussions helped to improve the quality of the paper.

The work of N. Bazarra and J.R. Fernández was supported by the Ministerio de Economía and Competitividad of the Spanish Government under the research project MTM2015-66640-P (with the participation of FEDER). The work of M.C. Leseduarte, A. Magaña and R. Quintanilla was supported by projects "Análisis Matemático de las Ecuaciones en Derivadas Parciales de la Termomecánica" (MT M2013-42004-P) and "Análisis Matemático de Problemas de la Termomecánica" (MTM2016-74934-P)(AEI/FEDER, UE) of the Spanish Ministry of Economy and Competitiveness.

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Departamento de Matemática Aplicada I, Universidade de Vigo, ETSI Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain

E-mail: `noabaza@hotmail.com and jose.fernandez@uvigo.es`

URL: `http:/webs.uvigo.es/jose.fernandez`

Departament de Matemàtiques, Universitat Politècnica de Catalunya

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa Colom 11, 08222 Terrassa, Barcelona, Spain

E-mail: `Mari.Carme.Leseduarte@upc.edu, antonio.magana@upc.edu,`

E-mail: `Ramon.Quintanilla@upc.edu`