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# POSTPROCESSING OF CONTINUOUS GALERKIN SOLUTIONS FOR DELAY DIFFERENTIAL EQUATIONS WITH NONLINEAR VANISHING DELAY

QIUMEI HUANG, KUN JIANG AND XIUXIU XU

**Abstract.** In this paper we propose several postprocessing techniques to accelerate the convergence of the continuous Galerkin solutions for delay differential equations with nonlinear vanishing delay. They are interpolation postprocessings (including integration type, Lagrange type, and polynomial preserving recovery type) and iteration postprocessing. The theoretical expectations are confirmed by numerical experiments.

**Key words.** Pantograph delay differential equations, quasi-graded mesh, continuous Galerkin methods, postprocessing, global superconvergence.

## 1. Introduction

Delay differential equations (DDEs) have a wide range of application in science and engineering. The nonlinear vanishing delay equation is an important type of delay differential equation and has received considerable attention in both theoretical analysis and numerical computation since the early 1970s (cf. [14, 15, 10]). Runge-Kutta and collocation methods are two popular numerical methods used to solve this kind functional differential equation, which can be found in the monographs by Bellen and Zennaro [1] and Brunner [3], the survey paper [2], and the recent papers [4, 6, 22, 27], etc..

Finite element methods (FEMs) are efficient numerical methods that extensively used in solving partial differential equations and integral equations. FEMs have also been introduced to solve ordinary differential equations (ODEs) and delay differential equations. See, for example, [8, 9, 20, 21] for ODEs, [7, 16] for DDEs with constant delay, [5, 13] for DDEs with proportional delay, and [26] for Volterra functional integro-differential equations with vanishing delays.

Superconvergence and supercloseness are two hot topics in FEMs. If the errors of numerical solutions $U$ at some points are far less than the global error, we call this phenomenon as superconvergence and the points are called superconvergence points. If the distance between the numerical solution $U$ and some interpolant $\Pi u$ of the exact solution $u$ is far less than that between the numerical solution $U$ and the exact solution $u$, that is, $\|U - \Pi u\| \ll \|u - U\|$, we call this phenomenon as supercloseness. Based on the superconvergence and supercloseness, one can put postprocessing techniques onto the numerical solution $U$ and get a new approximation $U^*$ of higher order convergence. There are several popular postprocessing techniques. In the early stages, Sloan iteration was proposed in [23, 24] to improve the convergence of solutions of integral equations. Zienkiewice and Zhu [28, 29] mentioned the postprocessing method of superconvergence patch recovery which leads to global superconvergence of the new approximate solution $U^*$ for partial differential equations (PDEs). The polynomial preserving recovery postprocessing

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method was proposed in [19, 30]. By combining the two adjacent elements and constructing higher order interpolation for one dimensional case (or combining all adjacent elements in high dimensional space), the interpolation postprocessing method ([17, 18]) was proposed to accelerate the numerical solutions.

The superconvergent points of CG and DG solutions are Lobatto [7, 13] and Radau II points [16, 12] respectively for DDEs of constant delay and proportional delay. For DDEs of pantograph type, Huang et al. [12] used two types of postprocessing techniques to improve the global convergence of DG solutions. They [25] obtained all the superconvergent points of CG solutions according to the supercloseness between the CG solution $U$ and the interpolation $\Pi_h u$ of the exact solution under uniform mesh and analyzed the optimal global convergence and local superconvergence of continuous Galerkin solutions for pantograph DDEs under quasi-geometric meshes (more general quasi-graded case).

As a sequel to papers [13, 25], we consider in this paper the delay differential equation with nonlinear vanishing delay,

$$\begin{aligned} &u'(t) = a(t)u(t) + b(t)u(\theta(t)) + f(t), \qquad t \in J = [0,T], \\ &u(0) = u_0. \end{aligned} \tag{1}$$

The delay item $\theta$ satisfies the conditions: (i) $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$, (ii) $\min\limits_{t\in J} \theta^{'}(t) =: q_0 > 0$. We study the superconvergence properties of the "postprocessed" CG solutions obtained by postprocessing for DDE (1). It will be shown that the convergence order of the CG solutions can be improved considerably by several postprocessing methods.

The outline of this paper is as follows. In section 2 we review the CG method for (1) and introduce the convergence results of the CG solutions. In section 3, we illustrate the supercloseness between the CG solution $U$ and a suitable interpolation $\Pi u$ of the exact solution $u$ and locate all the superconvergent points (subsection 3.1). Then we present two kinds of interpolation postprocessing methods, which respectively based on the supercloseness and the superconvergence points (subsections 3.2 & 3.3 ). In subsection 3.4, we present a type of postprocessing method using integral iteration to accelerate the convergence order of the CG solutions. In order to obtain higher order of convergence, in section 4, we propose another interpolation postprocessing method based on the superconvergence properties of the nodal points. Finally, we display numerical results to illustrate our theoretical analysis in section 5.

## 2. The CG method and convergence analysis

In this section, we introduce the CG method for DDE (1) with quasi-graded meshes and the global convergence properties of the CG solution.

**2.1. The CG method.** We assume that the given functions $a, b$ and $f$ in (1) are continuous on $J = [0,T]$. Suppose that on a small initial subinterval $J_0 = [0, t_0]$ ( $t_0 = \theta^k(T), k \in N$), for a suitable value of $k$, the approximation $\phi(t)$ of the exact solution $u$ is known. $\phi(t)$ can be obtained by the CG method or by the truncation of Taylor expansion of the exact solution $u(t)$. We then solve the following equation

$$\begin{aligned} &u'(t) = a(t)u(t) + b(t)u(\theta(t)) + f(t), \qquad t_0 \le t \le T, \\ &u(t) = \phi(t), \qquad \theta(t_0) \le t \le t_0. \end{aligned} \tag{2}$$

Selecting the appropriate mesh is crucial, which impact the computational complexity of numerical scheme. We introduce the macro-mesh $\{\xi_\mu\}$ of the interval $[t_0, T]$ by setting

$$t_0 = \xi_0 < \xi_1 < \cdots < \xi_k = T, \quad \xi_\mu := \theta^{k-\mu}(T) \;\; (0 \le \mu \le k),$$

with increasing size $H_\mu := \xi_\mu - \xi_{\mu-1} \;\; (\mu = 1, \cdots, k)$ denoting the macro-steps.

On the subinterval

$$I_\mu := [\xi_{\mu-1}, \xi_\mu] \;\; (\mu = 1, \cdots, k),$$

we insert $l-1$ nodes

$$t_0 = \xi_0 < t_1 < \cdots < t_l = \xi_1 < \cdots < t_{2l} = \xi_2 < \cdots < t_{kl} = \xi_k = T,$$

and the nodes satisfy the property $t_{n-l} = \theta(t_n) \;\; (n = l, \cdots, kl)$. In the last macro subinterval $I_k$, $t_{(k-1)l+1}, \cdots, t_{kl-1}$ can be chosen arbitrarily. The mesh $J_N : t_0 = \xi_0 < t_1 < \cdots < t_l = \xi_1 < \cdots < t_{2l} = \xi_2 < \cdots < t_{kl} = \xi_k = T$ is called the quasi-graded mesh. We will use the following notation

$$\begin{aligned} &I_n := [t_{n-1}, t_n], \quad h_n := t_n - t_{n-1}, \\ &N = kl, \quad \tilde{N} = N + 1, \quad h := \max_{1 \le n \le N} h_n \quad (1 \le n \le N). \end{aligned}$$

The corresponding CG finite space can be defined as

$$S_m^{(0)}(J_N) = \{v \in C(J) : v|_{I_n} \in P_m, 1 \le n \le N\}.$$

Where $P_m$ denotes the set of polynomials of degree not exceeding $m$, with $m \ge 1$. In the CG method, we expect to get an approximation solution $U \in P_m(I_n)$ for (2), satisfying

$$\begin{aligned} &\sum_{n=1}^{N} \int_{I_n} U'(t)v(t)dt = \\ &\sum_{n=1}^{N} \int_{I_n} [a(t)U(t) + b(t)U(\theta(t)) + f(t)]v(t)dt, \quad v(t) \in S_{m-1}^{(0)}(J_N). \end{aligned} \tag{3}$$

Here we set $U(t) = \phi(t)$, $\theta(t_0) \le t \le t_0$. Because of the continuity of $U(t)$, we have $U(t_n) = \lim_{t \to t_n^-} U(t) = \lim_{t \to t_n^+} U(t)$. Which implies $U(t) \in J_n$ has only $m$ degrees of freedom on each subinterval and $v(t) \in P_{m-1}(I_n)$ accordingly.

We also note that the CG method (3) can be interpreted, and thus formulated, as a time-stepping scheme. If $U$ is known on the time intervals $I_n$, we find $U|_{I_n} \in P_m(I_n)$ by solving

$$\int_{I_n} U'(t)v(t)dt = \int_{I_n} [a(t)U(t) + b(t)U(\theta(t)) + f(t)]v(t)dt, \forall v \in P_{m-1}(I_n). \tag{4}$$

**2.2. Convergence analysis.** Suppose the global convergence order of $U$ is $h^p$ and $U^*$ is a new approximation to $u$ derived by some postprocessing process for $U$, we will say that $U^*$ is global superconvergent on $J$, if

$$\|u - U^*\|_\infty = \mathcal{O}(h^{p^*}) \ll \|u - U\|_\infty = \mathcal{O}(h^p) \quad \text{with} \quad p^* > p.$$

To carry out the global superconvergence analysis of CG method, we introduce some useful lemmas.

**Lemma 2.1.** *([13]) Assume the following:*

(1) *The functions* $a(t), b(t), f(t)$ *describing the DDE (1) are in* $C^m(J)$.
(2) $u \in W^{m+1,\infty}([0,T])$ *is the exact solution of the DDE (1).*
(3) $\phi \in C^m(J_0)$, *and* $\|u-\phi\|_{J_0,\infty} \le Ch_0^{m+1}\|u\|_{J_0,m+1,\infty}$, *where* $h_0 = t_0$ *(choosing* $t_0$ *such that* $t_0 \le h$*).*
(4) $J_N$ *is a quasi-graded mesh for* $[t_0,T]$ *and* $h$ *is sufficiently small.*
(5) $U \in S_m^{(0)}(J_N)$ *is the CG solution in* $[t_0,T]$.
(6) $\overline{U}$ *is the approximation of* $u$ *in* $J$ *(i.e.,* $\overline{U} = \phi$ *in* $[0,t_0], \overline{U} = U$ *in* $[t_0,T]$*).*

*Then there follows the optimal global convergence estimate:*

$$\|u-\overline{U}\|_\infty \le Ch^{m+1}\|u\|_{m+1,\infty} \quad (m \ge 2).$$

We will discuss the superconvergent points of the CG approximation. Let $\{l_{n,i}(t): \ i \ge 0\}$ denote the set of Legendre polynomials defined on given subinterval $I_n$. Then, the zeros of the polynomial $M_{n,m}(t) = \int_{t_{n-1}}^t l_{n,m-1}(t)dt$ $(m \ge 2)$ are called the Lobatto points on $I_n$. They have the following property:

$$M_{n,m}(t) = l_{n,m}(t) - l_{n,m-2}(t) \quad (m \ge 2).$$

The following theorem shows that the superconvergence of CG solution occurs at the mesh points and other Lobatto points.

**Lemma 2.2.** ([13]) *Let the assumptions (1),(2),(4),(5),(6) in Lemma 2.1 hold and suppose that* $\|u-\phi\|_{\infty,J_0} \le Ch_0^{2m}\|u\|_{2m,\infty,J_0}$. *Then the nodal superconvergence is given by*

$$|(u-U)(t_n)| \le Ch^{2m}\|u\|_{2m,\infty}, \quad n = 1,\cdots,\widetilde{N} \quad (m \ge 2). \tag{5}$$

*If we have* $\|u-\phi\|_{\infty,J_0} \le Ch_0^{m+2}\|u\|_{m+2,\infty,J_0}$, *the attainable superconvergence order at other Lobatto points* $t_{nr}$ *is*

$$|(u-U)(t_{nr})| \le Ch^{m+2}\|u\|_{m+2,\infty}, \quad n = 1,\cdots,\widetilde{N}, \quad r = 1,\cdots,m-1 \quad (m \ge 2)$$

*where* $t_{nr} = t_{n-\frac{1}{2}} + h_n s_r$ *and* $s_r$ *are the zeros of the* $(m+1)$*-degree polynomial* $M_{m+1}(s)$ *in* $(-1,1)$.

**Remark 2.1.** We see from Lemma 2.2 that the Lobatto points $t_{nr}$ are superconvergent points of CG solutions. Thus, there arises the question as to whether we can improve the global convergence order of numerical solutions based on the superconvergent points and some postprocessing techniques.

## 3. Postprocessing Methods for CG Solution

In this section, we first get the supercloseness between the CG solution $U$ and a particular interpolation of the exact solution $u$ defined by (6)-(8). Based on the superconvergent points and supercloseness properties, we introduce several postprocessing methods to get the global superconvergence of numerical solutions.

**3.1. Supercloseness analysis.** We introduce the continuous interpolation operator $\Pi_h$ which is important in supercloseness analysis. The interpolation operator $\Pi_h: \ C[0,1] \to S_m^{(0)}(J_N)$ is defined by

$$\Pi_h u(t_{n-1}) = u(t_{n-1}); \tag{6}$$

$$\Pi_h u(t_n) = u(t_n); \tag{7}$$

$$\int_{I_n} \Pi_h uvdt = \int_{I_n} uvdt, \quad \forall v \in P_{m-2}(I_n), \quad m \ge 2. \tag{8}$$

($\Pi_h$ only satisfies (6), (7) when m=1). It is obvious that the interpolation operator satisfy the error estimate

$$\|u - \Pi_h u\|_{I_n,\infty} \leq Ch^{m+1}\|u\|_{I_n,m+1,\infty}, \tag{9}$$

$$\|u - \Pi_h u\|_{I_n,0} \leq Ch^{m+1}\|u\|_{I_n,m+1}. \tag{10}$$

In the following, we show the definition of "supercloseness" by citing from [17, 18].

**Definition 3.1.** *If the error between the finite element solution and some interplant of the exact solution is much smaller than that between the finite element solution and the exact solution, that is, if*

$$\|\Pi_h u - U\|_\infty \ll \|u - U\|_\infty$$

*then this phenomenon is called "supercloseness".*

**Remark 3.1.** If we obtained a new approximation $U^*$ by applying some type of postprocessing operations on the CG solution $U$, satisfying $\|U^* - u\|_\infty \ll \|U - u\|_\infty$, this means $U^*$ has higher convergence than $U$ itself.

We analyze the supercloseness between the CG solution U and the interpolant $\Pi_h u$ of the exact solution $u$ as follows.

**Theorem 3.1.** *Under the conditions in Lemma 2.1, the following global supercloseness result holds when $u \in W^{m+1,\infty}(J)$ :*

$$\|\Pi_h u - U\|_\infty \leq Ch^{m+2}\|u\|_{m+1,\infty}, \qquad (m \geq 2). \tag{11}$$

*Proof.* In order to prove this theorem, we also need to introduce the orthogonal projection operator $\mathcal{P}:\ L^2(J_N) \to S_{m-1}^{(0)}(J_N)$

$$\int_{I_n} \mathcal{P}uvdt = \int_{I_n} uvdt, \quad \forall v \in P_{m-1}(I_n), \quad m \geq 2.$$

It is easy to prove that $\int_{I_n}(\mathcal{P}u)^2du \leq \int_{I_n} u^2du$. The proof is similar to the idea of Theorem 2.1 of [25] in which the authors proved the same conclusion under uniform mesh. Under the quasi-graded mesh, $\theta(t)$ maps precisely the current nodes to some previous ones, we don't have to consider the case that $\theta(t)$ maps $t$ of $I_n$ to some two adjacent former subintervals. We leave this proof to the reader. □

**Remark 3.2.** Theorem 3.1 implies that the CG solution $U$ is closer to $\Pi_h u$ (the interpolation of the exact solution) than to $u$ itself. This is a supercloseness property of the CG solution for the DDE (2) under quasi-graded mesh.

**3.2. Interpolation postprocessing technique.** Based on the supercloseness and the superconvergence results in the last two subsections, we are ready now to use the interpolation postprocessing to improve the convergence order and analyze the resulting global superconvergence. We will define two different interpolation postprocessing methods, those are, an integration postprocessing method based on the integral conditions, and a Lagrange postprocessing method based on the superconvergent points.

We assume that $I_{2h}$, with mesh size $2h$, is obtained by combining two adjacent subintervals $I_i$ and $I_{i+1}$. Which means the number of elements $\tilde{N}$ for $J_h$ is even. The easiest and natural way of constructing a higher-order interpolation operator is to define $\Pi_{2h}^{2m}$ of degree $2m$ associated with $J_{2h}$ ,

$$\Pi_{2h}^{2m}u|_{I_i \cup I_{i+1}} \in P_{2m} \qquad (i = 1, 3, \ldots, \widetilde{N} - 1). \tag{12}$$

We first introduce the integration scheme. The interpolation operator $\Pi_{2h}^{2m}$ satisfies the conditions

$$\Pi_{2h}^{2m}u(t_i) = u(t_i), \quad i = n-1, n, n+1, \tag{13}$$

$$\int_{I_i} \Pi_{2h}^{2m}uvdt = \int_{I_i} uvdt, \quad \forall v \in P_{m-2}(I_i), \quad m \geq 2, \quad i = n, n+1. \tag{14}$$

For this type of interpolation definition, it is easy to check that

$$\|v - \Pi_{2h}^{2m}v\|_\infty \leq Ch^{m+2}\|u\|_{m+2,\infty}, \quad \forall v \in W^{m+2,\infty}(J), \tag{15}$$

$$\Pi_{2h}^{2m}\Pi_h v = \Pi_{2h}^{2m}v, \quad \forall v \in L^\infty(J), \tag{16}$$

$$\|\Pi_{2h}^{2m}v\|_\infty \leq C\|v\|_\infty, \quad \forall v \in L^\infty(J). \tag{17}$$

Based on the superconvergent points, we then introduce the Lagrange scheme. The interpolation operator $\Pi_{2h}^{2m}$ satisfies the conditions

$$\Pi_{2h}^{2m}u(t_i^*) = u(t_i^*), \quad (i = 1, \ldots, m+1), \tag{18}$$

where $t_i^*(i = 1, 2, \cdots, m+1)$ are the $m+1$ Lobatto points that lie in $I_n$ and $I_{n+1}(n = 1, 3, \ldots, \widetilde{N}-1)$. It can be readily verified that this type of interpolation operator also satisfies conditions (15)-(17).

The following theorem shows that one can obtain global superconvergence of order $m+2$ by the integration (or the Lagrange) postprocessing method.

**Theorem 3.2.** *Suppose $u \in W^{m+2,\infty}(J)$ and the conditions stated in Lemma 2.1 hold. If the interpolation operator $\Pi_{2h}^{2m}$ is defined either by* (13), (14) *or by* (18), *then the following global superconvergence estimate holds for $m \geq 2$ :*

$$\|u - \Pi_{2h}^{2m}U\|_\infty \leq Ch^{m+2}\|u\|_{m+2,\infty}. \tag{19}$$

*Proof.* We first prove the case that the interpolation operator $\Pi_{2h}^{2m}$ is defined by (13) and (14). From Theorem 3.1 and the properties (15)-(17) of the interpolation postprocessing operator $\Pi_{2h}^{2m}$, we have that

$$\begin{aligned}
\|u - \Pi_{2h}^{2m}U\|_\infty &\leq \|u - \Pi_{2h}^{2m}u\|_\infty + \|\Pi_{2h}^{2m}u - \Pi_{2h}^{2m}\Pi_h u\|_\infty \\
&\quad + \|\Pi_{2h}^{2m}\Pi_h u - \Pi_{2h}^{2m}U\|_\infty \\
&\leq Ch^{m+2}\|u\|_{m+2,\infty} + C\|\Pi_h u - U\|_\infty \\
&\leq Ch^{m+2}\|u\|_{m+2,\infty} + Ch^{m+2}\|u\|_{m+1,\infty} \\
&\leq Ch^{m+2}\|u\|_{m+2,\infty}.
\end{aligned}$$

For the interpolation operator $\Pi_{2h}^{2m}$ defined by (18), we can prove the corresponding superconvergence result (19) in a similar way. □

In fact, the degree $(2m)$ is not the only choice for the interpolation postprocessing operator, the superconvergence result (19) can be obtained by any interpolation operator $\Pi_{2h}^{p}$ $(p \geq m+1)$ which satisfies (15)-(17). We can also construct the operator $\Pi_{2h}^{p}$ based on the least-square theory.

### 3.3. Polynomial preserving recovery (PPR) postprocessing technique.

For this method, the interpolation operator $\Pi_{2h}^{m+1}U$ of degree $(m+1)$ associated with $J_{2h}$ is

$$\Pi_{2h}^{m+1}u|_{I_n \cup I_{n+1}} \in P_{m+1} \quad (n = 1, 3, \cdots, \widetilde{N}-1). \tag{20}$$

In each "bigger" subinterval $I_n \cup I_{n+1}$, $\Pi_{2h}^{m+1}u$ is the solution of the least-square problem

$$\sum_{i=1}^{2m+1} \left|u(t_i^*) - \Pi_{2h}^{m+1}u(t_i^*)\right|^2 = \min_{v\in P_{m+1}(I_n\cup I_{n+1})} \sum_{i=1}^{2m+1} \left|u(t_i^*) - v(t_i^*)\right|^2, \tag{21}$$

where $t_i^*$ $(i = 1, \cdots, m+1)$ are the $m+1$ Lobatto points in $I_n$ and $I_{n+1}$ $(n = 1, 3, \dots, \widetilde{N}-1)$. It can be verified that this type of interpolation also satisfies the conditions (15)-(17). We thus have the following theorem.

**Theorem 3.3.** *Under the conditions in Lemma 2.1 and assume that $u \in W^{m+2,\infty}(J)$. If the interpolation operator $\Pi_{2h}^{m+1}$ is defined by (20)-(21), then the following global superconvergence estimate holds for $m \geq 2$:*

$$\|u - \Pi_{2h}^{m+1}U\|_\infty \quad \leq \quad Ch^{m+2}\|u\|_{m+2,\infty}. \tag{22}$$

The proof of Theorem 3.3 is similar to that of Theorem 3.2. We leave the proof to the reader.

**3.4. Iteration postprocessing technique.** In this section, we introduce another postprocessing method. For any $t \in I_n$, we define a new numerical solution $\Pi_h^2U(t)$ satisfying

$$\Pi_h^2U(t) := U(t_{n-1}) + \int_{t_{n-1}}^t \big(a(s)U(s) + b(s)U(\theta(s)) + f(s)\big)ds, \tag{23}$$

We then obtain the following global superconvergence result for the new postprocessed solution "$\Pi_h^2U(t)$".

**Theorem 3.4.** *Let the conditions stated in Lemma 2.1 hold, under the assumption that $u \in W^{m+1,\infty}(J)$ with $m \geq 2$, if the iteration operator $\Pi_h^2U(t)$ is defined by (23). Then we have*

$$\|u - \Pi_h^2U\|_\infty \quad \leq \quad Ch^{m+2}\|u\|_{m+1,\infty}. \tag{24}$$

*Proof.* Integrating both sides of (1) from $t_{n-1}$ to $t$ $(t \in I_n)$, we find that

$$u(t) = u(t_{n-1}) + \int_{t_{n-1}}^t \big(a(s)u(s) + b(s)u(\theta(s)) + f(s)\big)ds. \tag{25}$$

Combining (23) with (25), there follows

$$\begin{aligned} u(t) - \Pi_h^2U(t) \quad &= \quad u(t_{n-1}) - U(t_{n-1}) \\ &\quad\; + \int_{t_{n-1}}^t (a(s)(u(s) - U(s)) + b(s)(u(\theta(s)) - U(\theta(s)))ds \\ &=: \quad I_1 + I_2. \end{aligned} \tag{26}$$

From Lemma 2.2

$$|I_1| \quad = \quad |u(t_{n-1}) - U(t_{n-1})| \leq Ch^{m+2}\|u\|_{m+1,\infty} \tag{27}$$

and

$$\begin{aligned}
\|I_2\|_{I_n,\infty} &= \Big\| \int_{t_{n-1}}^{t} \big(a(s)(u(s)-U(s)) + b(s)(u(\theta(s)) - U(\theta(s))\big)ds \Big\|_{I_n,\infty} \\
&\leq \max_{t_{n-1}\leq t\leq t_n} \big(|a(t)|, |b(t)|\big) \int_{I_n} \big(|u(s)-U(s)| + |u(\theta(s)) - U(\theta(s))|\big)ds \\
&\leq C \max_{t_{n-1}\leq t\leq t_n} \big(|a(t)|, |b(t)|\big) \|u-U\|_\infty \cdot h \\
&\leq Ch^{m+2}\|u\|_{m+1,\infty}, \qquad (28)
\end{aligned}$$

we complete the proof of Theorem 4.1. □

## 4. Higher Order Interpolation Postprocessing Method

We notice that the $2m$-order superconvergence (see (5)) holds at the mesh points of $J$. This section will show that one can obtain the global superconvergence order of $2m$ after suitable postprocessing based on the nodal superconvergence results.

For simplicity, we demonstrate our idea for the case of $m = 3$. Let $\tilde{N}$ (number of elements for $J$) be a multiple of 5 so that we can define an interpolation operator $\Pi_{5h}^5$ of degree 5 associated with $J_{5N}$ as follows:

$$\Pi_{5h}^5 u|_{I_n\cup I_{n+1}\cup I_{n+2}\cup I_{n+3}\cup I_{n+4}} \in P_5, \quad (n = 1, 6, 11, \ldots \tilde{N}-4),$$

and

$$\Pi_{5h}^5 u(t_i) = u(t_i), \quad (i = n-1, n, n+1, n+2, n+3, n+4). \tag{29}$$

It is easy to verify that

$$\|v - \Pi_{5h}^5 v\|_\infty \leq Ch^6\|u\|_{6,\infty}, \quad \forall v \in W^{6,\infty}(J), \tag{30}$$

$$\|\Pi_{5h}^5 v\|_\infty \leq C\|v\|_\infty, \quad \forall v \in L^\infty(J). \tag{31}$$

**Theorem 4.1.** *Under the conditions in Lemma 2 and assume that $u \in W^{2m,\infty}(J)$. We have*

$$\|u - \Pi_{5h}^5 U\|_\infty \leq Ch^6\|u\|_{6,\infty}. \tag{32}$$

*Proof.* Denoting the Lagrange basis function corresponding to $\{t_j\}$ by $\{\psi_j\}(1 \leq j \leq \tilde{N})$ , we have

$$\Pi_{5h}^5(U-u)(t) = \sum_{j=1}^{N} \Pi_{5h}^5(U-u)(t_j)\psi_j(t),$$

which together with (5), (31) and the uniform boundedness of $\{\psi_j\}_1^N$ lead to

$$\|\Pi_{5h}^5(U-u)\|_{0,\infty} \leq \sum_{j=1}^{N} Ch^6\|u\|_{6,\infty}\|\psi_j\|_{0,\infty} \leq Ch^6\|u\|_{6,\infty}. \tag{33}$$

Thus, using the interpolation property (30), we obtain that

$$\|\Pi_{5h}^5 U - u\|_{0,\infty} \leq \|\Pi_{5h}^5(U-u)\|_{0,\infty} + \|\Pi_{5h}^5 u - u\|_{0,\infty} \leq Ch^6\|u\|_{6,\infty}.$$

□

Similarly, we can define an interpolation operator $\Pi_{(2m-1)h}^{2m-1}$ of degree $2m-1$ associated with the mesh $J_{(2m-1)N}$,

$$\|u - \Pi_{(2m-1)h}^{2m-1} U\|_\infty \leq Ch^{2m}\|u\|_{2m,\infty}. \tag{34}$$

TABLE 1. Global errors and the supercloseness for $m = 2$.

| | | global error | | supercloseness | |
|---|---|---|---|---|---|
| $l$ | $\tilde{N}$ | err | R | er | $\mathrm{R_E}$ |
| 2 | 259 | 2.6158e-05 | | 1.1195e-06 | |
| 4 | 2081 | 3.3138e-06 | 2.9807 | 7.0795e-08 | 3.9839 |
| 6 | 7033 | 9.8623e-07 | 2.9890 | 1.4027e-08 | 3.9912 |
| 8 | 16673 | 4.1699e-07 | 2.9923 | 4.4474e-09 | 3.9928 |

**Remark 4.1.** It is clear that when $m \geq 3$ the global superconvergence order of $2m$ obtained by this postprocessing method is higher than the order of $m+2$ obtained by the postprocessing methods above.

## 5. Numerical Experiment

In this section, we use numerical examples to verify the global superconvergence order of CG solutions that proposed in the above theoretical analysis. We use the following notations:

$$err = \|u - U\|_\infty = \max_{t\in J} |u(t) - U(t)|, \qquad R = \frac{log(err_{N1}/err_{N2})}{log(h_{N1}/h_{N2})},$$

$$er = \|\Pi_h u - U\|_\infty = \max_{t\in J} |\Pi_h u(t) - U(t)|, \qquad R_E = \frac{log(er_{N1}/er_{N2})}{log(h_{N1}/h_{N2})},$$

$$Ierr = \|u - \Pi_{2h}^{2m} U\|_\infty = \max_{t\in J} |u(t) - \Pi_{2h}^{2m} U(t)|, \quad R_I = \frac{log(Ierr_{N1}/Ierr_{N2})}{log(h_{N1}/h_{N2})}.$$

Where the notations "$err, Ierr$" and "$R, R_I$" are errors and convergence orders of the CG solutions and the "post-processed' solutions by integration postprocessing respectively. In the following tables, we use similar notations "$Lerr$", "$Perr$", "$Nerr$", "$Herr$" to represent errors of "post-processed" CG solutions by Lagrange, PPR, iteration, and the higher order interpolation postprocessings respectively. "$R_L$", "$R_P$", "$R_N$", "$R_H$" represent the corresponding convergence orders. The notations"$er$" and "$R_E$" are supercloseness and the corresponding convergence order respectively.

**Example 5.1** We consider the DDE with nonlinear vanishing delay:

$$u'(t) = -2u(t) + u(\theta(t)) + 3e^t - e^{\theta(t)}, \quad 0 < t \leq 1, \tag{35}$$
$$u(0) = 1.$$

The delay function is $\theta(t) = \arctan(t)$ and the exact solution is $u(t) = e^t$. In the initial subinterval $J_0 = [0, t_0]$, we get the approximation $\phi(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}$ of the exact solution $u(t)$ by Taylor expansion. Here, we choose the quasi-graded mesh $J_h$ with the last $l+1$ nodes being chosen equidistantly. The total number of subsections is $\tilde{N} = N + 1 = kl + 1$. The numerical results are obtained by the piecewise quadratic CG approximation ($P_2^{CG}$) and piecewise cubic CG approximation ($P_3^{CG}$), respectively.

We plot a simple quasi-graded mesh in Figure 1 which shows that $\theta(t)$ maps precisely the current nodes to some previous ones.

(1) The convergence order of the CG solution for $P_2^{CG}$ and $P_3^{CG}$ and the corresponding supercloseness are presented in Tables 1-2.

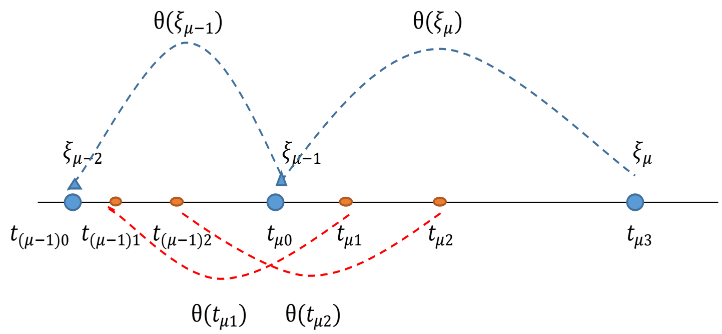


FIGURE 1. A simple quasi-graded mesh.

TABLE 2. Global errors and the supercloseness for $m = 3$.

| | | global error | | supercloseness | |
|---|---|---|---|---|---|
| $l$ | $\tilde{N}$ | err | R | er | $\mathrm{R_E}$ |
| 2 | 259 | 1.7772e-07 | | 4.1470e-09 | |
| 4 | 2081 | 1.1417e-08 | 3.9604 | 1.3241e-10 | 4.9690 |
| 8 | 7033 | 2.2759e-09 | 3.9774 | 1.8443e-11 | 4.9717 |
| 16 | 16673 | 7.2367e-10 | 3.9829 | 6.5326e-12 | 4.9992 |

TABLE 3. Errors of the interpolation postprocessing ($m = 2$).

| | | global error | | integration error | | Lagrange error | | PPR error | |
|---|---|---|---|---|---|---|---|---|---|
| $l$ | $\tilde{N}$ | err | R | Ierr | $\mathrm{R_I}$ | Lerr | $\mathrm{R_L}$ | Perr | $\mathrm{R_P}$ |
| 2 | 259 | 2.6158e-05 | | 1.4537e-06 | | 1.2954e-06 | | 2.6597e-06 | |
| 4 | 2081 | 3.3138e-06 | 2.9807 | 9.2330e-08 | 3.9768 | 8.3288e-08 | 3.9592 | 1.7205e-07 | 3.9503 |
| 6 | 7033 | 9.8623e-07 | 2.9890 | 1.8335e-08 | 3.9870 | 1.6602e-08 | 3.9777 | 3.4394e-08 | 3.9706 |
| 8 | 16673 | 4.1699e-07 | 2.9923 | 5.8177e-09 | 3.9901 | 5.2777e-09 | 3.9836 | 1.0949e-08 | 3.9788 |

TABLE 4. Errors of the interpolation postprocessing ($m = 3$).

| | | global error | | integration error | | Lagrange error | | PPR error | |
|---|---|---|---|---|---|---|---|---|---|
| $l$ | $\tilde{N}$ | err | R | Ierr | $\mathrm{R_I}$ | Lerr | $\mathrm{R_L}$ | Perr | $\mathrm{R_P}$ |
| 2 | 259 | 1.7772e-07 | | 5.2095e-09 | | 4.3440e-09 | | 2.3304e-08 | |
| 4 | 2081 | 1.1417e-08 | 3.9604 | 1.6593e-10 | 4.9725 | 1.3763e-10 | 4.9801 | 7.6121e-10 | 4.9361 |
| 6 | 7033 | 2.2759e-09 | 3.9774 | 2.1761e-11 | 5.0102 | 1.7982e-11 | 5.0194 | 1.0186e-10 | 4.9606 |
| 8 | 16673 | 7.2367e-10 | 3.9829 | 5.4068e-12 | 5.0403 | 4.5133e-12 | 4.8051 | 2.4740e-11 | 4.9916 |

We conclude from Tables 1-2 that

$$\|u - U\|_\infty = O(h^{m+1}), \quad \|\Pi_h u - U\|_\infty = O(h^{m+2}), (m = 2, 3).$$

(2) We present errors of the interpolation postprocessing approximations ( (13), (14), (18) and (20) ) for $P_2^{CG}$ and $P_3^{CG}$ in Tables 3-4.

TABLE 5. Errors of the iteration postprocessing ($m = 2$).

| | | global error | | iteration error | |
|---|---|---|---|---|---|
| $l$ | $\tilde{N}$ | err | R | Nerr | $\mathrm{R_N}$ |
| 2 | 259 | 2.6158e-05 | | 1.4537e-06 | |
| 4 | 2081 | 3.3138e-06 | 2.9807 | 9.2330e-08 | 3.9768 |
| 6 | 7033 | 9.8623e-07 | 2.9890 | 1.8335e-08 | 3.9870 |
| 8 | 16673 | 4.1699e-07 | 2.9923 | 5.8177e-09 | 3.9901 |

TABLE 6. Errors of the iteration postprocessing ($m = 3$).

| | | global error | | iteration error | |
|---|---|---|---|---|---|
| $l$ | $\tilde{N}$ | err | R | Nerr | $\mathrm{R_N}$ |
| 2 | 259 | 1.7772e-07 | | 5.4067e-09 | |
| 4 | 2081 | 1.1417e-08 | 3.9604 | 1.7203e-10 | 4.9740 |
| 8 | 7033 | 2.2759e-09 | 3.9774 | 2.2657e-11 | 4.9907 |
| 16 | 16673 | 7.2367e-10 | 3.9829 | 5.7292e-12 | 4.9992 |

TABLE 7. Errors of the higher order interpolation postprocessing.

| | | global error | | higher order interpolation error | |
|---|---|---|---|---|---|
| $l$ | $\tilde{N}$ | err | R | Herr | $\mathrm{R_H}$ |
| 5 | 4066 | 4.7020e-09 | | 3.6287e-10 | |
| 10 | 32561 | 2.9770e-10 | 3.9814 | 5.3140e-12 | 6.0935 |

We conclude from Tables 3-4 that

$$\|u - U\|_\infty = O(h^{m+1}), \quad \max_{1\le n\le N} |u - \Pi_{2h}^{2m} U| = O(h^{m+2}),$$
$$\max_{1\le n\le N} |u - \Pi_{2h}^{m+1} U| = O(h^{m+2}), \ (m = 2, 3).$$

(3) We show the errors of the iteration postprocessing approximations (23) for $P_2^{CG}$ and $P_3^{CG}$ in Tables 5-6.

We conclude from Tables 5-6 that

$$\|u - U\|_\infty = O(h^{m+1}), \qquad \max_{1\le n\le N} |u - \Pi_h^2 U| = O(h^{m+2}) \ (m = 2, 3).$$

The new numerical solution $\Pi_h^2 U$ by iteration postprocessing method gains higher global convergence order than the CG solution $U$. This confirms the correctness of Theorem 3.4.

(4) We show the errors of the higher order interpolation postprocessing approximations (29) for $P_3^{CG}$ in Table 7.

For $m = 3$, we conclude from Table 7 that

$$\|u - U\|_\infty = O(h^4), \qquad \max_{1\le n\le N} |u - \Pi_{5h}^5 U| = O(h^6).$$

The new numerical solution $\Pi_{(2m-1)h}^{2m-1} U$ obtained by the higher order interpolation postprocessing method gains higher global convergence than the normal postprocessed CG solution. This confirms the correctness of Theorem 4.1.

## 6. Concluding remarks

In this paper, we discuss the global superconvergence of the "postprocessed" CG solutions for DDEs of nonlinear vanishing delay under quasi-graded meshes by several postprocessing techniques. Based on the supercloseness between the CG solution $U$ and the interpolation $\Pi_h u$ of the exact solution $u$, we improve the global convergence by some postprocessing methods. All the results can be extended to the general nonlinear case. The numerical example illustrate the validation of the postprocessing methods.

In the future, we will research the Galerkin methods and postprocessing techniques of the nonlinear multiple delay differential equations and get numerical solutions of higher order global convergence.

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Beijing Institute for Scientific and Engineering Computing, Beijing University of Technology, Beijing, 100124, China

*E-mail*: qmhuang@bjut.edu.cn, jiangkun@emails.bjut.edu.cn

*E-mail*: xuxiuxiu@emails.bjut.edu.cn