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DISTRIBUTED LAGRANGE MULTIPLIER-FICTITIOUS DOMAIN FINITE ELEMENT METHOD FOR STOKES INTERFACE PROBLEMS

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Abstract. In this paper, the distributed Lagrange multiplier-fictitious domain (DLM/FD) finite element method is studied for a type of steady state Stokes interface problems with jump coefficients, and its well-posedness, stability and optimal convergence properties are analyzed by proving an *inf-sup* condition for a nested saddle-point problem that is induced by both Stokes equations and DLM/FD method in regard to Stokes variables (velocity and pressure) and Lagrange multipliers. Numerical experiments validate the obtained convergence theorem of DLM/FD finite element method for Stokes interface problems with respect to different jump ratios.

Key words. Stokes interface problems, jump coefficients, distributed Lagrange multiplier, fictitious domain method, mixed finite element, well-posedness, error estimates.

1. Introduction

Physicists and engineers use two phase flows to model a wide range of natural phenomena. One such application is a Stokes flow with jump in the viscosity across an interface. This can lead to kinks in the velocity field or discontinuities in pressure at the interface. Using standard finite element methods to capture the true nature of the solution near the interface presents a number of challenges. One method for handling an interface problem is to create a mesh which conforms to the interface [19]. If the interface changes with time, the mesh must be redrawn at each time step to conform with the moving interface. The Arbitrary Lagrangian-Eulerian method [15, 8] is able to adapt the mesh to small movements or changes in the interface, but larger movements or deformations of the interface require that the mesh be redrawn for the whole domain or part of the domain. However, it could be very complicated, time consuming, and less accurate. Furthermore, the transfer of solutions from the degenerated mesh to the new mesh may introduce artificial diffusions, causing loss of accuracy.

Therefore, methods which allow each sub-domain to extend beyond the interface have become increasingly popular. The extended finite element (XFEM) [9, 18] method allows for a mesh which does not conform to the interface. In the XFEM, the interface passes through elements. However, if the ratio of the areas or volumes on either side of the interface becomes too large in any one element, the system can become ill-conditioned and lead to breakdown problems with iterative linear solvers. More recently, the cut finite element method [14, 17] was developed to overcome this problem with the XFEM.

Fictitious domain methods were first developed to handle partial differential equations in a complex geometry [16, 24, 25, 20, 21, 23]. The idea behind the fictitious domain method is to extend the problem from a complicated domain to

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a larger, simpler domain where the problem can be solved more efficiently. When finite element method is used, this allows for a simpler, more regular mesh. In addition, the domain in which both the fluid and fictitious fluid are filled and its mesh are time independent even when the original fluid domain is time dependent.

Lagrange multipliers defined on the actual boundary were later added to implement the genuine boundary conditions [12, 13]. These boundary supported Lagrange multiplier based methods that were first developed for linear elliptic problems were later adapted to non-linear time dependent problems such as the Navier-Stokes equations. The distributed Lagrange multiplier fictitious domain (DLM/FD) method was developed for flows around rigid bodies and the particulate flow problem [10, 11]. In [28] the DLM/FD method is applied to fluid/flexible-body interactions, and a decoupled scheme is developed to solve for the fluid, solid, and Lagrange multiplier terms separately. While some interesting numerical results are provided in those papers, no theoretical analysis is given for DLM/FD finite element method until recently, this method is analyzed for the elliptic interface problem [1, 5] and the parabolic interface problem with a moving interface [27], where, the well-posedness and convergence theorems of the DLM/FD method are proved for those type of interface problems.

Most recently, the DLM/FD method is applied to fluid structure interaction (FSI) problems involving an incompressible viscous-hyperelastic solid [4]. Its stationary case which is defined at each discrete time step is analyzed and an optimal convergence theorem is obtained. In this setting the solid material exhibits both solid and fluid-like properties, with its Cauchy stress tensor given by $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^f + \boldsymbol{\sigma}_s^s$, the sum of a fluid-like part and an elastic part. Thus the influence of fictitious fluid to the structure equation is completely removed from the DLM/FD formulation. Such a specific choice of structure material significantly simplifies the DLM/FD formulation, which is, however, not for a general case of FSI problems.

In this paper, we will take DLM/FD method and apply it to the Stokes interface problem first, where we have a domain Ω which is divided into two sub-domains Ω_1 and Ω_2 with a jump in the viscosity term across the interface Γ. The idea behind the fictitious domain method is to create two non-matching meshes. A background mesh is created over the entire domain Ω. A second mesh is generated in the subdomain Ω_2 on one side of the interface. The mesh for Ω_2 then sits on top of the background mesh. The fluid in Ω_1 is extended into the entire domain Ω and then a distributed Lagrange multiplier (physically a pseudo body force) is used to enforce the fictitious fluid to satisfy the constraint of the structure motion in Ω_2.

Note that the DLM/FD method for the Stokes interface problem results one nested saddle-point problem, where the inner saddle-point problem arises from Stokes equations, and the outer saddle-point problem is induced by the DLM/FD method itself regarding Lagrange multiplier and Stokes variables. It is challenging how to accurately analyze the well-posedness, stability and optimal convergence properties for such a nested saddle-point structure based on the Babuška–Brezzi's theory [2, 6]. In this paper, we will develop a new analysis tool to tackle this problem. In fact, when analyzing the DLM/FD method for the Stokes interface problem, we require that the viscosity of the fluid in Ω_1 be less than the viscosity of the fictitious fluid in Ω_2. We are then able to prove that our DLM/FD formulation is well posed at both the continuous and discrete levels, and derive an optimal error estimate based on the assumed regularity of the true solution. Some numerical experiments reinforce our theoretical result.

We must stress that the key advantage of DLM/FD method is its ability to handle time dependent interface problems where the motion of the interface is not known a priori. The DLM/FD formulation of the steady Stokes interface problem and corresponding analysis presented in this paper will serve as a basis for tackling the unsteady Stokes interface problem which we will carry out in our next paper, and ultimately move towards multi-phase flows and a more generalized FSI model than is presented in [3, 4].

The structure of the paper is as follows. In Section 2 we give a description of the problem. In Section 3 we introduce the distributed Lagrange multiplier and the weak form of the problem. We define the discrete form of the problem by the mixed finite element method in Section 4, and show the present finite element discretization problem is well posed, then derive its error estimates. In Section 5 we present some cases of numerical tests and let the theoretical results be validated.

2. Model description

We define a steady state Stokes interface problem as follows

$$-\nabla\cdot(\beta_1\nabla\boldsymbol{u}_1)+\nabla p_1=\boldsymbol{f}_1,\qquad \text{in } \Omega_1, \tag{1}$$
$$\nabla\cdot\boldsymbol{u}_1=0,\qquad \text{in } \Omega_1, \tag{2}$$
$$-\nabla\cdot(\beta_2\nabla\boldsymbol{u}_2)+\nabla p_2=\boldsymbol{f}_2,\qquad \text{in } \Omega_2, \tag{3}$$
$$\nabla\cdot\boldsymbol{u}_2=0,\qquad \text{in } \Omega_2, \tag{4}$$
$$\boldsymbol{u}_1=\boldsymbol{u}_2,\qquad \text{on } \Gamma, \tag{5}$$
$$(\beta_1\nabla\boldsymbol{u}_1-p_1\boldsymbol{I})\boldsymbol{n}_1+(\beta_2\nabla\boldsymbol{u}_2-p_2\boldsymbol{I})\boldsymbol{n}_2=\boldsymbol{g}_0\qquad \text{on } \Gamma, \tag{6}$$
$$\boldsymbol{u}_1=0,\qquad \text{on } \partial\Omega_1\backslash\Gamma, \tag{7}$$
$$\boldsymbol{u}_2=0,\qquad \text{on } \partial\Omega_2\backslash\Gamma, \tag{8}$$

where, the domain $\Omega=\Omega_1\cup\Omega_2\in R^d$ $(d=2,3)$, the interface $\Gamma=\partial\Omega_1\cap\partial\Omega_2$, as shown in Fig. 1. We can also introduce the entire solution $(\boldsymbol{u},p)$ defined in Ω

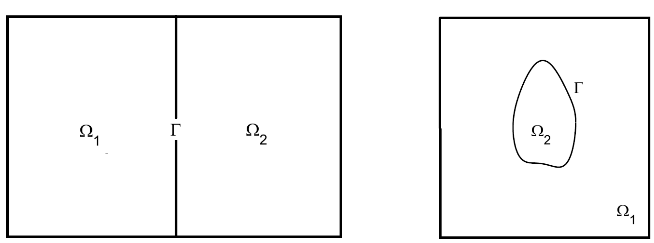


FIGURE 1. A domain decomposition with the interface Γ.

satisfying $\boldsymbol{u}|_{\Omega_1}=\boldsymbol{u}_1,\ \boldsymbol{u}|_{\Omega_2}=\boldsymbol{u}_2,\ p|_{\Omega_1}=p_1,\ p|_{\Omega_2}=p_2$, and is associated with the right hand side $\boldsymbol{f}$ satisfying $\boldsymbol{f}|_{\Omega_1}=\boldsymbol{f}_1,\ \boldsymbol{f}|_{\Omega_2}=\boldsymbol{f}_2$, and the Dirichlet boundary condition $\boldsymbol{u}=0$ on $\partial\Omega$. In order to make the problem more general, we allow for a nonzero flux jump $\boldsymbol{g}_0$ across the interface, which occurs when accounting for the surface tension of the fluids at the interface.

We introduce the equations of the fictitious Stokes fluid in Ω_2 for $(\tilde{\boldsymbol{u}}_2, \tilde{p}_2)$ as follows,

$$-\nabla\cdot(\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2)+\nabla\tilde{p}_2=\tilde{\boldsymbol{f}}_2, \qquad \text{in } \Omega_2, \tag{9}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}_2=0, \qquad \text{in } \Omega_2, \tag{10}$$

$$\tilde{\boldsymbol{u}}_2=\boldsymbol{u}_2, \qquad \text{on } \Gamma, \tag{11}$$

$$\tilde{\boldsymbol{u}}_2=0, \qquad \text{on } \partial\Omega_2\backslash\Gamma, \tag{12}$$

where $\tilde{\beta}_2$ and $\tilde{\boldsymbol{f}}_2$ are any sufficiently smooth extensions of β_1 and $\boldsymbol{f}_2$ into Ω_2, respectively. Note that in general $\tilde{\beta}_2\neq\beta_2$ and $\tilde{\boldsymbol{f}}_2\neq\boldsymbol{f}_2$. For the entire domain Ω we introduce the functions $\tilde{\beta}$ and $\tilde{\boldsymbol{f}}$ defined by $\tilde{\beta}|_{\Omega_1}=\beta_1$, $\tilde{\beta}|_{\Omega_2}=\tilde{\beta}_2$, $\tilde{\boldsymbol{f}}|_{\Omega_1}=\boldsymbol{f}_1$, and $\tilde{\boldsymbol{f}}|_{\Omega_2}=\tilde{\boldsymbol{f}}_2$.

3. Fictitious domain method and the weak formulations

Introduce the Sobolev spaces

$$\boldsymbol{V}=(H_0^1(\Omega))^d, \quad \boldsymbol{V}_2=(H^1(\Omega_2))^d, \quad Q=L_0^2(\Omega). \tag{13}$$

Let $(\cdot,\cdot)_\omega$ stand for L^2- product on a $n-$dimensional $(n=1,2,3)$ domain, ω. Also introduce the space $\boldsymbol{\Lambda}=\boldsymbol{V}_2^*$, the dual space of $\boldsymbol{V}_2$, and let $\langle\cdot,\cdot\rangle_{\Omega_2}$ denote the duality pairing between $\boldsymbol{\Lambda}$ and $\boldsymbol{V}_2$. In $\boldsymbol{\Lambda}$ we have the norm

$$\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}=\sup_{\boldsymbol{v}_2\in\boldsymbol{V}_2}\frac{\langle\boldsymbol{\lambda},\boldsymbol{v}_2\rangle_{\Omega_2}}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}}. \tag{14}$$

Let $\tilde{\boldsymbol{u}}$, $\tilde{p}$ satisfy $\tilde{\boldsymbol{u}}|_{\Omega_1}=\boldsymbol{u}_1$, $\tilde{\boldsymbol{u}}|_{\Omega_2}=\tilde{\boldsymbol{u}}_2$, $\tilde{p}|_{\Omega_1}=p_1$, $\tilde{p}|_{\Omega_2}=\tilde{p}_2$, then $\tilde{\boldsymbol{u}}|_{\partial\Omega}=0$, and $\tilde{\boldsymbol{u}}|_\Gamma=\boldsymbol{u}_1|_\Gamma=\boldsymbol{u}_2|_\Gamma$. If we add the fictitious Stokes equations (9)-(10) to the Stokes equations (1)-(2) defined in Ω_1, and work on their weak formulations, we then have

$$\begin{aligned}
&(\tilde{\beta}\nabla\tilde{\boldsymbol{u}},\nabla\boldsymbol{v})_\Omega-(\tilde{p},\nabla\cdot\boldsymbol{v})_\Omega\\
&=(\beta_1\nabla\boldsymbol{u}_1,\nabla\boldsymbol{v})_{\Omega_1}-(p_1,\nabla\cdot\boldsymbol{v})_{\Omega_1}+(\tilde{\beta}\nabla\tilde{\boldsymbol{u}},\nabla\boldsymbol{v})_{\Omega_2}-(\tilde{p},\nabla\cdot\boldsymbol{v})_{\Omega_2}\\
&=(-\nabla\cdot(\beta_1\nabla\boldsymbol{u}_1)+\nabla p_1,\boldsymbol{v})_{\Omega_1}+\Big(-\nabla\cdot(\tilde{\beta}\nabla\tilde{\boldsymbol{u}})+\nabla\tilde{p},\boldsymbol{v}\Big)_{\Omega_2}\\
&\quad+\Big((\beta_1\nabla\boldsymbol{u}_1-p_1\boldsymbol{I})\boldsymbol{n}_1+(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}-\tilde{p}\boldsymbol{I})\boldsymbol{n}_2,\boldsymbol{v}\Big)_\Gamma\\
&=(\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1}+(\tilde{\boldsymbol{f}},\boldsymbol{v})_{\Omega_2}+\Big((\beta_1\nabla\boldsymbol{u}_1-p_1\boldsymbol{I})\boldsymbol{n}_1+(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}-\tilde{p}\boldsymbol{I})\boldsymbol{n}_2,\boldsymbol{v}\Big)_\Gamma\\
&=(\tilde{\boldsymbol{f}},\boldsymbol{v})_\Omega+\Big((\beta_1\nabla\boldsymbol{u}_1-p_1\boldsymbol{I})\boldsymbol{n}_1+(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}-\tilde{p}\boldsymbol{I})\boldsymbol{n}_2,\boldsymbol{v}\Big)_\Gamma, \quad \forall\boldsymbol{v}\in\boldsymbol{V},
\end{aligned}\tag{15}$$

$$(\nabla\cdot\tilde{\boldsymbol{u}},q)_\Omega=(\nabla\cdot\boldsymbol{u}_1,q)_{\Omega_1}+(\nabla\cdot\tilde{\boldsymbol{u}},q)_{\Omega_2}=0, \quad \forall q\in Q. \tag{16}$$

On the other hand, we subtract the fictitious Stokes equations (9)-(10) from the Stokes equations defined in Ω_2 (3)-(4), respectively, and find their weak formulations defined in Ω_2, as

$$\begin{aligned}
&\Big(\beta_2\nabla\boldsymbol{u}_2-\tilde{\beta}\nabla\tilde{\boldsymbol{u}}|_{\Omega_2},\nabla\boldsymbol{v}\Big)_{\Omega_2}-(p_2-\tilde{p},\nabla\cdot\boldsymbol{v})_{\Omega_2}=(\boldsymbol{f}_2-\tilde{\boldsymbol{f}},\boldsymbol{v})_{\Omega_2}+(\boldsymbol{g}_0,\boldsymbol{v})_\Gamma\\
&\quad-\Big((\beta_1\nabla\boldsymbol{u}_1-p_1\boldsymbol{I})\boldsymbol{n}_1+(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}-\tilde{p}\boldsymbol{I})\boldsymbol{n}_2,\boldsymbol{v}\Big)_\Gamma,
\end{aligned}\tag{17}$$

$$(\nabla\cdot(\boldsymbol{u}_2-\tilde{\boldsymbol{u}}|_{\Omega_2}),q)_{\Omega_2}=0, \quad \forall(\boldsymbol{v},q)\in\boldsymbol{V}_2\times Q_2. \tag{18}$$

If we add (15)-(16) and (17)-(18) together, then the fictitious Stokes terms are all canceled, and the original weak formulations of (1)-(8) can be found back as follows.

Weak Form I. Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ with $\boldsymbol{u}|_{\Omega_1} = \boldsymbol{u}_1,\ \boldsymbol{u}|_{\Omega_2} = \boldsymbol{u}_2,\ p|_{\Omega_1} = p_1,\ p|_{\Omega_2} = p_2$, such that

$$(\beta\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_\Omega - (p, \nabla\cdot\boldsymbol{v})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega + (\boldsymbol{g}_0, \boldsymbol{v})_\Gamma, \tag{19}$$

$$(\nabla\cdot\boldsymbol{u}, q)_\Omega = 0, \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{V} \times Q. \tag{20}$$

Now we use a distributed Lagrange multiplier to weakly impose $\boldsymbol{u}_2 = \tilde{\boldsymbol{u}}|_{\Omega_2}$, i.e.,

$$\langle \boldsymbol{\xi}, \tilde{\boldsymbol{u}}|_{\Omega_2} - \boldsymbol{u}_2\rangle_{\Omega_2} = 0, \quad \forall\boldsymbol{\xi} \in \boldsymbol{\Lambda}. \tag{21}$$

The first term on the left hand side of (17) can be divided into two parts,

$$\left(\beta_2\nabla\boldsymbol{u}_2 - \tilde{\beta}\nabla\tilde{\boldsymbol{u}}|_{\Omega_2}, \nabla\boldsymbol{v}_2\right)_{\Omega_2} = \left((\beta_2 - \tilde{\beta})\nabla\boldsymbol{u}_2, \nabla\boldsymbol{v}_2\right)_{\Omega_2} + \left(\tilde{\beta}(\nabla\boldsymbol{u}_2 - \nabla\tilde{\boldsymbol{u}}|_{\Omega_2}), \nabla\boldsymbol{v}_2\right)_{\Omega_2}. \tag{22}$$

For the second part, we have

$$\begin{aligned}\left|\left(\tilde{\beta}(\nabla\boldsymbol{u}_2 - \nabla\tilde{\boldsymbol{u}}|_{\Omega_2}), \nabla\boldsymbol{v}_2\right)_{\Omega_2}\right| &\le \|\tilde{\beta}\|_\infty \|\nabla\boldsymbol{u}_2 - \nabla\tilde{\boldsymbol{u}}|_{\Omega_2}\|_{(L^2(\Omega_2))^d} \|\nabla\boldsymbol{v}_2\|_{(L^2(\Omega_2))^d} \\ &\le \|\tilde{\beta}\|_\infty \|\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_2}\|_{\boldsymbol{V}_2} \|\nabla\boldsymbol{v}_2\|_{(L^2(\Omega_2))^d}.\end{aligned} \tag{23}$$

Given any $\boldsymbol{w} \in \boldsymbol{V}_2$ there is a $\boldsymbol{\xi}_{\boldsymbol{w}} \in \boldsymbol{\Lambda}$ such that for all $\boldsymbol{v} \in \boldsymbol{V}_2$,

$$\langle\boldsymbol{\xi}_{\boldsymbol{w}}, \boldsymbol{v}\rangle_{\Omega_2} = (\boldsymbol{w}, \boldsymbol{v})_{\Omega_2} + (\nabla\boldsymbol{w}, \nabla\boldsymbol{v})_{\Omega_2}, \tag{24}$$

$$\|\boldsymbol{\xi}_{\boldsymbol{w}}\|_{\boldsymbol{\Lambda}} = \|\boldsymbol{w}\|_{\boldsymbol{V}_2}. \tag{25}$$

From (21) we attain $\|\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_2}\|_{\boldsymbol{V}_2} = 0$, leading to

$$\left(\tilde{\beta}(\nabla\boldsymbol{u}_2 - \nabla\tilde{\boldsymbol{u}}|_{\Omega_2}), \nabla\boldsymbol{v}_2\right)_{\Omega_2} = 0, \quad \forall\boldsymbol{v}_2 \in \boldsymbol{V}_2. \tag{26}$$

Thus,

$$\left(\beta_2\nabla\boldsymbol{u}_2 - \tilde{\beta}\nabla\tilde{\boldsymbol{u}}|_{\Omega_2}, \nabla\boldsymbol{v}_2\right)_{\Omega_2} = \left((\beta_2 - \tilde{\beta})\nabla\boldsymbol{u}_2, \nabla\boldsymbol{v}_2\right)_{\Omega_2}, \quad \forall\boldsymbol{v}_2 \in \boldsymbol{V}_2. \tag{27}$$

In addition, if we replace the flux jump term, $\Big((\beta_1\nabla\boldsymbol{u}_1 - p_1\boldsymbol{I})\boldsymbol{n}_1 + (\tilde{\beta}\nabla\tilde{\boldsymbol{u}} - \tilde{p}\boldsymbol{I})\boldsymbol{n}_2$ $, \boldsymbol{v})_\Gamma$ that arises in both (15) and (17), by the distributed Lagrange multiplier term defined in Ω_2, then we can define the weak formulation for the distributed Lagrange multiplier-fictitious domain (DLM/FD) method for (1)-(8) as follows.

Weak Form II (DLM/FD Form). Find $(\tilde{\boldsymbol{u}},\ \boldsymbol{u}_2,\ \tilde{p},\ \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{V}_2 \times Q \times \boldsymbol{\Lambda}$ such that

$$(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{v})_\Omega - (\tilde{p}, \nabla\cdot\boldsymbol{v})_\Omega + \langle\boldsymbol{\lambda}, \boldsymbol{v}|_{\Omega_2}\rangle_{\Omega_2} = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_\Omega, \quad \forall\boldsymbol{v} \in \boldsymbol{V} \tag{28}$$

$$(\nabla\cdot\tilde{\boldsymbol{u}}, q)_\Omega = 0, \quad \forall q \in Q \tag{29}$$

$$\left((\beta_2 - \tilde{\beta})\nabla\boldsymbol{u}_2, \nabla\boldsymbol{v}_2\right)_{\Omega_2} - \langle\boldsymbol{\lambda}, \boldsymbol{v}_2\rangle_{\Omega_2} = \left(\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}|_{\Omega_2}, \boldsymbol{v}_2\right)_{\Omega_2} + (\boldsymbol{w}, \boldsymbol{v}_2)_\Gamma, \forall\boldsymbol{v}_2 \in \boldsymbol{V}_2 \tag{30}$$

$$\langle\boldsymbol{\xi}, \tilde{\boldsymbol{u}}|_{\Omega_2} - \boldsymbol{u}_2\rangle_{\Omega_2} = 0, \quad \forall\boldsymbol{\xi} \in \boldsymbol{\Lambda}. \tag{31}$$

Theorem 3.1. *Given* $\boldsymbol{f} \in (L^2(\Omega))^d$ *with* $\boldsymbol{f}|_{\Omega_1} = \boldsymbol{f}_1$, $\boldsymbol{f}|_{\Omega_2} = \boldsymbol{f}_2$ *, and* $\beta \in L^\infty(\Omega)$ *with* $\beta|_{\Omega_1} = \beta_1$, $\beta|_{\Omega_2} = \beta_2$, *let* $\tilde{\boldsymbol{f}} \in (L^2(\Omega))^d$ *be any function that satisfies* $\tilde{\boldsymbol{f}}|_{\Omega_1} = \boldsymbol{f}_1$, *and let* $\tilde{\beta} \in L^\infty(\Omega)$ *be any function that satisfies* $\tilde{\beta}|_{\Omega_1} = \beta_1$.

(i). Suppose $(\tilde{\boldsymbol{u}},\ \boldsymbol{u}_2,\ \tilde{p},\ \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{V}_2 \times Q \times \boldsymbol{\Lambda}$, *is a solution of Weak Form II (28)-(31). Then* $(\tilde{\boldsymbol{u}}, \tilde{p}) \in \boldsymbol{V} \times Q$, *is a solution of Weak Form I (19)-(20).*

(ii). Conversely, let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ *be a solution of Weak Form I (19)-(20), and let* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ *satisfy*

$$\begin{aligned}\langle \boldsymbol{\lambda}, \boldsymbol{v}_2 \rangle_{\Omega_2} = &\Big((\beta_2 - \tilde{\beta}|_{\Omega_2}) \nabla \boldsymbol{u}_2, \nabla \boldsymbol{v}_2 \Big)_{\Omega_2} \\ &- \Big(\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}|_{\Omega_2}, \boldsymbol{v}_2 \Big)_{\Omega_2} - (\boldsymbol{w}, \boldsymbol{v}_2)_\Gamma, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_2, \end{aligned} \tag{32}$$

where $\boldsymbol{u}_2 := \boldsymbol{u}|_{\Omega_2}$. *Then,* $(\tilde{\boldsymbol{u}} := \boldsymbol{u},\ \boldsymbol{u}_2 := \boldsymbol{u}|_{\Omega_2},\ \tilde{p} := p, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{V}_2 \times Q \times \boldsymbol{\Lambda}$ *is a solution of Weak Form II (28)-(31).*

Proof. (i). (19) can be proved easily by using (27), taking $\boldsymbol{v} \in \boldsymbol{V}$ in (28) with $\boldsymbol{v}|_{\Omega_2} = \boldsymbol{v}_2$, and simply adding (28) and (30) together, where all the Lagrange multiplier terms are canceled. (20) is obvious.

(ii). We only need to prove (28) and (30) hold, all the others are trivial. By the definition of the duality pairing $\langle \cdot, \cdot \rangle_{\Omega_2}$ in Ω_2, we know there exists a unique $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ satisfying (32), which yields (30), and

$$\Big((\beta_2 - \tilde{\beta}) \nabla \boldsymbol{u}\big|_{\Omega_2}, \nabla \boldsymbol{v}_2 \Big)_{\Omega_2} - \langle \boldsymbol{\lambda}, \boldsymbol{v}_2 \rangle_{\Omega_2} = \Big(\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}|_{\Omega_2}, \boldsymbol{v}_2 \Big)_{\Omega_2} + (\boldsymbol{w}, \boldsymbol{v}_2)_\Gamma, \ \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{33}$$

Subtract (33) from (19), (28) is then obtained. □

4. DLM/FD finite element method for Stokes interface problems

4.1. DLM/FD finite element discretization. Let $T_h(\Omega)$ be a partition of Ω, independent of the location of the interface Γ, and $T_H(\Omega_2)$ be a partition of Ω_2. Based on these meshes, define the conforming finite element spaces $\boldsymbol{V}_h \subset \boldsymbol{V}$, $\boldsymbol{V}_{2,H} \subset \boldsymbol{V}_2$, $Q_h \subset Q$, $\boldsymbol{\Lambda}_H \subset \boldsymbol{\Lambda}$ where $\boldsymbol{\Lambda}_H = \{\boldsymbol{\lambda} \in \boldsymbol{\Lambda} : \exists \boldsymbol{u}_{2,H} \in \boldsymbol{V}_{2,H}, \langle \boldsymbol{\lambda}, \boldsymbol{v}_2 \rangle_{\Omega_2} = (\boldsymbol{u}_{2,H}, \boldsymbol{v}_2)_{\Omega_2}\ \forall \boldsymbol{v}_2 \in \boldsymbol{V}_2\}$. Note that the duality pairing between $\boldsymbol{\Lambda}_H$ and $\boldsymbol{V}_2$ is the L^2 inner product of elements in $\boldsymbol{V}_{2,H}$ and $\boldsymbol{V}_2$. The DLM/FD finite element method can be defined as follows: find $(\boldsymbol{u}_h,\ \boldsymbol{u}_{2,H},\ p_h,\ \boldsymbol{\lambda}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H$ such that

$$(\tilde{\beta} \nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_\Omega - (p_h, \nabla \cdot \boldsymbol{v}_h)_\Omega + \langle \boldsymbol{\lambda}_H, \boldsymbol{v}_h \rangle_{\Omega_2} = (\tilde{\boldsymbol{f}}, \boldsymbol{v}_h)_\Omega, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \tag{34}$$

$$(\nabla \cdot \boldsymbol{u}_h, q_h)_\Omega = 0, \quad \forall q_h \in Q_h \tag{35}$$

$$\begin{aligned}\Big((\beta_2 - \tilde{\beta}) \nabla \boldsymbol{u}_{2,H}, \nabla \boldsymbol{v}_{2,H} \Big)_{\Omega_2} - \langle \boldsymbol{\lambda}_H, \boldsymbol{v}_{2,H} \rangle_{\Omega_2} = &(\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}|_{\Omega_2}, \boldsymbol{v}_{2,H})_{\Omega_2} \\ &+ (\boldsymbol{w}, \boldsymbol{v}_{2,H})_\Gamma, \quad \forall \boldsymbol{v}_{2,H} \in \boldsymbol{V}_{2,H} \end{aligned} \tag{36}$$

$$\langle \boldsymbol{\xi}_H, \boldsymbol{u}_h - \boldsymbol{u}_{2,H} \rangle_{\Omega_2} = 0, \quad \forall \boldsymbol{\xi}_H \in \boldsymbol{\Lambda}_H. \tag{37}$$

4.2. Well-posedness, stability and convergence. In this section, we study properties of well-posedness, stability and convergence of the proposed DLM/FD weak form II (28)-(31) and of its finite element discretization (34)-(37) under the following assumptions: there exist constants $\underline{\beta}$, $\bar{\beta}$ such that

$$+\infty > \bar{\beta} \geq \beta_2 > \tilde{\beta} \geq \underline{\beta} > 0, \quad \beta_2 - \tilde{\beta} \geq \underline{\beta} > 0. \tag{38}$$

If we introduce the following bilinear and linear forms

$$a(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2; \boldsymbol{v}, \boldsymbol{v}_2) = (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{v})_\Omega + \Big((\beta_2 - \tilde{\beta})\nabla\boldsymbol{u}_2, \nabla\boldsymbol{v}_2\Big)_{\Omega_2}, \tag{39}$$

$$b(\boldsymbol{v}, \boldsymbol{v}_2; q, \boldsymbol{\xi}) = -(q, \nabla\cdot\boldsymbol{v})_\Omega + \langle\boldsymbol{\xi}, \boldsymbol{v}|_{\Omega_2} - \boldsymbol{v}_2\rangle_{\Omega_2}, \tag{40}$$

$$F(\boldsymbol{v}, \boldsymbol{v}_2) = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_\Omega + \Big(\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}|_{\Omega_2}, \boldsymbol{v}_2\Big)_{\Omega_2} + (\boldsymbol{w}, \boldsymbol{v}_2)_\Gamma, \tag{41}$$

then Weak Form II (28)-(31) can be rewritten as: find $(\tilde{\boldsymbol{u}},\ \boldsymbol{u}_2,\ \tilde{p},\ \boldsymbol{\lambda}) \in \boldsymbol{V}\times\boldsymbol{V}_2\times Q\times\boldsymbol{\Lambda}$ such that

$$L(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2, \tilde{p}, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}) = F(\boldsymbol{v}, \boldsymbol{v}_2), \qquad \forall(\boldsymbol{v},\ \boldsymbol{v}_2,\ q,\ \boldsymbol{\xi}) \in \boldsymbol{V}\times\boldsymbol{V}_2\times Q\times\boldsymbol{\Lambda}, \tag{42}$$

where

$$L(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2, \tilde{p}, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}) = a(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2; \boldsymbol{v}, \boldsymbol{v}_2) + b(\boldsymbol{v}, \boldsymbol{v}_2; \tilde{p}, \boldsymbol{\lambda}) - b(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2; q, \boldsymbol{\xi}).$$

Correspondingly, the DLM/FD finite element method (34)-(37) can also be rewritten as: find $(\boldsymbol{u}_h,\ \boldsymbol{u}_{2,H},\ p_h,\ \boldsymbol{\lambda}_H) \in \boldsymbol{V}_h\times\boldsymbol{V}_{2,H}\times Q_h\times\boldsymbol{\Lambda}_H$ such that

$$\begin{aligned} &L(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H) = F(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}),\\ &\qquad\qquad\qquad \forall(\boldsymbol{v}_h,\ \boldsymbol{v}_{2,H},\ q_h,\ \boldsymbol{\xi}_H) \in \boldsymbol{V}_h\times\boldsymbol{V}_{2,H}\times Q_h\times\boldsymbol{\Lambda}_H. \end{aligned} \tag{43}$$

We also introduce the norms

$$\|\boldsymbol{v}, \boldsymbol{v}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2} = \left(\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}^2\right)^{\frac{1}{2}}, \tag{44}$$

$$\|q, \boldsymbol{\xi}\|_{Q\times\boldsymbol{\Lambda}} = \left(\|q\|_Q^2 + \|\boldsymbol{\xi}\|_{\boldsymbol{\Lambda}}^2\right)^{\frac{1}{2}}, \tag{45}$$

$$|||\boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}||| = \left(\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}^2 + \|q\|_Q^2 + \|\boldsymbol{\xi}\|_{\boldsymbol{\Lambda}}^2\right)^{\frac{1}{2}}. \tag{46}$$

It is known from [1, 6, 7, 22] that

$$\sup_{(\boldsymbol{v},\boldsymbol{v}_2)\in\boldsymbol{V}\times\boldsymbol{V}_2} \frac{\langle\boldsymbol{\xi}, \boldsymbol{v}|_{\Omega_2} - \boldsymbol{v}_2\rangle_{\Omega_2}}{\|\boldsymbol{v}, \boldsymbol{v}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2}} \geq c_1\|\boldsymbol{\xi}\|_{\boldsymbol{\Lambda}}, \quad \forall\boldsymbol{\xi}\in\boldsymbol{\Lambda}, \tag{47}$$

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{-(q, \nabla\cdot\boldsymbol{v})_\Omega}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \geq c_2\|q\|_Q, \quad \forall q\in Q. \tag{48}$$

Here and thereafter, c or C with (or without) subscripts denotes a generic positive constant whose value may be different at different occurrences and is independent of mesh sizes h and H.

Lemma 4.1. *There exists a constant c_3 such that for any $(q, \boldsymbol{\xi}) \in Q\times\boldsymbol{\Lambda}$,*

$$\sup_{(\boldsymbol{v},\boldsymbol{v}_2)\in\boldsymbol{V}\times\boldsymbol{V}_2} \frac{b(\boldsymbol{v}, \boldsymbol{v}_2; q, \boldsymbol{\xi})}{\|\boldsymbol{v}, \boldsymbol{v}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2}} \geq c_3\|q, \boldsymbol{\xi}\|_{Q\times\boldsymbol{\Lambda}}. \tag{49}$$

Proof. Let $(\boldsymbol{w}, \boldsymbol{w}_2) \in \boldsymbol{V}\times\boldsymbol{V}_2$ satisfy

$$\|\boldsymbol{w}, \boldsymbol{w}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2} = \|\boldsymbol{\xi}\|_{\boldsymbol{\Lambda}} \text{ with } \langle\boldsymbol{\xi}, \boldsymbol{w}|_{\Omega_2} - \boldsymbol{w}_2\rangle \geq \alpha_1\|\boldsymbol{\xi}\|_{\boldsymbol{\Lambda}}^2,$$

and let $\boldsymbol{u}\in\boldsymbol{V}$ satisfy

$$\|\boldsymbol{u}\|_{\boldsymbol{V}} = \|q\|_Q \text{ with } -(q, \nabla\cdot\boldsymbol{u})_\Omega \geq \alpha_2\|q\|_Q^2.$$

Then using $\|\nabla\cdot\boldsymbol{v}\| \le \sqrt{d}\|\boldsymbol{v}\|_{\boldsymbol{V}} \le \sqrt{d}\|\boldsymbol{v},\boldsymbol{v}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2}$, we have

$$\begin{aligned} b(\boldsymbol{w},\boldsymbol{w}_2;q,\boldsymbol{\xi}) &= -(q,\nabla\cdot\boldsymbol{w})_\Omega + \langle\boldsymbol{\xi},\boldsymbol{w}|_{\Omega_2}-\boldsymbol{w}_2\rangle \\ &\ge \alpha_1\|\boldsymbol{\xi}\|^2_{\boldsymbol{\Lambda}} - \|q\|_Q\|\nabla\cdot\boldsymbol{w}\|_{0,\Omega} \\ &\ge \alpha_1\|\boldsymbol{\xi}\|^2_{\boldsymbol{\Lambda}} - \sqrt{d}\|q\|_Q\|\boldsymbol{w},\boldsymbol{w}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2} \\ &\ge \alpha_1\|\boldsymbol{\xi}\|^2_{\boldsymbol{\Lambda}} - \frac{d}{2\alpha_1}\|q\|^2_Q - \frac{\alpha_1}{2}\|\boldsymbol{w},\boldsymbol{w}_2\|^2_{\boldsymbol{V}\times\boldsymbol{V}_2} \\ &= \frac{\alpha_1}{2}\|\boldsymbol{\xi}\|^2_{\boldsymbol{\Lambda}} - \frac{d}{2\alpha_1}\|q\|^2_Q, \quad \forall\ (q,\boldsymbol{\xi})\in Q\times\boldsymbol{\Lambda}. \end{aligned} \tag{50}$$

Setting $\boldsymbol{u}_2 = \boldsymbol{u}|_{\Omega_2}$ gives

$$b(\boldsymbol{u},\boldsymbol{u}_2;q,\boldsymbol{\xi}) = -(q,\nabla\cdot\boldsymbol{u})_\Omega \ge \alpha_2\|q\|^2_Q, \quad \forall\ (q,\boldsymbol{\xi})\in Q\times\boldsymbol{\Lambda}. \tag{51}$$

Choosing $\boldsymbol{v} = \boldsymbol{w}+\eta\boldsymbol{u}$, $\boldsymbol{v}_2 = \boldsymbol{w}_2+\eta\boldsymbol{u}_2$, where $\eta = \frac{d}{\alpha_1\alpha_2}$, we have

$$b(\boldsymbol{v},\boldsymbol{v}_2;q,\boldsymbol{\xi}) \ge \frac{\alpha_1}{2}\|\boldsymbol{\xi}\|^2_{\boldsymbol{\Lambda}} + \frac{d}{2\alpha_1}\|q\|^2_Q \ge c\|q,\boldsymbol{\xi}\|^2_{Q\times\boldsymbol{\Lambda}}, \quad \forall\ (q,\boldsymbol{\xi})\in Q\times\boldsymbol{\Lambda}. \tag{52}$$

From the choice of $\boldsymbol{v}$ and $\boldsymbol{v}_2$, and using $\|\boldsymbol{u}_2\|_{\boldsymbol{V}_2} \le \|\boldsymbol{u}\|_{\boldsymbol{V}}$,

$$\begin{aligned} \|\boldsymbol{v},\boldsymbol{v}_2\|^2_{\boldsymbol{V}\times\boldsymbol{V}_2} \le& \|\boldsymbol{w}\|^2_{\boldsymbol{V}} + 2\eta\|\boldsymbol{w}\|_{\boldsymbol{V}}\|\boldsymbol{u}\|_{\boldsymbol{V}} + \eta^2\|\boldsymbol{u}\|^2_{\boldsymbol{V}} + \|\boldsymbol{w}_2\|^2_{\boldsymbol{V}_2} \\ &+ 2\eta\|\boldsymbol{w}_2\|_{\boldsymbol{V}_2}\|\boldsymbol{u}_2\|_{\boldsymbol{V}_2} + \eta^2\|\boldsymbol{u}_2\|^2_{\boldsymbol{V}_2} \\ \le& 2\|\boldsymbol{w}\|^2_{\boldsymbol{V}} + 2\|\boldsymbol{w}_2\|^2_{\boldsymbol{V}_2} + 2\eta^2\|\boldsymbol{u}\|^2_{\boldsymbol{V}} + 2\eta^2\|\boldsymbol{u}_2\|^2_{\boldsymbol{V}_2} \\ \le& 2\|\boldsymbol{w}\|^2_{\boldsymbol{V}} + 2\|\boldsymbol{w}_2\|^2_{\boldsymbol{V}_2} + 4\eta^2\|\boldsymbol{u}\|^2_{\boldsymbol{V}} \\ =& 2\|\boldsymbol{\xi}\|^2_{\boldsymbol{\Lambda}} + 4\eta^2\|q\|^2_Q \\ \le& c\|q,\boldsymbol{\xi}\|^2_{Q\times\boldsymbol{\Lambda}}, \quad \forall\ (q,\boldsymbol{\xi})\in Q\times\boldsymbol{\Lambda}. \end{aligned} \tag{53}$$

□

Lemma 4.2. *Assume that (38) holds. Then there exists a constant c_4 such that for any $(\boldsymbol{u},\boldsymbol{u}_2,p,\boldsymbol{\lambda})\in\boldsymbol{V}\times\boldsymbol{V}_2\times Q\times\boldsymbol{\Lambda}$,*

$$\sup_{(\boldsymbol{v},\boldsymbol{v}_2,q,\boldsymbol{\xi})\in\boldsymbol{V}\times\boldsymbol{V}_2\times Q\times\boldsymbol{\Lambda}} \frac{L(\boldsymbol{u},\boldsymbol{u}_2,p,\boldsymbol{\lambda};\boldsymbol{v},\boldsymbol{v}_2,q,\boldsymbol{\xi})}{|||\boldsymbol{v},\boldsymbol{v}_2,q,\boldsymbol{\xi}|||} \ge c_4|||\boldsymbol{u},\boldsymbol{u}_2,p,\boldsymbol{\lambda}|||. \tag{54}$$

Proof. It follows from (38) that

$$\begin{aligned} &L(\boldsymbol{u},\boldsymbol{u}_2,p,\boldsymbol{\lambda};\boldsymbol{u},\boldsymbol{u}_2,p,\boldsymbol{\lambda}) \\ =&(\tilde\beta\nabla\boldsymbol{u},\nabla\boldsymbol{u})_\Omega + ((\beta_2-\tilde\beta)\nabla\boldsymbol{u}_2,\nabla\boldsymbol{u}_2)_{\Omega_2} \\ \ge&\underline{\beta}(\|\nabla\boldsymbol{u}\|^2_{0,\Omega} + \|\nabla\boldsymbol{u}_2\|^2_{0,\Omega_2}) \\ \ge&C(\|\boldsymbol{u}\|^2_{\boldsymbol{V}} + \|\nabla\boldsymbol{u}_2\|^2_{0,\Omega_2}), \quad \forall\ (\boldsymbol{u},\boldsymbol{u}_2,p,\boldsymbol{\lambda})\in\boldsymbol{V}\times\boldsymbol{V}_2\times Q\times\boldsymbol{\Lambda}, \end{aligned} \tag{55}$$

where we have applied the Poincaré inequality, and the constant C depends only on the domain Ω and value of $\underline{\beta}$.

For $\boldsymbol{u}_2\in\boldsymbol{V}_2$, let $\boldsymbol{\xi}_{\boldsymbol{u}_2}\in\boldsymbol{\Lambda}$ satisfy

$$\langle\boldsymbol{\xi}_{\boldsymbol{u}_2},\boldsymbol{v}_2\rangle_{\Omega_2} = (\boldsymbol{u}_2,\boldsymbol{v}_2)_{\Omega_2} \text{ for all } \boldsymbol{v}_2\in\boldsymbol{V}_2,$$

then

$$
\begin{aligned}
L(\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}; 0,0,0, \boldsymbol{\xi}_{\boldsymbol{u}_2}) &= -\left\langle \boldsymbol{\xi}_{\boldsymbol{u}_2}, \boldsymbol{u}|_{\Omega_2} - \boldsymbol{u}_2 \right\rangle_{\Omega_2} \\
&= -(\boldsymbol{u}_2, \boldsymbol{u}|_{\Omega_2} - \boldsymbol{u}_2)_{\Omega_2} \\
&\geq \|\boldsymbol{u}_2\|_{0,\Omega_2}^2 - \|\boldsymbol{u}_2\|_{0,\Omega_2} \|\boldsymbol{u}|_{\Omega_2}\|_{\boldsymbol{V}_2} \\
&\geq \|\boldsymbol{u}_2\|_{0,\Omega_2}^2 - \frac{1}{2}\|\boldsymbol{u}_2\|_{0,\Omega_2}^2 - \frac{1}{2}\|\boldsymbol{u}|_{\Omega_2}\|_{\boldsymbol{V}_2}^2 \\
&\geq \frac{1}{2}\|\boldsymbol{u}_2\|_{0,\Omega_2}^2 - \frac{1}{2}\|\boldsymbol{u}\|_{\boldsymbol{V}}^2. \qquad (56)
\end{aligned}
$$

Let $(\boldsymbol{w}, \boldsymbol{w}_2) \in \boldsymbol{V} \times \boldsymbol{V}_2$ satisfy $\|\boldsymbol{w}, \boldsymbol{w}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2} = \|p, \boldsymbol{\lambda}\|_{Q\times\boldsymbol{\Lambda}}$ and $b(\boldsymbol{w}, \boldsymbol{w}_2; p, \boldsymbol{\lambda}) \geq c_3\|p, \boldsymbol{\lambda}\|_{Q\times\boldsymbol{\Lambda}}^2$. Then, we have

$$
\begin{aligned}
&L(\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}; \boldsymbol{w}, \boldsymbol{w}_2, 0, 0) \\
=&(\tilde{\beta}\nabla\boldsymbol{u}, \nabla\boldsymbol{w})_\Omega + ((\beta_2 - \tilde{\beta})\nabla\boldsymbol{u}_2, \nabla\boldsymbol{w}_2)_{\Omega_2} + b(\boldsymbol{w}, \boldsymbol{w}_2; p, \boldsymbol{\lambda}) \\
\geq& c_3\|p, \boldsymbol{\lambda}\|_{Q\times\boldsymbol{\Lambda}}^2 - \bar{\beta}(\|\boldsymbol{u}\|_{\boldsymbol{V}}\|\boldsymbol{w}\|_{\boldsymbol{V}} + \|\boldsymbol{u}_2\|_{\boldsymbol{V}_2}\|\boldsymbol{w}_2\|_{\boldsymbol{V}_2}) \\
\geq& c_3\|p, \boldsymbol{\lambda}\|_{Q\times\boldsymbol{\Lambda}}^2 - \bar{\beta}\left(\frac{\bar{\beta}}{2c_3}\left(\|\boldsymbol{u}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{u}_2\|_{\boldsymbol{V}_2}^2\right) + \frac{c_3}{2\bar{\beta}}\left(\|\boldsymbol{w}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{w}_2\|_{\boldsymbol{V}_2}^2\right)\right) \\
=&\frac{c_3}{2}\|p, \boldsymbol{\lambda}\|_{Q\times\boldsymbol{\Lambda}}^2 - \frac{\bar{\beta}^2}{2c_3}\|\boldsymbol{u}, \boldsymbol{u}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2}^2. \qquad (57)
\end{aligned}
$$

Let $\boldsymbol{v} = 2C^{-1}\boldsymbol{u} + c_3\bar{\beta}^{-2}\boldsymbol{w}, \boldsymbol{v}_2 = 2C^{-1}\boldsymbol{u}_2 + c_3\bar{\beta}^{-2}\boldsymbol{w}_2, q = 2C^{-1}p, \boldsymbol{\xi} = 2C^{-1}\boldsymbol{\lambda} + 2\boldsymbol{\xi}_{\boldsymbol{u}_2}$. Then we have

(58)

$$
L(\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}) \geq \frac{1}{2}\|\boldsymbol{u}, \boldsymbol{u}_2\|_{\boldsymbol{V}\times\boldsymbol{V}_2}^2 + \frac{c_3^2}{2\bar{\beta}^2}\|p, \boldsymbol{\lambda}\|_{Q\times\boldsymbol{\Lambda}}^2 \geq c^*|||\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}|||^2
$$

where the constant $c^* = \min\{\frac{1}{2}, \frac{c_3^2}{2\bar{\beta}^2}\}$, and

$$
\begin{aligned}
|||\boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}|||^2 &\leq c(\|\boldsymbol{u}\|_{\boldsymbol{V}}^2 + 2\|\boldsymbol{u}\|_{\boldsymbol{V}}\|\boldsymbol{w}\|_{\boldsymbol{V}} + \|\boldsymbol{w}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{u}_2\|_{\boldsymbol{V}_2}^2 \\
&\quad + 2\|\boldsymbol{u}_2\|_{\boldsymbol{V}_2}\|\boldsymbol{w}_2\|_{\boldsymbol{V}_2} + \|\boldsymbol{w}_2\|_{\boldsymbol{V}_2}^2 \\
&\quad + \|p\|_Q^2 + \|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}^2 + 2\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}\|\boldsymbol{u}_2\|_{\boldsymbol{\Lambda}} + \|\boldsymbol{\xi}_{\boldsymbol{u}_2}\|_{\boldsymbol{\Lambda}}^2) \\
&\leq c(2\|\boldsymbol{u}\|_{\boldsymbol{V}}^2 + 2\|\boldsymbol{w}\|_{\boldsymbol{V}}^2 + 2\|\boldsymbol{u}_2\|_{\boldsymbol{V}_2}^2 + 2\|\boldsymbol{w}_2\|_{\boldsymbol{V}_2}^2 \\
&\quad + \|p\|_Q^2 + 2\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}^2 + 2\|\boldsymbol{\xi}_{\boldsymbol{u}_2}\|_{\boldsymbol{\Lambda}}^2) \\
&\leq c|||\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}|||^2. \qquad (59)
\end{aligned}
$$

Then, the aimed result (54) is direct consequence of (58) and (59). □

Lemma 4.3. *Assume that (38) holds. Then there exists a constant c_5 such that for any $(\boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}) \in \boldsymbol{V} \times \boldsymbol{V}_2 \times Q \times \boldsymbol{\Lambda}$,*

$$
\sup_{(\boldsymbol{u},\boldsymbol{u}_2,p,\boldsymbol{\lambda})\in\boldsymbol{V}\times\boldsymbol{V}_2\times Q\times\boldsymbol{\Lambda}} \frac{L(\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi})}{|||\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}|||} \geq c_5|||\boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}|||. \qquad (60)
$$

The proof of Lemma 4.3 follow the same reasoning as the proof Lemma 4.2 and has been omitted. It is easy to see that

$$
L(\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}) \leq C|||\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}||| \, |||\boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}||| \qquad (61)
$$

holds for any $(\boldsymbol{u}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{V}_2 \times Q \times \boldsymbol{\Lambda}$ and $(\boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}) \in \boldsymbol{V} \times \boldsymbol{V}_2 \times Q \times \boldsymbol{\Lambda}$. Then along with Lemma 4.2 and Lemma 4.3, we can obtain the following theorem by using the arguments in [2].

Theorem 4.4. *Suppose that (38) is satisfied. Given $\tilde{\boldsymbol{f}} \in (L^2(\Omega))^d$, $\boldsymbol{f}_2 \in (L^2(\Omega_2))^d$ and $\boldsymbol{w} \in (L^2(\Gamma))^d$ there is a unique solution $(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{V}_2 \times Q \times \boldsymbol{\Lambda}$ to (42), with*

$$\|\tilde{\boldsymbol{u}}, \boldsymbol{u}_2, q, \boldsymbol{\lambda}\| \leq C(\|\tilde{\boldsymbol{f}}\|_{0,\Omega} + \|\boldsymbol{f}_2\|_{0,\Omega_2} + \|\boldsymbol{w}\|_{0,\Gamma}). \tag{62}$$

With the proper choice of finite element subspaces, existence and uniqueness of the solution to the discrete problem is proved in a similar manner. We assume that the spaces $\boldsymbol{V}_h, \boldsymbol{V}_{2,H}, Q_h$, and $\boldsymbol{\Lambda}_H$ satisfy the following inf-sup conditions

$$\sup_{(\boldsymbol{u}_h,\boldsymbol{u}_{2,H})\in \boldsymbol{V}_h\times\boldsymbol{V}_{2,H}} \frac{\langle \boldsymbol{\xi}_H, \boldsymbol{u}_h|_{\Omega_2} - \boldsymbol{u}_{2,H}\rangle_{\Omega_2}}{\|\boldsymbol{u}_h, \boldsymbol{u}_{2,H}\|_{\boldsymbol{V}\times\boldsymbol{V}_2}} \geq c_1\|\boldsymbol{\xi}_H\|_{\boldsymbol{\Lambda}}, \quad \forall \boldsymbol{\xi}_H \in \boldsymbol{\Lambda}_H \tag{63}$$

$$\sup_{\boldsymbol{u}_h\in\boldsymbol{V}_h} \frac{(q_h, \nabla\cdot\boldsymbol{u}_h)_\Omega}{\|\boldsymbol{u}_h\|_{\boldsymbol{V}}} \geq c_2\|q_h\|_Q, \quad \forall q_h \in Q_h. \tag{64}$$

One possible choice of spaces is

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{V} : \boldsymbol{v}|_K \in P_2(K),\ \forall K \in T_h\}, \tag{65}$$

$$\boldsymbol{V}_{2,H} = \{\boldsymbol{v}_2 \in \boldsymbol{V}_2 : \boldsymbol{v}_2|_K \in P_2(K),\ \forall K \in T_H\}, \tag{66}$$

$$Q_h = \{q \in Q \cap (C^0(\Omega))^d : q|_K \in P_1(K),\ \forall K \in T_h\}, \tag{67}$$

$$\boldsymbol{\Lambda}_H = \boldsymbol{V}_{2,H}. \tag{68}$$

Then, if the mesh sequences are quasi-uniform, the inf-sup conditions (63) and (64) hold for the Lagrange multiplier term [1, 5] and for the pressure terms [7, 22, 6], respectively.

By (63), we know there exists $(\boldsymbol{w}_h, \boldsymbol{w}_{2,H}) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H}$ which satisfies $\|\boldsymbol{w}_h, \boldsymbol{w}_{2,H}\|_{\boldsymbol{V}\times\boldsymbol{V}_2} = \|\boldsymbol{\xi}_H\|_{\boldsymbol{\Lambda}}$ with $\langle \boldsymbol{\xi}_h, \boldsymbol{w}_h|_{\Omega_{2,H}} - \boldsymbol{w}_{2,H}\rangle \geq c_1\|\boldsymbol{\xi}_H\|^2_{\boldsymbol{\Lambda}}$, and by (64) there exists $\boldsymbol{u}_h \in \boldsymbol{V}_h$ which satisfies $\|u_h\|_{\boldsymbol{V}} = \|q_h\|_Q$ with $(q_h, \nabla\cdot\boldsymbol{u}_h)_\Omega \geq c_2\|q_h\|_Q^2$. The proofs of the following lemmas and theorem then follow the proofs of Lemma 4.1, Lemma 4.2, Lemma 4.3 and Theorem 4.4.

Lemma 4.5. *Suppose that (38), (63) and (64) are satisfied. Then there is a constant c such that for all $(q_h, \boldsymbol{\xi}_H) \in Q_h \times \boldsymbol{\Lambda}_H$*

$$\sup_{(\boldsymbol{u}_h,\boldsymbol{u}_{2,H})\in \boldsymbol{V}_h\times\boldsymbol{V}_{2,H}} \frac{b(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}; q_h, \boldsymbol{\xi}_H)}{\|\boldsymbol{u}_h, \boldsymbol{u}_{2,H}\|_{\boldsymbol{V}\times\boldsymbol{V}_2}} \geq c\|q_h, \boldsymbol{\xi}_H\|_{Q\times\boldsymbol{\Lambda}}. \tag{69}$$

Lemma 4.6. *Suppose that (38), (63) and (64) are satisfied. Then there is a constant c such that for all $(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H$*

$$\begin{aligned}\sup_{(\boldsymbol{u}_h,\boldsymbol{u}_{2,H},p_h,\boldsymbol{\lambda}_H)\in \boldsymbol{V}_h\times\boldsymbol{V}_{2,H}\times Q_h\times\boldsymbol{\Lambda}_H} &\frac{L(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H)}{\|\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H\|} \\ &\geq c\|\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H\|.\end{aligned} \tag{70}$$

Lemma 4.7. *Suppose that (38), (63) and (64) are satisfied. Then there is a constant c such that for all $(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H$*

$$\begin{aligned}\sup_{(\boldsymbol{v}_h,\boldsymbol{v}_{2,H},q_h,\boldsymbol{\xi}_H)\in \boldsymbol{V}_h\times\boldsymbol{V}_{2,H}\times Q_h\times\boldsymbol{\Lambda}_H} &\frac{L(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, , \boldsymbol{\xi}_H)}{\|\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H\|} \\ &\geq c\|\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H\|.\end{aligned} \tag{71}$$

Theorem 4.8. *Suppose that (38), (63) and (64) are satisfied. Given* $\boldsymbol{f} \in (L^2(\Omega))^d$, $\boldsymbol{f}_2 \in (L^2(\Omega_2))^d$ *and* $\boldsymbol{w} \in (L^2(\Gamma))^d$ *there is a unique solution* $(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H$ *to (43), with*

$$\|\!|\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H|\!\|\le C(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{f}_2\|_{0,\Omega_2} + \|\boldsymbol{w}\|_{0,\Gamma}). \tag{72}$$

In the sequence, we will give a convergence analysis of the proposed DLM/FD finite element method.

Theorem 4.9. *Let* $(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2, p, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{V}_2 \times Q \times \boldsymbol{\Lambda}$ *be the solution of (42), and let* $(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H$ *be the solution of (43). Then there exists a constant C independent of h and H such that*

$$\begin{aligned} &\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\boldsymbol{V}} + \|\boldsymbol{u}_2 - \boldsymbol{u}_{2,H}\|_{\boldsymbol{V}_2} + \|p - p_h\|_Q + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{\boldsymbol{\Lambda}} \\ \le& C\big(\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\tilde{\boldsymbol{u}} - \boldsymbol{v}_h\|_{\boldsymbol{V}} + \inf_{\boldsymbol{v}_{2,H} \in \boldsymbol{V}_{2,H}} \|\boldsymbol{u}_2 - \boldsymbol{v}_{2,H}\|_{\boldsymbol{V}_2} \\ &+ \inf_{q_h \in Q_h} \|p - q_h\|_Q + \inf_{\boldsymbol{\xi}_H \in \boldsymbol{\Lambda}_H} \|\boldsymbol{\lambda} - \boldsymbol{\xi}_H\|_{\boldsymbol{\Lambda}}\big). \end{aligned} \tag{73}$$

Proof. Subtracting (43) from (42) for all $(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H$, we have

$$L(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h, \boldsymbol{u}_2 - \boldsymbol{u}_{2,H}, \tilde{p} - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_H; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H) = 0. \tag{74}$$

For any $(\boldsymbol{w}_h, \boldsymbol{w}_{2,H}, r_h, \boldsymbol{\zeta}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H$, denote $\boldsymbol{e} = \tilde{\boldsymbol{u}} - \boldsymbol{w}_h$, $\boldsymbol{e}_h = \boldsymbol{u}_h - \boldsymbol{w}_h$, $\boldsymbol{e}_2 = \boldsymbol{u}_2 - \boldsymbol{w}_{2,H}$, $\boldsymbol{e}_{2,H} = \boldsymbol{u}_{2,H} - \boldsymbol{w}_{2,H}$, $s = p - r_h$, $s_h = p_h - r_h$, $\boldsymbol{\epsilon} = \boldsymbol{\lambda} - \boldsymbol{\zeta}_H$, $\boldsymbol{\epsilon}_H = \boldsymbol{\lambda}_H - \boldsymbol{\zeta}_H$. Then we have

$$L(\boldsymbol{e}, \boldsymbol{e}_2, s, \boldsymbol{\epsilon}; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}, p_h, \boldsymbol{\xi}_H) = L(\boldsymbol{e}_h, \boldsymbol{e}_{2,H}, s_h, \boldsymbol{\epsilon}_H; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}, p_h, \boldsymbol{\xi}_H). \tag{75}$$

From lemma 4.6 and the continuity of $L(\cdot,\cdot,\cdot,\cdot;\cdot,\cdot,\cdot,\cdot)$,

$$\begin{aligned} &\|\!|\boldsymbol{e}_h, \boldsymbol{e}_{2,H}, s_h, \boldsymbol{\epsilon}_H|\!\| \\ \le& C \sup_{(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H} \frac{L(\boldsymbol{e}_h, \boldsymbol{e}_{2,H}, s_h, \boldsymbol{\epsilon}_H; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H)}{\|\!|\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H|\!\|} \\ =& C \sup_{(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H} \frac{L(\boldsymbol{e}, \boldsymbol{e}_2, s, \boldsymbol{\epsilon}; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H)}{\|\!|\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H|\!\|} \\ \le& C \sup_{(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H} \times Q_h \times \boldsymbol{\Lambda}_H} \frac{\|\!|\boldsymbol{e}, \boldsymbol{e}_2, s, \boldsymbol{\epsilon}|\!\| \|\!|\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H|\!\|}{\|\!|\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\xi}_H|\!\|} \\ =& C\|\!|\boldsymbol{e}, \boldsymbol{e}_2, s, \boldsymbol{\epsilon}|\!\|. \end{aligned} \tag{76}$$

Thus,

$$\begin{aligned} &\|\!|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h, \boldsymbol{u}_2 - \boldsymbol{u}_{2,H}, \tilde{p} - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_H|\!\| \\ =& \|\!|\boldsymbol{e} - \boldsymbol{e}_h, \boldsymbol{e}_2 - \boldsymbol{e}_{2,H}, s - s_h, \boldsymbol{\epsilon} - \boldsymbol{\epsilon}_H|\!\| \\ \le& \|\!|\boldsymbol{e}, \boldsymbol{e}_2, s, \boldsymbol{\epsilon}|\!\| + \|\!|\boldsymbol{e}_h, \boldsymbol{e}_{2,H}, s_h, \boldsymbol{\epsilon}_h|\!\| \\ \le& (1 + C)\|\!|\boldsymbol{e}, \boldsymbol{e}_2, s, \boldsymbol{\epsilon}|\!\| \\ =& (1 + C)\|\!|\tilde{\boldsymbol{u}} - \boldsymbol{w}_h, \boldsymbol{u}_2 - \boldsymbol{w}_{2,H}, \tilde{p} - r_h, \boldsymbol{\lambda} - \boldsymbol{\zeta}_H|\!\|, \end{aligned} \tag{77}$$

which completes the proof of this theorem. □

In order to get an estimate for $\inf_{\boldsymbol{\xi}_H \in \boldsymbol{\Lambda}_H} \|\boldsymbol{\lambda} - \boldsymbol{\xi}_H\|$ we need to have a characterization for the Lagrange multiplier $\boldsymbol{\lambda}$. First, in (28) we pick any $\boldsymbol{v} \in \boldsymbol{V}$ such that

$\boldsymbol{v} = 0$ outside Ω_1. Integrating by parts gives

$$\tilde{\boldsymbol{f}} = -\nabla \cdot (\tilde{\beta} \nabla \tilde{\boldsymbol{u}}) + \nabla \tilde{p}, \quad \text{in } \Omega_1. \tag{78}$$

Similarly, we can pick $(\boldsymbol{v}, \boldsymbol{v}_2) \in \boldsymbol{V} \times (H_0^1(\Omega_2))^d$ such that $\boldsymbol{v}|_{\Omega_2} = \boldsymbol{v}_2$ and $\boldsymbol{v} = 0$ outside Ω_2. Adding (28) to (30) and integrating by parts yields

$$\boldsymbol{f}_2 = -\nabla \cdot (\beta_2 \nabla \boldsymbol{u}_2) + \nabla \tilde{p}, \quad \text{in } \Omega_2. \tag{79}$$

Now, let $\boldsymbol{v} \in \boldsymbol{V}_0$ and take $\boldsymbol{v}_2 = \boldsymbol{v}|_{\Omega_2}$ in Ω_2. Adding (28) to (30) and imposing both $\boldsymbol{u}_2 = \tilde{\boldsymbol{u}}|_{\Omega_2}$ and $\nabla \boldsymbol{u}_2 = \nabla \tilde{\boldsymbol{u}}|_{\Omega_2}$ weakly in Ω_2 gives us

$$\begin{aligned} &(\tilde{\beta} \nabla \tilde{\boldsymbol{u}}, \nabla \boldsymbol{v})_{\Omega_1} - (\tilde{p}, \nabla \cdot \boldsymbol{v})_{\Omega_1} - (\tilde{p}, \nabla \cdot \boldsymbol{v})_{\Omega_2} + (\beta_2 \nabla \boldsymbol{u}_2, \nabla \boldsymbol{v})_{\Omega_2} \\ =&(\tilde{\boldsymbol{f}}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{f}_2, \boldsymbol{v})_{\Omega_2} + (\boldsymbol{w}, \boldsymbol{v})_\Gamma, \end{aligned} \tag{80}$$

where the integrals over Ω have been split into Ω_1 and Ω_2. Integrating by parts, we have

$$\begin{aligned} &-(\nabla \cdot (\tilde{\beta} \nabla \tilde{\boldsymbol{u}}), \boldsymbol{v})_{\Omega_1} + (\tilde{\beta} \nabla \tilde{\boldsymbol{u}}\, \boldsymbol{n}_1, \boldsymbol{v})_\Gamma + (\nabla \tilde{p}, \boldsymbol{v})_{\Omega_1} - (\tilde{p}, \boldsymbol{v} \cdot \boldsymbol{n}_1)_\Gamma + (\nabla \tilde{p}, \boldsymbol{v})_{\Omega_2} \\ &-(\tilde{p}, \boldsymbol{v} \cdot \boldsymbol{n}_2)_\Gamma - (\nabla \cdot (\beta_2 \nabla \boldsymbol{u}_2), \boldsymbol{v})_{\Omega_2} + (\beta_2 \nabla \boldsymbol{u}_2\, \boldsymbol{n}_2, \boldsymbol{v})_\Gamma = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{f}_2, \boldsymbol{v})_{\Omega_2} + (\boldsymbol{w}, \boldsymbol{v})_\Gamma. \end{aligned} \tag{81}$$

By applying (78), (79), and $\boldsymbol{n}_1 = -\boldsymbol{n}_2$, we obtain

$$(\tilde{\beta} \nabla \tilde{\boldsymbol{u}}\, \boldsymbol{n}_1, \boldsymbol{v})_\Gamma + (\beta_2 \nabla \boldsymbol{u}_2\, \boldsymbol{n}_2, \boldsymbol{v})_\Gamma = (\boldsymbol{w}, \boldsymbol{v})_\Gamma. \tag{82}$$

By applying (79) and (82) to (30), then

$$\begin{aligned} \langle \boldsymbol{\lambda}, \boldsymbol{v}_2 \rangle_{\Omega_2} =& -\Big(\tilde{\beta} \nabla \boldsymbol{u}_2, \nabla \boldsymbol{v}_2\Big)_{\Omega_2} + (\beta_2 \nabla \boldsymbol{u}_2, \boldsymbol{v}_2)_{\Omega_2} - (\boldsymbol{f}_2, \boldsymbol{v}_2)_{\Omega_2} + (\tilde{\boldsymbol{f}}, \boldsymbol{v}_2)_{\Omega_2} - (\boldsymbol{w}, \boldsymbol{v}_2)_\Gamma \\ =& \Big(\nabla \cdot (\frac{\tilde{\beta}}{\beta_2} \beta_2 \nabla \boldsymbol{u}_2), \boldsymbol{v}_2\Big)_{\Omega_2} - (\tilde{\beta} \nabla \boldsymbol{u}_2\, \boldsymbol{n}_2, \boldsymbol{v}_2)_\Gamma \\ &- (\nabla \cdot (\beta_2 \nabla \boldsymbol{u}_2), \boldsymbol{v}_2)_{\Omega_2} + (\beta_2 \nabla \boldsymbol{u}_2\, \boldsymbol{n}_2, \boldsymbol{v}_2)_\Gamma \\ &- (\boldsymbol{f}_2, \boldsymbol{v}_2)_{\Omega_2} + (\tilde{\boldsymbol{f}}, \boldsymbol{v}_2)_{\Omega_2} - (\boldsymbol{w}, \boldsymbol{v}_2)_\Gamma - (\nabla \tilde{p}, \boldsymbol{v}_2)_{\Omega_2} + (\nabla \tilde{p}, \boldsymbol{v}_2)_{\Omega_2} \\ =& \Big(\frac{\tilde{\beta}}{\beta_2} \nabla \cdot (\beta_2 \nabla \boldsymbol{u}_2), \boldsymbol{v}_2\Big)_{\Omega_2} - (\tilde{\beta} \nabla \boldsymbol{u}_2\, \boldsymbol{n}_2, \boldsymbol{v}_2)_\Gamma + (\beta_2 \nabla \boldsymbol{u}_2\, \boldsymbol{n}_2, \boldsymbol{v}_2)_\Gamma \\ &- (\boldsymbol{w}, \boldsymbol{v}_2)_\Gamma - (\nabla \tilde{p}, \boldsymbol{v}_2)_{\Omega_2} - (\nabla \cdot (\beta_2 \nabla \boldsymbol{u}_2), \boldsymbol{v}_2)_{\Omega_2} + (\nabla \tilde{p}, \boldsymbol{v}_2)_{\Omega_2} \\ &- (\boldsymbol{f}_2, \boldsymbol{v}_2)_{\Omega_2} + (\tilde{\boldsymbol{f}}, \boldsymbol{v}_2)_{\Omega_2} + \Big(\beta_2 \nabla \boldsymbol{u}_2 \nabla \frac{\tilde{\beta}}{\beta_2}, \boldsymbol{v}_2\Big)_{\Omega_2} \\ =& \Big(\frac{\tilde{\beta}}{\beta_2} \nabla \cdot (\beta_2 \nabla \boldsymbol{u}_2), \boldsymbol{v}_2\Big)_{\Omega_2} - (\tilde{\beta} \nabla \boldsymbol{u}_2\, \boldsymbol{n}_2, \boldsymbol{v}_2)_\Gamma - (\tilde{\beta} \nabla \tilde{\boldsymbol{u}}\, \boldsymbol{n}_1, \boldsymbol{v}_2)_\Gamma - (\nabla \tilde{p}, \boldsymbol{v}_2)_{\Omega_2} \\ &+ (\tilde{\boldsymbol{f}}, \boldsymbol{v}_2)_{\Omega_2} + \Big(\beta_2 \nabla \boldsymbol{u}_2 \nabla \frac{\tilde{\beta}}{\beta_2}, \boldsymbol{v}_2\Big)_{\Omega_2} \\ =& -\Big(\frac{\tilde{\beta}}{\beta_2} \boldsymbol{f}_2 - \tilde{\boldsymbol{f}}, \boldsymbol{v}_2\Big)_{\Omega_2} - \Big((1 - \frac{\tilde{\beta}}{\beta_2}) \nabla \tilde{p}, \boldsymbol{v}_2\Big)_{\Omega_2} \\ &- (\tilde{\beta}(\nabla \boldsymbol{u}_2\, \boldsymbol{n}_2 + \nabla \tilde{\boldsymbol{u}}\, \boldsymbol{n}_1), \boldsymbol{v}_2)_\Gamma + \Big(\beta_2 \nabla \boldsymbol{u}_2 \nabla \frac{\tilde{\beta}}{\beta_2}, \boldsymbol{v}_2\Big)_{\Omega_2}. \end{aligned} \tag{83}$$

We can write $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3$ where

$$\langle \boldsymbol{\lambda}_1, \boldsymbol{v}_2 \rangle_{\Omega_2} = -\Big(\frac{\tilde{\beta}}{\beta_2} \boldsymbol{f}_2 - \tilde{\boldsymbol{f}}, \boldsymbol{v}_2 \Big)_{\Omega_2}, \tag{84}$$

$$\langle \boldsymbol{\lambda}_2, \boldsymbol{v}_2 \rangle_{\Omega_2} = -(\tilde{\beta}(\nabla \boldsymbol{u}_2 - \nabla \tilde{\boldsymbol{u}})\boldsymbol{n}_2, \boldsymbol{v}_2)_\Gamma, \tag{85}$$

$$\langle \boldsymbol{\lambda}_3, \boldsymbol{v}_2 \rangle_{\Omega_2} = \Big(-(1 - \frac{\tilde{\beta}}{\beta_2})\nabla \tilde{p} + \beta_2 \nabla \boldsymbol{u}_2 \nabla \frac{\tilde{\beta}}{\beta_2}, \boldsymbol{v}_2 \Big)_{\Omega_2}. \tag{86}$$

For the immersed interface, if the interface is of class C^3 and the boundary $\partial\Omega$ is of class C^2, we have the following regularity result for the Stokes interface problem [26]

$$\boldsymbol{u} \in (H^2(\Omega_1 \cup \Omega_2))^d \cap (H^\tau(\Omega))^d, \quad p \in H^1(\Omega_1 \cup \Omega_2) \cap L^2(\Omega). \tag{87}$$

where, $1 < \tau < 1.5$. However, if the interface is only Lipschitz continuous then we may have the following reduced regularity results

$$\boldsymbol{u} \in (H^r(\Omega_1 \cup \Omega_2))^d \cap (H^\tau(\Omega))^d, \quad p \in H^s(\Omega_1 \cup \Omega_2) \cap L^2(\Omega). \tag{88}$$

Since the theory of regularity for the Stokes interface problem is not yet fully developed, we make the assumption that $1 < r \leq 2$ and $0 < s \leq 1$ in order to obtain an error convergence rate. The actual values for r and s will depend on the jump coefficient β as well as the regularity of the interface. The numerical results in Section 5 show such a reduced regularity for the problems we have chosen. Thanks to the equivalence between Weak Form I and Weak Form II, we further have

(89)
$$\tilde{\boldsymbol{u}} \in (H^r(\Omega_1 \cup \Omega_2))^d \cap (H^\tau(\Omega))^d, \quad \boldsymbol{u}_2 \in (H^r(\Omega_2))^d, \quad \tilde{p} \in H^s(\Omega_1 \cup \Omega_2) \cap L^2(\Omega).$$

Using these regularity results and the results from [1], we can see that there exist $\tilde{\boldsymbol{\xi}}_{1H} \in \boldsymbol{\Lambda}_H$ and $\tilde{\boldsymbol{\xi}}_{2H} \in \boldsymbol{\Lambda}_H$ such that

$$\|\boldsymbol{\lambda}_1 - \tilde{\boldsymbol{\xi}}_{1H}\|_{\boldsymbol{\Lambda}} \leq CH \|(\tilde{\beta}/\beta_2)\boldsymbol{f}_2 - \boldsymbol{f}\|_{(L^2(\Omega_2))^d}, \tag{90}$$

$$\|\boldsymbol{\lambda}_2 - \tilde{\boldsymbol{\xi}}_{2H}\|_{\boldsymbol{\Lambda}} \leq CH^{r-1}(\|\boldsymbol{u}|_{\Omega_1}\|_{(H^r(\Omega_1))^d} + \|\boldsymbol{u}_2\|_{(H^r(\Omega_2))^d}). \tag{91}$$

To get an error estimate for $\boldsymbol{\lambda}_3$, let π_H be the L^2 projection of $\boldsymbol{V}_2$ into $\boldsymbol{V}_{2,H}$, that is,

$$(\pi_H \boldsymbol{w}_2, \boldsymbol{v}_{2,H})_{\Omega_2} = (\boldsymbol{w}_2, \boldsymbol{v}_{2,H})_{\Omega_2}, \quad \forall \boldsymbol{v}_{2,H} \in \boldsymbol{V}_{2,H}. \tag{92}$$

Then let $\tilde{\boldsymbol{\xi}}_{3H} = P_H \boldsymbol{\lambda}_3 \in \boldsymbol{\Lambda}_H$ satisfy

$$\langle P_H \boldsymbol{\lambda}_3, \boldsymbol{v}_2 \rangle_{\Omega_2} = \Big(\pi_H(-(1 - \frac{\tilde{\beta}}{\beta_2})\nabla \tilde{p} + \beta_2 \nabla \boldsymbol{u}_2 \nabla \frac{\tilde{\beta}}{\beta_2}), \boldsymbol{v}_2 \Big)_{\Omega_2}, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_2, \tag{93}$$

so we have $\langle \boldsymbol{\lambda}_3 - P_H\boldsymbol{\lambda}_3, \boldsymbol{v}_{2,H}\rangle = 0$ for all $\boldsymbol{v}_{2,H} \in \boldsymbol{V}_{2,H}$. Then,

$$\begin{aligned}
&\|\boldsymbol{\lambda}_3 - P_H\boldsymbol{\lambda}_3\|_{\boldsymbol{\Lambda}} = \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_2} \frac{\langle \boldsymbol{\lambda}_3 - P_H\boldsymbol{\lambda}_3, \boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}}\\
&= \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_2} \frac{\langle \boldsymbol{\lambda}_3 - P_H\boldsymbol{\lambda}_3, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}}\\
&= \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_2} \frac{\langle \boldsymbol{\lambda}_3, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\rangle - (\pi_H(-(1-\frac{\tilde\beta}{\beta_2})\nabla\tilde p + \beta_2\nabla\boldsymbol{u}_2\nabla\frac{\tilde\beta}{\beta_2}), \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2)_{\Omega_2}}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}}\\
&= \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_2} \frac{(-(1-\frac{\tilde\beta}{\beta_2})\nabla\tilde p, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2)_{\Omega_2} + (\beta_2\nabla\boldsymbol{u}_2\nabla\frac{\tilde\beta}{\beta_2}, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2)_{\Omega_2}}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}}. \qquad (94)
\end{aligned}$$

By applying the Cauchy–Schwartz inequality and the standard L^2 error estimate for π_H, we have

$$\begin{aligned}
(-(1-\frac{\beta}{\beta_2})\nabla\tilde p, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2)_{\Omega_2} &\le C\|\nabla\tilde p\|_{0,\Omega_2}\|\boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\|_{0,\Omega_2}\\
&\le CH\|\tilde p\|_{H^1(\Omega_2)}\|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}, \qquad (95)\\
(\beta_2\nabla\boldsymbol{u}_2\nabla\frac{\tilde\beta}{\beta_2}, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2)_{\Omega_2} &\le C\|\nabla\frac{\tilde\beta}{\beta_2}\|_{\infty,\Omega_2}\|\nabla\boldsymbol{u}_2\|_{0,\Omega_2}\|\boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\|_{0,\Omega_2}\\
&\le CH\|\nabla\frac{\tilde\beta}{\beta_2}\|_{\infty,\Omega_2}\|\boldsymbol{u}_2\|_{\boldsymbol{V}_2}\|\boldsymbol{v}_2\|_{\boldsymbol{V}_2}. \qquad (96)
\end{aligned}$$

If $|\nabla(\tilde\beta/\beta_2)|$ is bounded a.e. on Ω_2, then there exists a constant C such that

$$\|\boldsymbol{\lambda}_3 - \tilde{\boldsymbol{\xi}}_{3H}\|_{\boldsymbol{\Lambda}} \le CH\left(\|\tilde p\|_{H^s(\Omega_2)} + \|\boldsymbol{u}_2\|_{\boldsymbol{V}_2}\right). \qquad (97)$$

By combining the error estimates for $\boldsymbol{\lambda}$ and using the standard interpolation error estimates for $\tilde{\boldsymbol{u}}, \boldsymbol{u}_2$, and $\tilde p$ in (73), we have the following a priori error estimate theorem.

Theorem 4.10. *Let $(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2, \tilde p, \boldsymbol{\lambda}) \in \boldsymbol{V}\times\boldsymbol{V}_2\times Q\times\boldsymbol{\Lambda}$ be the solution of (42), and let $(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\lambda}_H) \in \boldsymbol{V}_h\times\boldsymbol{V}_{2,H}\times Q_h\times\boldsymbol{\Lambda}_H$ be the solution of (43). If conditions (38), (63), and (64) are satisfied, and $|\nabla(\tilde\beta/\beta_2)|$ is bounded a.e. on Ω_2, then*

$$\begin{aligned}
&\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\boldsymbol{V}} + \|\boldsymbol{u}_2 - \boldsymbol{u}_{2,H}\|_{\boldsymbol{V}_2} + \|\tilde p - p_h\|_Q + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{\boldsymbol{\Lambda}}\\
\le& C(h^\sigma + H^\sigma)(\|\tilde{\boldsymbol{u}}\|_{(H^\tau(\Omega))^d} + \|\boldsymbol{u}_2\|_{(H^r(\Omega_2))^d}\\
&+ \|\tilde p\|_{H^s(\Omega_1\cup\Omega_2)} + \|(\tilde\beta/\beta_2)\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}\|_{(L^2(\Omega_2))^d}), \qquad (98)
\end{aligned}$$

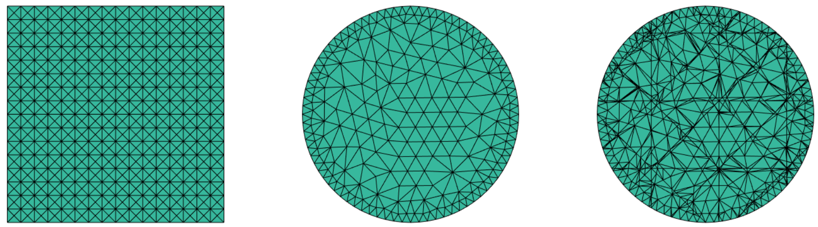
where the constants τ, r and s are from the reduced regularity (89) and $\sigma = \min\{\tau - 1, r-1, s\}$.

Proof. It follows from the standard finite element interpolation error estimates that

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\boldsymbol{V}} \le Ch^{\tau-1}\|\tilde{\boldsymbol{u}}\|_{(H^\tau(\Omega))^d} \qquad (99)$$

$$\inf_{\boldsymbol{v}_{2,H}\in\boldsymbol{V}_{2,H}} \|\boldsymbol{u}_2 - \boldsymbol{v}_{2,H}\|_{\boldsymbol{V}_2} \le Ch^{r-1}\|\boldsymbol{u}_2\|_{(H^r(\Omega_2))^d} \qquad (100)$$

$$\inf_{q_h\in Q_h} \|p - q_h\|_Q \le Ch^s\|\tilde p\|_{H^s(\Omega_1\cup\Omega_2)} \qquad (101)$$



(a) (b) (c)

FIGURE 2. A example of the meshes. (a) $T_h(\Omega)$, (b) $T_H(\Omega_2)$, (c) $T_r(\Omega_2)$.

Combining (90), (91) and (97), we see that there exists $\tilde{\boldsymbol{\xi}}_H = \tilde{\boldsymbol{\xi}}_{1H} + \tilde{\boldsymbol{\xi}}_{2H} + \tilde{\boldsymbol{\xi}}_{3H} \in \boldsymbol{\Lambda}_H$, such that

$$\begin{aligned}\inf_{\boldsymbol{\xi}_H \in \boldsymbol{\Lambda}_H} \|\boldsymbol{\lambda} - \boldsymbol{\xi}_H\|_{\boldsymbol{\Lambda}} \leq & \left\|\boldsymbol{\lambda} - \tilde{\boldsymbol{\xi}}_H\right\|_{\boldsymbol{\Lambda}} \\ \leq & CH^{r-1}\Big(\|(\tilde{\beta}/\beta_2)\boldsymbol{f}_2 - \boldsymbol{f}\|_{(L^2(\Omega_2))^d} + \|\boldsymbol{u}\|_{(H^r(\Omega))^d} \\ & + \|\boldsymbol{u}_2\|_{(H^r(\Omega_2))^d} + \|\tilde{p}\|_{H^s(\Omega_2)}\Big). \end{aligned} \tag{102}$$

Thus, the aimed result (98) is a direct result of Theorem 4.9 and estimates (99), (100), (101) and (102). □

5. Numerical experiments

In this section, we study the numerical performance of the DLM/FD finite element method, in which the P^2 finite element is used to discretize $\boldsymbol{u}_h$, $\boldsymbol{u}_{2,H}, \boldsymbol{\lambda}_H$, and the P^1 finite element is used to discretize p_h.

Let $\Omega = (0,1) \times (0,1)$, Ω_2 is a circle with center $(0.3, 0.3)$ and radius 0.1. The meshes $T_h(\Omega)$, $T_H(\Omega_2)$ are constructed independently, as depicted in Fig. 2 (a) and (b). Since $\boldsymbol{u}_h$ and $\boldsymbol{\lambda}_H$ are defined on different meshes, that is, $T_h(\Omega)$ and $T_H(\Omega_2)$, we employ the subgrid integration technique [27] to compute the Lagrange multiplier terms that appear in the presented DLM/FD finite element method (34)-(37), such as $\langle \boldsymbol{\lambda}_H, \boldsymbol{v}_h \rangle_{\Omega_2}$ and $\langle \boldsymbol{\xi}_H, \boldsymbol{u}_h \rangle_{\Omega_2}$. To that end, a submesh is also constructed based on $T_h(\Omega)$, $T_H(\Omega_2)$, see e.g. Fig. 2 (c).

5.1. Example 1. In this example, we first verify the theoretical convergence rates for the case of smooth exact solutions by appropriately choosing different parameters $\tilde{\beta} = \beta_1$, β_2 and functions $\tilde{f} = f_1$, f_2 such that the exact solution of the Stokes interface problem (1)-(8) in the entire domain Ω is

$$\boldsymbol{u} = \left(\sin(x-y), \sin(x-y)\right)^T, \quad p = e^x \cos(y).$$

Thus, $r = 2$ and $s = 1$ in the regularity result (89) for the above chosen solutions.

The numerical results of this example are reported in Tables 1-3 and Figures 3-5, where the values u and v denote the components of the velocity vector $\boldsymbol{u}$. From those tables and figures, we observe the first order convergence rate for errors of velocity in H^1 norm and for errors of pressure in L^2 norm, which is consistent with that predicted by Theorem 4.10 when the exact solutions are smooth. In addition, the convergence rate of velocity in L^2 norm is shown as higher than first order in

TABLE 1. Results of Example 1: $\beta_1 = 1$, $\beta_2 = 10$, $h/H = 4$.

h	H	$\|\|u-u_h\|\|_{1,\Omega}$	$\|\|u-u_h\|\|_{0,\Omega}$	$\|\|v-v_h\|\|_{1,\Omega}$	$\|\|v-v_h\|\|_{0,\Omega}$	$\|\|p-p_h\|\|_{0,\Omega}$
1/16	1/ 64	7.5603e-03	3.1589e-04	7.2792e-03	3.0614e-04	9.0696e-02
1/20	1/ 80	6.1280e-03	2.4491e-04	5.9890e-03	2.3886e-04	1.2997e-01
1/24	1/ 96	5.1337e-03	2.0517e-04	5.0780e-03	2.0024e-04	1.4636e-01
1/28	1/112	4.4040e-03	1.7375e-04	4.3398e-03	1.7151e-04	6.5847e-02
1/32	1/128	3.7961e-03	1.5159e-04	3.7528e-03	1.4978e-04	4.2513e-02
rate		0.99	1.05	0.95	1.02	1.13

TABLE 2. Results of Example 1: $\beta_1 = 1$, $\beta_2 = 100$, $h/H = 4$.

h	H	$\|\|u-u_h\|\|_{1,\Omega}$	$\|\|u-u_h\|\|_{0,\Omega}$	$\|\|v-v_h\|\|_{1,\Omega}$	$\|\|v-v_h\|\|_{0,\Omega}$	$\|\|p-p_h\|\|_{0,\Omega}$
1/16	1/ 64	1.4762e-02	5.7566e-04	1.4250e-02	5.5804e-04	3.3312e-01
1/20	1/ 80	9.5808e-03	3.7983e-04	9.3973e-03	3.7205e-04	2.9158e-01
1/24	1/ 96	1.0001e-02	3.4806e-04	9.7611e-03	3.3693e-04	4.2304e-01
1/28	1/112	7.4827e-03	2.8156e-04	7.4078e-03	2.7800e-04	2.4753e-01
1/32	1/128	6.6793e-03	2.4661e-04	6.5439e-03	2.4306e-04	1.6566e-01
rate		1.07	1.17	1.05	1.15	0.83

TABLE 3. Results of Example 1: $\beta_1 = 1$, $\beta_2 = 1000$, $h/H = 4$.

h	H	$\|\|u-u_h\|\|_{1,\Omega}$	$\|\|u-u_h\|\|_{0,\Omega}$	$\|\|v-v_h\|\|_{1,\Omega}$	$\|\|v-v_h\|\|_{0,\Omega}$	$\|\|p-p_h\|\|_{0,\Omega}$
1/16	1/ 64	2.0237e-02	7.2171e-04	1.9445e-02	6.7732e-04	3.1828e+00
1/20	1/ 80	1.0129e-02	4.0190e-04	9.9393e-03	3.9396e-04	2.7095e+00
1/24	1/ 96	1.5544e-02	4.0347e-04	1.4321e-02	3.8927e-04	2.2237e+00
1/28	1/112	1.0343e-02	3.2501e-04	1.0011e-02	3.1595e-04	1.9462e+00
1/32	1/128	8.4952e-03	2.7501e-04	8.1013e-03	2.6717e-04	1.6841e+00
rate		1.00	1.26	1.02	1.23	0.93

these tables, which is however not covered by our current theoretical results, will be explained at below together with Example 2.

5.2. Example 2. In this example, we choose the same geometric settings as that of Example 1, but $f_1 = f_2 = 1$, $\boldsymbol{w} = 0$ and different parameters $\tilde{\beta}$, $\tilde{\beta}_2$, thus the exact solution of this example is not prescribed and non-smooth. Then we take the P^2P^1 mixed finite element solution as the "numerical exact solution" of this example, which is obtained based on a locally refined and body-fitted mesh with 30329 nodes and 60304 triangular elements. Obviously, the regularities of such non-smooth solutions are reduced from the case of smooth solutions, i.e., $1 < r < 2$, $0 < s < 1$ in (89).

The numerical results of this example are reported in Tables 4-6 and Figures 6-8. We observe that the convergence rate in H^1 errors of velocity ranges between 0.25 and 0.6, which is consistent with the reduced regularity of the exact solution. It is also seen that the order of convergence shown in Table 5 is even lower than that shown in Table 6 in which the solution is expected to be less regular as a consequence of the larger jump ratio between the viscosities. To improve and also to explain this numerical phenomenon, a finer mesh will help considering that we are comparing with a "numerical exact solution" in order to obtain a reasonable convergence rate, which is however limited by our computer resources available. On the other hand, it is always crucial how to choose an optimal space for the Lagrange multiplier regarding a different jump ratio when the mesh is non-matching, especially. In this paper, we define the discrete space of Lagrange multiplier on $T_H(\Omega_2)$ instead of introducing a new finite element mesh for the Lagrange multiplier in Ω_2, which can produce a satisfied convergence rate to match with our theoretical results but may not be optimal corresponding to different jump ratios. Thus, our numerical experiments may show a possibly lower order of convergence for the case with

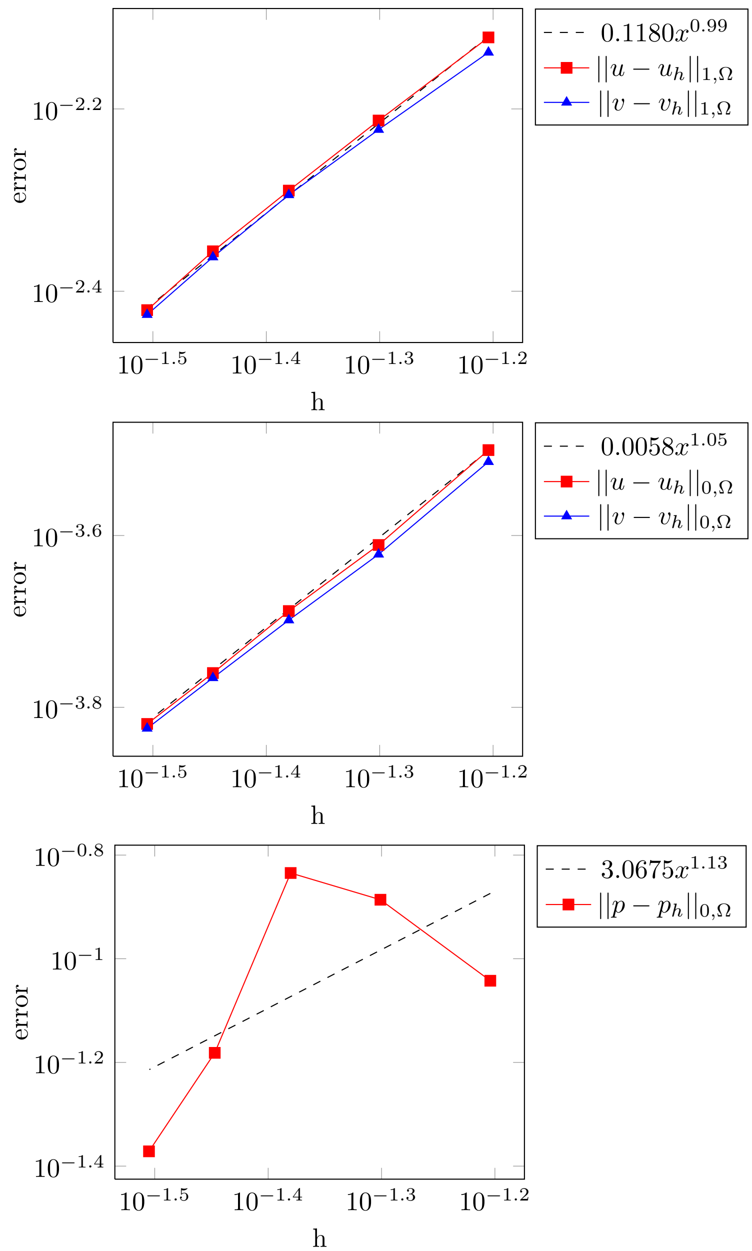


FIGURE 3. Loglog convergence plots of Example 1 ($\beta_1 = 1$, $\beta_2 = 10$, $h/H = 4$).

a smaller jump ratio of viscosities, but the overall numerical convergence results basically satisfy our theoretical results relative to the regularity constraints of the original model. We will continue our research on this direction in our future work to further improve the convergence behavior of the Lagrange multiplier in regard to different jump ratios of viscosities.

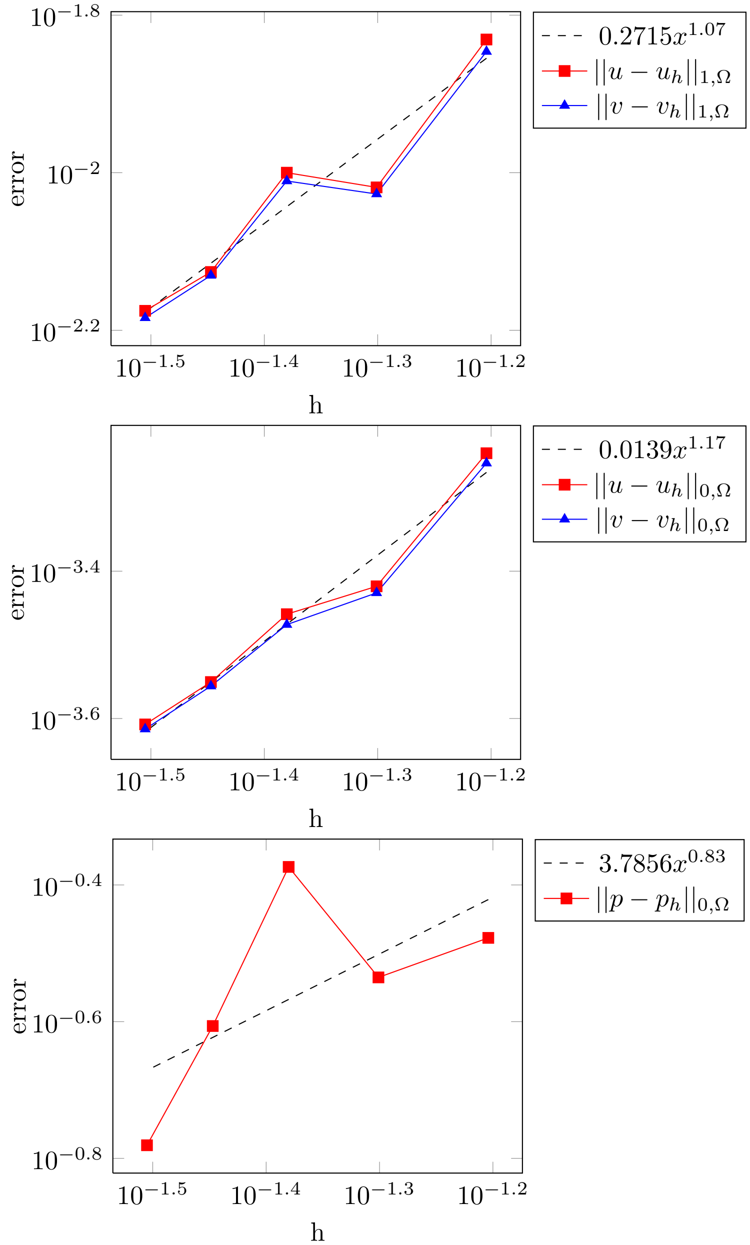


FIGURE 4. Loglog convergence plots of Example 1 ($\beta_1 = 1$, $\beta_2 = 100$, $h/H = 4$).

While not covered by the theoretical results given in this paper, it can be shown using Nitsche's technique that the convergence rate of velocity in L^2 norm is half an order higher than that in H^1 norm, as seen in the numerical results. Whereas, such higher-order convergence phenomena in L^2 norm is not reflected on Example 1, where the convergence rate of velocity in both H^1 norm and L^2 norm are all first

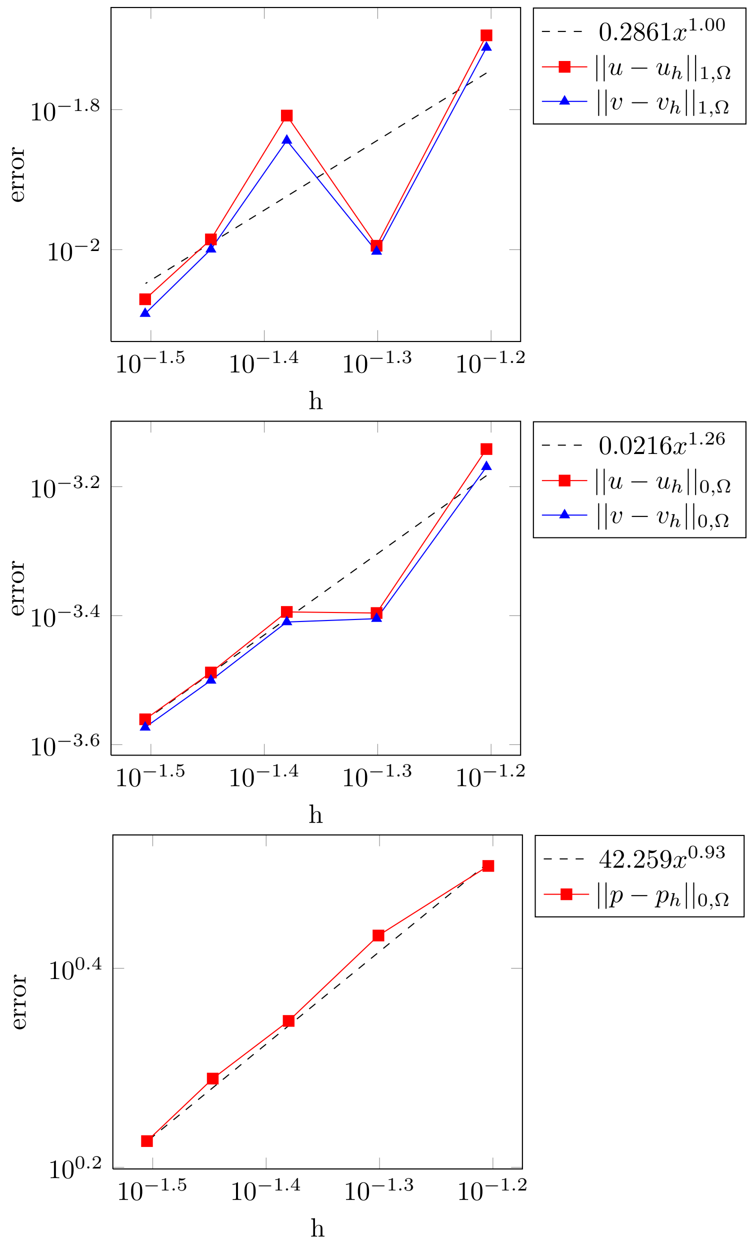


FIGURE 5. Loglog convergence plots of Example 1 ($\beta_1 = 1$, $\beta_2 = 1000$, $h/H = 4$).

order. Additionally, in Example 2 we also notice a significantly higher convergence rate in pressure than that is predicted by our theory.

Note that in both examples, we always let $H < h$ since this mesh ratio can give us better accuracy in numerical experiments. It is reasonable because a smaller mesh size H can define more Lagrange multipliers in Ω_2, further, introduce more

TABLE 4. Results of Example 2: $\beta_1 = 1$, $\beta_2 = 10$, $h/H = 4$.

h	H	$\|\|u - u_h\|\|_{1,\Omega}$	$\|\|u - u_h\|\|_{0,\Omega}$	$\|\|v - v_h\|\|_{1,\Omega}$	$\|\|v - v_h\|\|_{0,\Omega}$	$\|\|p - p_h\|\|_{0,\Omega}$
1/16	1/ 64	8.5369e-02	1.5585e-03	8.6160e-02	1.5568e-03	4.6481e+00
1/24	1/ 96	5.8084e-02	8.4369e-04	5.8416e-02	8.4364e-04	2.3801e+00
1/32	1/128	6.2376e-02	7.8786e-04	6.2016e-02	7.8807e-04	6.6086e-01
1/40	1/160	5.6614e-02	6.1394e-04	5.6600e-02	6.1400e-04	8.8549e-02
rate		0.41	0.97	0.42	0.96	4.13

TABLE 5. Results of Example 2: $\beta_1 = 1$, $\beta_2 = 100$, $h/H = 4$.

h	H	$\|\|u - u_h\|\|_{1,\Omega}$	$\|\|u - u_h\|\|_{0,\Omega}$	$\|\|v - v_h\|\|_{1,\Omega}$	$\|\|v - v_h\|\|_{0,\Omega}$	$\|\|p - p_h\|\|_{0,\Omega}$
1/16	1/ 64	2.2787e-01	5.9846e-03	2.2970e-01	5.9832e-03	5.1065e+00
1/24	1/ 96	1.7219e-01	3.7622e-03	1.7438e-01	3.7624e-03	3.1313e-01
1/32	1/128	1.8279e-01	3.4502e-03	1.8187e-01	3.4505e-03	2.1628e-01
1/40	1/160	1.8092e-01	2.9917e-03	1.8095e-01	2.9921e-03	1.8788e-01
rate		0.23	0.73	0.25	0.73	3.59

TABLE 6. Results of Example 2: $\beta_1 = 1$, $\beta_2 = 1000$, $h/H = 4$.

h	H	$\|\|u - u_h\|\|_{1,\Omega}$	$\|\|u - u_h\|\|_{0,\Omega}$	$\|\|v - v_h\|\|_{1,\Omega}$	$\|\|v - v_h\|\|_{0,\Omega}$	$\|\|p - p_h\|\|_{0,\Omega}$
1/16	1/ 64	3.4946e-01	1.0268e-02	3.5008e-01	1.0270e-02	1.9852e+00
1/24	1/ 96	2.7260e-01	6.5404e-03	2.7456e-01	6.5401e-03	5.5888e-01
1/32	1/128	2.7596e-01	5.7744e-03	2.7400e-01	5.7746e-03	5.4940e-01
1/40	1/160	2.4382e-01	4.2110e-03	2.4380e-01	4.2111e-03	2.4954e-01
rate		0.36	0.92	0.37	0.92	2.08

constraints to the DLM/FD scheme through Lagrange multipliers to guarantee the equality between the fictitious fluid velocity and structure velocity. The stability and the convergence of the DLM/FD scheme are then more ensured in contrast with a coarser mesh in Ω_2.

As a complement, considering that the ideal convergence rate of DLM/FD finite element method is about first order for velocity in H^1 norm and pressure in L^2 norm in the case of smooth solution, and one-half order lower in the case of non-smooth solution, we may also adopt a lower order stable Stokes-pair (such as the MINI element) instead of the P^2P^1 Taylor-Hood element to obtain the same convergence result but with less computational cost in our numerical experiments, which will be carried out in our next numerical study for DLM/FD method on other interface problems such as FSIs.

6. Conclusion and future work

In this paper we develop a type of DLM/FD finite element method for the Stokes interface problem, and define a mixed finite element approximation scheme to numerically solve the problem. In analyzing the problem, we look at the case where a sub-domain is completely immersed inside a larger domain, and the viscosity coefficient β_1 in the outer domain is less than the viscosity β_2 in the inner domain, as well as the case where the interface between the two fluids stretches across the whole domain. In either case, there is a jump in the viscosity coefficient across the interface. With the restriction $\beta_1 < \beta_2$, we are able to prove that the developed DLM/FD method is well posed at both the continuous and discrete levels, and the corresponding finite element method gives an optimal error estimate relative to the regularity constraints of the original model. Numerical experiments confirm the obtained theoretical results.

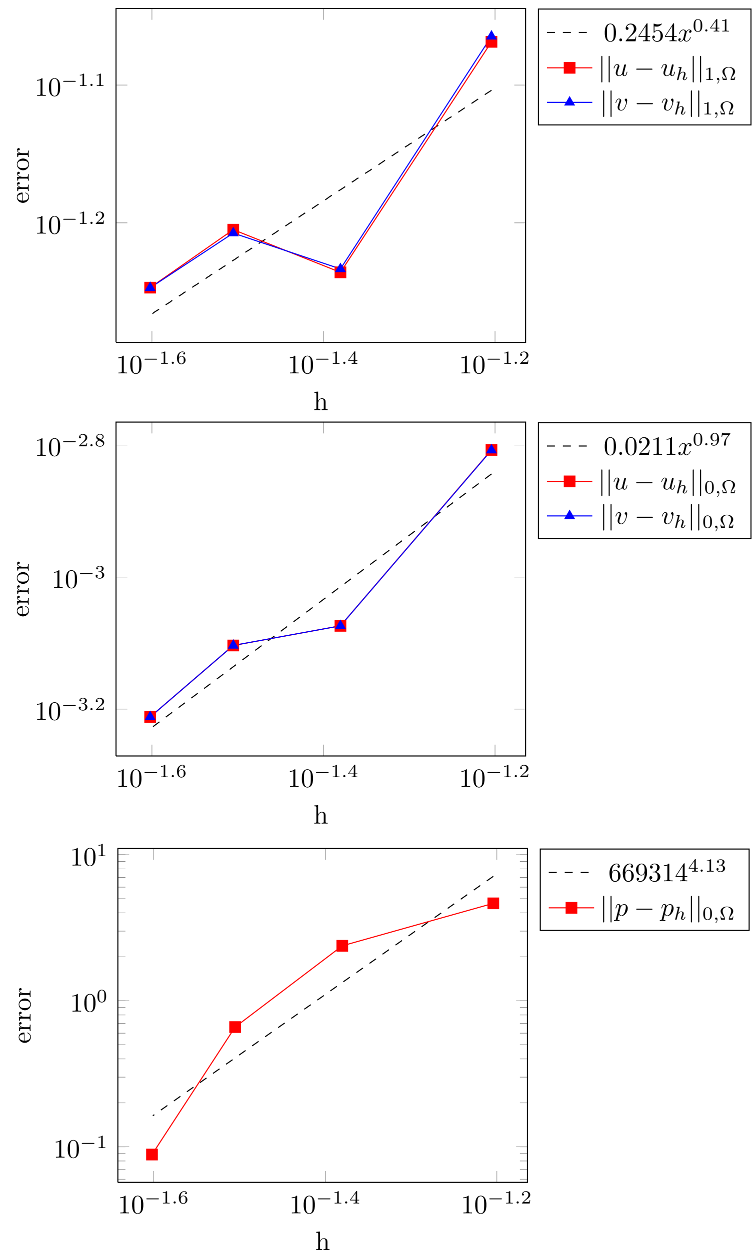


FIGURE 6. Loglog convergence plots of Example 2 ($\beta_1 = 1$, $\beta_2 = 10$, $h/H = 4$).

In future work, we will apply the DLM/FD method and analysis developed in this paper to the time dependent Stokes interface problem, with the goal of moving towards fluid structure interaction problems.

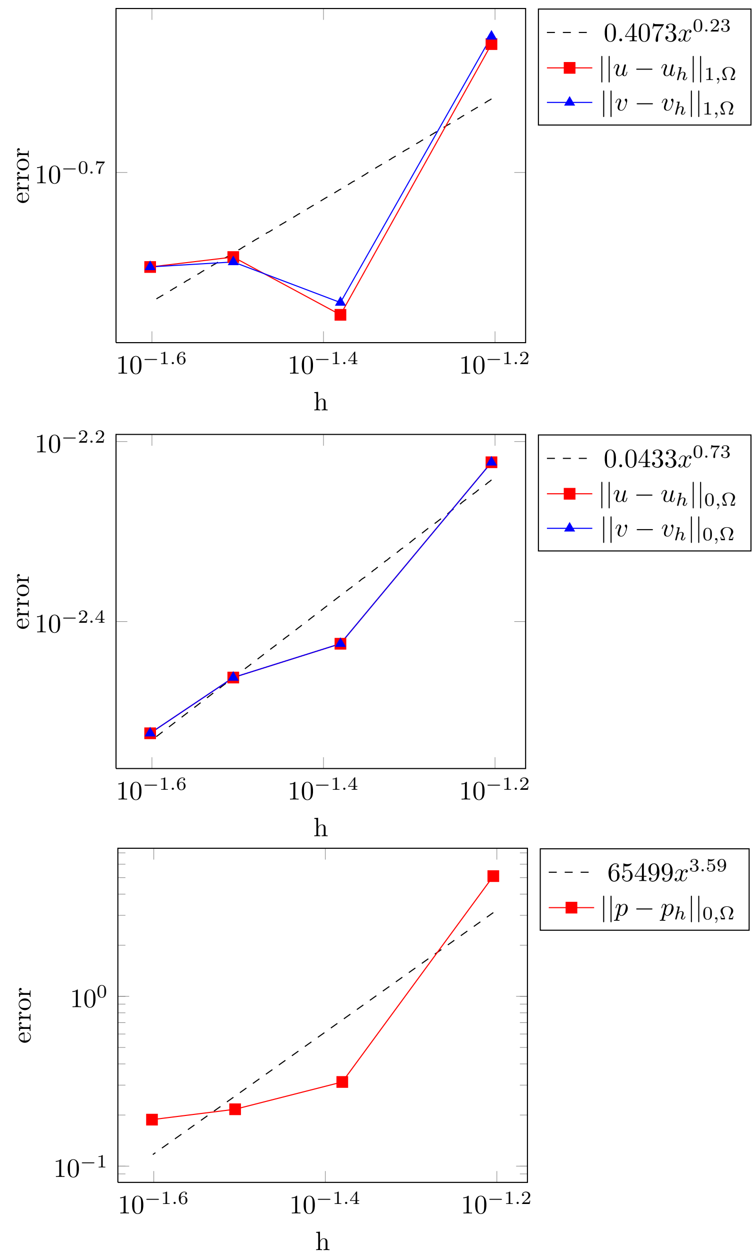


FIGURE 7. Loglog convergence plots of Example 2 ($\beta_1 = 1$, $\beta_2 = 100$, $h/H = 4$).

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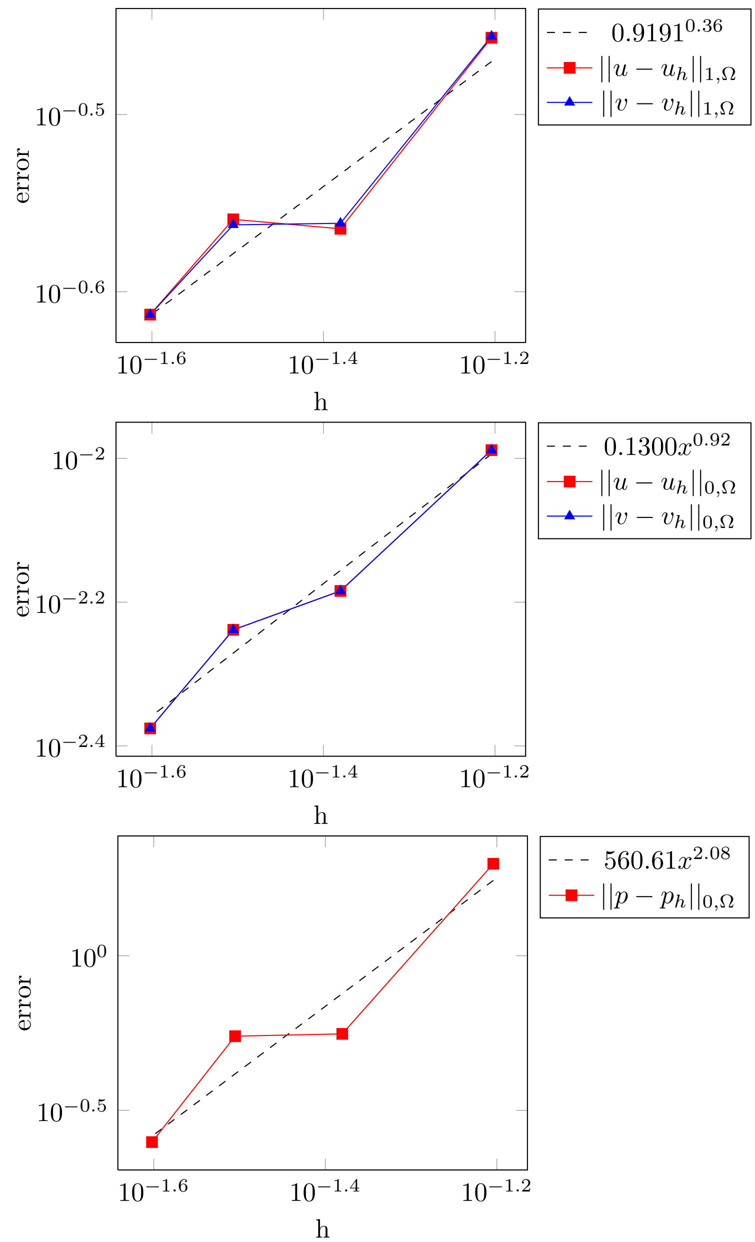

FIGURE 8. Loglog convergence plots of Example 2 ($\beta_1 = 1$, $\beta_2 = 1000$, $h/H = 4$).

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