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NONSTANDARD FINITE DIFFERENCE METHOD FOR NONLINEAR RIESZ SPACE FRACTIONAL REACTION-DIFFUSION EQUATION

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Abstract. In this paper, a modified nonstandard finite difference method for the two-dimensional Riesz space fractional reaction-diffusion equations is developed. The space fractional derivative is discretized by the shifted Grünwald-Letnikov method and the nonlinear reaction term is approximated by Taylor formula instead of Micken's. Multigrid method is introduced to reduce the computation time of the traditional Gauss-Seidal method. The stability and convergence of the nonstandard implicit difference scheme are strictly proved. The method is extended to simulate the fractional FitzHugh-Nagumo model. Numerical results are provided to verify the theoretical analysis.

Key words. Riesz fractional derivative, nonstandard finite difference method, shifted Grünwald-Letnikov method.

1. Introduction

Reaction-diffusion models are widely used in patten formulae in biology, chemistry, physics and engineering [26]. The computation of electrical wave propagation in the heart is one of the most important applications of reaction-diffusion models in physiology [28]. The simplest two-dimensional reaction-diffusion model can be described by

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathbf{K}\nabla u) + f(u), \tag{1}$$

where $\mathbf{K}$ is the diffusion coefficient and $f(u)$ is a nonlinear function representing the reaction source, u is a normalized transmembrane potential. If $f(u) = u(1-u)(u-a)$, Eq. (1) reduces to the Nagumo reaction-diffusion equation [6, 24].

Over the last few decades, fractional calculus has become famous of its ability to model anomalous diffusion phenomena, which has attracted more and more attention from researchers in various fields of science and engineering. Compared with the traditional integer order, fractional-derivative models has the advantages of describing the memory and hereditary properties of various processes. By applying the space Riesz fractional operator [8, 22] to the Eq. (1), the fractional system is given as following

$$\frac{\partial u}{\partial t} = \mathbf{K}\mathbf{R}^{\alpha} u + f(u). \tag{2}$$

Here $\mathbf{R}^{\alpha} = (R_x^{\alpha}, R_y^{\alpha}) = (\partial^{\alpha}/\partial|x|^{\alpha}, \partial^{\alpha}/\partial|y|^{\alpha})$ is the Riesz fractional order operator

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with fractional power $1 < \alpha \leq 2$. Due to the extensive applications of fractional-derivative models, it is becoming increasingly important to find the effective methods to solve them. The methods include several analytical techniques, such as the Fourier transform method, the Laplace transform method, and the Green function method [25]. Some numerical methods are also developed. For example, Meerschaert [16] obtained the solution of the one-component fractional reaction-diffusion equation by using a finite difference method; Liu [9, 10] proposed finite difference method (FDM) and alternating direction implicit (ADI) method for the two-dimension space fractional reaction-diffusion equation, and verified the stability as well as convergence; Zeng [31] developed a Crank-Nicolson ADI spectral method for fractional diffusion equations; Cai [4] proposed a high-resolution semi-discrete Hermite central-upwind scheme for multidimensional reaction-diffusion equation.

In addition to standard finite difference methods, numerical solution can also be obtained by applying the nonstandard finite difference method (NSFD) [18], which has the following advantages. Firstly, the NSFD can be applied to the structurally unstable planar dynamical system, for example, the Lotka-Volterra equations [17]. Secondly, the NSFD preserves the physical properties of the epidemic model and the numerical results are qualitatively equivalent to the real dynamics of the epidemic model [23]. Thirdly, a scheme based on NSFD is shown to be free of numerical instabilities and contrived behaviours regardless of the step-size used in the numerical simulations [7]. Finally, the NSFD has been applied to the fractional order ODE [29] and PDE [13], and the results are in good agreement with the already existing ones.

In this paper, we consider the following 2-D Riesz space fractional reaction-diffusion equation (2D-RFRDE) on a finite domain $\Omega = [a, b] \times [c, d]$ as

$$\frac{\partial u}{\partial t} = k_x \frac{\partial^{\alpha_1} u}{\partial |x|^{\alpha_1}} + k_y \frac{\partial^{\alpha_2} u}{\partial |y|^{\alpha_2}} + f(u, x, y, t) \qquad (x, y, t) \in \ \Omega \times (0, T), \tag{3}$$

with initial condition:

$$u(x, y, 0) = \phi(x, y) \qquad (x, y) \in \Omega, \tag{4}$$

and Dirichlet boundary conditions:

$$\begin{aligned} u(a, y, t) &= 0, \qquad u(b, y, t) = 0, \\ u(x, c, t) &= 0, \qquad u(x, d, t) = 0. \end{aligned} \tag{5}$$

Here $1 < \alpha_1, \alpha_2 \leq 2$, and k_x, $k_y > 0$ are diffusion coefficients. The space Riesz fractional derivative operator $\frac{\partial^{\alpha_1} u}{\partial |x|^{\alpha_1}}$ in [15] is defined as

$$\frac{\partial^{\alpha_1} u}{\partial |x|^{\alpha_1}} = -\frac{1}{2\cos(\pi\alpha_1/2)} \left(\frac{\partial^{\alpha_1} u}{\partial_+ x^{\alpha_1}} + \frac{\partial^{\alpha_1} u}{\partial_- x^{\alpha_1}} \right). \tag{6}$$

Here the left-handed $(+)$ and the right-handed $(-)$ fractional derivative are defined later. Similarly, we can define the space Riesz fractional derivative $\frac{\partial^{\alpha_2} u}{\partial |y|^{\alpha_2}}$ of order α_2 with respect to y. An improved nonstandard finite difference scheme is constructed to obtain the numerical solution of Eqs. (3)-(5).

The outline of this paper is showed as follows. In Section 2, we introduce some notations and lemmas which are needed later on. In Section 3, the nonstandard finite difference (NSFD) method for the 2D-RFRDE is proposed. The stability and convergence are discussed in Section 4. Some numerical results are given in Section 5. we draw the conclusions in Section 6.

2. Preliminaries

In this section, we introduce some notations and lemmas required later.

2.1. The Shifted Grünwald-Letnikov Derivative.

Fractional derivatives commonly include the Riemann-Liouville definition, the Grünwald-Letnikov definition and the Caputo definition. There exists a link between the first two definitions to differentiation of arbitrary real order. Let us suppose that the function $f(x)$ is $n-1$ continuously differentiable in the interval $[a,b]$, and that $f^{(n)}(x)$ is integrable in $[a,b]$. Then for every α $(0 \leq n-1 < \alpha < n)$ the Riemann-Liouville derivative exists and is equivalent to the Grünwald-Letnikov derivative [11, 12, 25].

The relationship between the Riemann-Liouville and Grünwald-Letnikov definitions is important for the numerical approximation of fractional-order differential equations and formulation of applied problems. Generally, the Riemann-Liouville definitions are used in the problem formulation, and then the Grünwald-Letnikov definitions are applied to obtain the numerical solution [11]. While the Caputo definition is often used to approximate the time derivative.

The left-handed (+) and the right-handed (−) fractional derivative in Eq. (6) are the Riemann-Liouville fractional derivatives of order α [25]

$$
\begin{aligned}
(D^{\alpha}_{a+}f)(x) &= \frac{d^{\alpha}f(x)}{d_{+}x^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(\xi)}{(x-\xi)^{\alpha+1-n}}d\xi,\\
(D^{\alpha}_{b-}f)(x) &= \frac{d^{\alpha}f(x)}{d_{-}x^{\alpha}} = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b \frac{f(\xi)}{(\xi-x)^{\alpha+1-n}}d\xi,
\end{aligned}
\tag{7}
$$

where n is an integer such that $n-1 < \alpha \leq n$ and $\Gamma(\cdot)$ is the gamma function. If $\alpha = m$ and m is an integer, then Eq. (7) reduce to the standard integer derivatives

$$
(D^{m}_{a+}f)(x) = \frac{d^m f(x)}{dx^m}, \qquad (D^{m}_{b-}f)(x) = (-1)^m\frac{d^m f(x)}{dx^m} = \frac{d^m f(x)}{d(-x)^m}.
$$

The Grünwald-Letnikov definitions for the left-handed and the right-handed fractional derivative are defined by

$$
\begin{aligned}
\frac{d^{\alpha}f(x)}{d_{+}x^{\alpha}} &= \lim_{M_{+}\to\infty}\frac{1}{{h_{+}}^{\alpha}}\sum_{k=0}^{M_{+}} g_{\alpha}^{(k)} f(x-kh),\\
\frac{d^{\alpha}f(x)}{d_{-}x^{\alpha}} &= \lim_{M_{-}\to\infty}\frac{1}{{h_{-}}^{\alpha}}\sum_{k=0}^{M_{-}} g_{\alpha}^{(k)} f(x+kh),
\end{aligned}
\tag{8}
$$

where M_{+}, M_{-} are positive integers, $h_{+} = (x-a)/M_{+}$ and $h_{-} = (b-x)/M_{-}$. In this paper, we introduced the shifted Grünwald-Letnikov schemes [14, 15] to discretize the space fractional derivative:

$$
\begin{aligned}
\frac{d^{\alpha}f(x)}{d_{+}x^{\alpha}} &= \lim_{M_{+}\to\infty}\frac{1}{{h_{+}}^{\alpha}}\sum_{k=0}^{M_{+}+p} g_{\alpha}^{(k)} f(x-(k-p)h),\\
\frac{d^{\alpha}f(x)}{d_{-}x^{\alpha}} &= \lim_{M_{-}\to\infty}\frac{1}{{h_{-}}^{\alpha}}\sum_{k=0}^{M_{-}+p} g_{\alpha}^{(k)} f(x+(k-p)h),
\end{aligned}
\tag{9}
$$

where p is a nonnegative integer and can be determined by the Theorem 2.4 in [15].

The normalized Grünwald weights are given by

$$g_\alpha^0 = 1, g_\alpha^{(k)} = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} = -g_\alpha^{(k-1)} \frac{\alpha - k + 1}{k} \qquad (k = 1, 2, 3, ...).$$

Lemma 2.1. *The coefficients* $g_\alpha^{(k)}$ *satisfy that*

(1) $g_\alpha^0 = 1, g_\alpha^{(1)} = -\alpha < 0$, *and* $g_\alpha^{(k)} > 0$ $(k \neq 1)$,
(2) $\sum_{k=0}^{\infty} g_\alpha^{(k)} = 0$, *and* $\sum_{k=0}^{l} g_\alpha^{(k)} < 0$ $(l = 1, 2, ...)$.

Proof of Lemma 2.1 is similar to [32]. The analytic definitions in Eq. (7) are often used in the formulation of the fractional PDE (FPDE), while the Grünwald-Letnikov definitions in Eq. (8) are used to obtain the discrete scheme of the FPDE. The shifted Grünwald estimation in Eq. (9) generally provides a more accurate approximation than the standard (unshifted) Grünwald estimate in Eq. (8).

2.2. The Nonstandard Finite Difference Method (NSFDM). These rules can provide guidance for the construction of nonstandard finite difference models of differential equations. There are two important points related to the application of these rules. Firstly, for a given differential equation, the rules generally permit a number of nonstandard schemes. In other words, at the present time, uniqueness does not exist in the determination of nonstandard schemes. Secondly, these schemes give difference equations which are superior to conventional ones for the purpose of providing numerical solutions.

Consider a first-order scalar equation of $\frac{du}{dt} = f(u)$. The corresponding nonstandard scheme is taken to be

$$\frac{u_{k+1} - u_k}{\varphi(\tau)} = f(u_k, u_{k+1}), \tag{10}$$

where φ is given by

$$\varphi(\tau, R^*) = \frac{1 - e^{-R^*\tau}}{R^*}. \tag{11}$$

Here τ is the time step, the value of R^* is determined as follows. Firstly, calculate the fixed-points of $f(\bar{u}) = 0$, assume it has $I-$real solutions and denote them by $\{\bar{u}^{(i)}; i = 1, 2, \ldots, I\}$. R^* is defined by

$$R^* = \max\{|R_i|\,; i = 1, 2, \ldots, I\}, \tag{12}$$

where $R_i = f'(\bar{u}^i)$. Note that $\varphi(\tau, R^*)$ has the properties

$$\varphi = \tau - R^*\tau^2 + O({R^*}^2\tau^3). \tag{13}$$

This is in agreement with the Rule 2. In order to satisfy the Rule 3, the nonlinear term in the right hand side of Eq. (1) must be replaced by nonlocal discrete form, such as

$$u^2 \to u_{k+1}u_k, u^3 \to u_{k+1}(u_k)^2. \tag{14}$$

Another nonlocal discrete approximations for these nonlinear terms were given by Micken et al. [1, 19, 27]. In this paper, the nonlocal discrete approximation of the nonlinear term $f(u, x, y, t)$ were obtained using the Taylor formula [30], which has advantage of specific extension compared with the Mincken's while satisfying the Rule 3.

3. The Discrete Scheme of The 2D-RFRDE

In this section, we present the details of NSFDM. In order to obtain the numerical discrete scheme, the space domain $\Omega = [0, D_x] \times [0, D_y]$ and the time interval $[0, T]$ are divided as following. For integers m_1 and m_2, let $h_x = D_x/m_1$ and $h_y = D_y/m_2$ be the spatial grid size in the x-direction and y-direction, respectively. Then $x_i = ih_x$ $(i = 0, 1, ..., m_1)$ and $y_j = jh_y$ $(j = 0, 1, ..., m_2)$. Let $\tau = T/n$ be the time step, then $t_k = k\tau$ $(k = 0, 1, ..., n)$. Define $u^k_{i,j}$ as the numerical approximation to $u(x_i, y_j, t_k)$. The initial conditions are $u^0_{i,j} = \phi(x_i, y_j)$.

To approximate the first order time derivative and satisfy the Rule 2, we have

$$\frac{du(t)}{dt} \to \frac{u^{k+1} - u^k}{\varphi(\tau)}. \tag{15}$$

Here the denominator function $\varphi(\tau)$ can be determined by Eqs. (12) in Section 2. Here we choose

$$\varphi(\tau) = 1 - e^{-\tau}. \tag{16}$$

By using the shifted Grünwald-Letnikov schemes in Section 2 on the finite domain Ω with $p = 1$, the Riesz fractional derivative can be discretized as

$$\begin{aligned} \frac{\partial^{\alpha_1} u}{\partial_+ x^{\alpha_1}}|_{(x_i,y_j,t_n)} &= \frac{1}{(h_x)^{\alpha_1}} \sum_{l=0}^{i+1} g^{(l)}_{\alpha_1} u(x_{i-l+1}, y_j, t_n) + O(h_x), \\ \frac{\partial^{\alpha_1} u}{\partial_- x^{\alpha_1}}|_{(x_i,y_j,t_n)} &= \frac{1}{(h_x)^{\alpha_1}} \sum_{l=0}^{m_1-i+1} g^{(l)}_{\alpha_1} u(x_{i+l-1}, y_j, t_n) + O(h_x). \end{aligned} \tag{17}$$

The similar expressions can be obtained in the y-direction.

Here, the Taylor formula is used to approximate the nonlinear term. Assuming $f(u) \in C^1(D)$ (D is a proper close domain), we have the following expression

$$\begin{aligned} f(u(x_i, y_j, t_{n+1})) = f(u(x_i, y_j, t_n)) + \\ f'(u(x_i, y_j, t_n))(u(x_i, y_j, t_{n+1}) - u(x_i, y_j, t_n)) + O(\tau^2). \end{aligned} \tag{18}$$

Substituting Eqs. (15)-(18) into Eq. (2), we obtain the discrete form of Eq. (2):

$$\begin{aligned} &\frac{u(x_i, y_j, t_{n+1}) - u(x_i, y_j, t_n)}{\varphi(\tau)} \\ &= -r_1 \left(\sum_{l=0}^{i+1} g^{(l)}_{\alpha_1} u(x_{i-l+1}, y_j, t_{n+1}) + \sum_{l=0}^{m_1-i+1} g^{(l)}_{\alpha_1} u(x_{i+l-1}, y_j, t_{n+1}) \right) \\ &- r_2 \left(\sum_{l=0}^{j+1} g^{(l)}_{\alpha_2} u(x_i, y_{j-l+1}, t_{n+1}) + \sum_{l=0}^{m_2-j+1} g^{(l)}_{\alpha_2} u(x_i, y_{j+l-1}, t_{n+1}) \right) \\ &+ f'(u(x_i, y_j, t_n), x_i, y_j, t_n) \left[u(x_i, y_j, t_{n+1}) - u(x_i, y_j, t_n) \right] \\ &+ f(u(x_i, y_j, t_n), x_i, y_j, t_n) + R_{i,j,n+1}, \end{aligned} \tag{19}$$

where $c_\alpha = 1/[2\cos(\pi\alpha/2)] < 0$, $r_1 = k_x c_{\alpha_1}/(h_x)^{\alpha_1}$, $r_2 = k_y c_{\alpha_2}/(h_y)^{\alpha_2}$, and $R_{i,j,n+1} = O(h_x + h_y + \tau)$. Hence, we have the nonstandard implicit difference scheme

$$
\begin{aligned}
u_{i,j}^{n+1} - \varphi(\tau)f'(u_{i,j}^n)u_{i,j}^{n+1} + \varphi(\tau)r_1 & \left(\sum_{l=0}^{i+1} g_{\alpha_1}^{(l)} u_{i-l+1,j}^{n+1} + \sum_{l=0}^{m_1-i+1} g_{\alpha_1}^{(l)} u_{i+l-1,j}^{n+1} \right) \\
+ \varphi(\tau)r_2 & \left(\sum_{l=0}^{j+1} g_{\alpha_2}^{(l)} u_{i,j-l+1}^{n+1} + \sum_{l=0}^{m_2-j+1} g_{\alpha_2}^{(l)} u_{i,j+l-1}^{n+1} \right) \\
&= u_{i,j}^n - \varphi(\tau)f'(u_{i,j}^n)u_{i,j}^n + \varphi(\tau)f_{i,j}^n,
\end{aligned} \tag{20}
$$

with the initial conditions

$$
\begin{aligned}
u_{i,j}^0 &= \phi_{i,j} = \phi(x_i, y_j), \\
u_{0,j}^n = u_{m_1,j}^n &= u_{i,0}^n = u_{i,m_2}^n = 0.
\end{aligned} \tag{21}
$$

Eqs. (20)-(21) can be rewritten in the matrix-vector form

$$
A\mathbf{u}^n = \mathbf{b}^n, \tag{22}
$$

where $\mathbf{u}^n = (u_{1,1}^n, u_{1,2}^n, ..., u_{1,m_2-1}^n, u_{2,1}^n, ..., u_{m_1-1,m_2-1}^n)$, and $\mathbf{b}^n$ is a $(m_1-1)(m_2-1)$ dimensional column vector. The coefficient matrix A is a $(m_1-1)(m_2-1) \times (m_1-1)(m_2-1)$ dense matrix because the fractional derivative is nonlocal. In this paper, the multigrid method [3] is introduced to solve the linear systerm (23) to reduce the computation time.

4. Stability and Convergence

In this section, we analyze the stability and convergence of the nonstandard implicit difference scheme, respectively. We strictly prove that the scheme (20)-(21) is stable and convergent.

Let

$$
\begin{aligned}
L_1 u_{i,j}^{n+1} = u_{i,j}^{n+1} + \varphi(\tau)r_1 & \left(\sum_{l=0}^{i+1} g_{\alpha_1}^{(l)} u_{i-l+1,j}^{n+1} + \sum_{l=0}^{m_1-i+1} g_{\alpha_1}^{(l)} u_{i+l-1,j}^{n+1} \right) \\
+ \varphi(\tau)r_2 & \left(\sum_{l=0}^{j+1} g_{\alpha_2}^{(l)} u_{i,j-l+1}^{n+1} + \sum_{l=0}^{m_2-j+1} g_{\alpha_2}^{(l)} u_{i,j+l-1}^{n+1} \right).
\end{aligned}
$$

Hence, Eq. (20) can be rewritten as

$$
\begin{aligned}
L_1 u_{i,j}^{n+1} - \varphi(\tau)f'(u_{i,j}^n)u_{i,j}^{n+1} = \\
u_{i,j}^n - \varphi(\tau)f'(u_{i,j}^n)u_{i,j}^n + \varphi(\tau)f(u_{i,j}^n), \qquad n = 0, 1, 2, ..., N.
\end{aligned} \tag{23}
$$

Let $\mathbf{u}^n = [u_{1,1}^n, u_{2,1}^n, ..., u_{m_1-1,m_2-1}^n]^T$ and $\|\mathbf{u}^n\|_\infty = \left|u_{i_0,j_0}^n\right|$ $= \underset{1\le i\le m_1-1, 1\le j\le m_2-1}{max} \left|u_{i,j}^n\right|$, we have the following theorems.

Theorem 4.1 (Stability). *Suppose that $u_{i,j}^n$ is the numerical solution computed by Eqs. (20)-(21), $f(u) \in C^1(D)$ with $H_1 = \max_{u\in\Omega} |f(u)|$, $H_2 = \max_{u\in\Omega} |f'(u)|$, and $r\beta = min\{r_1\beta_x, r_2\beta_y\}$. Assume $\tau < 1/2(H_2 - 2r\beta)$ with $2r\beta < H_2$. The nonstandard implicit difference scheme defined by (20)-(21) is stable:*

$$
\|\mathbf{u}^n\|_\infty \le C\left(\left\|\mathbf{u}^0\right\|_\infty + TH_1\right) \qquad (n = 0, 1, 2, ..., N),
$$

where C is a positive number independent of h_x, h_y, and τ.

Proof. From Lemma 2.1, $\sum_{l=0}^{n} g_{\alpha}^{(l)} < 0$ and $\sum_{l=0}^{n} g_{\alpha}^{(l)} \leq \sum_{l=0}^{m} g_{\alpha}^{(l)}$ $(n \leq m)$. Since $r_1, r_2 < 0$, we have

$$r_1\left(\sum_{l=0}^{i+1} g_{\alpha_1}^{(l)} + \sum_{l=0}^{m_1-i+1} g_{\alpha_1}^{(l)}\right) \geq 2r_1 \sum_{l=0}^{m_1} g_{\alpha_1}^{(l)} = r_1\beta_x,$$
$$r_2\left(\sum_{l=0}^{j+1} g_{\alpha_2}^{(l)} + \sum_{l=0}^{m_2-j+1} g_{\alpha_2}^{(l)}\right) \geq 2r_2 \sum_{l=0}^{m_2} g_{\alpha_2}^{(l)} = r_2\beta_y.$$

By using $f(u) \in C^1(D)$ with $|f(u)| \leq H_1$, $|f'(u)| \leq H_2$, and $r\beta = min\{r_1\beta_x, r_2\beta_y\}$, we deduce

$$\begin{aligned}
&\left|u_{i_0,j_0}^n\right| + \varphi(\tau)\left(2r\beta - H_2\right)\left|u_{i_0,j_0}^n\right| \\
&= \left|\left[1+\varphi(\tau)2r\beta\right]u_{i_0,j_0}^n\right| - \varphi(\tau)H_2\left|u_{i_0,j_0}^n\right| \\
&\leq \left|u_{i_0,j_0}^n + \varphi(\tau)\left(2r\beta - H_2\right)u_{i_0,j_0}^n\right| \\
&\leq \left|u_{i_0,j_0}^n + \varphi(\tau)(r_1\beta_x + r_2\beta_y)u_{i_0,j_0}^n - \varphi(\tau)H_2 u_{i_0,j_0}^n\right| \\
&\leq \left|L_1 u_{i_0,j_0}^n - \varphi(\tau)f'(u_{i_0,j_0}^{n-1})u_{i_0,j_0}^n\right| \\
&= \left|u_{i_0,j_0}^{n-1} - \varphi(\tau)f'(u_{i_0,j_0}^{n-1})u_{i_0,j_0}^{n-1} + \varphi(\tau)f(u_{i_0,j_0}^{n-1})\right|.
\end{aligned}$$

If $2r\beta \geq H_2$, we can see that

$$\left|u_{i_0,j_0}^n\right| \leq C\left(\left|u_{i_0,j_0}^{n-1}\right| + \varphi(\tau)\left|f'(u_{i_0,j_0}^{n-1})u_{i_0,j_0}^{n-1}\right| + \varphi(\tau)\left|f(u_{i_0,j_0}^{n-1})\right|\right).$$

If $2r\beta < H_2$, the inequality also holds for $\tau < 1/2(H_2 - 2r\beta)$, because of $\varphi(\tau) < \tau$.
Hence, we obtain

$$\begin{aligned}
\|\mathbf{u}^n\|_\infty &\leq C\left(\left|u_{i_0,j_0}^{n-1}\right| + \varphi(\tau)\left|f'(u_{i_0,j_0}^{n-1})u_{i_0,j_0}^{n-1}\right| + \varphi(\tau)\left|f(u_{i_0,j_0}^{n-1})\right|\right) \\
&\leq C[1+\varphi(\tau)H_2]\left\|\mathbf{u}^{n-1}\right\|_\infty + \varphi(\tau)H_1 \\
&\leq C[1+\varphi(\tau)H_2]^n\left\|\mathbf{u}^0\right\|_\infty + C\sum_{k=0}^{n-1}\left[1+\varphi(\tau)H_2\right]^k\varphi(\tau)H_1 \\
&\leq Ce^{n\varphi(\tau)H_2}\left\|\mathbf{u}^0\right\|_\infty + Cn\varphi(\tau)H_1[1+\varphi(\tau)H_2]^n \\
&\leq Ce^{TH_2}\left(\left\|\mathbf{u}^0\right\|_\infty + TH_1\right) \\
&\leq C\left(\left\|\mathbf{u}^0\right\|_\infty + TH_1\right).
\end{aligned}$$

□

The convergence of the nonstandard implicit difference scheme is verified as follows. We first introduce the theoretical concept about locally Lipschitz continuity [2].

Theorem 4.2 (Lagrange mean value theorem)**.** *Suppose f is a function defined on a closed interval $[a, b]$ (with $a < b$), such that the following two conditions hold:*

(1) *f is a continuous function on the closed interval $[a, b]$,*
(2) *f is a differentiable function on the open interval (a, b).*

Then, there exists ξ in the open interval (a, b) such that

$$f(b) - f(a) = f'(\xi)(b-a).$$

Assume $\tilde{u}_{i,j}^n = u(x_i, y_j, t_n)$ is the exact solution of Eq. (20) at mesh point (x_i, y_j, t_n), and $u_{i,j}^n$ is the numerical solution of Eqs. (20)-(21). Define $\eta_{i,j}^n =$

$u(x_i, y_j, t_n) - u_{i,j}^n$ and $\mathbf{y}^n = [\eta_{1,1}^n, \eta_{2,1}^n, ..., \eta_{m_1-1,m_2-1}^n]^T$. Let $\|\mathbf{y}^n\|_\infty = |\eta_{i_0,j_0}^n| = \max_{1\le i\le m_1-1, 1\le j\le m_2-1} |\eta_{i,j}^n|$, we obtain the following result.

Theorem 4.3 (Convergence)**.** *Suppose that $f(u) \in C^1(D)$ with $H_1 = \max_{u\in\Omega} |f(u)|$, $H_2 = \max_{u\in\Omega} |f'(u)|$, and $r\beta = \min\{r_1\beta_x, r_2\beta_y\}$.*

If we further assume $\tau < 1/2(H_2 - 2r\beta)$ with $2r\beta < H_2$, then there exists positive constants C_1 and C_2 independent of h_x, h_y, and τ, such that

$$\left|u(x_i, y_j, t_n) - u_{i,j}^n\right| \le C_1 \tau e^{(1+2\tau H_2)} + C_2(\tau + h_x + h_y)$$

for all i, j, n.

Proof. Since $\tilde{u}_{i,j}^n$ satisfies

$$L_1 \tilde{u}_{i,j}^n - \varphi(\tau) f'(\tilde{u}_{i,j}^{n-1}) \tilde{u}_{i,j}^n = \tilde{u}_{i,j}^{n-1} - \varphi(\tau) f'(\tilde{u}_{i,j}^{n-1}) \tilde{u}_{i,j}^{n-1} + \varphi(\tau) f(\tilde{u}_{i,j}^{n-1}) + \varphi(\tau) R_{i,j}^n, \qquad n = 0, 1, 2, ..., N.$$

And $u_{i,j}^n$ is the numerical solution of Eq. (23), the numerical error $\eta_{i,j}^n$ satisfies

$$\begin{aligned} &L_1 \eta_{i,j}^n - \varphi(\tau)\left[f'(\tilde{u}_{i,j}^{n-1})\tilde{u}_{i,j}^n - f'(u_{i,j}^{n-1})u_{i,j}^n\right] \\ &= \eta_{i,j}^{n-1} - \varphi(\tau)\left[f'(\tilde{u}_{i,j}^{n-1})\tilde{u}_{i,j}^{n-1} - f'(u_{i,j}^{n-1})u_{i,j}^{n-1}\right] \\ &\quad + \varphi(\tau)\left[f(\tilde{u}_{i,j}^{n-1}) - f(u_{i,j}^{n-1})\right] + \varphi(\tau) R_{i,j}^n. \end{aligned} \tag{24}$$

By using the triangle inequality, it yields

$$\begin{aligned} \left|f'(\tilde{u}_{i,j}^{n-1})\tilde{u}_{i,j}^n - f'(u_{i,j}^{n-1})u_{i,j}^n\right| &\le H_2 \left|\eta_{i,j}^n\right| + 2H_2 \left|u_{i,j}^n\right|, \\ \left|f'(\tilde{u}_{i,j}^{n-1})\tilde{u}_{i,j}^{n-1} - f'(u_{i,j}^{n-1})u_{i,j}^{n-1}\right| &\le 2H_2 \left|u_{i,j}^{n-1}\right| + H_2 \left|\eta_{i,j}^{n-1}\right|. \end{aligned}$$

Therefore, the left-side term of Eq. (24) can be written as

$$\begin{aligned} &\left|L_1 \eta_{i,j}^n - \varphi(\tau)\left[f'(\tilde{u}_{i,j}^{n-1})\tilde{u}_{i,j}^n - f'(u_{i,j}^{n-1})u_{i,j}^n\right]\right| \\ &\ge \left|L_1 \eta_{i,j}^n\right| - \varphi(\tau)\left|\left[f'(\tilde{u}_{i,j}^{n-1})\tilde{u}_{i,j}^n - f'(u_{i,j}^{n-1})u_{i,j}^n\right]\right| \\ &\ge \left|\eta_{i,j}^n + \varphi(\tau)(r_1\beta_x + r_2\beta_y)\eta_{i,j}^n\right| - \varphi(\tau)\left[H_2\left|\eta_{i,j}^n\right| + 2H_2\left|u_{i,j}^n\right|\right] \\ &\ge \left|\eta_{i,j}^n + 2\varphi(\tau) r\beta \eta_{i,j}^n\right| - \varphi(\tau) H_2 \left|\eta_{i,j}^n\right| - 2\varphi(\tau) H_2 \left|u_{i,j}^n\right| \\ &= \left|\eta_{i,j}^n\right| + \varphi(\tau)(2r\beta - H_2)\left|\eta_{i,j}^n\right| - 2\varphi(\tau) H_2 \left|u_{i,j}^n\right|. \end{aligned}$$

From Theorem 4.2, there exists a $\lambda \in D$ such that the right-side term of Eq. (24) satisfies

$$\begin{aligned} &\left|\eta_{i,j}^{n-1} - \varphi(\tau)\left[f'(\tilde{u}_{i,j}^{n-1})\tilde{u}_{i,j}^{n-1} - f'(u_{i,j}^{n-1})u_{i,j}^{n-1}\right]\right. \\ &\left.+\varphi(\tau)\left[f(\tilde{u}_{i,j}^{n-1}) - f(u_{i,j}^{n-1})\right] + \varphi(\tau) R_{i,j}^n\right| \\ &\le \left|\eta_{i,j}^{n-1}\right| + \varphi(\tau)\left|f'(\tilde{u}_{i,j}^{n-1})\tilde{u}_{i,j}^{n-1} - f'(u_{i,j}^{n-1})u_{i,j}^{n-1}\right| \\ &+ \varphi(\tau)\left|f'(\lambda)\eta_{i,j}^{n-1}\right| + \varphi(\tau) C^*(h_x + h_y + \tau) \\ &\le [1 + 2\varphi(\tau) H_2]\left|\eta_{i,j}^{n-1}\right| + 2\varphi(\tau) H_2 \left|u_{i,j}^{n-1}\right| + \varphi(\tau) C^*(h_x + h_y + \tau). \end{aligned}$$

Thus, by induction

$$\begin{aligned} &\left|\eta_{i,j}^n\right| + \varphi(\tau)(2r\beta - H_2)\left|\eta_{i,j}^n\right| - 2\varphi(\tau) H_2 \left|u_{i,j}^n\right| \\ &\quad \le [1 + 2\varphi(\tau) H_2]\left|\eta_{i,j}^{n-1}\right| + 2\varphi(\tau) H_2 \left|u_{i,j}^{n-1}\right| + \varphi(\tau) C^*(h_x + h_y + \tau). \end{aligned}$$

If $2r\beta \ge H_2$, we have

$$\begin{aligned} \left|\eta_{i_0,j_0}^n\right| &\le C[1 + 2\varphi(\tau) H_2]\left|\eta_{i_0,j_0}^{n-1}\right| + 2C\varphi(\tau) H_2 \left(\|\mathbf{u}^n\|_\infty + \left\|\mathbf{u}^{n-1}\right\|_\infty\right) \\ &\quad + \varphi(\tau) C^*(h_x + h_y + \tau). \end{aligned}$$

If $2r\beta < H_2$, the inequality also holds for $\tau < 1/2(H_2 - 2r\beta)$. By applying the result of Theorem 4.1, we have

$$\begin{aligned}\|\mathbf{y}^n\|_\infty &\le C\left[1+2\varphi(\tau)H_2\right]\left\|\mathbf{y}^{n-1}\right\|_\infty + \varphi(\tau)H_0 + \varphi(\tau)C^*(h_x+h_y+\tau)\\ &\le C[1+2\varphi(\tau)H_2]^n\left\|\mathbf{y}^0\right\|_\infty + C\varphi(\tau)H_0\sum_{k=0}^{n-1}[1+\varphi(\tau)(H_2{+}L_{\max})]^k\\ &+\sum_{k=0}^{n-1}[1+2\varphi(\tau)H_2]^k C^*\varphi(\tau)(\tau+h_x+h_y).\end{aligned}$$

Here, $H_0 = 4C^2H_2\left(\left\|\mathbf{u}^0\right\|_\infty + TH_1\right)$. Since $\mathbf{y}^0 = \mathbf{0}$ (because the initial conditions at the grid points match for the exact and the discretized equation), we deduce that

$$\begin{aligned}\|\mathbf{y}^n\|_\infty &\le C\varphi(\tau)H_0e^{[1+2\varphi(\tau)H_2]} + [1+2\varphi(\tau)H_2]^n n\tau C^*(\tau+h_x+h_y)\\ &\le CH_0\tau e^{[1+2\varphi(\tau)H_2]} + e^{2n\tau H_2}n\tau C^*(\tau+h_x+h_y)\\ &\le C_1\tau e^{(1+2\tau H_2)} + e^{2TH_2}TC^*(\tau+h_x+h_y)\\ &= C_1\tau e^{(1+2\tau H_2)} + C_2(\tau+h_x+h_y).\end{aligned}$$

□

Note that $\lim_{\tau\to 0}\tau e^{(1+2\tau H_2)} = 0$. Therefore, it can be concluded that if h_x, h_y and τ approach to zero, then $u_{i,j}^n \to u(x,y,t)$. Hence, the convergence of the nonstandard implicit difference scheme defined by Eqs. (20)-(21) has been proved.

5. Numerical Examples

We now present some numerical examples to verify the efficiency of the nonstandard implicit difference scheme proposed in Section 3.

Example 5.1. *Consider the following 2-D fractional reaction-diffusion model with nonlinear reaction term:*

$$\left\{\begin{array}{l}\frac{\partial u}{\partial t} = k_x\frac{\partial^{\alpha_1}u}{\partial|x|^{\alpha_1}} + k_y\frac{\partial^{\alpha_2}u}{\partial|y|^{\alpha_2}} + f(u) + f(x,y,t), (x,y,t)\in\Omega\times(0,T]\\ u(x,y,0){=}10x^2(1-x)^2y^2(1-y)^2, (x,y)\in\Omega\\ u(x,y,t)=0, (x,y,t)\in\partial\Omega\times(0,T]\end{array}\right. \tag{25}$$

where $\Omega = [0,1]\times[0,1]$, $f(u) = -u(1+u)$. Here

$$\begin{aligned}f(x,y,t) &= 100e^{-2t}x^4(1-x)^4y^4(1-y)^4\\ &\quad + 10k_xc_{\alpha_1}e^{-t}y^2(1-y)^2\left[g(x,\alpha_1)+g(1-x,\alpha_1)\right]\\ &\quad + 10k_yc_{\alpha_2}e^{-t}x^2(1-x)^2\left[g(y,\alpha_2)+g(1-y,\alpha_2)\right],\end{aligned} \tag{26}$$

with

$$g(x,\alpha) = \frac{-24}{\Gamma(5-\alpha)}x^{4-\alpha} + \frac{12}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{-2}{\Gamma(3-\alpha)}x^{2-\alpha}. \tag{27}$$

The exact solution of Eq. (25) is

$$u(x,y,t) = 10e^{-t}x^2(1-x)^2y^2(1-y)^2.$$

TABLE 1. Errors and space convergence orders of Example 5.1 with $\alpha_1 = \alpha_2 = 1.5$, $\tau = 10^{-3}$.

N	L^2 error	Order	L^∞ error	Order
10	1.7932e-03		5.6501e-03	
30	6.2142e-04	0.9646	1.8752e-03	1.0040
50	3.5843e-04	1.0772	1.1515e-03	0.9546
80	2.2342e-04	1.0057	7.2909e-04	0.9724

TABLE 2. Errors and space convergence orders of Example 5.1 with $\alpha_1 = 1.8, \alpha_2 = 1.5$, $\tau = 10^{-4}$.

N	L^2 error	Order	L^∞ error	Order
10	2.4694e-03		4.7862e-03	
30	6.8568e-04	1.1663	1.7785e-03	0.9011
50	4.1951e-04	0.9618	1.0908e-03	0.9570
80	2.7004e-04	0.9617	6.5517e-04	1.0846

TABLE 3. Errors and temporal convergence orders of Example 5.1 with $\alpha_1 = \alpha_2 = 1.6$, $N = 80$.

τ	L^2 error	Order	L^∞ error	Order
1/10	1.4374e-03		4.7099e-03	
1/30	5.0477e-04	0.9820	1.7446e-03	0.9320
1/50	3.0985e-04	0.9679	1.0677e-03	0.9739
1/80	2.0064e-04	0.9320	6.4992e-04	1.0647

In this paper, the convergence order of norm $\|\cdot\|_{L^2}$ (or norm $\|\cdot\|_{L^\infty}$) both in space and time is defined in [31] as

$$order = \begin{cases} \frac{\log(\|\eta(\varphi(\tau_1),N,t_n)\|/\|\eta(\varphi(\tau_2),N,t_n)\|)}{\log[\varphi(\tau_1)/\varphi(\tau_2)]}, & in \quad time, \\ \frac{log(\|\eta(\varphi(\tau),N_1,t_n)\|/\|\eta(\varphi(\tau),N_2,t_n)\|)}{\log(N_1/N_2)}, & in \quad space, \end{cases} \tag{28}$$

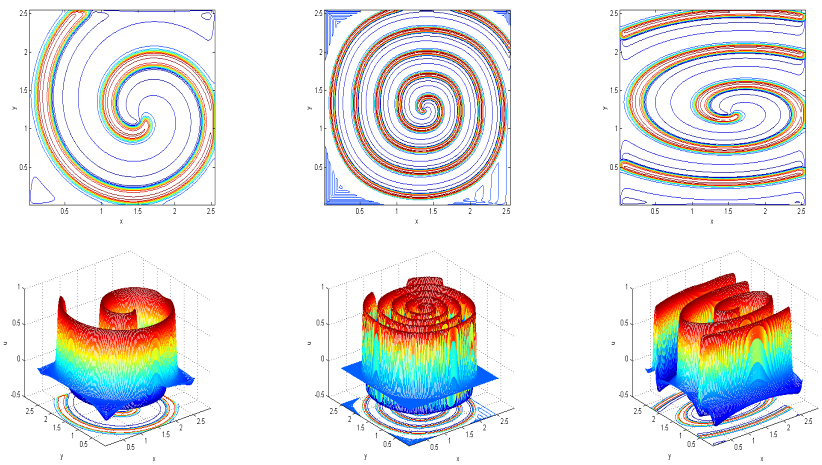
where $\eta(\varphi(\tau), N, t_n) = u(x, y, n\varphi(\tau)) - u_N^n$. Let $k_x = k_y = 1$, $h_x = h_y = h = 1/N$. We obtain the numerical solution at T=1 by applying the NSFDM with different values of α_1 and α_2. Table 1 and Table 2 summarize the space convergence orders of Example 5.1 with $\alpha_1 = \alpha_2 = 1.5$, $\tau = 10^{-3}$ and $\alpha_1 = 1.8, \alpha_2 = 1.5$, $\tau = 10^{-4}$, respectively. The time convergence order is also given in Table 3 with $\alpha_1 = \alpha_2 = 1.6$ and $N = 80$, which is in line with the theoretical analysis in Theorem 4.3.

Example 5.2. *Consider the two-dimension Riesz fractional FitzHugh-Nagumo (FHN) model*

$$\begin{cases} \frac{\partial u}{\partial t} = k_x \frac{\partial^{\alpha_1} u}{\partial |x|^{\alpha_1}} + k_y \frac{\partial^{\alpha_2} u}{\partial |y|^{\alpha_2}} + u(1-u)(u-a) - v \\ \frac{\partial v}{\partial t} = \varepsilon(\beta u - \gamma v - \delta), (x, y, t) \in \Omega \times (0, T] \end{cases} \tag{29}$$

where $\Omega = \{(x, y) : 0 \leq x \leq 2.5, 0 \leq y \leq 2.5\}$, and $a = 0.1$, $\varepsilon = 0.01$, $\beta = 0.5$, $\gamma = 1$, $\delta = 0$. The initial conditions are

$$\begin{aligned} u(x, y, 0) &= \begin{cases} 1.0, & x < 1.25, y < 1.25 \\ 0.0, & elsewhere \end{cases} \\ v(x, y, 0) &= \begin{cases} 0.1, & y \geq 1.25 \\ 0.0, & elsewhere \end{cases} \end{aligned} \tag{30}$$



(a) $k_x = k_y = 10^{-4}$ (b) $k_x = k_y = 10^{-5}$ (c) $k_x = 10^{-4}$, $k_y/k_x = 0.25$

FIGURE 1. The simulation results of the FHN model with $\alpha_1 = \alpha_2 = 2$ at $T = 1000$.

TABLE 4. CPU time of Liu's method and our method ($\alpha_1 = \alpha_2 = 1.5$, $k_x = k_y = 10^{-4}$).

CPU time (h)	Liu's method [10]	our method
Example 5.1 ($N = 20$, $\tau = 10^{-3}$, T=1)	4.27	0.01
Example 5.2 ($N = 256$, $\tau = 0.1$, T=1000)	10.75	0.52

with zero Dirichlet boundary conditions

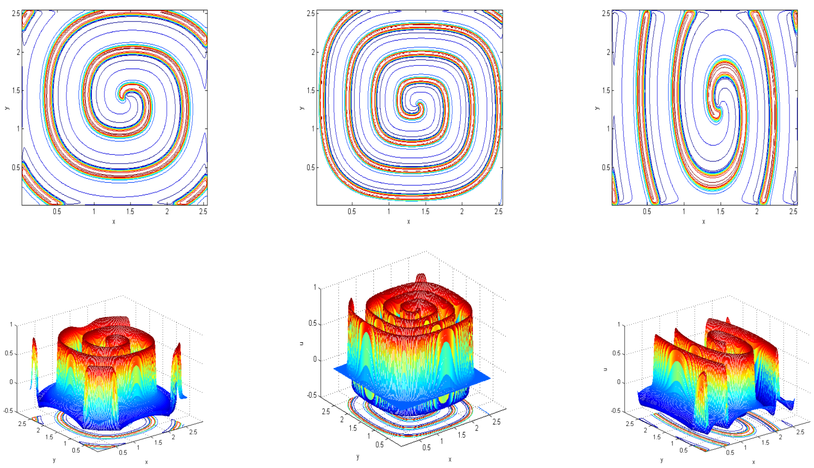
$$u(0, y, t) = u(2.5, y, t) = 0, \qquad u(x, 0, t) = u(x, 2.5, t) = 0. \tag{31}$$

Electrophysiological models of the heart describe how electrical currents flow through the heart effectively. In addition to the Riesz FHN model above, there exists cell potential model, c.f. [5].

The space domain Ω is discretized into $m_1 \times m_2 = 256 \times 256$ points, time step $\tau = 0.1$. We compute the fractional partial differential equation by using the NSFDM and solve the ODE by using the backward difference method. The simulation results at time T=1000 are shown in Fig. 1 and Fig. 2.

It is easy to derive that the spiral waves travel more slowly as the anisotropic diffusion ratios decrease from $k_x = k_y = 10^{-4}$ to $k_x = k_y = 10^{-5}$, which are shown in Fig. 1(a) and Fig. 1(b). By comparing Fig. 1(a) ($\alpha_1 = \alpha_2 = 2$), Fig. 2(a) ($\alpha_1 = \alpha_2 = 1.7$) and Fig. 2(b) ($\alpha_1 = \alpha_2 = 1.5$), we can see that the waves propagate more slowly as the fractional order decreasing when $k_x = k_y = 10^{-4}$. Results of the parameters $k_x = 10^{-4}$, $k_y/k_x = 0.25$ with $\alpha_1 = \alpha_2 = 2$ and $\alpha_2 = 2, \alpha_1 = 1.65$ with $k_x = k_y = 10^{-4}$ are shown in Fig. 1(c) and Fig. 2(c), which agree with the simulation results in [9, 10, 31].

Table 4 shows the CPU time for the numerical results of the above-mentioned problem. Compared with the SFDM with Gauss-Seidal method [10], the computational time of our NSFDM with Multigrid method is faster.



(a) $\alpha_1 = \alpha_2 = 1.7$ (b) $\alpha_1 = \alpha_2 = 1.5$ (c) $\alpha_1 = 1.65$, $\alpha_2 = 2$

FIGURE 2. The simulation results of the FHN model with $k_x = k_y = 10^{-4}$ at $T = 1000$.

6. Conclusion

In this paper, a nonstandard finite difference method for solving the 2-D Riesz space fractional reaction-diffusion equation is proposed. The implicit scheme is obtained on the basis of shifted Grünwald-Letnikov definition and Taylor formula. Multigrid method is introduced to solve proposed problems which is faster than the traditional Gauss-Seidal iterative method. We also prove that if $\tau < 1/2(H_2 - 2r\beta)$ with $2r\beta < H_2$, our NSFD scheme is stable and the convergence order is $O(h_x + h_y + \tau)$. Finally, the method is used to simulate the 2D Riesz fractional FitzHugh-Nagumo model, and stable spiral figures are obtained. The results are good proofs of the last advantage of the non-standard finite difference method mentioned in Section 1.

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