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AN ERROR ESTIMATE OF A EULERIAN-LAGRANGIAN LOCALIZED ADJOINT METHOD FOR A SPACE-FRACTIONAL ADVECTION DIFFUSION EQUATION

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Abstract. We derive a Eulerian-Lagrangian localized adjoint method (ELLAM) for a space-fractional advection diffusion equation that includes a fractional Laplacian operator for modeling such application as a superdiffusive advective transport. The method symmetrizes the numerical scheme and generates accurate numerical solutions even if large time steps and relatively coarse grid meshes are used. We also study the structure of the stiffness matrix to further reduce the computational complexity and memory requirement. We prove an error estimate for the ELLAM. Numerical experiments are presented to show the potential of the method.

Key words. Space-fractional advection diffusion, fractional Laplacian, characteristic method, error estimate, superdiffusive transport.

1. Introduction

Advection diffusion partial differential equations (PDEs) model advective d-iffusive transport in porous media, stochastic dynamics and other applications [3, 10, 12]. The traditional integer-order advection diffusion PDEs, which can be viewed as the Fokker-Planck PDEs of the Ito stochastic processes driven by Brownian motion, were shown to provide accurate description of Fickian diffusive transport in relatively homogeneous porous media. However, in strongly heterogeneous porous media, the underlying particle motions exhibit superdiffusive transport behavior that has an algebraic decaying heavy tail and so has a large deviation from the Brownian motion. Consequently, space-fractional advection diffusion PDEs were shown to provide an accurate description of the superdiffusive transport [13].

It is well known, even in the context of the traditional integer-order advection diffusion PDEs, conventional numerical methods tend to generate some combination of nonphysical oscillations and excessive numerical diffusion [5, 19]. Eulerian-Lagrangian methods provide a competitive means for accurately and efficiently solving these problems [2, 9, 8]. These methods exhibit the advantages of alleviating the Courant number restrictions and reducing the time truncation errors. Namely, they can produce accurate numerical solutions even if the mesh is coarse and the time step is large. There are two principal drawbacks of the Eulerian-Lagrangian method, i.e., it is failure to conserve mass and it is difficult to treat various boundary conditions. However, for advection-dominated problems, the ELLAM can overcome the two principal shortcomings of Eulerian-Lagrangian method, while maintaining their advantages [11]. In this paper we derive a ELLAM for a space-fractional advection-diffusion PDE and prove its error estimate. In the framework of the ELLAM [5], the advective component is treated by a characteristic tracking algorithm and the diffusive component is treated separately by using a more standard spatial approximation, i.e., the Eulerian-Lagrangian methods combine the convection and capacity terms in the governing equation to carry out the temporal discretization in

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a Lagrangian coordinate, and discretize the standard or anomalous diffusion term on a fixed mesh. In other words, the characteristic methods change an fractional advection diffusion equation into a fractional diffusion equation, which transports along with the characteristic curves. We also analyze the structure of its stiffness matrix to develop a fast solution method for the resulting linear algebraic system with a full stiffness matrix. Finally, we conduct some numerical examples to verify the accuracy of the ELLAM scheme and the efficiency of the fast solution method.

The remainder of this paper is organized as follows. We begin in section 2 by giving the nonlocal model and some preliminaries. In section 3, we derive the ELLAM scheme for the fractional equation. We provide an error estimate for the ELLAM scheme in section 4. Section 5 investigates the structure of the coefficient matrix and section 6 proves an auxiliary lemma used in section 4. In section 7, we conduct some numerical tests. Finally, we summarize some remarks.

2. Model Problem and Preliminaries

We consider the following space-fractional advection diffusion transport PDE

$$\begin{aligned} &p_t + (V(x,t)p)_x - d\, p_{xx} + \gamma(-\Delta)^{\frac{\alpha}{2}} p = f(x,t),\ x \in \mathbb{R},\ t \in (0,T], \\ &p(x,t) = 0, \quad x \notin (a,b),\ t \in (0,T], \quad p(x,0) = p_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} p(x,t) = C_\alpha \int_{\mathbb{R}} \frac{p(x,t) - p(y,t)}{|x-y|^{1+\alpha}} dy,\ \alpha \in (0,2), \tag{2}$$

with $C_\alpha = \frac{\alpha}{2^{1-\alpha}\sqrt{\pi}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}$. In such application as advective diffusive transport, $p(x,t)$ usually represents the concentration of the solute or solvent in the fluid, $V(x,t)$ refers to the velocity field of the fluid, $-dp_{xx}$ models the Fickian diffusive transport, $\gamma(-\Delta)^{\frac{\alpha}{2}} p(x,t)$ models the superdiffusive transport, and $f(x,t)$ represents the source term. Here d and γ are nonnegative constants. In stochastic dynamics, $p(x,t)$ is the probability density function that describes the ensemble of realizations of a Lévy process, $-dp_{xx}$ models the Brownian motion component and $V(x,t)$ is the drift. $p_0(x) \geq 0$ is the initial configuration of the model which satisfies the constraint

$$\int_{\mathbb{R}} p_0(x) dx = 1.$$

Since $p(x,t)$ is zero outside the interval (a,b) for any time $t \in (0,T]$, we just consider this model on the interval (a,b) in this paper.

2.1. Sobolev Spaces and Approximation Properties. First, let $W_p^k(a,b)$ consist of functions whose weak derivatives up to order-k are p-th Lebesgue integrable in (a,b). Let $H^k(a,b) := W_2^k(a,b)$

$$\|v\|_{H^k(a,b)} := \Big(\|v\|^2_{H^{k-1}(a,b)} + \Big\| \frac{d^k v}{dx^k} \Big\|^2_{L^2(a,b)} \Big)^{1/2}.$$

For any Banach space X, we introduce Sobolev spaces involving time

$$W_p^k(t_1,t_2;X) := \Big\{ f : \Big\| \frac{\partial^\beta f}{\partial t^\beta}(\cdot,t) \Big\|_X \in L^p(t_1,t_2),\ 0 \leq \beta \leq k,\ 1 \leq p \leq \infty \Big\},$$

$$\|f\|_{W_p^k(t_1,t_2;X)} := \begin{cases} \Big(\displaystyle\sum_{\beta=0}^{k} \int_{t_1}^{t_2} \Big\| \frac{\partial^\beta f}{\partial t^\beta}(\cdot,t) \Big\|_X^p dt \Big)^{1/p}, & 1 \leq p < \infty, \\ \displaystyle\max_{0\leq\beta\leq k} \text{ ess } \sup_{(t_1,t_2)} \Big\| \frac{\partial^\beta f}{\partial t^\beta}(\cdot,t) \Big\|_X, & p = \infty. \end{cases}$$

Let I, N be the positive integers. We define a uniform space-time partition on $[a,b] \times [0,T]$: $x_i := a + ih$ for $0 \le i \le I$ with $h := (b-a)/I$ and $t_n := n\Delta t$ for $0 \le n \le N$ with $\Delta t := T/N$. If a function $f(x,t)$ is defined only at discrete time steps t_n, we understand that the function f has been extended by constant to the time interval $(t_{n-1}, t_n]$. Thus, the preceding space-time norm reduce to the following equivalent discrete norm

$$\|f\|_{L^p(0,T;X)} := \begin{cases} \Big(\sum_{n=1}^{N} \|f(\cdot,t_n)\|_X^p \Delta t\Big)^{1/p}, & 1 \le p < \infty, \\ \max_{0 \le n \le N} \|f(\cdot,t_n)\|_X, & p = \infty. \end{cases}$$

Let $S_h(a,b) \subset H_0^1(a,b)$ be the finite element space that consists of continuous and piecewise-linear functions with respect to the spatial partition in (a,b). We let $\Pi_h v \in S_h(a,b)$ be the piecewise-linear interpolation of v for any $v \in H_0^1(a,b)$. The following estimate holds [6, 7]

$$\|\Pi_h v - v\|_{H^k(a,b)} \le C_1 h^{2-k} \, \|v\|_{H^2(a,b)}, \quad \forall v \in H^2(a,b), \; k = 0, 1. \tag{3}$$

Second, let $H^s(\mathbb{R})$ be the fractional Sobolev space, which is defined by

$$H^s(\mathbb{R}) = \{v \in L^2(\mathbb{R}) : |v|_{H^s(\mathbb{R})} < \infty, 0 < s < 1\},$$

where

$$|v|_{H^s(\mathbb{R})} = \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v(x)-v(y))^2}{|x-y|^{1+2s}} dxdy\Big)^{\frac{1}{2}} \tag{4}$$

denotes the Aronszajn-Slobodeckij seminorm. The space $H^s(\mathbb{R})$ is a Hilbert space, equipped with the norm

$$\|v\|_{H^s(\mathbb{R})} = \|v\|_{L^2(\mathbb{R})} + |v|_{H^s(\mathbb{R})}.$$

3. Derivation of the ELLAM Formulation

We develop a ELLAM to solve the nonlocal model (1). In the ELLAM formulation, the space-time test functions $w(x,t)$ are chosen to be continuous and piecewise smooth, and vanished outside the space-time strip $(a,b) \times (t_{n-1}, t_n]$. Specially, the test functions $w(x,t)$ satisfy that $w(x,t_n) = \lim_{t \to t_n - 0} w(x,t)$, but $w(x,t_{n-1}) \ne \lim_{t \to t_{n-1}+0} w(x,t)$ in general. In this paper, we use the notation $w(x,t_{n-1}^+) = \lim_{t \to t_{n-1}+0} w(x,t)$ to describe the possible discontinuity of $w(x,t)$ in time at time t_{n-1}.

We multiply Eq. (1) by test functions w and integrate the resulting equation on $\mathbb{R} \times (t_{n-1}, t_n]$. For the sake of convenience in writing, we put C_α into the coefficient γ. By using the boundary condition and the integration by parts formula, we obtain a weak formulation

$$\begin{aligned} &\int_{\mathbb{R}} p(x,t_n) w(x,t_n) dx + d \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}} p_x(x,t) w_x(x,t) dxdt \\ &\quad - \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}} p(x,t) \Big(w_t(x,t) + V(x,t) w_x(x,t)\Big) dxdt \\ &\quad + \gamma \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{p(x,t) - p(y,t)}{|x-y|^{1+\alpha}} dy \; w(x,t) dxdt \\ &\quad = \int_{\mathbb{R}} p(x,t_{n-1}) w(x,t_{n-1}^+) dx + \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}} f(x,t) w(x,t) dxdt. \end{aligned} \tag{5}$$

In the ELLAM framework [5], the test functions w are chosen to satisfy the adjoint equation of the hyperbolic part of Eq. (5), i.e.,

$$w_t + Vw_x = 0. \tag{6}$$

This implies that the test functions w is constant along the characteristic curve $r(t; x, t_n)$. Here $r(t; \bar{x}, \bar{t})$ refers to the characteristic curve passing $\bar{x}$ at time $\bar{t}$ defined by

$$\frac{dr}{dt} = V(r,t), \quad r(t;\bar{x},\bar{t})\Big|_{t=\bar{t}} = \bar{x}. \tag{7}$$

Thus, once the test functions $w(x,t)$ are specified in $[a,b]$ at time step t_n, they are determined completely in the space-time strip $[a,b] \times (t_{n-1}, t_n]$.

3.1. Evaluation of Source, Diffusion and Fractional Laplacian Terms. For convenience, in the evaluation of diffusion, source and fractional Laplacian terms, we reserve x for points in $[a,b]$ at time t_n representing the heads of the characteristics. We use the variable y to represent the spatial coordinate of an arbitrary point at time $t \in (t_{n-1}, t_n)$. Substituting $r(t; x, t_n)$ for the variable y, we evaluate the source term by the Euler quadrature as follows

$$\begin{aligned}
&\int_{t_{n-1}}^{t_n} \int_{\mathbb{R}} f(y,t)w(y,t)dydt \\
&\quad = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}} f(r(t;x,t_n),t)w(r(t;x,t_n),t)r_x(t;x,t_n)dxdt \\
&\quad = \int_a^b \Big[\int_{t_{n-1}}^{t_n} f(r(t;x,t_n),t)r_x(t;x,t_n)dt\Big]w(x,t_n)dx \\
&\quad = \Delta t \int_a^b f(x,t_n)w(x,t_n)dx + E_1(w).
\end{aligned} \tag{8}$$

Here $E_1(w)$ is the local truncation error defined by

$$E_1(w) := \int_a^b \int_{t_{n-1}}^{t_n} \Big[f(r(t;x,t_n),t)r_x(t;x,t_n) - f(x,t_n)\Big]dt \; w(x,t_n)dx. \tag{9}$$

We evaluate the diffusion term similarly

$$\begin{aligned}
&d\int_{t_{n-1}}^{t_n} \int_{\mathbb{R}} p_y(y,t)w_y(y,t)dydt \\
&\quad = d\int_{t_{n-1}}^{t_n} \int_{\mathbb{R}} p_y(r(t;x,t_n),t)w_y(x,t_n)r_x(t;x,t_n)dtdx \\
&\quad = d\int_a^b \int_{t_{n-1}}^{t_n} p_y(r(t;x,t_n),t)w_x(x,t_n)dtdx \\
&\quad = d\Delta t \int_a^b p_x(x,t_n)w_x(x,t_n)dx + dE_2(p,w).
\end{aligned} \tag{10}$$

Here $E_2(p,w)$ is the local truncation error defined by

$$E_2(p,w) := \int_a^b \int_{t_{n-1}}^{t_n} \Big[p_x(r(t;x,t_n),t) - p_x(x,t_n)\Big]dt \; w_x(x,t_n)dx. \tag{11}$$

Next, we evaluate the fractional Laplacian term similarly. For convenience, we use $(-\Delta)^{\frac{\alpha}{2}}p(y,t)$ to replace the integral form of it and obtain

$$
\begin{aligned}
&\gamma\int_{t_{n-1}}^{t_n}\int_{\mathbb{R}}(-\Delta)^{\frac{\alpha}{2}}p(y,t)w(y,t)dydt\\
&=\gamma\int_{t_{n-1}}^{t_n}\int_{\mathbb{R}}(-\Delta)^{\frac{\alpha}{2}}p(r(t;x,t_n),t)w(r(t;x,t_n),t)r_x(t;x,t_n)dxdt\\
&=\gamma\int_{t_{n-1}}^{t_n}\int_{\mathbb{R}}(-\Delta)^{\frac{\alpha}{2}}p(r(t;x,t_n),t)r_x(t;x,t_n)w(x,t_n)dxdt\\
&=\gamma\Delta t\int_{\mathbb{R}}(-\Delta)^{\frac{\alpha}{2}}p(x,t_n)w(x,t_n)dx+\gamma E_3(p,w),
\end{aligned}
\tag{12}
$$

where $E_3(p,w)$ are the local truncation errors and defined as

$$
\begin{aligned}
E_3(p,w):=\int_{t_{n-1}}^{t_n}\int_{\mathbb{R}}\Big[&(-\Delta)^{\frac{\alpha}{2}}p(r(t;x,t_n),t)r_x(t;x,t_n)\\
&-(-\Delta)^{\frac{\alpha}{2}}p(x,t_n)\Big]\ w(x,t_n)dxdt.
\end{aligned}
\tag{13}
$$

We can rewrite the first term on the right-hand side of the last equation in (12) as [14]

$$
\begin{aligned}
&\gamma\Delta t\int_{\mathbb{R}}(-\Delta)^{\frac{\alpha}{2}}p(x,t_n)w(x,t_n)dx\\
&=\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{p(x,t_n)-p(y,t_n)}{|x-y|^{1+\alpha}}dy\ w(x,t_n)dx\\
&=\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(p(x,t_n)-p(y,t_n))(w(x,t_n)-w(y,t_n))}{2|x-y|^{1+\alpha}}dydx.
\end{aligned}
\tag{14}
$$

3.2. ELLAM Formulation and Numerical Scheme. We substitute Eqs. (8), (10), (12) and (14) into Eq. (5) to obtain a ELLAM formulation for problem (1)

$$
\begin{aligned}
&\int_a^b p(x,t_n)w(x,t_n)dx+d\Delta t\int_a^b p_x(x,t_n)w_x(x,t_n)dx\\
&\quad+\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(p(x,t_n)-p(y,t_n))(w(x,t_n)-w(y,t_n))}{2|x-y|^{1+\alpha}}dydx\\
&=\int_a^b p(x^*,t_{n-1})w(x,t_n)r_x(t_{n-1};x,t_n)dx\\
&\quad+\Delta t\int_a^b f(x,t_n)w(x,t_n)dx+E_1(w)-dE_2(p,w)-\gamma E_3(p,w),\\
&\quad\text{for}\quad x\in\mathbb{R},\quad t\in(t_{n-1},t_n).
\end{aligned}
\tag{15}
$$

Here x^* is the foot of the characteristic curve $r(t;x,t_n)$ backtracking from x at time t_n. We also let $\tilde{x}$ be the head of the characteristic curve $r(t;\tilde{x},t_n)$ at time t_n which backtracks to x at time t_{n-1}. In other words,

$$
x^*=r(t_{n-1};x,t_n),\qquad x=r(t_{n-1};\tilde{x},t_n).
\tag{16}
$$

In (15) we have used the fact that w is constant along the characteristics to rewrite the first integral at time t_{n-1} on the right-hand side of (5) as an integral at time t_n in (15)

$$
\begin{aligned}
\int_{\mathbb{R}}p(y,t_{n-1})w(y,t_{n-1}^+)dy\quad&=\int_{\mathbb{R}}p(x^*,t_{n-1})w(x,t_n)r_x(t_{n-1};x,t_n)dx\\
&=\int_a^b p(x^*,t_{n-1})w(x,t_n)r_x(t_{n-1};x,t_n)dx.
\end{aligned}
$$

Note that it is not feasible to get the exact characteristic $r(t;x,t_n)$ in general, so we choose $r_h(t;x,t_n)$ to approximate the exact characteristic numerically, which is

$$r_h(t;\bar{x},\bar{t}) = \bar{x} + V(\bar{x},\bar{t})(t-\bar{t}). \tag{17}$$

Consequently, the numerical scheme states as follows: Find $p_h(x,t_n) \in S_h(a,b)$ for $n = 1,\ldots,N$, such that for any $w_h(x,t_n) \in S_h(a,b)$,

$$\begin{aligned}
&\int_a^b p_h(x,t_n)w_h(x,t_n)dx + d\Delta t\int_a^b p_{h,x}(x,t_n)w_{h,x}(x,t_n)dx \\
&\quad +\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(p_h(x,t_n)-p_h(y,t_n))(w_h(x,t_n)-w_h(y,t_n))}{2|x-y|^{1+\alpha}}dydx \\
&= \int_a^b p_h(x_h^*,t_{n-1})w_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\quad +\Delta t\int_a^b f(x,t_n)w_h(x,t_n)dx.
\end{aligned} \tag{18}$$

Here x_h^* and $\tilde{x}_h$ are defined by

$$x_h^* = r_h(t_{n-1};x,t_n), \qquad x = r_h(t_{n-1};\tilde{x}_h,t_n). \tag{19}$$

The ELLAM needs to impose the following type of constraint on the time step Δt (see [15])

$$\|V\|_{L^\infty(0,T;W^1_\infty)}\Delta t < 1.$$

This constraint guarantees that the approximate characteristics defined in (17), which are extended from different spatial points, do not intersect with each other during the time period $[t_{n-1},t_n]$. In other words, the traceback operator defined by the approximate characteristic tracking is a diffeomorphism.

4. An Error Estimate for the ELLAM Scheme

We prove an error estimate for the ELLAM scheme for the problem (1). First, we introduce the following lemma refers from the references [15, 18].

Lemma 4.1. *Let $r(s;x,t)$ and $r_h(s;x,t)$ be the exact and approximate characteristics defined in (7) and (17), respectively. Assume that $V, \frac{dV}{dt} \in L^\infty(0,T;W^1_\infty(a,b))$. Then the following estimates hold*

$$\begin{aligned}
&|x_h^* - x^*| \le O((\Delta t)^2), \\
&|r_x(t;x,t_n) - 1| \le O(t_n - t), \qquad \textit{and} \\
&|r_{h,x}(t_{n-1};x,t_n) - r_x(t_{n-1};x,t_n)| = O((\Delta t)^2).
\end{aligned} \tag{20}$$

Theorem 4.1. *Assume the exact solution $p \in L^\infty(0,T;H^s) \cap H^1(0,T;H^s)$ with $s \ge 0$. Then the following error estimate for the ELLAM scheme holds*

$$\begin{aligned}
&\|p_h - p\|_{L^\infty(0,T;L^2)} + d\Delta t\sum_{n=1}^{N_1}\|p_{h,x}(\cdot,t_n) - p_x(\cdot,t_n)\|_{L^2} \\
&\le C\Delta t\Big(\Big\|\frac{df}{dt}\Big\|_{L^2(0,T;L^2)} + \|f\|_{L^2(0,T;L^2)} + \Big\|\frac{d}{dt}(-\Delta)^{\frac{\alpha}{2}}p\Big\|_{L^2(0,T;L^2)} \\
&\quad + \|(-\Delta)^{\frac{\alpha}{2}}p\|_{L^2(0,T;L^2)} + \|\frac{dp}{dt}\|_{L^2(0,T;H^1)}\Big) \\
&\quad + Ch^s\|p\|_{H^1(0,T;H^s)} + C(\Delta t + h^s + h^{s+1-\alpha-\epsilon})\|p\|_{L^\infty(0,T;H^s)},
\end{aligned} \tag{21}$$

where the constant C depends on d and the parameter γ.

Proof. We let $e = p_h - p$ and choose the test function $w_h(\cdot, t_n) \in S_h(a, b)$. We then subtract Eq. (15) from the Eq. (18) to obtain a ELLAM error equation for any $w_h(x, t_n) \in S_h(a, b)$,

$$\begin{aligned}
&\int_a^b e(x,t_n)w_h(x,t_n)dx + d\Delta t\int_a^b e_x(x,t_n)w_{h,x}(x,t_n)dx \\
&\quad +\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(e(x,t_n)-e(y,t_n))(w_h(x,t_n)-w_h(y,t_n))}{2|x-y|^{1+\alpha}}dydx \\
&= \int_a^b e(x_h^*,t_{n-1})w_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\quad +\int_a^b \Big(p(x_h^*,t_{n-1})r_{h,x}(t_{n-1};x,t_n)-p(x^*,t_{n-1})r_x(t_{n-1};x,t_n)\Big)w_h(x,t_n)dx \\
&\quad -E_1(w_h)+dE_2(p,w_h)+\gamma E_3(p,w_h).
\end{aligned}$$

Let $\Pi_h p \in S_h(a, b)$ be the interpolation of the true solution p, $\xi_h = p_h - \Pi_h p \in S_h(a, b)$, and $\eta = \Pi_h p - p$. The error estimate for η is given in (3). We need to estimate ξ_h. We choose $w_h(x, t_n) = \xi_h(x, t_n)$ and rewrite the error equation in terms of ξ_h and η as follows

$$\begin{aligned}
&\int_a^b \xi_h^2(x,t_n)dx + d\Delta t\int_a^b \xi_{h,x}^2(x,t_n)dx \\
&\quad +\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\xi_h(x,t_n)-\xi_h(y,t_n))^2}{2|x-y|^{1+\alpha}}dydx \\
&= \int_a^b \xi_h(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\quad +\int_a^b \eta(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\quad -\int_a^b \eta(x,t_n)\xi_h(x,t_n)dx - d\Delta t\int_a^b \eta_x(x,t_n)\xi_{h,x}(x,t_n)dx \\
&\quad +\int_a^b p(x^*,t_{n-1})(r_{h,x}(t_{n-1};x,t_n)-r_x(t_{n-1};x,t_n))\xi_h(x,t_n)dx \\
&\quad +\int_a^b (p(x_h^*,t_{n-1})-p(x^*,t_{n-1}))r_{h,x}(t_{n-1};x,t_n)\xi_h(x,t_n)dx \\
&\quad -\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\eta(x,t_n)-\eta(y,t_n))(\xi_h(x,t_n)-\xi_h(y,t_n))}{2|x-y|^{1+\alpha}}dydx \\
&\quad -E_1(\xi_h)+dE_2(p,\xi_h)+\gamma E_3(p,\xi_h).
\end{aligned} \tag{22}$$

We bound the first term on the right-hand side of Eq. (22) by

$$\begin{aligned}
&\Big|\int_a^b \xi_h(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx\Big| \\
&\le \frac{1+C\Delta t}{2}\int_a^b \xi_h^2(x,t_n)dx + \frac{1+C\Delta t}{2}\int_a^b \xi_h^2(x_h^*,t_{n-1})dx \\
&\le \frac{1+C\Delta t}{2}\int_a^b \xi_h^2(x,t_n)dx + \frac{1+C\Delta t}{2}\int_{a_h^*}^{b_h^*} \xi_h^2(x_h^*,t_{n-1})\Big|\frac{dx_h^*}{dx}\Big|^{-1}dx_h^* \\
&\le \frac{1+C\Delta t}{2}\|\xi_h(\cdot,t_n)\|_{L^2}^2 + \frac{1+C\Delta t}{2}\|\xi_h(\cdot,t_{n-1})\|_{L^2}^2.
\end{aligned} \tag{23}$$

Here the constant C depends on $\|V\|_{L^\infty(0,T;W^1_\infty)}$. In the second term after the second inequality, we used the substitution of variables from x to x_h^* given by the first equation in Eq. (19). We also use Eq. (17) to get

$$\begin{aligned} r_{h,x}(t_{n-1};x,t_n) &= 1 - V_x(x,t_n)\Delta t, \\ r_{h,x}^{-1}(t_{n-1};x,t_n) &= (1 - V_x(x,t_n)\Delta t)^{-1} = 1 + O(\Delta t). \end{aligned} \tag{24}$$

A delicate analysis shows an optimal-order error estimate of the second and third terms on the right-hand side of Eq. (22). For clarity of exposition, the proof is presented in Lemma 6.1, where we obtain

$$\begin{aligned} &\Big|\int_a^b \eta(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx - \int_a^b \eta(x,t_n)\xi_h(x,t_n)dx\Big| \\ &\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + \varepsilon_1 d\Delta t\|\xi_{h,x}(\cdot,t_n)\|_{L^2}^2 + Ch^{2s}\|p\|_{H^1(t_{n-1},t_n;H^s)}^2 \\ &\qquad + C\Delta t h^{2s}\|p\|_{L^\infty(0,T;H^s)}^2. \end{aligned} \tag{25}$$

Let $x_{i-1/2}$ be the middle point of the interval $[x_{i-1},x_i]$. Note that $\xi_{h,x}(x,t_n)$ is constant on each interval $[x_{i-1},x_i]$ and that η satisfies $\eta(x_{i-1},t_n) = \eta(x_i,t_n) = 0$ for $i = 1,\ldots,I$, we bound the fourth term on the right-hand side of Eq. (22)

$$\begin{aligned} &\Big|d\Delta t\int_a^b \eta_x(x,t_n)\xi_{h,x}(x,t_n)dx\Big| \\ &\quad = \Big|d\Delta t\sum_{i=1}^I \xi_{h,x}(x_{i-1/2},t_n)\int_{x_{i-1}}^{x_i}\eta_x(x,t_n)dx\Big| \\ &\quad = \Big|d\Delta t\sum_{i=1}^I \xi_{h,x}(x_{i-1/2},t_n)(\eta(x_i,t_n) - \eta(x_{i-1},t_n))\Big| = 0. \end{aligned} \tag{26}$$

We use the estimate (20) to bound the fifth and sixth terms on the right side of Eq. (22) by

$$\begin{aligned} &\Big|\int_a^b p(x^*,t_{n-1})(r_{h,x}(t_{n-1};x,t_n) - r_x(t_{n-1};x,t_n))\xi_h(x,t_n)dx\Big| \\ &\quad \le C(\Delta t)^2\|\xi_h(\cdot,t_n)\|_{L^2}\Big(\int_a^b p^2(x^*,t_{n-1})dx\Big)^{1/2} \\ &\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^3\|p\|_{L^\infty(0,T;L^2)}^2, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\Big|\int_a^b (p(x_h^*,t_{n-1}) - p(x^*,t_{n-1}))r_{h,x}(t_{n-1};x,t_n)\xi_h(x,t_n)dx\Big| \\ &= \Big|\int_a^b\int_{x^*}^{x_h^*} p_y(y,t_{n-1})dy\ r_{h,x}(t_{n-1};x,t_n)\xi_h(x,t_n)dx\Big| \\ &\quad \le C\int_a^b\Big|\int_{x^*}^{x_h^*}|p_y(y,t_{n-1})|dy\Big|\ |\xi_h(x,t_n)|dx \\ &\quad \le C\|\xi_h(\cdot,t_n)\|_{L^2}\Big(\int_a^b(\int_{x^*}^{x_h^*}|p_y(y,t_{n-1})|dy)^2dx\Big)^{\frac12} \\ &\quad \le C(\Delta t)^2\|\xi_h(\cdot,t_n)\|_{L^2}\|p_y\|_{L^\infty(0,T;L^\infty)} \\ &\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^3\|p\|_{L^\infty(0,T;W_\infty^1)}^2. \end{aligned} \tag{28}$$

We bound the seventh term on the right-hand side of Eq. (22). Generally, we use the Sobolev's interpolation inequality $\|u\|_{H^{\frac{\alpha}{2}}} \le \|u\|_{L^2}^{1-\frac{\alpha}{2}}\|u\|_{H^1}^{\frac{\alpha}{2}}$ to estimate and get

$$
\begin{aligned}
&\Big|\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\eta(x,t_n)-\eta(y,t_n))(\xi_h(x,t_n)-\xi_h(y,t_n))}{2|x-y|^{1+\alpha}}dydx\Big|\\
&\le \frac{1}{4}\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\eta(x,t_n)-\eta(y,t_n))^2}{|x-y|^{1+\alpha}}dydx\\
&\quad+\frac{1}{4}\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\xi_h(x,t_n)-\xi_h(y,t_n))^2}{|x-y|^{1+\alpha}}dydx\\
&\le \frac{1}{4}\gamma\Delta t\Big[\|\eta(\cdot,t_n)\|^2_{H^{\frac{\alpha}{2}}}+\|\xi_h(\cdot,t_n)\|^2_{H^{\frac{\alpha}{2}}}\Big]\\
&\le \frac{1}{4}\gamma\Delta t\Big[\|\eta(\cdot,t_n)\|^{2-\alpha}_{L^2}\|\eta(\cdot,t_n)\|^{\alpha}_{H^1}+\|\xi_h(\cdot,t_n)\|^2_{H^{\frac{\alpha}{2}}}\Big]\\
&\le C\Delta t\ h^{2(s-\frac{\alpha}{2})}\|p\|^2_{L^\infty(0,T;H^s)}+C\Delta t\|\xi_h(\cdot,t_n)\|^2_{H^{\frac{\alpha}{2}}}.
\end{aligned}
$$

Here the constant C depends on the parameter γ. This estimate is of order $O(h^{s-\frac{\alpha}{2}}+\Delta t)$. Another estimate with higher order can be bounded by

$$
\begin{aligned}
&\Big|\gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\eta(x,t_n)-\eta(y,t_n))(\xi_h(x,t_n)-\xi_h(y,t_n))}{2|x-y|^{1+\alpha}}dydx\Big|\\
&=\frac{1}{2}\gamma\Delta t\Big|\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\eta(x,t_n)-\eta(y,t_n))(\xi_h(x,t_n)-\xi_h(y,t_n))}{|x-y|^{\frac{1+2(\alpha-(1-\epsilon))}{2}}|x-y|^{\frac{1+2(1-\epsilon)}{2}}}dydx\\
&\le C\Delta t\|\eta(\cdot,t_n)\|_{H^{\alpha-1+\epsilon}}\|\xi_h(\cdot,t_n)\|_{H^{1-\epsilon}}\\
&\le C\Delta t\|\eta(\cdot,t_n)\|_{H^{\alpha-1+\epsilon}}\|\xi_{h,x}(\cdot,t_n)\|_{L^2}\\
&\le \frac{C\Delta t}{\varepsilon_1 d}h^{2(s+1-\alpha-\epsilon)}\|p\|^2_{L^\infty(0,T;H^s)}+\varepsilon_1 d\Delta t\|\xi_{h,x}(\cdot,t_n)\|^2_{L^2}\\
&\le C\Delta t\ h^{2(s+1-\alpha-\epsilon)}\|p\|^2_{L^\infty(0,T;H^s)}+\varepsilon_1 d\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2}.
\end{aligned}
\tag{29}
$$

Here the constant C depends on d and the parameter γ. The parameter ε_1 is a constant and the parameter $0<\epsilon\ll 1$. In the second inequality, we use the following estimate (see [1])

$$\|\xi_h(\cdot,t_n)\|_{H^{1-\epsilon}}\le C\|\xi_h(\cdot,t_n)\|_{H^1}\le C\|\xi_{h,x}(\cdot,t_n)\|_{L^2}.$$

We decompose the eighth term on the right side of Eq. (22) in two terms

$$
\begin{aligned}
&\Big|\int_a^b\int_{t_{n-1}}^{t_n}\Big[f(r(t;x,t_n),t)r_x(t;x,t_n)-f(x,t_n)\Big]dt\ \xi_h(x,t_n)dx\Big|\\
&\le\int_a^b\int_{t_{n-1}}^{t_n}|f(x,t_n)-f(r(t;x,t_n),t)|dt\ |\xi_h(x,t_n)|dx\\
&\quad+\int_a^b\int_{t_{n-1}}^{t_n}|f(r(t;x,t_n),t)|\ |1-r_x(t;x,t_n)|dt\ |\xi_h(x,t_n)|dx.
\end{aligned}
$$

The first term on the right-hand side is bounded by

$$
\begin{aligned}
&\int_a^b\int_{t_{n-1}}^{t_n}|f(x,t_n)-f(r(t;x,t_n))|dt\ |\xi_h(x,t_n)|dx\\
&\le\int_{t_{n-1}}^{t_n}\int_a^b|\int_t^{t_n}\frac{d}{d\theta}f(r(\theta;x,t_n),\theta)d\theta||\xi_h(x,t_n)|dxdt\\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2}+C\Delta t\int_{t_{n-1}}^{t_n}\int_a^b\int_t^{t_n}(\frac{d}{d\theta}f(r(\theta;x,t_n),\theta))^2d\theta dxdt\\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2}+C(\Delta t)^2\|\frac{df}{dt}\|^2_{L^2(t_{n-1},t_n;L^2)}.
\end{aligned}
\tag{30}
$$

We use the estimate (20) to bound the second term

$$\begin{aligned}
&\int_a^b \int_{t_{n-1}}^{t_n} |f(r(t;x,t_n),t)|\ |1-r_x(t;x,t_n)|dt\ |\xi_h(x,t_n)|dx \\
&\quad \le \int_{t_{n-1}}^{t_n} \int_a^b \Delta t|f(r(t;x,t_n),t)|\ |\xi_h(x,t_n)|dxdt \\
&\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^2\|f\|^2_{L^2(t_{n-1},t_n;L^2)}.
\end{aligned} \tag{31}$$

We similarly bound the ninth term on the right-hand side of Eq. (22) by

$$\begin{aligned}
&\Big|d\int_a^b \int_{t_{n-1}}^{t_n} [p_x(x,t_n) - p_x(r(t;x,t_n),t)]dt\ \xi_{h,x}(x,t_n)dx\Big| \\
&= \Big|d\int_a^b \xi_{h,x}(x,t_n)\Big[\int_{t_{n-1}}^{t_n}\int_t^{t_n} \frac{d}{d\theta}p_x(r(\theta;x,t_n),\theta)d\theta dt\Big]dx \\
&\quad \le \varepsilon_1 d\Delta t\|\xi_{h,x}(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^2\Big\|\frac{dp}{dt}\Big\|^2_{L^2(t_{n-1},t_n;H^1)},
\end{aligned} \tag{32}$$

where the constant C depends on d and ε_1 is a positive constant.

We estimate the tenth term by decomposing it into two terms, i.e.,

$$\begin{aligned}
&\gamma\Big|\int_{t_{n-1}}^{t_n}\int_a^b [(-\Delta)^{\frac{\alpha}{2}}p(r(t;x,t_n),t)r_x(t;x,t_n) \\
&\quad -(-\Delta)^{\frac{\alpha}{2}}p(x,t_n)]\xi_h(x,t_n)dxdt\Big| \\
&\le \gamma\Big|\int_{t_{n-1}}^{t_n}\int_a^b [(-\Delta)^{\frac{\alpha}{2}}p(x,t_n) - (-\Delta)^{\frac{\alpha}{2}}p(r(t;x,t_n),t)]\ \xi_h(x,t_n)dxdt\Big| \\
&\quad +\gamma\Big|\int_{t_{n-1}}^{t_n}\int_a^b \Delta t(-\Delta)^{\frac{\alpha}{2}}p(r(t;x,t_n),t)\xi_h(x,t_n)dxdt\Big| \\
&\le C\Big|\int_{t_{n-1}}^{t_n}\int_a^b \Big(\int_t^{t_n}\frac{d}{d\theta}(-\Delta)^{\frac{\alpha}{2}}p(r(\theta;x,t_n),\theta)d\theta\Big)^2 dxdt\Big| \\
&\quad +C\int_{t_{n-1}}^{t_n}\int_a^b (\Delta t(-\Delta)^{\frac{\alpha}{2}}p(r(\theta;x,t_n),\theta))^2dxdt + C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} \\
&\le C\Delta t\int_{t_{n-1}}^{t_n}\int_a^b\int_t^{t_n} (\frac{d}{d\theta}(-\Delta)^{\frac{\alpha}{2}}p(r(\theta;x,t_n),\theta))^2d\theta dxdt \\
&\quad +C\int_{t_{n-1}}^{t_n}\int_a^b (\Delta t(-\Delta)^{\frac{\alpha}{2}}p(r(\theta;x,t_n),\theta))^2dxdt + C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} \\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} + C(\Delta t)^2\|\frac{d}{dt}(-\Delta)^{\frac{\alpha}{2}}p\|^2_{L^2(t_{n-1},t_n;L^2)} \\
&\quad +C(\Delta t)^2\|(-\Delta)^{\frac{\alpha}{2}}p\|^2_{L^2(t_{n-1},t_n;L^2)},
\end{aligned} \tag{33}$$

where the constant C depends on the parameter γ.

We substitute estimates (23)–(33) for the corresponding terms in Eq. (22) to obtain the following estimate

$$
\begin{aligned}
&\|\xi_h(\cdot,t_n)\|_{L^2}^2 + d\Delta t\|\xi_{h,x}(\cdot,t_n)\|_{L^2}^2 + \gamma\Delta t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\xi_h(x,t_n)-\xi_h(y,t_n))^2}{|x-y|^{1+\alpha}}dydx \\
&\le \frac{1+C\Delta t}{2}\Big(\|\xi_h(\cdot,t_n)\|_{L^2}^2 + \|\xi_h(\cdot,t_{n-1})\|_{L^2}^2\Big) + 3\varepsilon_1 d\Delta t\|\xi_{h,x}(\cdot,t_n)\|_{L^2}^2 \\
&+C(\Delta t)^2\Big(\|\frac{df}{dt}\|_{L^2(t_{n-1},t_n;L^2)}^2 + \|f\|_{L^2(t_{n-1},t_n;L^2)}^2 + \|\frac{d}{dt}(-\Delta)^{\frac{\alpha}{2}}p\|_{L^2(t_{n-1},t_n;L^2)}^2 \\
&+\|(-\Delta)^{\frac{\alpha}{2}}p\|_{L^2(t_{n-1},t_n;L^2)}^2 + \|\frac{dp}{dt}\|_{L^2(t_{n-1},t_n;H^1)}^2\Big) + Ch^{2s}\|p\|_{H^1(t_{n-1},t_n;H^s)}^2 \\
&+C\Delta t\Big((\Delta t)^2 + h^{2s} + h^{2(s+1-\alpha-\epsilon)}\Big)\|p\|_{L^\infty(0,T;H^s)}^2.
\end{aligned}
$$

We choose $\varepsilon_1 = \frac{1}{4}$ and sum the estimate for $n = 1, \ldots, N_1(\le N)$, and cancel like terms to obtain

$$
\begin{aligned}
&\|\xi_h(\cdot,t_{N_1})\|_{L^2}^2 + d\Delta t\sum_{n=1}^{N_1}\|\xi_{h,x}(\cdot,t_n)\|_{L^2}^2 + \gamma\Delta t\sum_{n=1}^{N_1}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\xi_h(x,t_n)-\xi_h(y,t_n))^2}{|x-y|^{1+\alpha}}dydx \\
&\le C\Delta t\sum_{n=0}^{N_1-1}\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^2\Big(\|\frac{df}{dt}\|_{L^2(0,T;L^2)}^2 + \|f\|_{L^2(0,T;L^2)}^2 \\
&+\|\frac{d}{dt}(-\Delta)^{\frac{\alpha}{2}}p\|_{L^2(0,T;L^2)}^2 + \|(-\Delta)^{\frac{\alpha}{2}}p\|_{L^2(0,T;L^2)}^2 + \|\frac{dp}{dt}\|_{L^2(0,T;H^1)}^2\Big) \\
&+Ch^{2s}\|p\|_{H^1(0,T;H^s)}^2 + C\Big((\Delta t)^2 + h^{2s} + h^{2(s+1-\alpha-\epsilon)}\Big)\|p\|_{L^\infty(0,T;H^s)}^2.
\end{aligned}
$$

We then apply Gronwall inequality to conclude

$$
\begin{aligned}
&\|\xi_h\|_{L^\infty(0,T;L^2)} + d\Delta t\sum_{n=1}^{N_1}\|\xi_{h,x}(\cdot,t_n)\|_{L^2} \\
&\le C\Delta t\Big(\Big\|\frac{df}{dt}\Big\|_{L^2(0,T;L^2)} + \|f\|_{L^2(0,T;L^2)} + \Big\|\frac{d}{dt}(-\Delta)^{\frac{\alpha}{2}}p\Big\|_{L^2(0,T;L^2)} \\
&+\|(-\Delta)^{\frac{\alpha}{2}}p\|_{L^2(0,T;L^2)} + \|\frac{dp}{dt}\|_{L^2(0,T;H^1)}\Big) \\
&+Ch^s\|p\|_{H^1(0,T;H^s)} + C(\Delta t + h^s + h^{s+1-\alpha-\epsilon})\|p\|_{L^\infty(0,T;H^s)}.
\end{aligned}
$$

The general constant C depends on d, γ, and also depends exponentially on the final time T in problem (1) due to the application of Gronwall inequality. We combine this estimate with (3) to finish the proof. □

5. The Structure of Coefficient Matrix and A Fast Solution Method

In the numerical scheme (18), we choose the piecewise linear hat function $\{\phi_i(x)\}_{i=1}^{I-1}$ as the test function $w_h(x,t_n) \in S_h(a,b)$ for $1 \le n \le N$. Then $p_h(x,t_n) \in S_h(a,b)$ can be represented in a unique way as a linear combination of the hat function $\phi_i(x)$. We define several $(I-1)\times(I-1)$ matrices A, B, H, D and vectors

$$P_h^n = (p_{h,1}^n, p_{h,2}^n, \cdots, p_{h,I-1}^n)^T,\ F = (f_1^n, f_2^n, \cdots, f_{I-1}^n)^T,$$

where $p_{h,i}^n = p_h(x_i,t_n)$ for $i = 1,2,\ldots,I-1,\ n = 0,1,\ldots,N$. Then we obtain the matrix form for (18) as follows

$$(A + d\Delta tB + \gamma\Delta tH)P_h^n = DP_h^{n-1} + \Delta tF,\ 1 \le n \le N, \tag{34}$$

where

$$A_{i,j} = \int_a^b \phi_j(x)\phi_i(x)dx,\ B_{i,j} = \int_a^b \phi_{j,x}(x)\phi_{i,x}(x)dx,$$
$$H_{i,j} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\phi_j(x)-\phi_j(y))(\phi_i(x)-\phi_i(y))}{2|x-y|^{1+\alpha}}dydx, \tag{35}$$
$$D_{i,j} = \int_a^b \phi_j(x_h^*)\phi_i(x)dx,\ f_i^n = \int_a^b f(x,t_n)\phi_i(x)dx,$$

for $1 \le i,j \le I-1,\ 1 \le n \le N$.

We find the coefficient matrices A, B are tridiagonal and symmetric Toeplitz matrices. Then we investigate the structure of the matrix H in the following theorem.

Theorem 5.1. *The matrix H described in (35) is a Toeplitz matrix for any $\alpha \in (0,2)$.*

Proof. We rewrite the expression of $H_{i,j}$ in (35) as

$$\begin{aligned} H_{i,j} =& \int_a^b \phi_j(x)\phi_i(x)\Big(\int_{-\infty}^a \frac{1}{|x-y|^{1+\alpha}}dy + \int_b^\infty \frac{1}{|x-y|^{1+\alpha}}dy\Big)dx \\ &+ \frac{1}{2}\int_a^b\int_a^b \frac{(\phi_j(x)-\phi_j(y))(\phi_i(x)-\phi_i(y))}{|x-y|^{1+\alpha}}dydx \\ =& \int_a^b \phi_j(x)\phi_i(x)\Big[\frac{1}{\alpha}(x-a)^{-\alpha} + \frac{1}{\alpha}(b-x)^{-\alpha}\Big]dx \\ &+ \frac{1}{2}\int_a^b\int_a^b \frac{(\phi_j(x)-\phi_j(y))(\phi_i(x)-\phi_i(y))}{|x-y|^{1+\alpha}}dydx \end{aligned} \tag{36}$$

for $1 \le i,j \le I-1$. Since the matrix H is symmetric, we just need to evaluate the upper triangular part of it. For the convenience of proof, we divide the upper triangular part of the matrix H into three parts, i.e., the entries on main diagonal (I), superdiagonal (II) and any other diagonals (III).

We first consider the part III with $1 \le i \le I-3,\ j \ge i+2$. Since the integration $\int_a^b \phi_j(x)\phi_i(x)dx$ vanishes for $j-i \ge 2$, the entries $H_{i,j}$ represent as follows

$$\begin{aligned} H_{i,j} =& -\int_{x_{i-1}}^{x_{i+1}}\int_{x_{j-1}}^{x_{j+1}} \frac{\phi_j(y)}{|x-y|^{1+\alpha}}dy\ \phi_i(x)dx \\ =& -\int_{x_{i-1}}^{x_i}\frac{x-x_{i-1}}{h}\int_{x_{j-1}}^{x_j}\frac{\frac{y-x_{j-1}}{h}}{|x-y|^{1+\alpha}}dydx - \int_{x_{i-1}}^{x_i}\frac{x-x_{i-1}}{h}\int_{x_j}^{x_{j+1}}\frac{\frac{x_{j+1}-y}{h}}{|x-y|^{1+\alpha}}dydx \\ &- \int_{x_i}^{x_{i+1}}\frac{x_{i+1}-x}{h}\int_{x_{j-1}}^{x_j}\frac{\frac{y-x_{j-1}}{h}}{|x-y|^{1+\alpha}}dydx - \int_{x_i}^{x_{i+1}}\frac{x_{i+1}-x}{h}\int_{x_j}^{x_{j+1}}\frac{\frac{x_{j+1}-y}{h}}{|x-y|^{1+\alpha}}dydx \\ =& -h^{1-\alpha}\Big(\int_0^1 s\int_0^1 \frac{t}{(t-s+j-i)^{1+\alpha}}dtds + \int_0^1 s\int_0^1 \frac{t}{(j-i+2-t-s)^{1+\alpha}}dtds \\ &+ \int_0^1 s\int_0^1 \frac{t}{(j-i-2+t+s)^{1+\alpha}}dtds + \int_0^1 s\int_0^1 \frac{t}{(j-i-t+s)^{1+\alpha}}dtds\Big). \end{aligned}$$

On the right-hand side of the third equation, $s = (x-x_{i-1})/h$ in the first two integrals and $s = (x_{i+1}-x)/h$ in the last two integrals; $t = (y-x_{j-1})/h$ in the first and third integrals and $t = (x_{j+1}-y)/h$ in the second and fourth integrals. From the above equations, we observe that, once $j-i = m, 2 \le m \le I-2$, is specified, the entries on that diagonal are the same.

Next, we consider the entries on the main diagonal (part I). For $1 \leq i \leq I-1$,

$$H_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \phi_i^2(x)\Big(\frac{1}{\alpha}(x-a)^{-\alpha} + \frac{1}{\alpha}(b-x)^{-\alpha}\Big)dx + \int_a^b \phi_i(x)\int_a^b \frac{\phi_i(x)-\phi_i(y)}{|x-y|^{1+\alpha}}dydx. \tag{37}$$

We rewrite the second term on the right-hand side of (37) as follows

$$\begin{aligned}
&\int_a^b \phi_i(x)\int_a^b \frac{\phi_i(x)-\phi_i(y)}{|x-y|^{1+\alpha}}dydx\\
&= \int_{x_{i-1}}^{x_{i+1}} \phi_i^2(x)\Big(\int_a^{x_{i-1}} \frac{1}{|x-y|^{1+\alpha}}dy + \int_{x_{i+1}}^b \frac{1}{|x-y|^{1+\alpha}}dy\Big)dx\\
&\quad + \int_{x_{i-1}}^{x_{i+1}} \phi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{\phi_i(x)-\phi_i(y)}{|x-y|^{1+\alpha}}dydx\\
&= \int_{x_{i-1}}^{x_{i+1}} \frac{\phi_i^2(x)}{\alpha}\Big((x-x_{i-1})^{-\alpha}-(x-a)^{-\alpha}+(x_{i+1}-x)^{-\alpha}-(b-x)^{-\alpha}\Big)dx\\
&\quad + \int_{x_{i-1}}^{x_{i+1}} \phi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{\phi_i(x)-\phi_i(y)}{|x-y|^{1+\alpha}}dydx,
\end{aligned} \tag{38}$$

where the second term after the second equation can be represented by

$$\begin{aligned}
&\int_{x_{i-1}}^{x_{i+1}} \phi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{\phi_i(x)-\phi_i(y)}{|x-y|^{1+\alpha}}dydx\\
&= \int_{x_{i-1}}^{x_i} \frac{x-x_{i-1}}{h}\int_{x_{i-1}}^{x_i} \frac{\frac{x-x_{i-1}}{h}-\frac{y-x_{i-1}}{h}}{|x-y|^{1+\alpha}}dydx\\
&\quad + \int_{x_{i-1}}^{x_i} \frac{x-x_{i-1}}{h}\int_{x_i}^{x_{i+1}} \frac{\frac{x-x_{i-1}}{h}-\frac{x_{i+1}-y}{h}}{|x-y|^{1+\alpha}}dydx\\
&\quad + \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{h}\int_{x_{i-1}}^{x_i} \frac{\frac{x_{i+1}-x}{h}-\frac{y-x_{i-1}}{h}}{|x-y|^{1+\alpha}}dydx\\
&\quad + \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{h}\int_{x_i}^{x_{i+1}} \frac{\frac{x_{i+1}-x}{h}-\frac{x_{i+1}-y}{h}}{|x-y|^{1+\alpha}}dydx\\
&= 2h^{1-\alpha}\Big(\int_0^1 s\int_0^1 \frac{s-t}{|s-t|^{1+\alpha}}dtds + \int_0^1 s\int_0^1 \frac{s-t}{(2-t-s)^{1+\alpha}}dtds\Big).
\end{aligned} \tag{39}$$

On the right-hand side of the second equation, we apply $s = (x-x_{i-1})/h$ in the first two integrals, $s = (x_{i+1}-x)/h$ in the lats two integrals and $t = (y-x_{i-1})/h$ in the first and third integrals, $t = (x_{i+1}-y)/h$ in the second and forth integrals. Combining equations (37) – (39), we obtain

$$\begin{aligned}
H_{i,i} =& 2h^{1-\alpha}\Big(\frac{1}{\alpha(3-\alpha)} - \frac{1}{\alpha(3-\alpha)}(1-2^{3-\alpha})\\
&+ \frac{4}{\alpha(2-\alpha)}(1-2^{2-\alpha}) - \frac{4}{\alpha(1-\alpha)}(1-2^{1-\alpha})\\
&+ \int_0^1 s\int_0^1 \frac{s-t}{|s-t|^{1+\alpha}}dtds + \int_0^1 s\int_0^1 \frac{s-t}{(2-t-s)^{1+\alpha}}dtds\Big),
\end{aligned}$$

for any $1 \le i \le I-1$ and for any $\alpha \in (0,2), \alpha \ne 1$. For the case of $\alpha = 1$,

$$H_{i,i} = 2\Big(4\ln 2 - 2 + \int_0^1 s \int_0^1 \frac{s-t}{|s-t|^2} dtds + \int_0^1 s \int_0^1 \frac{s-t}{(2-t-s)^2} dtds\Big).$$

We observe that the entries on the main diagonal are equal for any $\alpha \in (0,2)$.

Finally, we consider the entries on the superdiagonal (part II). Using the similar techniques for computing the entries on the main diagonal, we obtain that

$$\begin{aligned}
H_{i,i+1} &= \int_{x_i}^{x_{i+1}} \phi_{i+1}(x)\phi_i(x)\Big[\frac{1}{\alpha}(x-a)^{-\alpha} + \frac{1}{\alpha}(b-x)^{-\alpha}\Big]dx \\
&\quad + \int_a^b \phi_i(x) \int_a^b \frac{\phi_{i+1}(x) - \phi_{i+1}(y)}{|x-y|^{1+\alpha}} dydx \\
&= h^{1-\alpha}\Big(\frac{1-2^{3-\alpha}}{\alpha(3-\alpha)} + \frac{1}{\alpha(2-\alpha)} - \frac{1}{\alpha(3-\alpha)} - \frac{3(1-2^{2-\alpha})}{\alpha(2-\alpha)} + \frac{2(1-2^{1-\alpha})}{\alpha(1-\alpha)} \\
&\quad - \int_0^1 s \int_0^1 \frac{t}{(1+t-s)^{1+\alpha}} dtds - \int_0^1 s \int_0^1 \frac{t}{(3-t-s)^{1+\alpha}} dtds \\
&\quad - \int_0^1 (1-s) \int_0^1 \frac{t-s}{|t-s|^{1+\alpha}} dtds + \int_0^1 (1-s) \int_0^1 \frac{s-t}{(2-t-s)^{1+\alpha}} dtds\Big),
\end{aligned}$$

for any $1 \le i \le I-2$ and for any $\alpha \in (0,2), \alpha \ne 1$. For the case of $\alpha = 1$,

$$\begin{aligned}
H_{i,i+1} = 2 - 2\ln 2 &- \int_0^1 s \int_0^1 \frac{t}{(1+t-s)^2} dtds - \int_0^1 s \int_0^1 \frac{t}{(3-t-s)^2} dtds \\
&- \int_0^1 (1-s) \int_0^1 \frac{t-s}{|t-s|^2} dtds + \int_0^1 (1-s) \int_0^1 \frac{s-t}{(2-t-s)^2} dtds.
\end{aligned}$$

Thus we observe that the entries on the superdiagonal are the same. □

5.1. A fast solution method. In general, we use iterative methods to solve the numerical scheme in (34) for each time step. However, the coefficient matrix is dense for the nonlocal nature of the fractional Laplacian. The computational cost of the matrix-vector multiplication with the stiffness matrix and the memory requirement are enormous. From the above theorem, we notice that the coefficient matrix is well-structured and is a Toeplitz matrix. Thus a fast solution method given in [4, 14, 17] will be applied. We consider use the conjugate gradient method cooperating with the fast matrix-vector multiplication to solve the numerical scheme. Therefore, the computational cost will be decreased from $O(I^2)$ to $O(IlogI)$ for each iteration, while the memory requirement will be reduced from $O(I^2)$ to $O(I)$. Comparing with the general iterative methods, this fast solution method is efficient and losses no accuracy.

6. Auxiliary Lemma

We prove one auxiliary lemma in this section. This lemma proves the optimal-order error bound in (25).

Lemma 6.1. *Assume* $p \in L^\infty(0,T;H^s) \cap H^1(0,T;H^s)$*. Let* $\pi_h p \in S_h(a,b)$ *be the interpolation of* p *and* $\eta = \pi_h p - p$*. Then the following estimate holds*

$$
\begin{aligned}
&\Big| \int_a^b \eta(x_h^*, t_{n-1}) \xi_h(x,t_n) r_{h,x}(t_{n-1}; x, t_n) dx - \int_a^b \eta(x,t_n)\xi_h(x,t_n) dx \Big| \\
&\leq C\Delta t \|\xi_h(\cdot,t_n)\|_{L^2}^2 + \varepsilon_1 d\Delta t \|\xi_{h,x}(\cdot,t_n)\|_{L^2}^2 + Ch^{2s}\|p\|^2_{H^1(t_{n-1},t_n;H^s)} \\
&\quad + C\Delta t h^{2s} \|p\|^2_{L^\infty(0,T;H^s)}.
\end{aligned}
\tag{40}
$$

Proof. We rewrite the left-hand side of (40) as

$$
\begin{aligned}
&\int_a^b \eta(x,t_n)\xi_h(x,t_n) dx - \int_a^b \eta(x_h^*, t_{n-1}) \xi_h(x,t_n) r_{h,x}(t_{n-1}; x, t_n) dx \\
&= \int_a^b (\eta(x,t_n) - \eta(x_h^*, t_{n-1}))\xi_h(x,t_n) dx \\
&\quad + \Delta t \int_a^b \eta(x_h^*, t_{n-1}) \xi_h(x,t_n) V_x(x,t_n) dx.
\end{aligned}
\tag{41}
$$

We bound the second term on the right-hand side in a similar way to (23) by

$$
\begin{aligned}
&\Big| \Delta t \int_a^b \eta(x_h^*, t_{n-1}) \xi_h(x,t_n) V_x(x,t_n) dx \Big| \\
&\leq C\Delta t \|\xi_h(\cdot,t_n)\|_{L^2} \ (\int_a^b \eta^2(x_h^*, t_{n-1}) dx)^{\frac{1}{2}} \\
&\leq C\Delta t \|\xi_h(\cdot,t_n)\|_{L^2} \ \|\eta(\cdot, t_{n-1})\|_{L^2} \\
&\leq C\Delta t \|\xi_h(\cdot,t_n)\|_{L^2}^2 + C\Delta t \ h^2 s \|p\|^2_{L^\infty(0,T;H^s)}.
\end{aligned}
$$

Next, we decompose the first term on the right-hand side of (41) as follows

$$
\begin{aligned}
&\int_a^b (\eta(x,t_n) - \eta(x_h^*, t_{n-1}))\xi_h(x,t_n) dx \\
&= \int_a^b \int_{t_{n-1}}^{t_n} \eta_t(x,t) dt \ \xi_h(x,t_n) dx \\
&\quad + \int_a^b (\eta(x,t_{n-1}) - \eta(x_h^*, t_{n-1}))\xi_h(x,t_n) dx.
\end{aligned}
\tag{42}
$$

The first term on the right-hand side is bounded by

$$
\begin{aligned}
\Big| \int_a^b \int_{t_{n-1}}^{t_n} \eta_t(x,t) dt \ \xi_h(x,t_n) dx \Big| &\leq (\Delta t)^{1/2} \|\xi_h(\cdot,t_n)\|_{L^2} \ \|\eta\|_{H^1(t_{n-1},t_n;L^2)} \\
&\leq C\Delta t \|\xi_h(\cdot,t_n)\|_{L^2}^2 + Ch^{2s}\|p\|^2_{H^1(t_{n-1},t_n;H^s)}.
\end{aligned}
$$

We rewrite the second term on the right-hand side of (42) as

$$\begin{aligned}
&\int_a^b (\eta(x_h^*, t_{n-1}) - \eta(x, t_{n-1}))\xi_h(x, t_n)dx \\
&= \int_a^b \eta(x_h^*, t_{n-1})\xi_h(x, t_n)dx - \int_a^b \eta(x, t_{n-1})\xi_h(x, t_n)dx \\
&= \int_{a_h^*}^{b_h^*} \eta(x, t_{n-1})\xi_h(\tilde{x}, t_n)(\frac{dx}{d\tilde{x}})^{-1}dx - \int_a^b \eta(x, t_{n-1})\xi_h(x, t_n)dx \\
&= \int_a^b \eta(x, t_{n-1})\xi_h(\tilde{x}, t_n)dx - \int_a^b \eta(x, t_{n-1})\xi_h(x, t_n)dx \\
&\quad + \int_{a_h^*}^{b_h^*} O(\Delta t)\eta(x, t_{n-1})\xi_h(\tilde{x}, t_n)dx.
\end{aligned} \tag{43}$$

Here we use the definition of η and ξ_h, i.e., $\eta(x) = \xi_h(x) = 0,\ x \notin (a, b)$. Then we first bound the third term on the right-hand side of (43) as

$$\begin{aligned}
&\Big|\int_{a_h^*}^{b_h^*} O(\Delta t)\eta(x, t_{n-1})\xi_h(\tilde{x}, t_n)dx\Big| \\
&\le C\Delta t\|\eta(\cdot, t_{n-1})\|_{L^2}\Big(\int_{a_h^*}^{b_h^*} \xi_h(\tilde{x}, t_n)^2 dx\Big)^{\frac{1}{2}} \\
&\le C\Delta t\|\eta(\cdot, t_{n-1})\|_{L^2}\Big(\int_a^b \xi_h(x, t_n)^2(\frac{d\tilde{x}}{dx})^{-1}d\tilde{x}\Big)^{\frac{1}{2}} \\
&\le C\Delta t\|\eta(\cdot, t_{n-1})\|_{L^2}\|\xi_h(\cdot, t_n)\|_{L^2} \\
&\le C\Delta t\|\xi_h(\cdot, t_n)\|_{L^2}^2 + C\Delta t\ h^{2s}\|p\|_{L^\infty(0,T;H^s)}^2.
\end{aligned}$$

The first and the second terms are bounded by

$$\begin{aligned}
&\Big|\int_a^b \eta(x, t_{n-1})(\xi_h(\tilde{x}, t_n) - \xi_h(x, t_n))dx\Big| \\
&\le C(\Delta t)^{\frac{1}{2}}\|\eta(\cdot, t_{n-1})\|_{L^2}\Big(\int_a^b\int_x^{\tilde{x}} \xi_{h,x}^2(\zeta, t_n)d\zeta\ dx\Big)^{\frac{1}{2}} \\
&\le C\Delta t\|\eta(\cdot, t_{n-1})\|_{L^2}\|\xi_{h,x}(\cdot, t_n)\|_{L^2} \\
&\le \varepsilon_1 d\Delta t\|\xi_{h,x}(\cdot, t_n)\|_{L^2}^2 + \frac{C\Delta t\ h^{2s}}{\varepsilon_1 d}\|p\|_{L^\infty(0,T;H^s)}^2 \\
&\le \varepsilon_1 d\Delta t\|\xi_{h,x}(\cdot, t_n)\|_{L^2}^2 + C\Delta t\ h^{2s}\|p\|_{L^\infty(0,T;H^s)}^2,
\end{aligned}$$

where ε_1 is a positive constant and the constant C depends on d.

Combining all these estimates we have proved the Lemma (6.1). □

7. Numerical Experiments

In this section, we conduct two numerical examples for the nonlocal advection diffusion equation (1) by changing the values of d and γ to observe the accuracy of the ELLAM. We also investigate the efficiency of the fast solution method.

Example 7.1. *We consider the nonlocal advection diffusion equation (1) with* $d = 10^{-4}, \gamma = 10^{-4}$*. The spatial interval* (a, b) *is* $(-1, 2)$ *and the time interval is* $(0, T] = (0, 1]$*. We choose the velocity* $V = 1$ *and the source term* $f = 0$*. In addition, we*

TABLE 1. Spatial L^2-errors for the Example 7.1 solved by the ELLAM scheme and the BE method with several spatial partitions h and $\alpha = 1.5, \Delta t = \frac{1}{50}, \frac{1}{5000}$.

Δt	h	L^2-error(ELLAM)	L^2-error(BE)
$\frac{1}{50}$	2^{-3}	3.4665E-01	6.1438E-01
	2^{-4}	7.5024E-02	5.6699E-01
	2^{-5}	1.2709E-02	5.5916E-01
	2^{-6}	2.7746E-03	5.5680E-01
	2^{-7}	6.8768E-04	5.5680E-01
$\frac{1}{5000}$	2^{-7}	—	6.8473E-04

choose a general Gaussian pulse as the initial condition, which is given by

$$p_0(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{x^2}{2\sigma^2}),$$

where the parameter σ is the standard deviation and chosen as $\sigma^2 = 0.0125$.

First, we run the numerical experiments by using the ELLAM scheme and the finite element method with the Backward-Eulerian (BE) method in time discretization, and compare the corresponding numerical results. In terms of the practical application, we choose $\alpha \in (1, 2)$ in numerical experiments. Since the exact solution of this problem is difficult to obtain, we use the numerical solution with finer spatial partition ($h = \frac{1}{2048}$) as the reference solution to obtain the spatial L^2-errors in this paper. Table 1 shows the spatial L^2-errors obtained by the two different numerical methods with several spatial partitions h and $\alpha = 1.5, \Delta t = \frac{1}{50}, \frac{1}{5000}$. We observe that the ELLAM scheme is more accurate than the BE method, because the latter generates excessive numerical diffusion for large time steps. When we refine the time step to $\Delta t = \frac{1}{5000}$ and spatial partition to $h = 2^{-7}$, the L^2-errors obtained by the BE method become smaller. This comparison illustrates that the ELLAM is more accurate for larger spatial and time steps. In Table 2, we present the spatial L^2-errors and the corresponding convergence rates for problem (1), which are obtained by the ELLAM scheme with several α and spatial partitions h, and $\Delta t = \frac{1}{5}$. The convergence rates show that the ELLAM scheme has a second-order accuracy in space.

Furthermore, we apply Gaussian elimination, conjugate gradient (CG) method and the fast conjugate gradient (FCG) method to solve the linear algebraic system and investigate their performance. Table 3 shows the spatial L^2-errors and corresponding CPU times for several spatial partitions with $\alpha = 1.5, \Delta t = \frac{1}{500}$. We observe that the CG method performs better than the Gaussian elimination, while the FCG method performs the best efficiently. In addition, the fast solution method looses no accuracy.

Example 7.2. *We consider the same numerical experiment as that in the Example 7.1. Besides, we change the diffusion coefficients as $d = 0, \gamma = 10^{-4}$.*

TABLE 2. Spatial L^2-errors and corresponding convergence rates for the Example 7.1 obtained by the ELLAM scheme with several fractional orders α and spatial partitions h, and $\Delta t = \frac{1}{5}$.

h	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$	
	L^2-error	order	L^2-error	order	L^2-error	order
2^{-4}	4.4180E-02	——	4.3843E-02	——	4.2547E-02	——
2^{-5}	1.0987E-02	2.01	1.0902E-02	2.01	1.0594E-02	2.01
2^{-6}	2.7092E-03	2.02	2.6915E-03	2.02	2.6290E-03	2.01
2^{-7}	6.8020E-04	1.99	6.8017E-04	1.98	6.8017E-04	1.95
2^{-8}	1.7002E-04	2.00	1.7122E-04	1.99	1.7365E-04	1.97

TABLE 3. Spatial L^2-errors and the CPU times consumed by Gaussian elimination, CG method, and FCG method for Example 7.1 with different spatial partitions and $\alpha = 1.5, \Delta t = \frac{1}{500}$.

h	Gauss		CG		FCG	
	L^2-error	CPU(s)	L^2-error	CPU(s)	L^2-error	CPU(s)
2^{-5}	1.2241E-02	7.83	1.2241E-02	1.10	1.2241E-02	1.27
2^{-6}	2.9678E-03	29.3	2.9678E-03	2.58	2.9678E-03	2.19
2^{-7}	7.2634E-04	2m 27s	7.2634E-04	6.62	7.2634E-04	3.96
2^{-8}	1.7908E-04	30m 18s	1.7908E-04	18.5	1.7908E-04	7.25
2^{-9}	4.8446E-05	4h 24m	4.8446E-05	1m 19s	4.8446E-05	14.6

Similarly, we investigate the accuracy of the ELLAM scheme and the efficiency of the fast solution method. We also use the numerical solution with the finer spatial partition ($h = \frac{1}{2048}$) as the reference solution. Table 4 shows the spatial L^2-errors obtained by the ELLAM and the BE method for (1) with different spatial partitions and $\alpha = 1.5, \Delta t = \frac{1}{50}, \frac{1}{5000}$. The contrastive results show that the ELLAM scheme is more accurate than the BE method for large spatial and time steps. Table 5 shows the spatial L^2-errors and the corresponding convergence rates for Example 7.2 solved by the ELLAM scheme for several fractional orders with $\Delta t = \frac{1}{5}$. From the numerical results, we observe that this scheme has a second-order accuracy in space numerically. In Table 6, we present the spatial L^2-errors and the CPU times consumed by Gaussian elimination, CG method and the FCG method. The numerical results illustrate that the CG method co-operating with the fast matrix-vector multiplication algorithm performs the best and losses no accuracy.

8. Summary

In this paper, we derive a ELLAM scheme to solve a nonlocal advection diffusion equation. In addition, we make an error estimate for this numerical scheme and prove it of order $O(\Delta t + h^s + h^{s+1-\alpha-\epsilon})$, where $0 < \alpha < 2,\ s \geq 0$ and $0 < \epsilon \ll 1$.

TABLE 4. Spatial L^2-errors for the Example 7.2 solved by the ELLAM scheme and the BE method for several spatial partitions h with $\alpha = 1.5, \Delta t = \frac{1}{50}, \frac{1}{5000}$.

Δt	h	L^2 -error(ELLAM)	L^2- error(BE)
$\frac{1}{50}$	2^{-3}	3.5339E-01	6.1178E-01
	2^{-4}	7.6505E-02	5.6391E-01
	2^{-5}	1.2840E-02	5.5608E-01
	2^{-6}	2.7948E-03	5.5419E-01
	2^{-7}	6.9247E-04	5.5372E-01
$\frac{1}{5000}$	2^{-7}	—	6.7036E-04

TABLE 5. Spatial L^2-errors and corresponding convergence rates for the Example 7.2 obtained by the ELLAM scheme for several fractional orders α and spatial partitions h with $\Delta t = \frac{1}{5}$.

h	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$	
	L^2-error	order	L^2-error	order	L^2-error	order
2^{-4}	4.4555E-02	—	4.4213E-02	—	4.2904E-02	—
2^{-5}	1.1067E-02	2.01	1.0982E-02	2.01	1.0671E-02	2.01
2^{-6}	2.7277E-03	2.02	2.7098E-03	2.02	2.6469E-03	2.01
2^{-7}	6.8484E-04	1.99	6.8172E-04	1.99	6.8532E-04	1.95
2^{-8}	1.7084E-04	2.00	1.7503E-04	1.96	1.9801E-04	1.79

TABLE 6. Spatial L^2-errors and the CPU times consumed by Gaussian elimination, CG method, and the FCG method for Example 7.2 with different spatial partitions and $\alpha = 1.5, \Delta t = \frac{1}{500}$.

h	Gauss		CG		FCG	
	L^2-error	CPU(s)	L^2-error	CPU(s)	L^2-error	CPU(s)
2^{-5}	1.2368E-02	7.44	1.2368E-02	1.07	1.2368E-02	1.25
2^{-6}	2.9933E-03	29.2	2.9933E-03	2.49	2.9933E-03	2.30
2^{-7}	7.3212E-04	2m 26s	7.3212E-04	6.57	7.3212E-04	4.25
2^{-8}	1.8039E-04	30m 2s	1.8039E-04	19.1	1.8039E-04	7.94
2^{-9}	4.8797E-05	4h 23m	4.8797E-05	1m 42s	4.8797E-05	15.6

Furthermore, we prove that the coefficient matrix of this numerical scheme is a Toeplitz matrix. Then a CG method co-operating with the fast matrix-vector multiplication algorithm can be applied to accelerate the computing. We conduct two numerical examples to observe that the ELLAM scheme has a second-order accuracy in space numerically and the FCG method is efficient without losing any accuracy. We conclude this paper by the following remarks: (i) Although the order of the L^2-error in numerical examples is optimal, it does not reach the optimal-order in the proof of the error estimate. (ii) In this paper, we assume that the diffusion coefficient d is positive. Then the error estimate holds. But from the numerical results, it seems to indicate that the estimate also holds for $d = 0$. (iii) For the convection diffusion equations without fractional Laplacian in two dimensions, the uniform estimates of Eulerian-Lagrangian methods for transient convection-diffusion equations and an optimal-order error estimate to ELLAM schemes for transient advection-diffusion equations on unstructured meshes [16, 19] have already been analyzed.

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