INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 1, Pages 42–53

THE PROPERTY OF THE BRANCH OF NONSINGULAR FINITE ELEMENT/FINITE VOLUME SOLUTIONS TO THE STATIONARY NAVIER-STOKES EQUATIONS AND ITS APPLICATION

JIAN LI AND YINNIAN HE

Abstract. In this paper, a branch of nonsingular solutions of the stationary Navier-Stokes equations are investigated, which are unique on a neighborhood, and mostly isolated without relying on very stringent requirement on the small data. We summarize and develop an equivalent definition of nonsingular solutions of finite element/finite volume methods in the same framework. Furthermore, we establish the equivalent definition of a branch of singular solutions of finite element methods for the coupled Navier-Stokes/Darcy equations.

Key words. Stationary Navier-Stokes equations, finite element methods, finite volume methods, the branch of nonsingular solutions, *inf-sup* condition, large data.

1. Introduction

The Navier-Stokes equations usually have more than one solution unless the data satisfy the stringent requirement of uniqueness condition of the solutions, which required that the data be small enough in certain norms [17, 33]. However, this uniqueness condition is rarely satisfied in the real world. In many practical examples, the solutions are mostly isolated, and depend continuously on the viscosity. As the viscosity varies along an interval, each solution of the Navier-Stokes equations describes an isolated branch, which means the bifurcation phenomenon is rare. This situation is expressed mathematically by the notion of branches of nonsingular solutions.

Finite element approximations of nonsingular solutions have been investigated in [4, 17, 18], where optimal order error estimates have been obtained for the stationary Navier-Stokes equations with large data. Also, an analysis of the nonsingular finite volume solutions to the Navier-Stokes equations is not direct to establish where the whole system lacks symmetry in the context of a petrov-Galerkin method [2, 6, 7, 8, 14, 15, 16, 19, 25, 34].

For both finite element/finite volume approximations of the stationary Navier-Stokes equations, the original definition of nonsingular solutions is difficult to be applied and developed for the further research in this field. Here, we apply the definition of an isomorphism between two spaces to obtain the equivalent definition of discrete nonsingular solutions to the stationary Navier-Stokes equations and its coupled system.

This paper is organized as follows: In the next section, we introduce notations of a branch of nonsingular solutions to the stationary Navier-Stokes equations. Then, in the third section, the property of a branch of nonsingular finite element solutions to the stationary Navier-Stokes equations is derived. Also, the corresponding property of nonsingular finite volume solutions is investigated in the fourth section. Finally, we investigate the property of a branch of nonsingular finite element

Received by the editors October 01, 2014 and in revised form, April 18, 2019.
2000 *Mathematics Subject Classification.* 65N30, 76D06.

solutions to the coupled Navier-Stokes/Darcy model with heterogeneous porous medium.

2. A branch of nonsingular solutions to the stationary Navier-Stokes equations

Let Ω be a bounded domain in $\Re^d, d = 2, 3$, assumed that Γ is of C^2 or if Ω is a two-dimensional convex polygon. The stationary Navier-Stokes equations are

$$-\Delta u + \lambda \nabla p = \lambda f - \lambda((u \cdot \nabla)u + \frac{1}{2}(\mathrm{div} u)u), \qquad \text{in } \Omega, \tag{1}$$

$$\mathrm{div}\, u = 0, \qquad \text{in } \Omega, \tag{2}$$

$$u|_\Gamma = 0, \qquad \text{on } \Gamma, \tag{3}$$

where $u = u(x)$ represents the velocity vector, $p = p(x)$ the pressure, $f = f(x)$ the prescribed body force, $\lambda = \mu^{-1}$, and $\mu > 0$ the viscosity.

For simplicity, some useful Sobolev spaces can be defined by:

$$X = [H_0^1(\Omega)]^d,\ M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\},\ Z = [L^{3/2}(\Omega)]^d,$$

$$\bar{X} = X \times M,\ Y = [H^{-1}(\Omega)]^d,\ V = \{v \in X : \mathrm{div}\, v = 0\},$$
$$H = \left\{ v \in [L^2(\Omega)]^d : \mathrm{div}\, v = 0 \right\},\ D(A) = [H^2(\Omega)]^d \cap V,$$

where the Stokes operator $A : D(A) \to H$ is defined by $A = -P\Delta$ and $P : [L^2(\Omega)]^d \to H$ is the standard L^2-orthogonal projection. The spaces $[L^2(\Omega)]^m$, $m = 1, 2$, or 4, are endowed with the L^2-scalar product $(\cdot, \cdot)$ and the L^2-norm $\|\cdot\|_{L^2}$, as appropriate. In addition, $\|\cdot\|_{L^r},\ 1 \le r \le \infty$, denotes the norm of the space $L^r(\Omega)$. The space X is equipped with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\|u\|_{H^1}$ (or equivalently $\|\nabla u\|_{L^2}$), $u, v \in X$. In particular, define the norm on $\bar{X}$:

$$|||(v, q)||| = (\|\nabla v\|_{L^2}^2 + \lambda^2 \|q\|_{L^2}^2)^{1/2},\ (v, q) \in \bar{X}.$$

In this paper standard definitions are used for the Sobolev spaces $W^{m,r}(\Omega)$ [1], with the norm $\|\cdot\|_{W^{m,r}}$ and the seminorm $|\cdot|_{W^{m,r}}$, $m, r \ge 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_{H^m}$ for $\|\cdot\|_{W^{m,2}(\Omega)}$.

First, we consider the linear Stokes equations in order to introduce some mathematical theory of the nonsingular solutions of the stationary Navier-Stokes equations. A linear operator $T : Y \to \bar{X}$ is defined as follows: Given $g \in Y$, the solution of the Stokes problem

$$\begin{aligned} -\Delta v + \lambda \nabla q &= g, && \text{in } \Omega, \\ \mathrm{div}\, v &= 0, && \text{in } \Omega, \\ v|_\Gamma &= 0, && \text{on } \Gamma, \end{aligned} \tag{4}$$

is denoted by $\tilde{v}(\lambda) = (v, \lambda q) = Tg \in \bar{X}$. Furthermore, a C^2-mapping $\mathcal{G} : R^+ \times \bar{X} \to Y$ is defined by

$$\mathcal{G}(\lambda, \tilde{v}(\lambda)) = \lambda \left((v \cdot \nabla)v + \frac{1}{2}(\mathrm{div}\, v)v - f \right)$$

since the term $\mathrm{div} u = 0$. Finally, we define

$$F(\lambda, \tilde{v}(\lambda)) = \tilde{v}(\lambda) + T\mathcal{G}(\lambda, \tilde{v}(\lambda)), \quad \lambda \in R^+,\ \tilde{v}(\lambda) \in \bar{X}.$$

In this section, a branch of nonsingular solutions of the stationary Navier-Stokes equations, as introduced in [4, 17, 18, 22, 23], are studied.

Definition 2.1. Let Λ be a compact interval in R^+; $\{(\lambda, \tilde{u}(\lambda))\}$, with $\tilde{u}(\lambda) = (u, \lambda p)$, is a branch of nonsingular solutions to the equation

$$F(\lambda, \tilde{u}(\lambda)) = 0, \tag{5}$$

if $D_u F(\lambda, \tilde{u}(\lambda))$ is an isomorphism from $\bar{X}$ onto Y for all $\lambda \in \Lambda$.

Remark. As mentioned above, the further results on Ω can be obtained for the unique solution $\tilde{v}(\lambda) = (v, \lambda q) = Tg \in \bar{X}$ of the stationary Stokes problem for a prescribed $g \in [L^r(\Omega)]^d$ exists and satisfies

$$\|v\|_{W^{2,r}} + \lambda\|q\|_{W^{1,r}} \le C\|g\|_{L^r}, \tag{6}$$

where $C > 0$ is a constant depending on Ω. As usual, the letter C represents any constants independent of h and of all the functions appearing in a given inequality. Here and later, $C_0,\ C_1, \ldots$ are positive constants depending only on the data (λ, Ω, f).

Using integration by parts, the weak formulation of the stationary Navier-Stokes equations (1)-(3) is: Find $(u, p) \in \bar{X}$ such that

$$a(u, v) - \lambda d(v, p) + \lambda d(u, q) + \lambda b(u, u, v) = \lambda(f, v), \quad \forall (v, q) \in \bar{X}, \tag{7}$$

where the bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are defined as follows:

$$a(u, v) = \mu(\nabla u, \nabla v) \quad \forall u, v \in X,\ d(v, q) = (\operatorname{div} v, q) \quad \forall (v, q) \in \bar{X}.$$

Obviously, the bilinear form $a(\cdot,\cdot)$ is continuous and coercive on the space pair $X \times X$; the bilinear form $d(\cdot,\cdot)$ is continuous and satisfies the *inf-sup* condition: There exists a positive constant $\beta_1 > 0$ such that, for all $q \in M$,

$$\sup_{v \in X} \frac{d(v, q)}{\|\nabla v\|_{L^2}} \ge \beta_1 \|q\|_{L^2}. \tag{8}$$

Moreover, the trilinear form $b(\cdot,\cdot,\cdot)$ is continuous on the space triplet $X \times X \times X$

$$\begin{aligned} b(u, v, w) &= (B(u, v), w) \\ &= ((u \cdot \nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w) \\ &= \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v)\ \forall u, v, w \in X, \end{aligned}$$

and satisfies

$$b(u, v, w) = -\, b(u, w, v), \quad \forall u,\ v,\ w \in X, \tag{9}$$

$$|b(u, v, w)| \le C_0\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}\|\nabla w\|_{L^2}, \forall u,\ v,\ w \in X. \tag{10}$$

Furthermore, the existence and uniqueness results of (5) can be referred in [17, 18, 22, 23].

Lemma 2.1. ([17, 18, 22, 23]) *If λ satisfies the following uniqueness condition:*

$$\lambda < \lambda_0 = \frac{1}{\sqrt{C_0\|f\|_{-1}}},$$

then (5) *admits a unique solution* (u, p). *Moreover, the pair* $(u, p) \in \bar{X}$ *is a solution of the problem* (7) *if and only if* $\tilde{u}(\lambda) \in \bar{X}$ *is a solution of* (5).

Similarly, we can apply the same approach as in [18] to obtain the following stability of (5):

Lemma 2.2. *Assume that* $f \in [L^2(\Omega)]^d$, *and the pair* $\tilde{u}(\lambda) = (u, \lambda p) \in \bar{X}$ *is a solution to problem* (5). *Then* $\tilde{u}(\lambda) \in D(A) \times [H^1(\Omega) \cap M]$ *and* $G(\lambda, \tilde{u}(\lambda)) \in Y$ *satisfies*

$$\|u\|_{H^2} + \lambda\|p\|_{H^1} \quad \le C, \tag{11}$$

where the positive constant C depends on the data (λ, Ω, f).

3. A branch of nonsingular finite element solutions to the stationary Navier-Stokes equations

Let us now return to the finite element approximation of the problem presented. We shall consider the families of finite element spaces (X_h, M_h), for which we shall need the following assumptions: Let K_h be a regular, quasi-uniform triangulation of the polygonal domain Ω [5, 10]. Associated with K_h, we consider the finite element spaces for the velocity and pressure: $X_h \subset X$ and $M_h \subset M$.

Usually, we assume that the finite element spaces satisfy the discrete *inf-sup* condition:

$$\sup_{v_h \in X_h} \frac{d(v_h, q_h)}{\|\nabla v_h\|_{L^2}} \geq \beta_2 \|q_h\|_{L^2}, \ \forall q_h \in M_h, \tag{12}$$

where the constant $\beta_2 > 0$ is independent of h. Examples of the spaces that satisfy these assumptions include the following examples in 2d case [17, 32, 33]: Macro-element pairs $P_1 - P_i, i = 0, 1$ with a way [30] to approximate the P_2 velocity field defined on a macro-element mesh by refining K_h uniformly to obtain the mesh $K_{h/2}$, famous Taylor-Hood element, Mini element pair $P_1 b - P_1$, and the popular lowest equal-order or lowest order finite element spaces $P_1 - P_i$, which do not satisfy the inf-sup condition and can be stabilized by a symmetric and semidefinite term $G(p_h, q_h)$ [11, 12, 24].

Then, a bilinear form on $\bar{X}_h \times \bar{X}_h$ for the finite element methods introduced is defined by

$$\mathcal{B}_h((\bar{u}_h, \lambda \bar{p}_h), (v_h, \lambda q_h)) = a(\bar{u}_h, v_h) - \lambda d(v_h, \bar{p}_h) + \lambda d(\bar{u}_h, q_h) + \lambda \chi_i G(\bar{p}_h, q_h) \tag{13}$$

for $\forall (\bar{u}_h, \bar{p}_h), (v_h, q_h) \in \bar{X}_h$, where the characteristic constant

$$\chi_i = \begin{cases} 0, & \text{finite element methods(i=0)}, \\ 1, & \text{stabilized finite element methods(i=1)}. \end{cases}$$

Furthermore, this bilinear form satisfies the continuity and weak coercivity properties [11, 12, 17, 18, 24, 33]:

$$|\mathcal{B}_h((\bar{u}_h, \lambda \bar{p}_h), (v_h, \lambda q_h))| \leq C \|\!|(\bar{u}_h, \bar{p}_h)|\!\| \|\!|(v_h, q_h)|\!\|, \tag{14}$$

$$\sup_{(v_h, q_h) \in \bar{X}_h} \frac{|\mathcal{B}_h((u_h, \lambda p_h), (v_h, \lambda q_h))|}{\|\!|(v_h, q_h)|\!\|} \geq \beta_3 \|\!|(\bar{u}_h, \bar{p}_h)|\!\|, \tag{15}$$

where the constant $\beta_3 > 0$ is independent of h.

Using the above notations, the corresponding finite element formulation of system (1)-(3) reads: Find $(\bar{u}_h, \bar{p}_h) \in \bar{X}_h$, such that, for all $(v_h, q_h) \in \bar{X}_h$,

$$\mathcal{B}_h((\bar{u}_h, \lambda \bar{p}_h), (v_h, \lambda q_h)) + \lambda b(\bar{u}_h, \bar{u}_h, v_h) = \lambda (f, v_h); \tag{16}$$

i.e.,

$$F(\lambda, \tilde{\bar{u}}_h(\lambda)) \equiv \tilde{\bar{u}}_h(\lambda) + T_h \mathcal{G}(\lambda, \tilde{\bar{u}}_h(\lambda)) = 0, \tag{17}$$

where T_h is the discrete counterpart of the operator T.

Similar to the continuous case, $\{(\lambda, \tilde{\bar{u}}_h(\lambda))\}$ with $\tilde{\bar{u}}_h(\lambda) = (\bar{u}_h, \lambda \bar{p}_h)$ is a branch of nonsingular solutions to (17) if

$$D_{\bar{u}_h} F(\lambda, \tilde{\bar{u}}_h(\lambda)) \text{ is an isomorphism from } \bar{X}_h \text{ onto } Y \text{ for all } \lambda \in \Lambda. \tag{18}$$

Recall that $T_h : Y \rightarrow \bar{X}_h$ is the solution operator of the discrete Stokes equations. This operator yields the solution $\tilde{\bar{u}}_h(\lambda) = (\bar{u}_h, \lambda \bar{p}_h)$ to problem (17). Apparently, this solution is also a solution of the discrete Navier-Stokes equation (16) if and only if it is a solution of (17).

For convenience, set

$$\bar{B}_\lambda((\bar{w}_h, \lambda\bar{\chi}_h); (v_h, \lambda q_h)) =: A_\lambda(\bar{u}_h; \bar{w}_h, v_h) - \lambda d(v_h, \bar{\chi}_h) + \lambda d(\bar{w}_h, q_h) + \lambda\chi_i G(\bar{\chi}_h, q_h),$$

and $A_\lambda(\bar{u}_h; w_h, v_h) = a(w_h, v_h) + \lambda b(\bar{u}_h, w_h, v_h) + \lambda b(w_h, \bar{u}_h, v_h)$.

Furthermore, based on the results in [4, 17, 18, 36], we obtain the following result.

Theorem 3.1. *$\tilde{\bar{u}}_h(\lambda) \in \bar{X}_h$ is a branch of non-singular solutions to equation (16) or (17) if and only if there exists a constant $\gamma > 0$ dependent of the data (λ, f, Ω), such that*

$$(19)\quad \sup_{(v_h, q_h)\in \bar{X}_h} \frac{|\bar{B}_\lambda((\bar{w}_h, \lambda\bar{\chi}_h); (v_h, \lambda q_h))|}{|\!|\!|(v_h, q_h)|\!|\!|} \geq \gamma |\!|\!|(\bar{w}_h, \bar{\chi}_h)|\!|\!|, \quad \forall (\bar{w}_h, \bar{\chi}_h) \in \bar{X}_h,$$

where the parameter γ is determined by the following

$$\gamma = \frac{\min\{\frac{3\gamma_0}{4}, \frac{3\lambda\sigma}{4}\left(1 - \frac{4C_1^2(1+2C_0\lambda^2\|f\|_{-1})\sigma}{3\lambda\gamma_0}\right)\}}{\max\{1, \frac{C_1\sigma+\lambda}{\lambda}\}}.$$

Proof. Since $D_{\bar{u}_h}F(\lambda, \tilde{\bar{u}}_h(\lambda))$ is an isomorphism from $\bar{X}_h$ onto Y for all $\lambda \in \Lambda$, it first is a surjective linear mapping, thus for each $\tilde{\bar{w}}_1 =: (\bar{w}_1, \lambda\bar{\chi}_1) \in \bar{X}_h \subset \bar{X}$ there exists a unique $\tilde{\bar{w}}_h =: (\bar{w}_h, \lambda\bar{\chi}_h) \in \bar{X}_h \subset \bar{X}$ such that

$$(20)\qquad D_{\bar{u}_h}F_h(\lambda, \tilde{\bar{u}}_h)\tilde{\bar{w}}_h = \tilde{\bar{w}}_1.$$

Thus, $\tilde{\bar{w}}_h = (\bar{w}_h, \lambda\bar{\chi}_h)$ satisfies

$$\tilde{\bar{w}}_h - \tilde{\bar{w}}_1 = (\bar{w}_h - \bar{w}_1, \lambda(\bar{\chi}_h - \bar{\chi}_1)) = -T_h(D_{\bar{u}_h}\mathcal{G}(\lambda, \tilde{\bar{u}}_h)\bar{w}_h).$$

where

$$\begin{aligned} D_{\bar{u}_h}\mathcal{G}(\lambda, \tilde{\bar{u}}_h)\bar{w}_h &= \lambda\left(((\bar{u}_h\cdot\nabla)\bar{w}_h + (\bar{w}_h\cdot\nabla)\bar{u}_h) + \frac{1}{2}\left((\mathrm{div}\bar{u}_h)\bar{w}_h + (\mathrm{div}\bar{w}_h)\bar{u}_h\right)\right) \\ &= \lambda(B(\bar{u}_h, \bar{w}_h) + B(\bar{w}_h, \bar{u}_h)). \end{aligned}$$

Then we obtain by a direct calculation

$$\begin{aligned} a(\bar{w}_h - \bar{w}_1, v_h) - \lambda d(v_h, \bar{\chi}_h - \bar{\chi}_1) + \lambda d(\bar{w}_h - \bar{w}_1, q_h) + \lambda\chi_i G(\bar{\chi}_h - \bar{\chi}_1, q_h) \\ = -\lambda b(\bar{u}_h, \bar{w}_h, v_h) - \lambda b(\bar{w}_h, \bar{u}_h, v_h). \end{aligned}$$

Setting $\lambda(\tilde{f}, v_h) = a(\bar{w}_1, v_h) - \lambda d(v_h, \bar{\chi}_1) + \lambda d(\bar{w}_1, q_h) + \lambda\chi_i G(\bar{\chi}_1, q_h)$, the following equation holds

$$\begin{aligned} a(\bar{w}_h, v_h) + \lambda b(\bar{u}_h, \bar{w}_h, v_h) + \lambda b(\bar{w}_h, \bar{u}_h, v_h) \\ -\lambda d(v_h, \bar{\chi}_h) + \lambda d(\bar{w}_h, q_h) + \lambda\chi_i G(\bar{\chi}_h, q_h) = \lambda(\tilde{f}, v_h) \end{aligned}$$

has a unique solution $(\bar{w}_h, \bar{\chi}_h) \in \bar{X}_h$ for any given $\tilde{\bar{w}}_1 = (\bar{w}_1, \lambda\bar{\chi}_1) \in \bar{X}_h \subset \bar{X}$.

In addition, $D_{\bar{u}_h}F_h(\lambda, \tilde{\bar{u}}_h(\lambda))$ is an injective mapping from $\bar{X}_h$ onto Y for each $\lambda \in \Lambda$. Then there exists a constant $\gamma_0 > 0$ such that

$$(21)\qquad ||D_{\bar{u}_h}F(\lambda, \tilde{\bar{u}}_h(\lambda))\bar{w}_h||_Z \geq \gamma_0\|\nabla\bar{w}_h\|_0 \quad \forall \bar{w}_h \in \bar{X}_h,$$

where $||D_{\bar{u}_h}F(\lambda, \tilde{\bar{u}}_h(\lambda))\bar{w}_h||_Z = \sup\limits_{v_h\in X_h} \langle D_{\bar{u}_h}F(\lambda, \tilde{\bar{u}}_h(\lambda))\bar{w}_h, v_h\rangle / \|\nabla v_h\|_0$.

In the following, we mainly determine the parameter $\gamma > 0$. First, taking $v_h = u_h$ in (16) or (17), we obtain

$$\|\nabla\bar{u}_h\|_0^2 \leq \lambda(f, \bar{u}_h) \leq \lambda\|f\|_{-1}\|\nabla\bar{u}_h\|_0,$$

which implies that $\|\nabla \bar{u}_h\|_0 \le \lambda \|f\|_{-1}$. Using the definition of $A_\lambda(\cdot,\cdot,\cdot)$ and (21), we get

$$\begin{aligned} A_\lambda(\bar{u}_h; w_h, w_h) &= a(w_h, w_h) + \lambda b(\bar{u}_h, w_h, w_h) + \lambda b(w_h, \bar{u}_h, w_h) \\ &\ge \gamma_0 \|\nabla w_h\|_0^2. \end{aligned} \tag{22}$$

Obviously, there holds

$$A_\lambda(\bar{u}_h, \bar{w}_h, v_h) \le \gamma_1 \|\nabla \bar{w}_h\|_0 \|\nabla \bar{v}_h\|_0, \tag{23}$$

where $\gamma_1 = (1 + 2C_0\lambda^2\|f\|_{-1})$. Taking $(v_h, q_h) = (\bar{w}_h, \bar{\chi}_h)$ in $B_\lambda((\cdot,\cdot);(\cdot,\cdot))$, we obtain

$$\begin{aligned} B_\lambda((\bar{w}_h, \lambda\bar{\chi}_h); (\bar{w}_h, \lambda\bar{\chi}_h)) &= A_\lambda(\bar{u}_h, \bar{w}_h, \bar{w}_h) + \lambda G(\bar{\chi}_h, \bar{\chi}_h) \\ &\ge \gamma_0 \|\nabla \bar{w}_h\|_0^2 + \lambda \chi_i G(\bar{\chi}_h, \bar{\chi}_h). \end{aligned}$$

Note that for any $\bar{\chi}_h \in M_h \subset M$, there exists $\xi \in X$ and a positive constant C_1 such that

$$d(\xi, \bar{\chi}_h) = \|\bar{\chi}_h\|_0^2, \ \|\nabla \xi_h\|_0 \le C\|\xi\|_1 \le C_1 \|\bar{\chi}_h\|_0, \tag{24}$$

where $\xi_h = I_h \xi$ satisfies

$$d(\xi - \xi_h, q_h) = 0, \tag{25}$$

and

$$\|\xi - \xi_h\|_0 \le Ch\|\xi\|_1. \tag{26}$$

Setting $\Pi_h p dx = \frac{1}{K}\int_K p dx, \ \forall q \in L^2(\Omega)$, using (25)-(26), the inverse inequality, and the Young inequality it follows that

$$\begin{aligned} d(\xi_h, \bar{\chi}_h) &= d(\xi, \bar{\chi}_h) - d(\xi - \xi_h, \bar{\chi}_h) \\ &\ge \|\bar{\chi}_h\|_0^2 - C\chi_i h \|\xi\|_1 \|\nabla(\bar{\chi}_h - \Pi_h \bar{\chi}_h)\|_0 \\ &\ge \|\bar{\chi}_h\|_0^2 - C_1 \chi_i h \|\bar{\chi}_h\|_0 \|\nabla(\bar{\chi}_h - \Pi_h \bar{\chi}_h)\|_0, \\ &\ge \frac{3}{4}\|\bar{\chi}_h\|_0^2 - C_2 \chi_i G(\bar{\chi}_h, \bar{\chi}_h). \end{aligned}$$

Also, using the Young inequality, (23) and (24) yields that

$$\begin{aligned} & B_\lambda((\bar{w}_h, \lambda\bar{\chi}_h); (-\xi_h, 0)) = -A_\lambda(\bar{u}_h, \bar{w}_h, \xi_h) + \lambda d(\xi_h, \bar{\chi}_h) \\ \ge\ & -\gamma_1 \|\nabla \bar{w}_h\|_0 \|\nabla \xi_h\|_0 + \frac{3}{4}\lambda \|\bar{\chi}_h\|_0^2 - C_2 \chi_i \lambda G(\bar{\chi}_h, \bar{\chi}_h) \\ \ge\ & -C_1\gamma_1 \|\nabla \bar{w}_h\|_0 \|\bar{\chi}_h\|_0 + \frac{3}{4}\lambda \|\bar{\chi}_h\|_0^2 - C_2 \chi_i \lambda G(\bar{\chi}_h, \bar{\chi}_h) \\ \ge\ & -\frac{\gamma_0}{4}\|\nabla \bar{w}_h\|_0^2 - \frac{C_1^2\gamma_1^2}{\gamma_0}\|\bar{\chi}_h\|_0^2 + \frac{3}{4}\lambda\|\bar{\chi}_h\|_0^2 - C_2\chi_i\lambda G(\bar{\chi}_h, \bar{\chi}_h). \end{aligned}$$

Therefore, combining these two inequalities and setting the positive constant

$$\sigma = \min\left\{\frac{1}{C_2}, \frac{3\lambda\gamma_0}{4C_1^2\gamma_1^2}\right\},$$

we obtain

$$\begin{aligned} & B_\lambda((\bar{w}_h, \lambda\bar{\chi}_h); (\bar{w}_h - \sigma\xi_h, \lambda\bar{\chi}_h)) \\ \ge\ & \frac{3\gamma_0}{4}\|\nabla \bar{w}_h\|_0^2 + \frac{3}{4}\lambda\sigma\left(1 - \frac{4C_1^2\gamma_1^2\sigma}{3\lambda\gamma_0}\right)\|\chi_h\|_0^2 + \lambda\chi_i(1 - C_2\sigma)G(\bar{\chi}_h, \bar{\chi}_h) \\ \ge\ & C_3\left(\|\nabla \bar{w}_h\|_0^2 + \lambda^2\|\chi_h\|_0^2\right), \end{aligned}$$

where $C_3 = \min\left\{\frac{3\gamma_0}{4}, \frac{3\lambda\sigma}{4}\left(1 - \frac{4C_1^2(1+2C_0\lambda^2\|f\|_{-1})^2\sigma}{3\lambda\gamma_0}\right)\right\} > 0$ is independent of the mesh scale h. Using the triangle inequality, we can easily see that

$$|||(\bar{w}_h - \sigma\xi_h, \bar{\chi}_h)|||_{\bar{X}} \leq C_4(\|\nabla \bar{w}_h\|_0 + \lambda\|\chi_h\|_0),$$

where $C_4 = \max\{1, \frac{C_1\sigma+\lambda}{\lambda}\}$. Finally, setting $\gamma = C_3/C_4$ and using a straightforward computation, we complete the proof of (19). []

4. A branch of nonsingular stabilized finite volume solutions to the stationary Navier-Stokes equations

In the coming purpose, the finite volume methods are developed and presented. Let N_h be the set containing all the interior nodes associated with the triangulation K_h, and N be the total number of the nodes. A dual mesh $\tilde{K_h}$ is introduced based on K_h; the elements in $\tilde{K_h}$ are called control volumes. The dual mesh can be constructed by the following rule: For each element $K \in K_h$ with vertices P_j, $j = 1, 2, \cdots N$, select its barycenter Q_j and the midpoint M_j on each of the edges of K, and construct the control volumes in $\tilde{K_h}$ by connecting Q_j to M_j. Then, the dual finite element space is defined by

$$\tilde{X_h} = \left\{v \in [L^2(\Omega)]^d : v|_{\tilde{K}} \in [P_0(\tilde{K})]^d \quad \forall \tilde{K} \in \tilde{K_h};\ v|_{\partial\Omega} = 0\right\},$$

which has the same dimensions as the corresponding finite element space X_h. Here, we only consider the popular lower order and lower equal-order finite element spaces [20, 21, 22, 30, 32, 34, 35] in order to analyze a branch of nonsingular finite volume solutions to the stationary Navier-Stokes equations.

Now, the (stabilized) finite volume variational formulation for the stationary Navier-Stokes equations (1)-(3) is: Find $\tilde{u}_h(\lambda) = (u_h, \lambda p_h) \in \bar{X}_h \subset \bar{X}$ such that

$$F_h(\lambda, \tilde{u}_h(\lambda)) =: \tilde{u}_h(\lambda) + T_h\mathcal{G}(\lambda, \tilde{u}_h(\lambda)) = 0; \tag{27}$$

i.e.,

$$\mathcal{C}_h((u_h, \lambda p_h), (v_h, \lambda q_h)) + \lambda b(u_h, u_h, \Gamma_h v_h) = \lambda(f, \Gamma_h v_h), \quad \forall (v_h, q_h) \in \bar{X}_h, \tag{28}$$

where the liner mapping $\Gamma_h : X_h \rightarrow \tilde{X}_h$ introduced in [27] satisfies the following properties:

$$\Gamma_h v_h(x) = \sum_{j=1}^{N} v_h(P_j)\chi_j(x), \qquad x \in \Omega,\ v_h \in X_h, \tag{29}$$

and $\chi_i(x)|_{K_j} = \delta_{ij}$ is the characteristic function associated with $x \in \tilde{K}_j \in \tilde{K}_h$. Here, these bilinear terms and the trilinear term are defined as follows:

$$A(u_h, \Gamma_h v_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial\tilde{K}_j} \frac{\partial u_h}{\partial \vec{n}} ds, \quad u_h,\ v_h \in X_h,$$

$$D(\Gamma_h v_h, p_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial\tilde{K}_j} p_h \vec{n}\ ds, \quad p_h \in M_h,$$

$$\mathcal{C}_h((u_h, \lambda p_h), (v_h, \lambda q_h)) = A(u_h, \Gamma_h v_h) + \lambda D(\Gamma_h v_h, p_h) + \lambda d(u_h, q_h),$$

$$(f, \Gamma_h v_h) = \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\tilde{K}_j} f\ dx, \quad v_h \in X_h,$$

where $\vec{n}$ is the unit normal outward to $\partial\tilde{K}_j$.

The same theory is valid for the stabilized finite volume methods approximated by the lower order finite element pairs because of the equivalence [9, 21, 26, 35] between the finite element methods and the finite volume methods for the bilinear terms $A(u_h, \Gamma_h v_h) = a(u_h, v_h)$ and $D(\Gamma_h v_h, q_h) = -d(v_h, q_h)$. Then, we directly verify the corresponding continuity and weak coercivity of the bilinear form $\mathcal{C}_h(\cdot,\cdot)$:

(30)
$$|\mathcal{C}_h((u_h, \lambda p_h), (v_h, \lambda q_h))| \le C |||(u_h, p_h)||| \, |||(v_h, q_h)|||, \qquad \forall (u_h, p_h),\ (v_h, q_h) \in \bar{X}_h,$$

and

$$\sup_{(v_h, q_h) \in \bar{X}_h} \frac{|\mathcal{C}_h((u_h, \lambda p_h), (v_h, \lambda q_h))|}{|||(v_h, q_h)|||} \ge \beta_4 |||(u_h, p_h)|||, \qquad \forall (u_h, p_h) \in \bar{X}_h, \tag{31}$$

where the constant $\beta_4 > 0$ is independent of h.

Furthermore, using the same approach as for Theorem 3.1, we can obtain the following property.

Theorem 4.1. *$\tilde{u}_h(\lambda) \in \bar{X}_h$ is a branch of non-singular solutions to equation (27) if and only if there exists a constant $\gamma^* > 0$ dependent of the data (λ, f, Ω), such that*

$$\sup_{(v_h, q_h) \in \bar{X}_h} \frac{B_\lambda((w_h, \lambda \chi_h); (v_h, \lambda q_h))}{|||(v_h, q_h)|||} \ge \gamma^* |||(w_h, \chi_h)|||, \quad (w_h, \chi_h) \in \bar{X}_h. \tag{32}$$

where

$$\begin{aligned} B_\lambda((w_h, \lambda\chi_h); (v_h, \lambda q_h)) =: A_\lambda(u_h; w_h, \Gamma_h v_h) + \lambda D(\Gamma_h v_h, \chi_h) + \lambda d(w_h, q_h) \\ + \lambda \chi_i G(\chi_h, q_h), \end{aligned}$$

with $A_\lambda(u_h; w_h, \Gamma_h v_h) = A(w_h, \Gamma_h v_h) + \lambda b(u_h, w_h, \Gamma_h v_h) + \lambda b(w_h, u_h, \Gamma_h v_h)$.

Proof. Using the same process as for the corresponding non-singular finite element solutions to the Navier-Stokes equations, and applying the equivalence of these bilinear terms between two different methods, the desired result follows.

Remark. γ^* is different from the previous constant γ because of the definition and boundness of the trilinear term.

5. A branch of nonsingular solutions of finite element methods for the coupled Navier-Stokes/Darcy model

In this section we will introduce the branch of the nonsingular finite element solutions to the stationary Navier-Stokes/Darcy equations and then utilize it to show the well-posedness of the coupled weak formulation.

To specify the problem considered, we try to use the corresponding mathematical symbol of the Navier-Stokes equations defined above. Let the two domains be denoted by Ω_p and Ω_f and lie across an interface Γ from each other. The Darcy problem is stated as follows:

$$-\nabla \cdot (\mathbb{K}(x) \nabla \phi) = f_p, \qquad \text{in } \Omega_p, \tag{33}$$

where (33) implies $u_p = -\mathbb{K}(x)\nabla\phi$; u_p is the fluid velocity discharge rate; f_p is the sink/source term; the hydraulic head $\phi = z + \frac{p_D}{\rho g}$ is determined by the dynamic pressure p_D, the height z, the density ρ, and the gravity constant g. Especially, $\mathbb{K}(x)$ is the hydraulic conductivity tensor and the coefficient has the property $\mathbb{K} \in W^{1,\infty}(D_j)$ with respect to a nonoverlapping partitioning of Ω_f into open, connected Lipschitz polytopes $D := \{D_j : j = 1, \cdots n\}$, that is, $\bar{\Omega}_f = \cup_{j=1}^n \bar{D}_j$. Moreover, we assume that $|\mathbb{K}(x)|_{1,\infty,D_j} \le C$ for $j = 1, \cdots n$ and that $K_0 < |\mathbf{e}|^2 K_{ij}(x) e_i e_j < K_0^*$ for some positive constants K_0 and K_0^* [29].

In the free-flow region Ω_f, the Navier-Stokes equations is modeled by

$$-\nabla\cdot\mathbb{S}(u,p)+(u\cdot\nabla)u=f,\qquad \nabla\cdot u=0,\ \text{in } \Omega_f, \tag{34}$$

where u and p respectively denote the fluid velocity and pressure; $\mathbb{S}(u,p)=2\mu\mathbb{D}(u)-p\mathbb{I}$ denotes the stress tensor, $\mathbb{D}(u)=\frac{1}{2}(\nabla u+\nabla^T u)$ the rate of deformation tensor, $\mathbb{I}$ the identity tensor, f the external body force, and μ the kinematic viscosity of the fluid.

Assume homogeneous Dirichlet boundary conditions for the hydraulic head and fluid velocity on the outer boundaries $\partial\Omega_p$ and $\partial\Omega_f$, respectively, i.e., we have $\phi=0$ on $\partial\Omega_D\backslash\Gamma$, and $u=0$ on $\partial\Omega_S\backslash\Gamma$. Along the interface Γ, we impose the the Beavers-Joseph-Saffmn-Jones (BJSJ) interface conditions [3, 20, 31]

$$u\cdot n_f=u_p\cdot n_f=-\mathbb{K}(x)\nabla\phi\cdot n_f, \tag{35}$$

$$-\tau_j\cdot(\mathbb{S}(u,p)\cdot n_f)=\alpha_{BJSJ}\tau_j\cdot u_f,\qquad j=1,\cdots,d-1, \tag{36}$$

$$-n_f\cdot(\mathbb{S}(u,p)\cdot n_f)=g(\phi-z), \tag{37}$$

where α_{BJSJ} is the BJSJ coefficient; n_f denotes the unit outer normal from the fluid to the porous media regions at the interface Γ; τ_j $(j=1,\ldots,d-1)$ denotes mutually orthogonal unit tangential vectors to the interface Γ. For simplicity, we usually set $z=0$.

For the mathematical setting of problem (33)-(37), the following Hilbert spaces are introduced [1]:

$$X_f=\{v\in[H^1(\Omega_f)]^d:\ v=0 \text{ on } \partial\Omega_f\backslash\Gamma\},\ X_p=\{\psi\in[H^1(\Omega_p)]^d:\ \psi=0 \text{ on } \partial\Omega_f\backslash\Gamma\},$$

$$\bar{Y}=[H^{-1}(\Omega_f)]^d\times H^{-1}(\Omega_p),\ M_f=L^2(\Omega_f),\ \overline{N}=X_f\times M\times X_p.$$

For the domain D, $(\cdot,\cdot)_D$ denotes the L^2 inner product and $<\cdot,\cdot>$ denotes the L^2 inner product on the interface Γ.

Multiplying (33) and (34), by $g\psi\in X_p$, $v\in X_f$ and $q\in M_f$, integrating over the corresponding domains and applying the divergence theorem, is to find $\bar{U}(\lambda)=(u,\lambda p,\lambda\phi)\in\bar{N}$ for given $F=(f,f_p)\in\bar{Y}$ such that

$$\begin{aligned}&D((u,\phi);(v,\psi))-\lambda d(v,p)+\lambda d(u,q)\\ =&\lambda(F,(v,\psi))-\lambda(B(u,u),v),\ \forall\ (v,q,\psi)\in\bar{N},\end{aligned} \tag{38}$$

where

$$\begin{aligned}D((u,\phi);(v,\psi)) &= a(u,v)+a_\Gamma(u,v)+\lambda g a_p(\phi,\psi)+g<\phi,v\cdot n_f>\\ &\quad -\lambda g<u\cdot n_f,\psi>,\\ (F,(v,\psi)) &= (f,v)_{\Omega_f}+g(f_p,\psi)_{\Omega_p}+<gz,v\cdot n_f>.\end{aligned}$$

Here, the same definition of the bilinear terms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ in previous section is used. Moreover, the additional bilinear terms is defined by

$$\begin{aligned}a(u,v)=\mu(\nabla u,\nabla v) &= 2\mu(D(u),D(v)),\\ a_p(\phi,\psi) &= g(\mathbb{K}(x)\nabla\phi,\nabla\psi)_{\Omega_p},\\ a_\Gamma(u,v) &= \alpha_{BJSJ}<u\cdot\tau,v\cdot\tau>.\end{aligned}$$

Taking $(v,\psi)=(u,\phi)$ in the definition of $D((u,\phi);(v,\psi))$, we obtain [13]

$$\begin{aligned}D((u,\phi);(u,\phi)) &= a(u,u)+a_\Gamma(u,u)+\lambda g a_p(\phi,\phi)\\ &\geq \mu\|\nabla u\|_0^2+\lambda gK_0\|\nabla\phi\|_0^2+\alpha_{BSJS}\|u\cdot\tau\|_\Gamma^2\\ &\geq C_a(\|\nabla u\|^2+\|\nabla\phi\|_0),\end{aligned} \tag{39}$$

where $C_a=\min(\mu,\lambda gK_0,\alpha_{BJSJ})>0$. Thus, the system of (38) is well-posed.

Furthermore, from a point of view of nonsingular solutions of the coupled Navier-Stokes/Darcy system, a C^2-mapping $G : R^+ \times \bar{N} \to \bar{Y}$ is introduced as follows:

$$G(\lambda, \bar{U}(\lambda)) = \lambda \left[\begin{pmatrix} B(u,u) \\ 0 \end{pmatrix} - F \right].$$

Then we define a linear operator $\mathbb{T} \in L(\bar{Y}; \bar{N})$ such that the above equation (38) is denoted by

$$\bar{U}(\lambda) = -\mathbb{T}G(\lambda, \bar{U}(\lambda)). \tag{40}$$

Then, the system (38) can be presented as follows:

$$\mathcal{F}(\lambda, \bar{U}(\lambda)) =: \bar{U}(\lambda) + \mathbb{T}G(\lambda, \bar{U}(\lambda)) = 0. \tag{41}$$

That is, $\bar{U}(\lambda) = (u, \lambda p, \lambda\phi)$ is a solution of the problem (41) if and only if it is a solution of (38). Furthermore, $\{\lambda, \bar{U}(\lambda)\}$ is called a branch of nonsingular solutions if $D_{\bar{U}}\mathcal{F}(\lambda, \bar{U}(\lambda))$ is an isomorphism between $\bar{N}$ and $\bar{Y}$ for all $\lambda \in R^+$. Then for the branch of nonsingular solutions $\{\lambda, \bar{U}(\lambda)\}$, given any $\bar{w}_1 = (w_1, r_1, \chi_1) \in \bar{N}$, there exists a unique $\bar{w} = (w, r, \chi) \in \bar{N}$ satisfying

$$D_{\bar{U}}\mathcal{F}(\lambda, \bar{U}(\lambda))\bar{w} = \bar{w}_1. \tag{42}$$

Thanks to (41) and (42), we obtain

$$\bar{w} - \bar{w}_1 = -\mathbb{T}D_{\bar{U}}G(\lambda, \bar{U}(\lambda))\bar{w}. \tag{43}$$

Hence

$$\begin{aligned} & D((w - w_1, \chi - \chi_1); (v, \psi)) - \lambda d(v, r - r_1) + \lambda d(w - w_1, q) \\ = \ & -\lambda b(u, w, v) - \lambda b(w, u, v), \ \forall \ (v, q, \psi) \in \bar{N}. \end{aligned} \tag{44}$$

Define

$$\lambda(\tilde{F}, (v, \psi, q)) = D((w_1, \chi_1); (v, \psi)) + \lambda d(v, r_1) - \lambda d(w_1, q), \tag{45}$$

and

$$\begin{aligned} D_\lambda((w, r, \chi); (v, q, \psi)) = & D((w, \chi); (v, \psi)) + \lambda d(v, r) - \lambda d(w, q) + \lambda b(u, w, v) \\ & + \lambda b(w, u, v). \end{aligned} \tag{46}$$

Then, for any $\tilde{F}$, there exists a unique $\bar{w} = (w, r, \chi) \in \bar{N}$ such that

$$D_\lambda((w, r, \chi); (v, q, \psi)) = \lambda(\tilde{F}, (v, q, \psi)), \ \forall \ (v, q, \psi) \in \bar{N}. \tag{47}$$

In addition, $D_{\bar{U}}\mathcal{F}(\lambda, \bar{U}(\lambda))$ is an injective mapping from $\bar{N}$ onto $\bar{Y}$ for each $\lambda \in R^+$. Then there exists a positive constant $\eta_0 > 0$ depending on Ω such that

$$||D_{\bar{U}}\mathcal{F}(\lambda, \bar{U}(\lambda))\bar{w}||_{\bar{Y}} \geq \eta_0 \|\nabla \bar{w}\|_0, \quad \forall w \in X_f, \tag{48}$$

where $||D_{\bar{U}}\mathcal{F}(\lambda, \bar{U}(\lambda))\bar{w}||_Z = \sup\limits_{v \in X_f} \langle D_{\bar{U}}\mathcal{F}(\lambda, \bar{U}(\lambda))\bar{w}, v\rangle / \|\nabla v\|_0$.

Following [4, 17, 18], we can obtain the following theorem.

Theorem 5.1. $\{\lambda, \bar{U}(\lambda)\}$ is a branch of nonsingular solutions to equation (41) if and only if there exists a constant $\eta > 0$ such that

$$\sup_{(v,q,\psi) \in \bar{N}} \frac{D_\lambda((w, r, \chi); (v, q, \psi))}{\|(v, q, \psi)\|} \geq \eta \|(w, r, \chi)\|, \quad \forall (w, r, \chi) \in \bar{N}, \tag{49}$$

where $\eta > 0$ is dependent of the data $(\lambda, f, f_D, g, \alpha_{BSJS}, \Omega)$.

Then, we can apply the same approach of finite element discretization as above for the coupled Navier-Stokes/Darcy system to derive the following theorem.

Theorem 5.2. $\{\lambda, \bar{U}_h(\lambda)\}$ with $\bar{U}_h(\lambda) = (u_h, \lambda p_h, \lambda\phi_h)$ is a branch of nonsingular solutions to the corresponding finite element approximation of equation (41) if and only if there exists a constant $\eta > 0$ such that

(50)
$$\sup_{(v_h,q_h,\psi_h)\in\bar{N}_h} \frac{D_\lambda((w_h, r_h, \chi_h); (v_h, q_h, \psi_h))}{\|\!|(v_h, q_h, \psi_h)\|\!|} \geq \eta\|\!|(w_h, r_h, \chi_h)\|\!|, \quad \forall (w_h, r_h, \chi_h) \in \bar{N}_h,$$

where $\bar{N}_h = X_f^h \times M^h \times X_p^h$ is the corresponding finite element space of $\bar{N}$.

Remark. In this paper, we only consider the stationary coupled Navier-Stokes /Darcy equations. In fact, the property of a branch of nonsingular finite element solutions can also been extent to the other coupled problems related to the stationary Navier-Stokes equations.

Acknowledgments

This research was supported by NSFs of China (No. 11771259 and 11771348).

References

[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] R. E. Bank and D. J. Rose, Some error estimates for the box method, SIAM J. Numer. Anal. 24 (1987), 777–787.

[3] G. Beavers and D. Joseph, Boundary conditions at a naturally permeable wall, J. Fluid Mech., 30 (1967), 197–207.

[4] F. Brezzi, J. Rappaz and P.-A. Raviart, Finite element approximation of nonlinear problems. Part I: Branch of nonsingular solutions, Numer. Math. 36 (1980), 1–25.

[5] Z. Chen, Finite Element Methods and Their Applications, Spring-Verlag, Heidelberg, 2005.

[6] Z. Chen, R. Li, and A. Zhou, A note on the optimal L^2-estimate of finite volume element method, Adv. Comput. Math. 16 (2002), 291–303.

[7] S. H. Chou and D. Y. Kwak, Analysis and convergence of a MAC scheme for the generalized Stokes problem, Numer. Meth. Partial Diff. Equ. 13 (1997), 147–162.

[8] S. H. Chou and D. Y. Kwak, A covolume method based on rotated bilinears for the generalized Stokes problem, SIAM J. Numer. Anal. 35 (1998), 494–507.

[9] S. H. Chou and Q. Li, Error estimates in L^2, H^1 and L^∞ in co-volume methods for elliptic and parabolic problems: a unified approach, Math Comp. 69 (2000), 103–120.

[10] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[11] E. Burman, Pressure projection Stabilizations for Galerkin approximations of Stokes' and Darcy's problem, Numer. Meth. Part. D. E., 24 (2008), 127-143.

[12] P. Bochev,C. R. Dohrmann, and M. D. Gunzburger, Stabilization of low-order mixed finite elements for the Stokes equations, SIAM J. Numer. Anal., 44 (2006), 82–101.

[13] Y. Cao, M. Gunzburger, F. Hua, and X. Wang, Coupled Stokes-Darcy model with Beavers-Joseph interface boundary condition, Comm. Math. Sci., 8 (2010), 1–25.

[14] R. E. Ewing, T. Lin, and Y. Lin, On the accuracy of the finite volume element method based on piecewise linear polynomials, SIAM J. Numer. Anal. 39 (2002), 1865–1888.

[15] R. Eymard, R. Herbin, and J. C. Latché, Convergence Analysis of a Colocated Finite Volume Scheme for the Incompressible Navier-Stokes Equations on General 2D or 3D Meshes, SIAM J. Numer. Anal. 45 (2007), 1–36.

[16] J. Frehse and R. Rannacher, Finite volume element approximations of nonlocal reactive flows in porous media, Numer. Methods Partial Differential Eq 16 (2000), 285–311.

[17] V. Girault and P. A. Raviart, Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms, Springer-Verlag, Berlin, Heidelberg, 1986.

[18] Y. He and J. Li, A stabilized finite element method based on local polynomial pressure projection for the stationary Navier-Stokes equation, Appl. Numer. Math. 58 (2008), 1503–1514.

[19] J. Huang and S. Xi, On the finite volume element method for general self-adjoint elliptic problems, SIAM J. Numer. Anal. 35 (1998), 1762–1774.

[20] I. Jones, Low Reynolds number flow past a porous spherical shell, Proc. Camb. Phil. Soc., 73 (1973), 231–238.

[21] J. Li and Z. Chen, A New Stabilized Finite Volume Method for the Stationary Stokes Equations, Adv. Comp. Math. 30 (2009), 141–152.
[22] J. Li and Z. Chen, Optimal L^2, H^1 and L^∞ Analysis of Finite Volume Methods for the Stationary Navier-Stokes Equations with Large Data, Numer. Math. 126 (2014), 75-101.
[23] J. Li, Z. Chen and Y. He, A Stabilized Multi-level Method for Non-singular Finite Volume Solutions of the Stationary 3D Navier-Stokes Equations, Numer. Math. 122 (2012), 279–304.
[24] J. Li and Y. He, A new stabilized finite element method based on two local Gauss integration for the Stokes equations, J. Comp. Appl Math., 214 (2008), 58–65.
[25] J. Li, L. Shen and Z. Chen, Convergence and stability of a stabilized finite volume method for the stationary Navier-Stokes equations, BIT Numerical Mathematics, 50 (2010), 823–842.
[26] J. Li, X. Zhao and J. Wu, Study of Stabilization of the Lower Order Finite Volume Methods for the incompressible flows, Acta Mathematica Sinica, 56(2013). (In Chinese)
[27] R. Li and P. Zhu, Generalized difference methods for second order elliptic partial differential equations (I)-triangle grids, Numer. Math. J. Chinese Universities, 2 (1982), 140–152.
[28] R. Li, Generalized difference methods for a nonlinear Dirichlet problem, SIAM J. Numer. Anal. 24 (1987), 77–88.
[29] A. Loneland, L. Marcinkowski and T. Rahman, Additive average Schwarz method for a Crouzeix-Raviart Finite Volume Element Discretization of Elliptic Problems with Heterogeneous Coefficients, submitted.
[30] D. J. Silvester, Stabilised Mixed Finite Element Methods, Numerical Analysis Report No. 262, 1995.
[31] P. Saffman, On the boundary condition at the interface of a porous medium, Stud. Appl. Math., 1 (1971), 77–84.
[32] R. Stenberg, Analysis of Mixed Finite Element Methods for the Stokes Problem: A Unified Approach , Math. Comp. 42 (1984), 9–23.
[33] R. Temam, Navier-Stokes Equations, North-Holland: Amsterdam, New York, 1984.
[34] J. Wang, Y. Wang, and X. Ye, A new finite volume method for the stokes problems, Int. J. Numer. Anal. Mod. 7 (2010), 281–302.
[35] X. Ye, On the relationship between finite volume and finite element methods applied to the Stokes equations, Numer. Methods Partial Diff. Equ. 5 (2001), 440–453.
[36] K. Yosida, Functional Analysis, Springer Verlag, 1994.

Department of Mathematics, School of Arts and Sciences, Shaanxi University of Science and Technology, Xian 710021, P. R. China
E-mail: `jiaaanli@gmail.com`

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China.
E-mail: `heyn@mail.xjtu.edu.cn`