INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 5, Pages 731–744

A RELAXATION APPROACH TO DISCRETIZATION OF BOUNDARY OPTIMAL CONTROL PROBLEMS OF SEMILINEAR PARABOLIC EQUATIONS

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Abstract. We consider an optimal boundary control problem described by a semilinear parabolic partial differential equation, with control and state constraints. Since this problem may have no classical solutions, it is reformulated in the relaxed form. The relaxed control problem is discretized by using a finite element method in space and a partially implicit scheme in time, while the controls are approximated by piecewise constant relaxed controls. We first state the necessary conditions for optimality for the continuous problem and the discrete relaxed problem. Next, under appropriate assumptions, we prove that accumulation points of sequences of optimal (resp. admissible and extremal) discrete relaxed controls are optimal (resp. admissible and extremal) for the continuous relaxed problem.

Key words. Boundary optimal control, semilinear parabolic systems, state constraints, relaxed controls, discretization.

1. Introduction

It is well known that optimal control problems, without any convexity assumptions on the data, have no classical solutions in general. These problems are usually studied by considering their corresponding relaxed formulations, where at each time, the control variable is not a vector in some set but instead a probability measure on that set. Relaxation theory has been introduced, initially, in order to prove existence of optimal controls and later to derive necessary conditions for optimality. There exist an extensive literature concerning relaxation of control problems, see e.g. Warga [19], Roubíček [16], Fattorini [11] and the references therein.

In this paper we consider an optimal boundary control problem for systems governed by a semilinear parabolic partial differential equation, with control and state constraints. The problem is motivated, for example, by the control of a heat (or other) diffusion process whose source is nonlinear in the heat and temperature, with nonconvex cost and control constraint set (e.g. on-off type control). This class of problems has been extensively studied by several authors, among them Ahmed et al. [1], Casas [5], Barbu [2], Fattorini et al. [10], Tröltzsch [18] etc. We first state the existence of optimal controls and the necessary conditions for optimality for the continuous relaxed problem. Then, the relaxed problem is discretized by using a Galerkin finite element method with continuous piecewise linear basis functions in space for space approximation, and a partially implicit scheme in time, while the controls are approximated by piecewise constant relaxed controls. The discretization is motivated by the fact that in practice optimization methods are usually applied to the problem after some discretization. Then, we prove the existence of optimal controls and derive necessary conditions for optimality for the discrete relaxed problem. Finally, we study the behaviour in the limit of the above approximation. More precisely, we prove, under appropriate assumptions, that accumulation points of sequences of optimal (resp. admissible and extremal) discrete

Received by the editors March 15, 2018 and, in revised form, October 21, 2018.
2000 *Mathematics Subject Classification.* 49K20, 49M25, 65K10, 65M60.

relaxed controls are optimal (resp. admissible and extremal) for the continuous relaxed problem. The novelty of the present paper is in the finite element approximation of a boundary optimal control problem using, as a tool, relaxed controls, which can be further used in optimization algorithms (see [8]). For a different approach, using differential inclusions and approximations in abstract spaces, of a Mayer type optimal control problem, see Mordukhovich et al. [14], where existence theory, necessary optimality conditions and convergence are considered.

For approximation of nonconvex optimal control and variational problems, and of Young measures, see e.g. [4, 6, 9, 13, 15] and the references therein.

2. The continuous optimal control problems

Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $I = (0,T)$, $T < \infty$, an interval, and set $Q := \Omega \times I$, $\Sigma_0 := \Gamma_0 \times I$, $\Sigma_1 := \Gamma_1 \times I$ and $\Sigma := \Gamma \times I$. Consider the parabolic state equation

$$y_t + A(t)y = f_0(x,t,y(x,t)) \text{ in } Q, \tag{1}$$

$$y(x,t) = 0 \text{ on } \Sigma_0, \tag{2}$$

$$\frac{\partial y}{\partial \nu_A} = f_1(x,t,w(x,t)) \text{ on } \Sigma_1, \tag{3}$$

$$y(x,0) = y^0(x) \text{ in } \Omega, \tag{4}$$

where $A(t)$ is the second order elliptic differential operator

$$A(t)y := -\sum_{j=1}^{d}\sum_{i=1}^{d} (\partial/\partial x_i)[a_{ij}(x,t)\partial y/\partial x_j] \tag{5}$$

and

$$\frac{\partial y}{\partial \nu_A} = \sum_{j=1}^{d}\sum_{i=1}^{d} a_{ij}(x,t)\frac{\partial y}{\partial x_j}\nu_j, \quad \text{with } (x,t) \in \Sigma_1, \tag{6}$$

where $\nu(x)$ is the outwards unit vector to Γ at the point x.

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in $L^2(\Omega)$, by $(\cdot,\cdot)_{\Gamma_1}$ and $\|\cdot\|_{\Gamma_1}$ the inner product and norm in $L^2(\Gamma_1)$, by $(\cdot,\cdot)_1$ and $\|\cdot\|_1$ the inner product and norm in the Sobolev space $H^1(\Omega)$ and by $<\cdot,\cdot>$ the duality bracket between $V := \{v \in H^1(\Omega) : v|_{\Gamma_0} = 0\}$, where $v|_{\Gamma_0}$ is the trace function on Γ_0 and its dual space V^*. The state equation will be interpreted in the following weak form

$$\begin{aligned} &< y_t, v > + a(t,y,v) = (f_0(t,y),v) + (f_1(t,w),v)_{\Gamma_1}, \ \forall v \in V, \text{ a.e. in } I, \\ &y(t) \in V \text{ a.e. in } I, y(0) = y^0, \end{aligned} \tag{7}$$

where the derivative y_t is understood in the sense of V-vector valued distributions, and $a(t,\cdot,\cdot)$ denotes the usual bilinear form on $V \times V$ associated with $A(t)$

$$a(t,y,v) := \sum_{j=1}^{d}\sum_{i=1}^{d} \int_\Omega a_{ij}(x,t)\frac{\partial y}{\partial x_i}\frac{\partial v}{\partial x_j}dx. \tag{8}$$

We define the set of *classical controls*

$$W := \{w : \Sigma_1 \to U \mid w \text{ measurable}\} \subset L^\infty(\Sigma_1),$$

where U is a compact subset of $\mathbb{R}^{d'}$, and the functionals

$$G_m(w) := \int_Q g_{0m}(x,t,y)dxdt + \int_{\Sigma_1} g_{1m}(x,t,y,w)d\gamma dt, \quad m = 0,...,q. \tag{9}$$

The continuous classical optimal control problem is

$$\text{minimize } G_0(w) \tag{10}$$

subject to the state equation (1)-(4), the control constraints $w \in W$ and the state constraints

$$G_m(w) = 0, \quad m = 1, ..., p, \tag{11}$$

$$G_m(w) \leq 0, \quad m = p+1, ..., q. \tag{12}$$

In what follows, we shall make some of the following assumptions.
(H_1) $\Gamma = \Gamma_0 \cup \Gamma_1$ is Lipschitz (e.g. appropriately piecewise C^1, or polyhedral), where Γ_0 has a positive $(d-1)$-dimensional measure (see e.g. [18]).
(H_2) The coefficients a_{ij} satisfy the ellipticity condition

$$\sum_{j=1}^{d}\sum_{i=1}^{d} a_{ij}(x,t) z_i z_j \geq \alpha_0 \sum_{i=1}^{d} z_i^2, \quad \forall z_i, z_j \in \mathbb{R}, \text{ a.e. in } Q,$$

with $\alpha_0 > 0$, $a_{ij} \in L^\infty(Q)$, which implies that
$|a(t,y,v)| \leq \alpha_1 \|y\|_1 \|v\|_1, \quad a(t,v,v) \geq \alpha_2 \|v\|_1^2, \quad \forall y, v \in V, \quad t \in I$, for some $\alpha_1 \geq 0,\ \alpha_2 > 0$.
(H_3) The function f_0 is defined on $Q \times \mathbb{R}$, measurable for fixed y, continuous for fixed x, t, and satisfies
$|f_0(x,t,y)| \leq \psi_0(x,t) + \gamma |y|, \quad (x,t,y) \in Q \times \mathbb{R}$, with $\psi_0 \in L^2(Q)$, $\gamma \geq 0$,
$|f_0(x,t,y_1) - f_0(x,t,y_2)| \leq L |y_1 - y_2|, \quad (x,t,y_1,y_2) \in Q \times \mathbb{R}^2$.
(H_4) The function f_1 is defined on $\Sigma_1 \times U$, measurable for fixed u, continuous for fixed x, t, and satisfies
$|f_1(x,t,u)| \leq \psi_1(x,t), \quad (x,t,u) \in \Sigma_1 \times U$, with $\psi_1 \in L^2(\Sigma_1)$.
(H_5) The functions g_{0m} (resp. g_{1m}) are defined on $Q \times \mathbb{R}$ (resp. $\Sigma_1 \times \mathbb{R} \times U$), measurable for fixed y (resp. y, u), continuous for fixed x, t, and satisfy
$|g_{0m}(x,t,y)| \leq \zeta_{0m}(x,t) + \delta_{0m} y^2, \quad (x,t,y) \in Q \times \mathbb{R}$,
with $\zeta_{0m} \in L^1(Q)$, $\delta_{0m} \geq 0$,
$|g_{1m}(x,t,y,u)| \leq \zeta_{1m}(x,t) + \delta_{1m} y^2, \quad (x,t,y,u) \in \Sigma_1 \times \mathbb{R} \times U$,
with $\zeta_{1m} \in L^1(\Sigma_1)$, $\delta_{1m} \geq 0$.
(H_6) The function f_{0y} is defined on $Q \times \mathbb{R}$, measurable on Q for fixed $y \in \mathbb{R}$ and continuous on $\mathbb{R}$ for fixed $(x,t) \in Q$ and satisfies $|f_{0y}(x,t,y)| \leq L_1, (x,t,y) \in Q \times \mathbb{R}$.
(H_7) The functions g_{0my} (resp. g_{1my}) are defined on $Q \times \mathbb{R}$ (resp. $\Sigma_1 \times \mathbb{R} \times U$), measurable on Q for fixed $y \in \mathbb{R}$ (resp. on Σ_1 for fixed $y \in \mathbb{R}$ and $u \in U$) and continuous on $\mathbb{R}$ for fixed $(x,t) \in Q$ (resp. on $\mathbb{R} \times U$ for fixed $(x,t) \in \Sigma_1$), and satisfy
$|g_{0my}(x,t,y)| \leq \eta_{0m}(x,t) + \delta'_{0m} |y|, \quad (x,t,y) \in Q \times \mathbb{R}$,
with $\eta_{0m} \in L^2(Q)$, $\delta'_{0m} \geq 0$,
$|g_{1my}(x,t,y,u)| \leq \eta_{1m}(x,t) + \delta'_{1m} |y|, \quad (x,t,y,u) \in \Sigma_1 \times \mathbb{R} \times U$,
with $\eta_{1m} \in L^2(\Sigma_1)$, $\delta'_{1m} \geq 0$.

It is well known that, even if the set U is convex, the classical problem may have no solutions. The existence of such a solution is usually proved under strong, often unrealistic (for nonlinear systems) convexity assumptions (such as the Cesari property). Reformulated in the so-called relaxed form, the problem is convexified in some sense and has a solution in a larger space under weaker assumptions.

Next, we define the set of *relaxed controls* (Young measures; for the relevant theory, see [19])

(13) $R := \{r: \overline{\Sigma}_1 \to M_1(U)\,|\,r \text{ weakly measurable}\} \subset L^\infty_w(\Sigma_1, M(U)) \equiv L^1(\Sigma_1, C(U))^*$,

where $M(U)$ (resp. $M_1(U)$) is the set of Radon (resp. probability) measures on U. The set R is endowed with the relative weak star topology of $L^1(\Sigma_1, C(U))^*$, and R is convex, metrizable and compact. If we identify every classical control $w(\cdot)$ with its associated Dirac relaxed control $r(\cdot) = \delta_{w(\cdot)}$, then W may be also regarded as a subset of R, and W is thus dense in R. For $\phi \in L^1(\Sigma_1, C(U))$ (or $\phi \in B(\overline{\Sigma}_1, U; \mathbb{R})$, where $B(\overline{\Sigma}_1, U; \mathbb{R})$ is the set of Caratheodory functions, see Warga [19]) and $r \in L^\infty_w(\Sigma_1, M(U))$ (in particular, for $r \in R$), we shall use the simplified notation

$$\varphi(x,t,r(x,t)) := \int_U \varphi(x,t,u) r(x,t)(du), \tag{14}$$

where $\varphi(x,t,r(x,t))$ is thus linear (under convex combinations, for $r \in R$) in r. A sequence (r_k) converges to $r \in R$ in R iff

$$\lim_{k\to\infty} \int_{\Sigma_1} \phi(x,t,r_k(x,t)) d\gamma dt = \int_{\Sigma_1} \phi(x,t,r(x,t)) d\gamma dt, \tag{15}$$

for every $\phi \in L^1(\Sigma_1; C(U))$, or $\phi \in B(\overline{\Sigma}_1, U; \mathbb{R})$, or $\phi \in C(\overline{\Sigma}_1 \times U)$.

The continuous relaxed optimal control problem is

$$\text{minimize } G_0(r) \tag{16}$$

subject to the relaxed state equation

$$\begin{aligned} &< y_t, v > + a(t,y,v) = (f_0(t,y), v) + (f_1(t,r), v)_{\Gamma_1}, \ \forall v \in V, \text{ a.e. in } I, \\ &y(t) \in V \text{ a.e. in } I, y(0) = y^0, \end{aligned} \tag{17}$$

the control constraints $r \in R$ and the state constraints

$$G_m(r) = 0, \quad m = 1,...,p, \tag{18}$$

$$G_m(r) \le 0, \quad m = p+1,...,q. \tag{19}$$

The following theorem can be proved by standard compactness arguments (see [12])

Theorem 2.1. *Under Assumptions (H_1-H_4), for every control $r \in R$ and $y^0 \in L^2(\Omega)$ (or $y^0 \in V$), the relaxed state equation has a unique solution $y := y_r$ such that $y \in L^2(I,V)$, $y_t \in L^2(I,V^*)$; moreover, y is essentially equal to a function in $C(\bar{I}, L^2(\Omega))$, and thus the initial condition is well defined.*

The following proposition generalizes Proposition 2.1 in [6], with a simpler proof, and will be very useful in what follows.

Proposition 2.1. *For $i = 1,...,K$, $K \ge 0$, let $s_i \ge 1$, $\sigma_i \in [0, s_i]$ if $s_i < +\infty$, $\sigma_i := 0$ if $s_i = +\infty$, with $\frac{1}{s_0} + \sum_{i=1}^K \frac{\sigma_i}{s_i} = 1$, $\frac{1}{s_i} := 0$ if $s_i = +\infty$. For simplicity reasons we denote in this Proposition by $\|\cdot\|$ some norm in $\mathbb{R}^N$. Let Z be a compact subset of $\mathbb{R}^P$, $P \ge 1$ and let F be a function defined on $Z \times (\mathbb{R}^N)^K \times U$, measurable for every y, u fixed, continuous for every z fixed, and satisfying*

$$|F(z,y,u)| \le \Phi(z) + \Psi(z) \prod_{i=1}^K \xi_i(\|y_i\|), \text{ for every } (z,y,u) \in Z \times (\mathbb{R}^N)^K \times U,$$

with $\|y_i\| \le C_i$ if $s_i = +\infty$, where $y := (y_1,...,y_K)$, $\Phi \in L^1(Z)$, $\Psi \in L^{s_0}(Z)$,

$\xi_i(\|y_i\|) := \|y_i\|^{\sigma_i}$ if $s_i < +\infty$, $\xi_i(\|y_i\|) := 1$ if $s_i = +\infty$. If (y_i^k) converges to y_i in $L^{s_i}(Z;\mathbb{R}^N)$ strongly, $i = 1, ..., K$, with $\|y_i^k\| \le C_i$ (for k sufficiently large) if $s_i = +\infty$, and (r^k) converges to r in R, then

$$\lim_{k\to\infty} \int_Z F(z, y^k(z), r^k(z))dz = \int_Z F(z, y(z), r(z))dz.$$

Proof. We have

$$\int_Z F(z, y^k, r^k)dz - \int_Z F(z, y, r)dz = A_k + B_k,$$

where

$$A_k := \int_Z F(z, y^k, r^k)dz - \int_Z F(z, y, r^k)dz,$$

$$B_k := \int_Z F(z, y, r^k)dz - \int_Z F(z, y, r)dz.$$

Since (r_k) converges to r in R, we have $B_k \to 0$. Since $y_i^k \to y_i$ in $L^{s_i}(Z;\mathbb{R}^N)$ strongly, $i = 1, ..., K$, we have also $\|y_i^k\| \to \|y_i\|$ in $L^{s_i}(Z)$ strongly. Hence there exist (see Theorem IV.9 in [3]) subsequences (same notation) and functions $\bar{y}_i \in L^{s_i}(Z)$ such that $y_i^k(z) \to y_i(z)$, $i = 1, ..., K$ a.e. in Z and
$\|y_i^k(z)\| \le \bar{y}_i(z)$, in $Z - S_i^k$, $i = 1, ..., K$, if $s_i < +\infty$,
with $\text{meas}(S_i^k) = 0$. If $s_i = +\infty$, we have also, for $k \ge k_0$ (for some k_0)
$\|y_i^k(z)\| \le C_i$, in $Z - S_i^k$,
with $\text{meas}(S_i^k) = 0$. We then have, for every $k \ge k_0$

$$\left|F(z, y^k(z), r^k(z))\right| \le \Phi(z) + \Psi(z) \prod_{i=1}^{K} \xi_i(\|\bar{y}_i(z)\|) := \bar{F}(z), \text{ in } Z - \bigcup_{1\le i\le K} S_i^k,$$

i.e. a.e. in Z, where $\bar{F} \in L^1(Z)$, by the multiple Hölder inequality. By the uniform continuity of F, for z fixed, on the compact set $B(z) \times U$, where $B(z)$ is a closed ball in $\mathbb{R}^{NK}$ with center $y(z)$ and containing $y^k(z)$ for every k (or for $k \ge k'$), we have, since $r^k \in M_1(U)$

$$\left|F(z,y^k(z), r^k(z)) - F(z, y(z), r^k(z))\right| = \left|\int_U [F(z, y^k(z), u) - F(z, y(z), u)] r^k(du)\right| \le$$

$$\int_U \left|F(z, y^k(z), u) - F(z, y(z), u)\right| r^k(du) \le \max_{u\in U} \left|F(z, y^k(z), u) - F(z, y(z), u)\right| \to 0$$

a.e. in Z. The result follows then from Lebesgue's dominated convergence theorem and the uniqueness of the limit. □

The following lemma can be proved by using techniques similar to [7] and [16].

Lemma 2.1. *Under Assumptions (H_1-H_4), the operator $r \mapsto y_r$ from R to $L^2(I, V)$ is continuous. Under Assumptions (H_1-H_5), the functionals $r \mapsto G_m(r)$ on R, are continuous.*

The following Theorems 2.2 and 2.3 have the advantage (as compared to classical ones) of avoiding various convexity assumptions (e.g. Cesari property) in proving existence and necessary optimality conditions.

Theorem 2.2. *Under Assumptions (H_1-H_5), if the relaxed problem is feasible, then it has a solution.*

Proof. The theorem follows from Lemma 2.1 and the compacteness of R. □

Since $W \subset R$, we generally have

$$(20) \qquad c_R := \min_{\text{constraints on } r} G_0(r) \leq \inf_{\text{constraints on } w} G_0(w) := c_W,$$

where the equality holds, in particular, if there are no state constraints, as W is dense in R. Since usually approximation methods slightly violate the state constraints, approximating an optimal relaxed control by a relaxed or a classical one, hence the possibly lower relaxed optimal cost c_R, is not a drawback in practice (see [19], p. 259).

Next, we give some useful results concerning necessary/sufficient conditions for optimality, which can be proved by using the techniques of [19], [7] (see also [11]).

Lemma 2.2. *Under Assumptions (H_1-H_7), dropping the index m in the functionals, the directional derivative of G defined on R is given by*

(21)

$$DG(r, r' - r) := \lim_{\varepsilon \to 0^+} \frac{G(r + \varepsilon(r' - r)) - G(r)}{\varepsilon} = \int_{\Sigma_1} H(x, t, y, z, r'(x,t) - r(x,t)) d\gamma dt,$$

for $r, r' \in R$, where the Hamiltonian H is defined by

$$(22) \qquad H(x, t, y, z, u) := z f_1(x, t, u) + g_1(x, t, y, u),$$

and the adjoint state $z := z_r$ satisfies the linear adjoint equation

(23)

$$- < z_t, v > + a(t, v, z) = (z f_{0y}(y) + g_{0y}(y), v) + (g_{1y}(y, r), v)_{\Gamma_1}, \forall v \in V, \text{ a.e. in } I,$$
$$z(t) \in V \text{ a.e. in } I, \ z(T) = 0, \text{with } y := y_r.$$

The mappings $r \mapsto z_r$, from R to $L^2(Q)$, and $(r, r') \mapsto DG(r, r' - r)$, from $R \times R$ to $\mathbb{R}$, are continuous.

The following theorem states necessary conditions for optimality.

Theorem 2.3. *Under Assumptions (H_1-H_7), if $r \in R$ is optimal for either the relaxed or the classical optimal control problem, then r is strongly extremal relaxed, i.e. there exist multipliers $\lambda_m \in \mathbb{R}$, $m = 0, ..., q$, with $\lambda_0 \geq 0$, $\lambda_m \geq 0$, $m = p+1, ..., q$, $\sum_{m=0}^{q} |\lambda_m| = 1$, such that*

$$(24) \qquad \sum_{m=0}^{q} \lambda_m DG_m(r, r' - r) \geq 0, \ \forall r' \in R,$$

$$(25) \qquad \lambda_m G_m(r) = 0, m = p+1, ..., q \ \text{(complementary slackness conditions).}$$

The condition (24) is equivalent to the strong relaxed pointwise minimum principle

$$(26) \qquad H(x, t, y(x,t), z(x,t), r(x,t)) = \min_{u \in U} H(x, t, y(x,t), z(x,t), u), \text{ a.e. in } \Sigma_1,$$

where the complete Hamiltonian H is defined with $g_1 := \sum_{m=0}^{q} \lambda_m g_{1m}$ and the adjoint z is defined with $g_0 := \sum_{m=0}^{q} \lambda_m g_{0m}$ and $g_1 := \sum_{m=0}^{q} \lambda_m g_{1m}$.

The next theorem gives sufficient conditions for optimality.

Theorem 2.4. *Under Assumptions (H_1-H_7) and the additional assumption that the data are such that $G_0, G_{p+1}, ..., G_q$ are convex and $G_1, G_2, ..., G_p$ are affine, if $r \in R$ is admissible and extremal for the control problem, with $\lambda_0 > 0$, then r is optimal for this problem.*

Proof. The assumptions imply that the functional $G(r) := \sum_{m=0}^{q} \lambda_m G_m(r)$ is convex. The condition (24) is then satisfied if and only if r minimizes G on R. Suppose now that r is not optimal for the relaxed problem, in which case there exists $r' \in R$ satisfying the state constraints and such that $G_0(r') < G_0(r)$. We then have, using the state constraints and the complementary slackness conditions (25),

$$\begin{aligned} G(r') &= \lambda_0 G_0(r') + \sum_{m=1}^{p} \lambda_m G_m(r') + \sum_{m=p+1}^{q} \lambda_m G_m(r') \\ &= \lambda_0 G_0(r') + \sum_{m=p+1}^{q} \lambda_m G_m(r') \\ &< \lambda_0 G_0(r) = \lambda_0 G_0(r) + \sum_{m=1}^{p} \lambda_m G_m(r) + \sum_{m=p+1}^{q} \lambda_m G_m(r) = G(r), \end{aligned}$$

which is a contradiction. Therefore, r is optimal for the relaxed problem. □

Remark 2.1. *In the absence of equality state constraints, it can be shown that if the optimal control r is regular, i.e. there exists $r' \in R$, such that*

$$G_m(r) + DG_m(r, r' - r) < 0,\ m = p+1, ..., q, \ \textit{(Slater condition)},$$

then $\lambda_0 \neq 0$ for any set of multipliers as in Theorem 2.3.

3. The discrete optimal control problems

We introduce the following additional assumptions.

(H_8) For simplicity reasons we consider that Γ is polyhedral, a is independent of t, $f_0, f_{0y}, f_1, g_{0m}, g_{0my}, g_{1m}, g_{1my}$ are continuous (or continuous in (x, t), piecewise w.r.t t) on the closure of their domains of definition and $y^0 \in V$.

Under Assumptions (H_8), for each integer $n \geq 0$, let $\{E_i^n\}_{i=1}^{M^n}$ be an admissible regular quasi-uniform triangulation of Ω into closed d-simplices (finite elements), with $h^n = \max_i[\text{diam}(E_i^n)] \to 0$ as $n \to \infty$, and $\{I_j^n\}_{j=1}^{N^n}$, a subdivision of the interval $\bar{I}$ into intervals $I_j^n = [t_{j-1}^n, t_j^n)$, $j = 1, ..., N^n - 1$, $I_{N^n}^n = [t_{N^n-1}^n, t_{N^n}^n]$ with $t_0^n = 0$ and $t_{N^n}^n = T$. The intervals are of equal length Δt^n, with $\Delta t^n \to 0$ as $n \to \infty$. We define the *panels* $S_{kj}^n = F_k^n \times I_j^n$, where $\{F_k^n\}_{k=1}^{P^n}$ are the boundary edges on Γ_1 of the triangles E_i^n that reach the boundary Γ_1. Let $V^n \subset V$ be the subspace of functions that are continuous on $\bar{\Omega}$ and linear (i.e. affine) on each E_i^n. The set of *discrete classical controls* $W^n \subset W$ is the subset of classical controls that are constant on the interior of each panel S_{kj}^n. The set of *discrete relaxed controls* $R^n \subset R$ is the subset of relaxed controls of the form r_{kj}^n, $k = 1, 2, ..., P^n$, $j = 1, ..., N^n$, that are equal to a constant measure in $M_1(U)$ on each panel S_{kj}^n. The set R^n is endowed with the relative weak star topology of $M(U)^{P^n N^n}$.

For a given discrete control $r^n \in R^n$, the corresponding discrete state $y^n := (y_0^n, ..., y_{N^n}^n)$ is given by the discrete relaxed state equation (partially implicit scheme)

$$(27)\quad \begin{aligned} &\tfrac{1}{\Delta t^n}\left(y_j^n - y_{j-1}^n, v\right) + a(y_j^n, v) = \left(f_0(t_{j-1}^n, y_{j-1}^n), v\right) + \left(f_1(t_{j-1}^n, r_{j-1}^n), v\right)_{\Gamma_1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in V^n, j = 1, ..., N^n, \\ &(y_0^n - y^0, v)_1 = 0, \text{ for every } v \in V^n, \\ &y_j^n \in V^n,\ j = 1, ..., N^n. \end{aligned}$$

The implicit discrete scheme (27) reduces to a regular linear system which has a unique solution for every control.

The discrete control constraint is $r^n \in R^n$ and the discrete functionals are

(28)
$$G^n_m(r^n) := \Delta t^n \sum_{j=1}^{N^n} \left(\int_\Omega g_{0m}(t^n_{j-1}, y^n_{j-1}) dx + \int_{\Gamma_1} g_{1m}(t^n_{j-1}, y^n_{j-1}, r^n_j) d\gamma \right), m = 0, ..., q.$$

The discrete state constraints are *either* of the *two* following ones

(29) Case (a) $$|G^n_m(r^n)| \le \varepsilon^n_m, \quad m = 1, ..., p,$$

(30) Case (b) $$G^n_m(r^n) = \varepsilon^n_m, \quad m = 1, ..., p,$$

and

(31) $$G^n_m(r^n) \le \varepsilon^n_m, \quad \varepsilon^n_m \ge 0, \quad m = p+1, ..., q,$$

where the feasibility perturbations ε^n_m are given numbers converging to zero, to be defined later. The discrete cost functional to be minimized is $G^n_0(r^n)$.

Theorem 3.1. *Under Assumptions (H_2-H_5) and (H_8), the mappings $r^n \mapsto y^n_j$ and $r^n \mapsto G^n_m(r^n)$, defined on R^n, are continuous. If any of the discrete problems is feasible, then it has a solution.*

Proof. The continuity of the operators $r^n \mapsto y^n_j$ is easily proved by induction on j (or by using the discrete Bellman-Gronwall inequality, see [17]). The continuity of $r^n \to G^n_m(r^n)$ follows from the continuity of g_{0m}, g_{1m}. The existence of an optimal control follows then from the compactness of R^n. □

The proofs of the following lemma and theorem parallel the continuous case and are omitted.

Lemma 3.1. *We drop the index m in the functionals. Under Assumptions (H_2-H_8), the directional derivative of the functional G^n is given by*

(32) $$DG^n(r^n, r'^n - r^n) = \Delta t^n \sum_{j=1}^{N^n} \int_{\Gamma_1} H(t^n_{j-1}, y^n_{j-1}, z^n_j, r'^n_j - r^n_j) d\gamma, \ r^n, r'^n \in R^n,$$

where the discrete adjoint z^n is given by the linear adjoint scheme

(33)
$$\begin{aligned} -\tfrac{1}{\Delta t^n}(z^n_j - z^n_{j-1}, v) + a(v, z^n_{j-1}) &= \big(z^n_j f_{0y}(t^n_{j-1}, y^n_{j-1}) + g_{0y}(t^n_{j-1}, y^n_{j-1}), v\big) \\ &+ \big(g_{1y}(t^n_{j-1}, y^n_{j-1}, r^n_j), v\big)_{\Gamma_1}, \ \forall v \in V^n, \ j = N^n, ..., 1, \\ z^n_{N^n} = 0, \ z^n_j \in V^n, \ j = 1, ..., N^n, & \end{aligned}$$

which has a unique solution z^n_{j-1} for each $j = N^n, ..., 1$, (regular system). Moreover, the mappings $r^n \mapsto z^n$ and $(r^n, r'^n) \mapsto DG^n(r^n, r'^n - r^n)$ are continuous.

Theorem 3.2. *(i) Under Assumptions (H_2-H_8), if $r^n \in R^n$ is optimal for the discrete problem with state constraints, Case (b), then it is extremal, i.e. there exist multipliers $\lambda^n_m \in \mathbb{R}$, $m = 0, ..., q$, with $\lambda^n_0 \ge 0$, $\lambda^n_m \ge 0$, $m = p+1, ..., q$, $\sum_{m=0}^{q} |\lambda^n_m| = 1$, such that*

(34)
$$\sum_{m=0}^{q} \lambda^n_m DG^n_m(r^n, r'^n - r^n) =$$
$$\Delta t^n \sum_{j=1}^{N^n} \int_{\Gamma_1} H(t^n_{j-1}, y^n_{j-1}, z^n_j, r'^n_j - r^n_j) d\gamma \ge 0, \ \forall r'^n \in R^n,$$

(35) $$\lambda^n_m [G_m(r^n) - \varepsilon^n_m] = 0, \ m = p+1, ..., q,$$

where the Hamiltonian H is defined with $g_1 := \sum_{m=0}^{q} \lambda_m^n g_{1m}$ and the adjoint z^n is defined with $g_0 := \sum_{m=0}^{q} \lambda_m^n g_{0m}$ and $g_1 := \sum_{m=0}^{q} \lambda_m^n g_{1m}$. The global condition (34) is equivalent to the strong discrete panelwise minimum principle

$$\int_{F_k^n} H(t_{j-1}^n, y_{j-1}^n, z_j^n, r_{kj}^n) d\gamma = \min_{u\in U} \int_{F_k^n} H(t_{j-1}^n, y_{j-1}^n, z_j^n, u) d\gamma, \quad k = 1, ..., P^n,\ j = 1, ..., N^n. \tag{36}$$

(ii) With Assumptions (H_2-H_8), we suppose in addition that the data are such that $G_0, G_{p+1}, ..., G_q$ are convex and $G_1, ..., G_p$ are affine. If $r^n \in R^n$ is admissible and extremal for the discrete problem, Case (b), with $\lambda_0^n > 0$, then r^n is optimal for this problem.

4. Behavior in the limit

The following control approximation result is proved in [7].

Proposition 4.1. *Under Assumptions (H_8) on Γ, for every $r \in R$, there exists a sequence ($w^n \in W^n \subset R^n$) that converges to r in R.*

Lemma 4.1 (Stability). *Under Assumptions (H_2-H_4) and (H_8), if Δt^n is sufficiently small, for every $r^n \in R^n$, we have the following inequalities, where c denotes various constants independent of n and r^n*

$$\left\| y_j^n \right\| \le c, \quad j = 0, ..., N^n, \tag{37}$$

$$\sum_{j=1}^{N^n} \left\| y_j^n - y_{j-1}^n \right\|^2 \le c, \tag{38}$$

$$\Delta t^n \sum_{j=1}^{N^n} \left\| y_j^n \right\|_1^2 \le c. \tag{39}$$

Proof. y_0^n is clearly bounded by definition. Dropping the index n for simplicity of notation, setting $v = y_j \Delta t$ in the discrete equation (27) we have

$$\begin{aligned} &\tfrac{1}{2} \left\| y_j - y_{j-1} \right\|^2 + \tfrac{1}{2} \left\| y_j \right\|^2 - \tfrac{1}{2} \left\| y_{j-1} \right\|^2 + \Delta t\, a(y_j, y_j) \\ &\qquad \le \Delta t \left| (f_0(t_{j-1}, y_{j-1}), y_j) \right| + \Delta t \left| (f_1(t_{j-1}, r_{j-1}), y_j)_{\Gamma_1} \right| . \end{aligned} \tag{40}$$

Using the Trace Theorem and assumptions (H_2-H_4) we then take

$$\begin{aligned} &\tfrac{1}{2} \left(\left\| y_j - y_{j-1} \right\|^2 + \left\| y_j \right\|^2 - \left\| y_{j-1} \right\|^2 \right) + \Delta t\, a_2 \left\| y_j \right\|_1^2 \\ &\le \Delta t \left((\left\| f_0(t_{j-1}, y_{j-1}) \right\| \, \left\| y_j \right\| + \left\| f_1(t_{j-1}, r_{j-1}) \right\|_{\Gamma_1} \left\| y_j \right\|_{\Gamma_1} \right) \\ &\le \Delta t \left((\left\| \psi_0 \right\| + \gamma \left\| y_{j-1} \right\|) \left\| y_j \right\| + c \left\| \psi_1 \right\|_{\Gamma_1} \left\| y_j \right\|_1 \right) \\ &\le \Delta t \left(\left\| \psi_0 \right\| + \gamma \left\| y_{j-1} \right\| \, \left\| y_j - y_{j-1} \right\| + \gamma \left\| y_{j-1} \right\|^2 + c \left\| \psi_1 \right\|_{\Gamma_1} \left\| y_j \right\|_1 \right) \\ &\le \Delta t \left(\left\| \psi_0 \right\| + \tfrac{\gamma}{\beta_1} \left\| y_{j-1} \right\|^2 + \gamma\beta_1 \left\| y_j - y_{j-1} \right\|^2 \right) \\ &\qquad + \Delta t \left(\gamma \left\| y_{j-1} \right\|^2 + c \tfrac{1}{\beta_2} \left\| \psi_1 \right\|_{\Gamma_1}^2 + c\beta_2 \left\| y_j \right\|_1^2 \right) \\ &\le \Delta t \left(\left\| \psi_0 \right\| + c \left\| y_{j-1} \right\|^2 + \gamma\beta_1 \left\| y_j - y_{j-1} \right\|^2 + c \tfrac{1}{\beta_2} \left\| \psi_1 \right\|_{\Gamma_1}^2 + c\beta_2 \left\| y_j \right\|_1^2 \right) . \end{aligned} \tag{41}$$

For Δt sufficiently small, it follows

$$\begin{aligned} &(1 - 2\Delta t\, \gamma\beta_1) \left\| y_j - y_{j-1} \right\|^2 + \left\| y_j \right\|^2 - \left\| y_{j-1} \right\|^2 + 2\Delta t\, (a_2 - c\beta_2) \left\| y_j \right\|_1^2 \\ &\qquad \le 2\Delta t \left\| \psi_0 \right\| + 2c\, \Delta t \left\| y_{j-1} \right\|^2 + 2c \tfrac{\Delta t}{\beta_2} \left\| \psi_1 \right\|_{\Gamma_1}^2 . \end{aligned} \tag{42}$$

By summation over $j = 1, \dots, k$, where $1 \leq k \leq N$, we obtain

$$c\sum_{j=1}^{k} \|y_j - y_{j-1}\|^2 + \|y_k\|^2 + c\,\Delta t \sum_{j=1}^{k} \|y_j\|_1^2 \leq \|y_0\|^2 + c\,\Delta t \sum_{j=1}^{k} \|y_{j-1}\|^2 + cT. \tag{43}$$

Dropping the first and third term and using the discrete Bellman-Gronwall inequality (see [17]) we obtain (37). The inequalities (38) and (39) follow then from (43). □

For given values $v_0, ..., v_N$ in a vector space, we define the piecewise constant and continuous piecewise linear functions

$$v_-(t) := v_{j-1}, \quad v_+(t) := v_j, \quad t \in \overset{o}{I_j^n},\ j = 1, ..., N^n,$$

$$v_\wedge(t) := v_{j-1} + \frac{t - t_{j-1}^n}{\Delta t^n}(v_j - v_{j-1}), \quad t \in I_j^n,\ j = 1, ..., N^n.$$

Lemma 4.2 (Consistency of states and functionals)**.** *Under Assumptions (H_2-H_4) and (H_8), if $r^n \to r$ in R, then the corresponding discrete states $y_\wedge^n, y_-^n, y_+^n$ converge to y_r in $L^2(I, L^2(\Omega)) = L^2(Q)$ strongly and*

$$\lim_{n\to\infty} G_m^n(r^n) = G_m(r), \quad m = 0, ..., q. \tag{44}$$

Proof. Since, by inequality (39), y_-^n and y_+^n are bounded in $L^2(I, V)$, it follows that $y_\wedge^n$ is also bounded in $L^2(I, V)$. By extracting subsequences, we can suppose that $y_\wedge^n \to y$ in $L^2(I, V)$ weakly (hence in $L^2(Q)$ weakly). The discrete state equation (27) can be written in the form

$$\frac{d}{dt}(y_\wedge^n(t), v) = (\Phi^n(t), v)_1, \quad \forall v \in V^n, \text{ a.e. in } (0, T),$$

in the scalar distribution sense, where the piecewise constant function Φ^n is defined, for $j = 1, ..., N^n$, using Riesz's representation theorem, by

$$(\Phi_j^n(t), v)_1 := -a(y_j^n, v) + (f_0(t_{j-1}^n, y_{j-1}^n), v) + (f_1(t_{j-1}^n, r_{j-1}^n), v)_{\Gamma_1}, \text{ in } \overset{o}{I_j^n}.$$

By Assumptions (H_2-H_4), we have, for $j = 1, ..., N^n$

$$\left|(\Phi_j^n, v)_1\right| \leq c\left[\left\|y_j^n\right\|_1 \|v\|_1 + \left(1 + \left\|y_{j-1}^n\right\|\right)\|v\|_1\right].$$

Therefore, using inequality (39) it follows

$$\int_0^T \|\Phi^n(t)\|_1\, dt \leq c\left(1 + \int_0^T \left\|y_+^n\right\|_1^2 dt + \int_0^T \left\|y_-^n\right\|^2 dt\right) \leq c, \tag{45}$$

which shows that Φ^n belongs to $L^1(I, V)$. Now, let, $\tilde{\Phi}^n$ denote the extension of Φ^n by 0 outside $[0, T]$. We then have, on $\mathbb{R}$

$$\frac{d}{dt}(y_\wedge^n(t), v) = (\tilde{\Phi}^n(t), v)_1 + (y_0^n, v)\delta_0 - (y_N^n, v)\delta_T, \forall v \in V^n,$$

where δ_0, δ_T are the Dirac distributions at 0 and T. Taking the Fourier transforms ($\hat{\Phi}^n$ Fourier transform of $\tilde{\Phi}^n$), we have

$$2i\pi\tau(\hat{y}_\wedge^n, v) = (\hat{\Phi}^n(\tau), v)_1 + (y_0^n, v) - (y_N^n, v)e^{-2i\pi\tau T}.$$

Setting $v = \hat{y}_\wedge^n(\tau)$ and taking absolute values we get, since y_0^n, y_N^n are bounded in $L^2(\Omega)$,

$$2\pi|\tau|\,\|\hat{y}_\wedge^n(\tau)\|^2 \leq \left\|\hat{\Phi}^n(\tau)\right\|_1 \|\hat{y}_\wedge^n(\tau)\|_1 + c\,\|\hat{y}_\wedge^n(\tau)\|. \tag{46}$$

By the definition of the Fourier transform and (45) we obtain

$$\left\|\hat{\Phi}^n(\tau)\right\|_1 \leq \int_0^T \|\Phi^n(t)\|_1 \, dt \leq c,$$

therefore from (46)

$$|\tau| \, \|\hat{y}_\wedge^n(\tau)\|^2 \leq c \, \|\hat{y}_\wedge^n(\tau)\|_1 . \tag{47}$$

Following Temam [17], for $\rho \in [0, 1/4)$, the next inequality holds on $\mathbb{R}$

$$|\tau|^{2\rho} \leq c \frac{1+|\tau|}{1+|\tau|^{1-2\rho}}. \tag{48}$$

Using (47), (48) and Cauchy-Schwarz inequality we have

$$\int_{-\infty}^{+\infty} |\tau|^{2\rho} \|\hat{y}_\wedge^n(\tau)\|^2 \, d\tau \leq c \int_{-\infty}^{+\infty} \|\hat{y}_\wedge^n(\tau)\|^2 \, d\tau + c' \int_{-\infty}^{+\infty} \frac{\|\hat{y}_\wedge^n(\tau)\|_1}{1+|\tau|^{1-2\rho}} d\tau$$

$$\leq c \int_{-\infty}^{+\infty} \|\hat{y}_\wedge^n(\tau)\|^2 \, d\tau + c' \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\rho})^2} \right)^{1/2} \left(\int_{-\infty}^{+\infty} \|\hat{y}_\wedge^n(\tau)\|_1^2 \, d\tau \right)^{1/2}. \tag{49}$$

The integral $\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\rho})^2}$ is finite for $\rho < 1/4$ and by the Parseval's identity we have

$$\int_{-\infty}^{+\infty} \|\hat{y}_\wedge^n(\tau)\|^2 \, d\tau \leq c \int_{-\infty}^{+\infty} \|\hat{y}_\wedge^n(\tau)\|_1^2 \, d\tau = c \int_0^T \|y_\wedge^n(t)\|_1^2 \, dt \leq c.$$

Therefore, from (49) we obtain

$$\int_{-\infty}^{+\infty} |\tau|^{2\rho} \|\hat{y}_\wedge^n(\tau)\|^2 \, d\tau \leq c.$$

By the Compactness Theorem 2.2, Chapter III, in [17], and since the injection of $H^1(\Omega)$ into $H^{1-\varepsilon}(\Omega)$, $\varepsilon \in (0, 1]$, is compact, and the injection of $H^{1-\varepsilon}(\Omega)$ into $L^2(\Omega)$ is continuous, there exists a subsequence (same notation) such that $y_\wedge^n \to \tilde{y}$ in $L^2(I, H^{1-\varepsilon}(\Omega))$ strongly and in $L^2(Q)$ strongly, for some $\tilde{y}$, and we must have $\tilde{y} = y$, since $\hat{y}_\wedge^n \to y$ also in $L^2(Q)$ weakly. From Lemma 4.1 (inequality (38) multiplied by Δt) follows that $y_+^n - y_-^n \to 0$ in $L^2(Q)$ strongly. Therefore $y_+^n \to y$ and $y_-^n \to y$ in $L^2(Q)$ strongly and in $L^2(I, V)$ weakly.

Now, to show that $y = y_r$, we proceed similarly to the proof of Lemma 4.3 in [7], i.e. we pass to the limit in the discrete equation, integrated in t, with appropriate interpolating test functions $\varphi^n(t)v^n(x)$; for the passage to the limit in the nonlinear terms containing f_0 and f_1 we use Proposition 2.1.

Finally, convergences (44) follow from Proposition 2.1. □

We suppose in the sequel that the continuous relaxed problem is feasible. The following (theoretical, in the presence of state constraints) theorem addresses the behavior in the limit of optimal discrete controls.

Theorem 4.1. *We suppose that Assumptions (H_2-H_5) and (H_8) are satisfied. In the presence of state constraints, we suppose in addition that the sequences (ε_m^n) in the discrete state constraints, Case (a), converge to zero as $n \to \infty$ and satisfy*

$$|G_m^n(\tilde{r}^n)| \leq \varepsilon_m^n, \; m = 1, ..., p, \quad G_m^n(\tilde{r}^n) \leq \varepsilon_m^n, \; \varepsilon_m^n \geq 0, \; m = p+1, ..., q,$$

for every n, where $(\tilde{r}^n \in R^n)$ is a sequence converging in R to an optimal control $\tilde{r} \in R$ of the relaxed problem. For each n, let r^n be optimal for the discrete problem, Case (a). Then every accumulation point of the sequence (r^n) is optimal for the continuous relaxed problem.

Proof. Note that our assumption implies that the discrete problems are feasible for every n. Let (r^n) be a subsequence (same notation) that converges to some $r \in R$. Since r^n is optimal, hence admissible, and $\tilde{r}^n$ is admissible, for the discrete problem, we have

$$G_0^n(r^n) \le G_0^n(\tilde{r}^n) \text{ and } |G_m^n(r^n)| \le \varepsilon_m^n,\ m = 1, ..., p,$$

$$G_m^n(r^n) \le \varepsilon_m^n, \quad m = p+1, ..., q.$$

Passing to the limit and using Lemma 4.2, we see that r is optimal for the continuous relaxed problem. If there are no state constraints, by taking a sequence converging to some continuous optimal control, we arrive directly to the same conclusion. □

Lemma 4.3 (Consistency of adjoint and functional derivatives)**.** *Under Assumptions (H_2-H_8), if $r^n \to r$ in R, then the corresponding discrete adjoint states $z_-^n, z_+^n, z_\wedge^n$ converge to z_r in $L^2(Q)$ strongly. If $r^n \to r$ and $r'^n \to r'$, then*

$$\lim_{n\to\infty} DG_m^n(r^n, r'^n - r^n) = DG_m(r, r' - r),\ m = 0, ..., q.$$

Proof. The proof is similar to that of Lemma 4.2, using also the consistency of states. □

Next, we study the behavior in the limit of extremal discrete controls. Consider the discrete problem with state constraints, Case (b). We shall construct sequences of perturbations (ε_m^n) converging to zero and such that the discrete problem is feasible for every n. Let $r'^n \in R^n$ be any solution of the problem without state constraints

$$c^n := \min_{r^n \in R^n} \{\sum_{m=1}^{p} [G_m^n(r^n)]^2 + \sum_{m=p+1}^{q} [\max(0, G_m^n(r^n))]^2\}, \tag{50}$$

and set

$$\varepsilon_m^n := G_m^n(r'^n),\ m = 1, ..., p,\ \varepsilon_m^n := \max(0, G_m^n(r'^n)),\ m = p+1, ..., q. \tag{51}$$

Let $\tilde{r}$ be an admissible control for the continuous relaxed problem, and $(\tilde{r}^n \in R^n)$ a sequence converging to $\tilde{r}$ in R (Proposition 4.1). We have

$$\lim_{n\to\infty} [G_m^n(\tilde{r}^n)]^2 = [G_m(\tilde{r})]^2 = 0, \quad m = 1, ..., p,$$

$$\lim_{n\to\infty} [\max(0, G_m^n(\tilde{r}^n))]^2 = [\max(0, G_m(\tilde{r}))]^2 = 0, \quad m = p+1, ..., q,$$

which imply a fortiori that $c^n \to 0$, hence $\varepsilon_m^n \to 0$, $m = 1, ..., q$. Then clearly the discrete problem, Case (b), is feasible for every n, for these perturbations ε_m^n. We suppose in the sequel that the perturbations ε_m^n are chosen as in the above minimum feasibility procedure. Note that in practice we usually have $c^n = 0$, for sufficiently large n, due to sufficient discrete controllability, in which case the perturbations ε_m^n are equal to zero, i.e. the discrete problem with zero perturbations is feasible.

Theorem 4.2. *Under Assumptions (H_2-H_8), for each n, let r^n be admissible and extremal for the discrete problem, Case (b). Then every accumulation point of the sequence (r^n) is admissible and extremal for the continuous relaxed problem.*

Proof. Since R is compact and $\sum_{m=0}^{q} |\lambda_m^n| = 1$, let (r^n), (λ_m^n), $m = 0, ..., q$, be subsequences such that $r^n \to r$ in R and $\lambda_m^n \to \lambda_m$, $m = 0, ..., q$. Let any $r' \in R$ and

(r'^n) be a sequence such that $r'^n \to r'$. Consider the discrete principle in global form, which can be written as

$$\int_{\Sigma_1} H(x,t,y^n_{j-1},z^n_j,r'^n - r^n)d\gamma dt \geq 0, \quad \forall n. \tag{52}$$

Passing to the limit in (52), by Lemmas 4.2, 4.3 and Proposition 2.1, we obtain

$$\int_{\Sigma_1} H(x,t,y,z,r'(x,t) - r(x,t))d\gamma dt \geq 0, \quad \forall r' \in R. \tag{53}$$

On the other hand, we have similarly

$$\lambda_m G_m(r) = \lim_{n\to\infty} \lambda^n_m [G^n_m(r^n) - \varepsilon^n_m] = 0, \; m = p+1,...,q,$$

$$G_m(r) = \lim_{n\to\infty} [G^n_m(r^n) - \varepsilon^n_m] = 0, \; m = 1,...,p,$$

$$G_m(r) = \lim_{n\to\infty} [G^n_m(r^n) - \varepsilon^n_m] \leq 0, \; m = p+1,...,q,$$

and $\lambda_0 \geq 0$, $\lambda_m \geq 0$, $m = p+1,...,q$, $\sum\limits_{m=0}^{q} |\lambda_m| = 1$, which show with (53) that r is admissible and extremal for the continuous relaxed problem. □

5. Conclusion

In the absence of convexity assumptions on the control set and the state constraints and due to the non-linear state equations, the optimal control problem considered here does not have in general classical solutions. Introducing relaxed controls, the existence of optimal controls was proven here under weaker assumptions. In addition, it was shown that necessary and sufficient conditions for relaxed optimality can also be derived in the form of a relaxed pointwise Pontryagin minimum principle. Finally, the continuous problem has been discretized in space and time, and the behaviour in the limit of sequences of optimal and admissible extremal controls has been studied.

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