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WONG–ZAKAI APPROXIMATIONS OF STOCHASTIC ALLEN–CAHN EQUATION

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Abstract. We establish an unconditional and optimal strong convergence rate of Wong–Zakai type approximations in Banach space norm for a parabolic stochastic partial differential equation with monotone drift, including the stochastic Allen–Cahn equation, driven by an additive Brownian sheet. The key ingredient in the analysis is the full use of additive nature of the noise and monotonicity of the drift to derive a priori estimation for the solution of this equation. Then we use the factorization method and stochastic calculus in martingale type 2 Banach spaces to deduce sharp error estimation between the exact and approximate Ornstein–Uhlenbeck processes, in Banach space norm. Finally, we combine this error estimation with the aforementioned a priori estimation to deduce the desired strong convergence rate of Wong–Zakai type approximations.

Key words. Stochastic Allen–Cahn equation, Wong–Zakai approximations, strong convergence rate.

1. Introduction

Consider the following parabolic stochastic partial differential equation (SPDE) driven by an additive Brownian sheet W:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)) + \frac{\partial^2 W(t,x)}{\partial t \partial x}, \quad (t,x) \in (0,T] \times (0,1), \tag{1}$$

with the following initial value and homogeneous Dirichlet boundary conditions:

$$u(t,0) = u(t,1) = 0, \ u(0,x) = u_0(x), \quad (t,x) \in [0,T] \times (0,1). \tag{2}$$

Here f satisfies certain monotone and polynomial growth conditions (see Assumption 2.1). We remark that if $f(x) = x - x^3$, then Eq. (1) is called the stochastic Allen–Cahn equation. This type of stochastic equation, arising from phase transition in materials science by stochastic perturbation such as impurities of the materials, has been extensively studied in the literatures; see, e.g., [4, 12] for one-dimensional white noises and [9, 13] for possibly high-dimensional colored noises.

The main concern in this paper is to derive an unconditional and optimal strong convergence rate of Wong–Zakai–Galerkin approximations to simulate the Brownian sheet in Eq. (1). Specifically, we simulate the space-time white noise by temporal piecewise constant approximation and then make the spectral projection to this temporal approximation (see Eq. (19)). This type of approximation and its versions, such as the spatiotemporal piecewise constant approximation, have been investigated by many researchers in mathematical and numerical settings. See, e.g., [1, 8] for mathematical applications to support theorem in Hölder norm for parabolic SPDEs and the existence of stochastic flow for a stochastic differential equation without Lipschitz conditions; see, e.g., [3, 7, 16] for numerical applications to construct Galerkin approximations for SPDEs with Lipschitz coefficients and the

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convergence of the Wong–Zakai approximate attractors to the original attractor of stochastic reaction-diffusion equations.

We note that the same simulation method had been used in [10] for the stochastic Burgers equation, where the authors derived the strong convergence of the proposed simulation method without any algebraic rate. On the other hand, the authors of [12] regularized the white noise by a spatiotemporal Wong–Zakai approximations and apply to a practical Monte–Carlo method combined with an Euler–Galerkin scheme for the stochastic Allen–Cahn equation. They used a probabilistic maximum principle which leads to the assumption that $u_0 \in L^\infty(0,1)$ to prove the conditional convergence rate

$$\Big(\mathbb{E}\Big[\chi_{\Omega_{\tau,h}}\|u-\widehat{u}\|^2_{L^2((0,T)\times(0,1))}\Big]\Big)^{\frac{1}{2}}=\mathcal{O}\Big(\tau^{\frac{1}{4}}+h/\tau^{\frac{1}{4}}\Big),$$

in a large subset $\Omega_{\tau,h}\subset\Omega$ such that $\mathbb{P}(\Omega_{\tau,h})\to 1$ as the temporal and spatial step sizes τ, h tend to 0, where u and $\widehat{u}$ denote the exact and Wong–Zakai approximate solution of the stochastic Allen–Cahn equation, respectively.

These problems are main motivations for this study to give an unconditional and optimal strong convergence rate of Wong–Zakai-type approximations of Eq. (1) with a monotone drift which grows polynomially. Our approach shows that, to derive a strong convergence rate of the proposed Wong–Zakai–Galerkin approximations under the $L^\infty(0,T;L^2(\Omega;L^2(0,1)))$-norm, it is necessary to bound the exact solution and derive the strong convergence rate of the associated exact and approximate Ornstein–Uhlenbeck processes in the $L^p(0,T;L^p(\Omega;L^p(0,1)))$-norm and the $L^\infty(0,T;L^l(\Omega;L^l(0,1)))$-norm, respectively, for possibly large indices $p,l>2$ (see (26)). This is mainly due to the appearance of the polynomial growth in the nonlinearity and quite different from that of [3, 7] where these authors only needed to deal with the $L^\infty(0,T;L^2(\Omega;L^2(0,1)))$-norm.

To derive the aforementioned a priori estimation for the solution of Eq. (1), the key ingredient in our analysis is by making full use of the additive nature of the noise which allows the transformation of Eq. (1) to the equivalent random partial differential equation (PDE) (13) and the monotonicity of f (see Proposition 2.1). Then we combine the factorization method with stochastic calculus in martingale type 2 Banach spaces to bounded uniformly the exact and approximate Ornstein–Uhlenbeck processes and derive a sharp strong convergence rate for them in Banach setting (see Lemma 2.1 and Theorem 3.1).

The main result is the following unconditional strong convergence rate of the aforementioned Wong–Zakai–Galerkin approximations applied to Eq (1):

$$\sup_{t\in[0,T]}\Big(\mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|^p_{L^p(0,1)}\Big]\Big)^{\frac{1}{p}}=\mathcal{O}\Big[\Big(\frac{1}{m}\Big)^{\frac{1}{4}}\wedge\Big(\frac{1}{n}\Big)^{\frac{1}{2}}\Big], \tag{3}$$

for any $1\le p<\frac{p_*}{2}+1$ provided that $u_0\in L^{p_*}(\Omega;L^{p_*}(0,1))$ (see Theorem 3.2 and Remark 3.2). Here m, n are the number of temporal steps and dimension of spectral Galerkin space, and u and $u^{m,n}$ denote the exact solution of Eq. (1) and the Wong–Zakai–Galerkin approximate solution of Eq. (19), respectively. Note that we generalize, in a separate paper [14], the approach of the present paper in combination with new techniques to derive a strong convergence rate of a fully discrete approximation for Eq. (1) under certain regularity condition on the initial datum.

The rest of this article is organized as follows. Some frequently used notations and preliminaries of stochastic calculus in martingale type 2 Banach settings are given in the next section, where we derive a priori estimation for the solution of Eq. (1). Finally, we deduce the optimal strong convergence rate for the Wong–Zakai–Galerkin approximation (19) of Eq. (1) in the last section.

2. Preliminaries

In this section, we give some commonly used notations and preliminaries of the stochastic calculus in martingale type 2 Banach setting, as well as a priori estimation for the solution of Eq. (1).

2.1. Notations and Assumption. Let $p \geq 1$, $r \in [1, \infty]$, $q \in [2, \infty]$, $\theta \geq 0$ and $\delta, \kappa \in [0, 1]$. Here and after we denote by $L_x^q := L_x^q(0, 1)$ and $\mathbb{H} := L_x^2$. Similarly, L_ω^p and L_t^r denote the related Lebesgue spaces on Ω and $[0, T]$, respectively. For convenience, sometimes we use the temporal, sample path and spatial mixed norm $\|\cdot\|_{L_\omega^p L_t^r L_x^q}$ in different orders, such as

$$\|X\|_{L_\omega^p L_t^r L_x^q} := \left(\int_\Omega \left(\int_0^T \left(\int_0^1 |u(t, x, \omega)|^q \mathrm{d}x \right)^{\frac{r}{q}} \mathrm{d}t \right)^{\frac{p}{r}} \mathrm{d}\mathbb{P}(\omega) \right)^{\frac{1}{p}}$$

for $u \in L_\omega^p L_t^r L_x^q$, with the usual modification for $r = \infty$ or $q = \infty$.

Denote by A the Dirichlet Laplacian on L_x^q for $q \geq 2$. Then A is the infinitesimal generator of an analytic C_0-semigroup $S(\cdot)$ on L_x^q, and thus one can define the fractional powers $(-A)^\theta$ for $\theta \in \mathbb{R}$ of the self-adjoint and positive definite operator $-A$. Let $\theta \geq 0$ and $\mathbb{W}_x^{\theta,q}$ be the domain of $(-A)^{\theta/2}$ equipped with the norm $\|\cdot\|_{\mathbb{W}_x^{\theta,q}}$ (denote $\dot{\mathbb{H}}_x^\theta := \mathbb{W}_x^{\theta,2}$ and $\|\cdot\|_\theta := \|\cdot\|_{\mathbb{W}_x^{\theta,q}}$):

$$\|X\|_{\mathbb{W}_x^{\theta,q}} := \|(-A)^{\frac{\theta}{2}} X\|_{L_x^q}, \quad X \in \mathbb{W}_x^{\theta,q}.$$

For a Banach space $(B, \|\cdot\|_B)$ and a bounded subset $\mathscr{O} \subset \mathbb{R}$, we use $\mathcal{C}(\mathscr{O}; B)$ to denote the Banach space consisting of B-valued continuous functions f such that $\|f\|_{\mathcal{C}(\mathscr{O};B)} := \sup_{x \in \mathscr{O}} \|f(x)\|_B < \infty$, and $\mathcal{C}^\kappa(\mathscr{O}; B)$ with $\kappa \in (0, 1]$ to denote the B-valued function f such that

$$\|f\|_{\mathcal{C}^\kappa(\mathscr{O};B)} := \sup_{x \in \mathscr{O}} \|f(x)\|_B + \sup_{x,y \in \mathscr{O}, x \neq y} \frac{\|f(x) - f(y)\|_B}{|x - y|^\kappa} < \infty.$$

In the following we simply denote $\mathcal{C}^\kappa([0, 1]; \mathbb{R}) = \mathcal{C}^\kappa$. Similarly, $L^p(\Omega; \mathcal{C}([0, T]; B))$ is used to denote the Banach space consisting of B-valued a.s. continuous stochastic processes $u = \{u(t) : \ t \in [0, T]\}$ such that

$$\|X\|_{L^p(\Omega;\mathcal{C}([0,T];B))} := \left(\mathbb{E}\left[\sup_{t \in [0,T]} \|u(t)\|_B^p \right] \right)^{\frac{1}{p}} < \infty,$$

and $L^p(\Omega; \mathcal{C}^\delta([0, T]; B))$ with $\delta \in (0, 1]$ to denote B-valued stochastic processes $u = \{u(t) : \ t \in [0, T]\}$ such that

$$\|X\|_{L^p(\Omega;\mathcal{C}^\delta([0,T];B))} : = \left(\mathbb{E}\left[\sup_{t \in [0,T]} \|u(t)\|_B^p \right] \right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\left(\sup_{t,s \in [0,T], t \neq s} \frac{\|u(t) - u(s)\|_B}{|t - s|^\delta} \right)^p \right] \right)^{\frac{1}{p}} < \infty.$$

Throughout we assume that the drift coefficient f of Eq. (1) satisfies the following condition.

Assumption 2.1. *f is continuously differentiable and there exist constants $b \in \mathbb{R}$, $L_f, \widetilde{L_f} \in \mathbb{R}_+$ and $q \geq 2$ such that*

$$(f(x) - f(y))(x - y) \leq b|x - y|^2 - L_f |x - y|^q, \quad x, y \in \mathbb{R}; \tag{4}$$

$$|f(0)| < \infty, \quad |f'(x)| \leq \widetilde{L_f}(1 + |x|^{q-2}), \quad x \in \mathbb{R}. \tag{5}$$

It is clear from (5) that f grows as most polynomially of order $(q-1)$ by mean value theorem:

$$|f(x)| \leq C(1 + |x|^{q-1}), \quad x \in \mathbb{R}, \tag{6}$$

where $C = C(|f(0)|, \widetilde{L_f})$ is a positive constant. Here and what follows we use C to denote a universal constant independent of various discrete parameters which may be different in each appearance. A motivating example of f such that Assumption 2.1 holds true is a polynomial of odd order $(q-1)$ with negative leading coefficient perturbed with a Lipschitz continuous function (for the stochastic Allen–Cahn equation, $q = 4$); see, e.g., [6, Exmple 7.8].

2.2. Stochastic Calculus. In order to apply the theory of stochastic analysis in Banach setting, we need to transform the original SPDE (1) into an infinite dimensional stochastic evolution equation. To this purpose, let us define $F : L_x^{q'} \to L_x^q$ by the Nymiskii operators associated with f, respectively:

$$F(u)(x) := f(u(x)), \quad u \in L_x^{q'}, \ x \in [0, 1].$$

where q' denote the conjugation of q, i.e., $1/q' + 1/q = 1$. Then by Assumption 2.1,

$$F \text{ has a continuous extension from } L_x^{q'} \text{ to } L_x^q, \quad \text{and} \tag{7}$$

$${}_{L_x^{q'}}\langle F(x) - F(y), x - y\rangle_{L_x^q} \leq b\|x - y\|^2 - L_f \|x - y\|_{L_x^q}^q, \quad x, y \in L_x^q, \tag{8}$$

where ${}_{L_x^{q'}}\langle \cdot, \cdot \rangle_{L_x^q}$ denotes the dual between $L_x^{q'}$ and L_x^q. Denote by $W_{\mathbb{H}}$ the $\mathbb{H}$-valued cylindrical Wiener process in a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in [0,T]}, \mathbb{P})$, i.e., there exists an orthonormal basis $\{h_k\}_{k=1}^\infty$ of $\mathbb{H}$ and a sequence of mutually independent Brownian motions $\{\beta_k\}_{k=1}^\infty$ such that

$$W_{\mathbb{H}}(t) = \sum_{k=1}^\infty h_k \beta_k(t), \quad t \in [0, T]. \tag{9}$$

Then Eq. (1) with initial-boundary value condition (2) is equivalent to the following stochastic evolution equation:

$$\mathrm{d}u(t) = (Au(t) + F(u(t)))\mathrm{d}t + \mathrm{d}W_{\mathbb{H}}(t), \ t \in (0, T]; \quad u(0) = u_0. \tag{SACE}$$

Note that for any $q \geq 2$ and $\theta \geq 0$, the function space $\mathbb{W}_x^{\theta,q}$ is a martingale type 2 Banach space. We need the following Burkholder-Davis-Gundy inequality in martingale type 2 Banach space (see, e.g., [2, Theorem 2.4] and [11, Section 2]):

$$\left\| \int_0^t \Phi(r) \mathrm{d}W_{\mathbb{H}}(r) \right\|_{L_\omega^p L_t^\infty L_x^q} \leq C \|\Phi\|_{L^p(\Omega; L^2(0,T; \gamma(\mathbb{H}, L_x^q)))}, \tag{10}$$

for any $\Phi \in L^p(\Omega; L^2(0,T;\gamma(\mathbb{H}, L_x^q)))$ with $p, q \geq 2$, where $\gamma(\mathbb{H}, L_x^q)$ denotes the radonifying operator norm:

$$\|\Phi\|_{\gamma(\mathbb{H},L_x^q)} := \Big\| \sum_{k=1}^{\infty} \gamma_k \Phi h_k \Big\|_{L^2(\Omega';L_x^q)}.$$

Here $\{h_k\}_{k=1}^{\infty}$ is any orthonormal basis of $\mathbb{H}$ and $\{\gamma_n\}_{n\geq 1}$ is a sequence of independent $\mathcal{N}(0,1)$-random variables on a probability space $(\Omega', \mathscr{F}', \mathbb{P}')$, provided that the above series converges. We also note that L_x^q with $q \geq 2$ is a Banach function space with finite cotype, then $\Phi \in \gamma(\mathbb{H}; L_x^q)$ if and only if $(\sum_{k=1}^{\infty} (\Phi h_k)^2)^{1/2}$ belongs to L_x^q for any orthonormal basis $\{h_k\}_{k=1}^{\infty}$ of $\mathbb{H}$; see [15, Lemma 2.1]. Moreover, in this situation,

$$\|\Phi\|_{\gamma(\mathbb{H};L_x^q)}^2 \simeq \Big\| \sum_{k=1}^{\infty} (\Phi h_k)^2 \Big\|_{L_x^{\frac{q}{2}}}, \quad \Phi \in \gamma(\mathbb{H}; L_x^q). \tag{11}$$

2.3. Ornstein–Uhlenbeck Process. Recall that a predictable stochastic process $u : [0,T] \times \Omega \to \mathbb{H}$ is called a mild solution of Eq. (SACE) if $u \in L^{\infty}(0,T;\mathbb{H})$ a.s. and it holds a.s. that

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r))\mathrm{d}r + W_A(t), \quad t \in [0,T], \tag{12}$$

where $\{W_A(t) := \int_0^t S(t-r)\mathrm{d}W_{\mathbb{H}}(r) : \ t \in [0,T]\}$ is the so-called Ornstein–Uhlenbeck process. The uniqueness of the mild solution of Eq. (SACE) is understood in the sense of stochastic equivalence. Set $z(t) := u(t) - W_A(t)$, $t \in [0,T]$. Due to the additive nature, it is clear that u is the unique mild solution of Eq. (SACE) if and only if z is the unique solution of the following random PDE:

$$\dot{z}(t) = Az(t) + F(z(t) + W_A(t)), \quad t \in [0,T]; \quad z(0) = u_0. \tag{13}$$

We begin with the following sharp Hölder regularity and $L_\omega^p L_t^\infty L_x^\infty$-estimation of the Ornstein–Uhlenbeck process W_A. Our main tool is the following factorization formula, which is valid by stochastic Fubini theorem:

$$\int_0^t S(t-r)\mathrm{d}W_{\mathbb{H}}(r) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t (t-r)^{\alpha-1} S(t-r) W_\alpha(r)\mathrm{d}r,$$

where $\alpha \in (0,1)$ and $W_\alpha(t) := \int_0^t (t-r)^{-\alpha} S(t-r)\mathrm{d}W_{\mathbb{H}}(r)$, $t \in [0,T]$. It is known that, when $p > 1$ and $1/p < \alpha < 1$, the linear operator R_α defined by

$$R_\alpha f(t) := \int_0^t (t-r)^{\alpha-1} S(t-r) f(r)\mathrm{d}r, \quad t \in [0,T],$$

is bounded from $L^p(0,T;L_x^q)$ to $\mathcal{C}^\delta([0,T];\mathbb{W}_x^{\theta,q})$ for any $q \geq 2$ with $\delta < \alpha - 1/p$ when $\theta = 0$ or $\delta = \alpha - 1/p - \theta/2$ when $\theta > 0$ and $\alpha > \theta/2 + 1/p$; see, e.g., [6, Proposition 5.14] or [11, Proposition 4.1].

Lemma 2.1. *For any $p \geq 1$, $W_A \in L^p(\Omega; \mathcal{C}^\delta([0,T];\mathcal{C}^\kappa))$ for any $\delta, \kappa \geq 0$ with $\delta + \kappa/2 < 1/4$. In particular, there exists a constant $C = C(p)$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|W_A(t)\|_{L_x^\infty}^p \Big] \leq C. \tag{14}$$

Proof. Let $p, q \geq 2$. Applying Fubini theorem and the Burkholder-Davis-Gundy inequality (10) implies that

$$\begin{aligned}\|W_\alpha\|^p_{L^p_\omega L^p_t L^q_x} &= \int_0^T \mathbb{E}\Big[\Big\|\int_0^t (t-r)^{-\alpha} S(t-r) \mathrm{d}W_{\mathbb{H}}(r)\Big\|^p_{L^q_x}\Big] \mathrm{d}t \\ &\leq C \int_0^T \Big(\int_0^t r^{-2\alpha} \|S(r)\|^2_{\gamma(\mathbb{H}; L^q_x)} \mathrm{d}r\Big)^{\frac{p}{2}} \mathrm{d}t.\end{aligned}$$

By (11) and the uniform boundedness of $\{e_k = \sqrt{2}\sin(k\pi \cdot)\}_{k\in\mathbb{N}_+}$ (which vanishes on the boundary 0 and 1), we have

$$\|S(t)\|^2_{\gamma(\mathbb{H}; L^q_x)} \simeq \Big\|\sum_{k=1}^\infty (S(t)e_k)^2\Big\|_{L^{\frac{q}{2}}_x} \leq \sum_{k=1}^\infty e^{-2\lambda_j t}\|e_k\|^2_{L^q_x} \leq C t^{-\frac{1}{2}}, \quad t \in (0, T],$$

where we have used the elementary inequality $\sum_{k=1}^\infty e^{-2\lambda_j t} \leq C t^{-\frac{1}{2}}$. Then

$$\|W_\alpha\|_{L^p_\omega L^p_t L^q_x} \leq C\Big(\int_0^T \Big(\int_0^t r^{-(2\alpha+\frac{1}{2})} \mathrm{d}r\Big)^{\frac{p}{2}} \mathrm{d}t\Big)^{\frac{1}{p}},$$

which is finite if and only if $\alpha \in (0, 1/4)$. As a result of the Hölder continuity characterization, $W_A \in L^p(\Omega; \mathcal{C}^\delta([0, T]; \mathbb{W}^{\theta,q}_x))$ for any $\delta, \theta \geq 0$ with $\delta + \theta/2 < 1/4$. We conclude by the Sobolev embedding $\mathbb{W}^{\theta,q}_x \hookrightarrow \mathcal{C}^\kappa$ with $\kappa \in [0, \theta - 1/q)$ and taking q sufficiently large. □

2.4. A Priori Moments' Estimation. The existence of a unique mild solution of Eq. (12) which belongs to $\mathcal{C}([0, T]; \mathbb{H}) \cap L^q((0, T) \times (0, 1))$ a.s. under the conditions (7)-(8), and thus Eq. (SACE) under Assumption 2.1, had been established in [6, Theorem 7.17]. In the following, we give a priori estimation of the moments of this solution, which plays a key role in our analysis. A weak moments' estimation had been given in [5, Theorem 4.8] for Eq. (SACE) (with non-random initial datum) where f is a polynomial whose derivative is nonpositive perturbed by a linear function (see [5, Hypothesis 4.4]), i.e., $f(x) = \lambda x - \sum_{k=1}^K a_{2k+1} x^{2k+1}$, $x \in \mathbb{R}$, with $\lambda \in \mathbb{R}$ and $a_{2k+1} > 0$, $k = 1, \cdots, K \in \mathbb{N}_+$.

Proposition 2.1. *Let $p \geq 2$ and Assumption 2.1 hold. Assume that $u_0 \in L^p(\Omega; L^p_x)$. Then Eq.* (SACE) *exists a unique mild solution $u = \{u(t) : t \in [0, T]\}$ which is in $L^p(\Omega; \mathcal{C}(0, T; L^p_x)) \cap L^{p+q-2}(\Omega; L^{p+q-2}(0, T; L^{p+q-2}_x))$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u(t)\|^p_{L^p_x}\Big] + \int_0^T \mathbb{E}\Big[\|u(t)\|^{p+q-2}_{L^{p+q-2}_x}\Big] \mathrm{d}t \leq C\Big(1 + \mathbb{E}\Big[\|u_0\|^p_{L^p_x}\Big]\Big). \tag{15}$$

Proof. Let $t \in [0, T]$. Testing both sides of Eq. (13) by $|z|^{p-2}z$ and integrating by parts yield that

$$\begin{aligned}&\frac{1}{p}\|z(t)\|^p_{L^p_x} + (p-1)\int_0^t \langle |z(r)|^{p-2}, |\nabla z(r)|^2\rangle \mathrm{d}r \\ &= \frac{1}{p}\|u_0\|^p_{L^p_x} + \int_0^t \langle (F(u(r)), |z(r)|^{p-2} z(r)\rangle \mathrm{d}r.\end{aligned}$$

It follows from the condition (4) and Young inequality that

$$\begin{aligned}
&\int_0^t \langle (F(u(r)), |z(r)|^{p-2} z(r)\rangle \mathrm{d}r \\
&= \int_0^t \langle (F(z(r)+W_A(r)) - F(W_A(r)), |z(r)|^{p-2} z(r)\rangle \mathrm{d}r \\
&\quad - \int_0^t \langle W_A(r), |z(r)|^{p-2} z(r)\rangle \mathrm{d}r \\
&\leq C \int_0^t \Big(\|z(r)\|^p_{L^p_x} + \|W_A(r)\|^p_{L^p_x} \Big) \mathrm{d}r - L_f \int_0^t \|z(r)\|^{p+q-2}_{L^{p+q-2}_x} \mathrm{d}r.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\frac{1}{p}\|z(t)\|^p_{L^p_x} + L_f \int_0^t \|z(r)\|^{p+q-2}_{L^{p+q-2}_x} \mathrm{d}r \\
&\leq \frac{1}{p}\|u_0\|^p_{L^p_x} + C\int_0^t \|z(r)\|^p_{L^p_x} \mathrm{d}r + C\int_0^t \|W_A(r)\|^p_{L^p_x} \mathrm{d}r. \qquad (16)
\end{aligned}$$

Now taking $L^{p/p}_\omega L^\infty_t$-norm, we conclude from the estimation (14) and Hölder and Grönwall inequalities that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|z(t)\|^p_{L^p_x}\Big] + \mathbb{E}\Big[\int_0^T \|z(t)\|^{p+q-2}_{L^{p+q-2}_x} \mathrm{d}t\Big] \leq C\Big(1+\mathbb{E}\Big[\|u_0\|^p_{L^p_x}\Big]\Big).$$

This estimation, in combination with the fact that $u = z + W_A$ and the estimation (14), shows (15). □

Remark 2.1. *Using the arguments in Proposition 2.1, one can also show the well-posedness of Eq.* (SACE) *in* $L^p(\Omega; \mathcal{C}(0,T; L^\rho_x)) \cap L^{p(\rho+q-2)/\rho}(\Omega; L^{\rho+q-2}(0,T; L^{\rho+q-2}_x))$ *for any* $p \geq \rho \geq 2$*, provided* $u_0 \in L^p(\Omega; L^\rho_x)$ *and Assumption 2.1 hold. Moreover, the following estimation holds true:*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u(t)\|^p_{L^\rho_x}\Big] + \mathbb{E}\Big[\Big(\int_0^T \|u(t)\|^{\rho+q-2}_{L^{\rho+q-2}_x} \mathrm{d}t\Big)^{\frac{p}{\rho}}\Big] \leq C\Big(1+\mathbb{E}\Big[\|u_0\|^p_{L^\rho_x}\Big]\Big).$$

3. Wong–Zakai–Galerkin Approximations

This section is devoted to establishing the optimal strong convergence rate for Wong–Zakai type approximations.

Let $m, n \in \mathbb{N}_+$. Let $\{I_i := (t_i, t_{i+1}] : \ i = 0, 1, \cdots, m-1\}$ be an equal length subdivision of the time interval $(0,T]$, and $\mathcal{P}_n$ denote the orthogonal projection from $\mathbb{H}$ to its finite dimensional subspace $V_n := \mathrm{span}\{e_1, e_2, \cdots, e_n\}$, where $\{e_k(\cdot) = \sqrt{2}\sin(k\pi\cdot)\}_{k=1}^n$ are the n eigenvectors corresponding to the first eigenvalues $\{\lambda_j = (k\pi)^2\}_{k=1}^n$ of the Dirichlet Laplacian A.

Let β_k^m be the piecewise linear approximation

$$\beta_k^m(t) = \beta_k(t_i) + \frac{m}{T}(\beta_k(t_{i+1}) - \beta_k(t_i))(t - t_i), \quad t \in I_i, \qquad (17)$$

with initial datum $\beta_k^m(0) = 0$, $i = 0, 1, \cdots, m-1$. Since $W_\mathbb{H}$ can be formally represented as (9), the resulting approximation of $W_\mathbb{H}$ can be formally given by

$$W^m_\mathbb{H}(t) = W_\mathbb{H}(t_i) + \frac{m}{T}(W_\mathbb{H}(t_{i+1}) - W_\mathbb{H}(t_i))(t - t_i), \quad t \in I_i. \qquad (18)$$

Denote by $u^{m,n}$ the mild solution of

$$\begin{aligned} &\mathrm{d}u^{m,n}(t) = (Au^{m,n}(t) + F(u^{m,n}(t)))\mathrm{d}t + \mathcal{P}_n \mathrm{d}W^m_{\mathbb{H}}(t),\ t \in (0,T]; \\ &u^{m,n}(0) = u_0. \end{aligned} \tag{19}$$

Then the related approximate Ornstein–Uhlenbeck process is

$$W_A^{m,n}(t) := \int_0^t S(t-r)\mathcal{P}_n \mathrm{d}W^m_{\mathbb{H}}(r), \quad t \in (0,T].$$

Since W^m is piecewise linear and therefore of bounded variation, $W_A^{m,n}$ is indeed a classical Riemann-Stieltjes integral whose sample paths can be simulated:

$$W_A^{m,n}(t) = \frac{m}{T}\sum_{i=0}^{m-1}\int_{I_i}\Big[S(t-r)\mathcal{P}_n\int_{I_i}\mathrm{d}W_{\mathbb{H}}(\tau)\Big]\mathrm{d}r, \quad t \in [0,T]. \tag{20}$$

Here and in the rest part of the paper we set $S(t-r)=0$ for any $0 \le t < r \le T$ to lighten the notations.

We note that such simulation method had been studied in [10, Lemma 2.2] where the authors derived strong convergence in $L^p_\omega L^\infty_{t,x}$-norm for any $p \ge 1$ but without any algebraic rate:

$$\lim_{m,n\to\infty}\|W_A - W_A^{m,n}\|_{L^p_\omega L^\infty_{t,x}} = 0.$$

The following result shows the strong error estimation, between the exact and approximate Ornstein–Uhlenbeck processes, under a weak $L^\infty_t L^p_\omega L^q_x$-norm for any $p \ge 1$ and $q \ge 1$.

Theorem 3.1. *Let $p \ge 1$ and $\rho \ge 1$. There exists a constant $C = C(p,\rho)$ such that*

$$\|W_A - W_A^{m,n}\|_{L^\infty_t L^p_\omega L^\rho_x} \le C\Big[\Big(\frac{1}{m}\Big)^{\frac14} \wedge \Big(\frac{1}{n}\Big)^{\frac12}\Big]. \tag{21}$$

Proof. Due to the monotonicity of the L^p-space with respect to p, to prove (21) for any $p \ge 1$ and $\rho \ge 1$ it suffices to show (21) for any $p = \rho = 2k$ which is a even number.

Fix $t \in [0,T]$. By stochastic Fubini theorem, the approximate Ornstein–Uhlenbeck process $W_A^{m,n}$ in (20) can be rewritten as

$$W_A^{m,n}(t) = \frac{m}{T}\sum_{i=0}^{m-1}\int_{I_i}\Big[\int_{I_i} S(t-\tau)\mathcal{P}_n \mathrm{d}\tau\Big]\mathrm{d}W_{\mathbb{H}}(r).$$

Then we have

$$\begin{aligned} &\mathbb{E}\Big[\|W_A(t) - W_A^{m,n}(t)\|^{2k}_{L^{2k}_x}\Big] \\ &= \mathbb{E}\Big[\Big\|\frac{m}{T}\sum_{i=0}^{m-1}\int_{I_i}\Big[\int_{I_i}(S(t-r) - S(t-\tau)\mathcal{P}_n)\mathrm{d}\tau\Big]\mathrm{d}W_{\mathbb{H}}(r)\Big\|^{2k}_{L^{2k}_x}\Big] \\ &= \Big(\frac{m}{T}\Big)^{2k}\int_0^1 \mathbb{E}\Big[\Big|\sum_{i=0}^{m-1}\int_{I_i}\Big[\int_{I_i}(S(t-r) - S(t-\tau)\mathcal{P}_n)\mathrm{d}\tau\Big]\mathrm{d}W_{\mathbb{H}}(r)\Big|^{2k}\Big]\mathrm{d}x. \end{aligned}$$

It is not difficult to show that

$$\mathbb{E}\Big[\Big|\sum_{i=0}^{m-1} a_i\Big|^{2k}\Big] = \mathbb{E}\Big[\Big(\sum_{i=0}^{m-1} |a_i|^2\Big)^k\Big], \tag{22}$$

for any independent centered random variable a_i, $i = 0, 1 \cdots, m-1$. Due to the independence of the Wiener integral in disjoint temporal intervals, we can use (22) with $a_i = \int_{I_i}\big[\int_{I_i}(S(t-r) - S(t-\tau)\mathcal{P}_n)\mathrm{d}\tau\big]\mathrm{d}W_{\mathbb{H}}(r)$, $i = 0, 1, \cdots, m-1$, and Minkovskii inequality to deduce that

$$\begin{aligned}\mathbb{E}\Big[\|W_A(t) - W_A^{m,n}(t)\|_{L_x^{2k}}^{2k}\Big] &= \Big(\frac{m}{T}\Big)^{2k}\int_0^1 \mathbb{E}\Big[\Big(\sum_{i=0}^{m-1}|a_i|^2\Big)^k\Big]\mathrm{d}x \\ = \Big(\frac{m}{T}\Big)^{2k}\Big\|\sum_{i=0}^{m-1}|a_i|^2\Big\|_{L_{\omega,x}^k}^k &\le \Big(\frac{m}{T}\Big)^{2k}\Big(\sum_{i=0}^{m-1}\|a_i\|_{L_{\omega,x}^{2k}}^2\Big)^k.\end{aligned}$$

It follows from the Burkholder-Davis-Gundy inequality (10), the estimation (11) and Minkovskii inequality that

$$\begin{aligned}&\mathbb{E}\Big[\|W_A(t) - W_A^{m,n}(t)\|_{L_x^{2k}}^{2k}\Big] \\ &\le Cm^{2k}\Big(\sum_{i=0}^{m-1}\int_{I_i}\Big\|\int_{I_i}(S(t-r) - S(t-\tau)\mathcal{P}_n)\mathrm{d}\tau\Big\|_{\gamma(\mathbb{H},L_x^{2k})}^2\mathrm{d}r\Big)^k \\ &\le Cm^{2k}\Big(\sum_{i=0}^{m-1}\int_{I_i}\sum_{j=1}^{\infty}\Big\|\int_{I_i}(S(t-r) - S(t-\tau)\mathcal{P}_n)e_j\mathrm{d}\tau\Big\|_{L_x^{2k}}^2\mathrm{d}r\Big)^k.\end{aligned}$$

As a result of Minkovskii inequality and Fubini Theorem, we get

$$\mathbb{E}\Big[\|W_A(t) - W_A^{m,n}(t)\|_{L_x^{2k}}^{2k}\Big] \le Cm^{2k}\Big(\sum_{j=n+1}^{\infty}\sum_{i=0}^{m-1}\Psi_k^i(t)\Big)^k,$$

where

$$\Psi_k^i(t) := \int_{I_i}\Big[\int_{I_i}\Big(\chi_{r<t}e^{-\lambda_j(t-r)} - \chi_{\tau<t}e^{-\lambda_j(t-\tau)}\Big)d\tau\Big]^2\mathrm{d}r,$$

for $t \in [0, T)$ and $i = 0, 1, \cdots, m-1$. Here χ denotes the indicative function, i.e., $\chi_{r<t} = 1$ when $r < t$ and vanishes otherwise.

If follows from [3, Lemma 3.1] that

$$\sum_{i=0}^{m-1}\Psi_k^i(t) \le 8\Big(\frac{1}{m}\Big)^2\frac{1 - e^{\lambda_j m^{-1}}}{\lambda_j},$$

from which we get

$$\sum_{j=n+1}^{\infty}\sum_{i=0}^{m-1}\Psi_k^i(t) \le 8\Big(\frac{1}{m}\Big)^2\Big(\sum_{j=n+1}^{\infty}\frac{1 - e^{-\lambda_j m^{-1}}}{\lambda_j}\Big) \le C\Big(\frac{1}{m}\Big)^2\Big[\Big(\frac{1}{m}\Big)^{\frac{1}{2}} \wedge \Big(\frac{1}{n}\Big)\Big].$$

Collecting the above estimations, we conclude (21) with $p = \rho = 2k$ being a even number and complete the proof □

Remark 3.1. *The strong error estimation* (21) *is optimal. The temporal strong convergence rate* $\mathscr{O}(m^{-1/4})$ *under the* $L_t^\infty L_{\omega,x}^2$*-norm had been derived in* [3, (31) of Theorem 3.1] *for white noise which is a fractional noise with Hurst parameter* $H = 1/2$*. To illustrate the optimality of the spatial convergence rate* $\mathscr{O}(n^{-1/2})$*, we use the elementary estimation* $e^x \geq 1 + x$ *for any* $x \geq 0$ *to show the reverse estimation*

$$\mathbb{E}\Big[\Big\|\int_0^t S(t-r)\mathrm{d}W_{\mathbb{H}}(r) - \int_0^t S(t-r)\mathcal{P}_n \mathrm{d}W_{\mathbb{H}}(r)\Big\|^2\Big] \\ = \sum_{j=n+1}^{\infty} \frac{1-e^{-2\lambda_j t}}{2\lambda_j} \geq \frac{t}{2(1+2\pi^2 t)} \cdot \frac{1}{n}, \quad t > 0.$$

Now we can give the optimal strong convergence rate of the Wong–Zakai–Galerkin approximation (19) for Eq. (SACE).

Theorem 3.2. *Let* $p_* \geq 2$*,* $u_0 \in L^{p_*}(\Omega; L_x^{p_*})$ *and Assumption 2.1 hold. Let* u *and* $u^{m,n}$ *be the solutions of Eq.* (SACE) *and* (19)*, respectively. Then for any* $p \in [1, \frac{p_*}{q-2} + 1)$*, there exists a constant* $C = C(T, p, p_*, b, q, L_f, L_f')$ *such that*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\|u(t) - u^{m,n}(t)\|_{L_x^p}^p\Big] + \int_0^T \mathbb{E}\Big[\|u(t) - u^{m,n}(t)\|_{L_x^{p+q-2}}^{p+q-2}\Big]\mathrm{d}t \\ \leq C\Big(1 + \|u_0\|_{L_{\omega,x}^{p_*}}^{\frac{p_* p(q-2)}{p_*+q-2}}\Big)\Big[\Big(\frac{1}{m}\Big)^{\frac{1}{4}} \wedge \Big(\frac{1}{n}\Big)^{\frac{1}{2}}\Big]^p. \tag{23}$$

Proof. Define $z^{m,n} := u^{m,n} - W_A^{m,n}$. Then $z^{m,n}$ satisfies

$$\dot{z}^{m,n}(t) = Az^{m,n}(t) + F(z^{m,n}(t) + W_A^{m,n}(t)), \ t \in (0,T]; \ z^{m,n}(0) = u_0.$$

Let $t \in (0,T]$ and denote by $e^{m,n}(t) := z(t) - z^{m,n}(t)$. Then

$$\begin{aligned} &\dot{e}^{m,n}(t) = Ae^{m,n}(t) + F(z(t) + W_A(t)) - F(z^{m,n}(t) + W_A^{m,n}(t)), \ t \in (0,T]; \\ &e^{m,n}(0) = 0. \end{aligned} \tag{24}$$

Testing both sides of Eq. (24) by $|e^{m,n}(t)|^{p-2}e^{m,n}(t)$ and integrating by parts, similarly to the proof of Proposition 2.1, yield that

$$\frac{1}{p}\|e^{m,n}(t)\|_{L_x^p}^p + (p-1)\int_0^t \langle |e^{m,n}|^{p-2}, |\nabla e^{m,n}|^2\rangle \mathrm{d}r \\ = \int_0^t \langle F(z + W_A) - F(z + W_A^{m,n}), |e^{m,n}|^{p-2}e^{m,n}\rangle \mathrm{d}r \\ + \int_0^t \langle F(z + W_A^{m,n}) - F(z^{m,n} + W_A^{m,n}), |e^{m,n}|^{p-2}e^{m,n}\rangle \mathrm{d}r. \tag{25}$$

Using mean value theorem and the conditions (4)-(5), and applying Young and Hölder inequalities, we can bound the two terms in the right-hand side of the above equality by

$$
\begin{aligned}
&\le C\int_0^t \|F(z+W_A)-F(z+W_A^{m,n})\|_{L_x^p}^p \mathrm{d}r + C\int_0^t \|e^{m,n}\|_{L_x^p}^p \mathrm{d}r\\
&\quad - L_f\int_0^t \|e^{m,n}\|_{L_x^{p+q-2}}^{p+q-2}\mathrm{d}r\\
&\le C\int_0^t \Big\|\Big(1+|z|^{q-2}+|W_A|^{q-2}+|W_A^{m,n}|^{q-2}\Big)|W_A-W_A^{m,n}|\Big\|_{L_x^p}^p \mathrm{d}r\\
&\quad + C\int_0^t \|e^{m,n}\|_{L_x^p}^p \mathrm{d}r - \frac{L_f}{2}\int_0^t \|e^{m,n}\|_{L_x^{p+q-2}}^{p+q-2}\mathrm{d}r.
\end{aligned}
$$

Now taking $L_\omega^1 L_t^\infty$-norm on both sides of (25), we have

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|e^{m,n}(t)\|_{L_x^p}^p\Big] + \int_0^T \mathbb{E}\Big[\|e^{m,n}(t)\|_{L_x^{p+q-2}}^{p+q-2}\Big]\mathrm{d}t\\
&\le \Big\|\Big(1+|z|^{q-2}+|W_A|^{q-2}+|W_A^{m,n}|^{q-2}\Big)|W_A-W_A^{m,n}|\Big\|_{L_{t,\omega,x}^p}^p\\
&\quad + C\int_0^T \mathbb{E}\Big[\|e^{m,n}(t)\|_{L_x^p}^p\Big]\mathrm{d}t.
\end{aligned}
$$

Let $\epsilon>0$ and denote $p_\epsilon := \frac{p(p+\epsilon)}{\epsilon}$ such that $\frac{1}{p+\epsilon}+\frac{1}{p_\epsilon}=\frac{1}{p}$. By Hölder inequality, we get

$$
\begin{aligned}
&\Big\|\Big(1+|z|^{q-2}+|W_A|^{q-2}+|W^{m,n}|^{q-2}\Big)|W_A-W_A^{m,n}|\Big\|_{L_{t,\omega,x}^p}^p\\
&\le \|W_A-W_A^{m,n}\|_{L_{t,\omega,x}^{p_\epsilon}}^p\cdot\Big(1+\|z\|_{L_{t,\omega,x}^{(p+\epsilon)(q-2)}}^{p(q-2)}+\|W_A\|_{L_{t,\omega,x}^{(p+\epsilon)(q-2)}}^{p(q-2)}+\|W^{m,n}\|_{L_{t,\omega,x}^{(p+\epsilon)(q-2)}}^{p(q-2)}\Big).
\end{aligned}
$$

Since $p\in[1,\frac{p_*}{q-2}+1)$, one can choose $0<\epsilon<\frac{p_*}{q-2}+1-p$ such that $(p+\epsilon)(q-2)<p_*+q-2$. It follows that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|e^{m,n}(t)\|_{L_x^p}^p\Big] + \int_0^T \mathbb{E}\Big[\|e^{m,n}(t)\|_{L_x^{p+q-2}}^{p+q-2}\Big]\mathrm{d}t\\
&\le C\int_0^T \mathbb{E}\Big[\|e^{m,n}(t)\|_{L_x^p}^p\Big]\mathrm{d}t + C\|W_A-W_A^{m,n}\|_{L_{t,\omega,x}^{p_\epsilon}}^p\\
&\quad\times\Big(1+\|z\|_{L_{t,\omega,x}^{p_*+q-2}}^{p(q-2)}+\|W_A\|_{L_{t,\omega,x}^{p_*+q-2}}^{p(q-2)}+\|W^{m,n}\|_{L_{t,\omega,x}^{p_*+q-2}}^{p(q-2)}\Big).
\end{aligned}
\tag{26}
$$

The error estimation of W_A and $W_A^{m,n}$ in Theorem 3.1, combining with the regularity of W_A in Lemma 2.1 and the estimation (15), ensures that

$$
\|z\|_{L_{t,\omega,x}^{p_*+q-2}}^{p(q-2)}+\|W_A\|_{L_{t,\omega,x}^{p_*+q-2}}^{p(q-2)}+\|W^{m,n}\|_{L_{t,\omega,x}^{p_*+q-2}}^{p(q-2)}\le C\Big(1+\|u_0\|_{L_{\omega,x}^{p_*}}^{\frac{p_*p(q-2)}{p_*+q-2}}\Big).
$$

Substituting the above estimation into (26), we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|e^{m,n}(t)\|_{L_x^p}^p\Big] + \int_0^T \mathbb{E}\Big[\|e^{m,n}(t)\|_{L_x^{p+q-2}}^{p+q-2}\Big]\mathrm{d}t\\
&\le C\Big(1+\|u_0\|_{L_{\omega,x}^{p_*}}^{\frac{p_*p(q-2)}{p_*+q-2}}\Big)\|W_A-W_A^{m,n}\|_{L_{t,\omega,x}^{p_\epsilon}}^p.
\end{aligned}
$$

It follows from the relations $u = z + W_A$ and $u^{m,n} = z^{m,n} + W_A^{m,n}$ and triangle inequality that

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|_{L_x^p}^p\Big] + \int_0^T \mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|_{L_x^{p+q-2}}^{p+q-2}\Big]\mathrm{d}t \\
&\le \|e^{m,n}\|_{L_t^\infty L_{\omega,x}^p}^p + \|e^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{p+q-2} + \|W_A - W_A^{m,n}\|_{L_t^\infty L_{\omega,x}^p}^p + \|W_A - W_A^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{p+q-2} \\
&\le C\Big(1+\|u_0\|_{L_{\omega,x}^{p_*}}^{\frac{p_*p(q-2)}{p_*+q-2}}\Big)\Big(\Big\|W_A - W_A^{m,n}\Big\|_{L_t^\infty L_{\omega,x}^{p\epsilon}}^p + \Big\|W_A - W_A^{m,n}\Big\|_{L_t^\infty L_{\omega,x}^{p+q-2}}^{p+q-2}\Big).
\end{aligned}$$

Applying Theorem 3.1, we get (23). □

Remark 3.2. *In the case of stochastic Allen–Cahn equation, i.e., Eq.* (SACE) *with $f(x) = x - x^3$ for $x \in \mathbb{R}$, then Assumption 2.1 holds with $q = 4$. Applying the estimation* (23) *of Theorem 3.2, the Wong–Zakai–Galerkin approximation* (19) *applied to this equation possesses the strong convergence rate*

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|_{L_x^p}^p\Big] + \int_0^T \mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|_{L_x^{p+2}}^{p+2}\Big]\mathrm{d}t \\
&\le C\Big(1+\|u_0\|_{L_{\omega,x}^{p_*}}^{\frac{2p_*p}{p_*+2}}\Big)\Big[\Big(\frac{1}{m}\Big)^{\frac{p}{4}} \wedge \Big(\frac{1}{n}\Big)^{\frac{p}{2}}\Big],
\end{aligned}$$

for any $2 \le p < \frac{p_*}{2} + 1$ provided that $u_0 \in L^{p_*}(\Omega; L_x^{p_*})$.

One can also use a modified argument as in Theorem 3.2 to derive a strong convergence rate which might not optimal when $q > 2$ under minimal assumptions on the initial datum.

Corollary 3.1. *Let $p \ge 2$, $u_0 \in L^p(\Omega; L_x^p)$ and Assumption 2.1 hold. Let u and $u^{m,n}$ be the solutions of Eq.* (SACE) *and* (19)*, respectively. Then there exists a constant $C = C(T, p, b, q, L_f, L_f')$ such that*

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|_{L_x^p}^p\Big] + \int_0^T \mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|_{L_x^{p+q-2}}^{p+q-2}\Big]\mathrm{d}t \\
&\le C\Big(1+\mathbb{E}\Big[\|u_0\|_{L_x^p}^p\Big]\Big)\Big[\Big(\frac{1}{m}\Big)^{\frac{1}{4}} \wedge \Big(\frac{1}{n}\Big)^{\frac{1}{2}}\Big]^{\frac{p+q-2}{q-1}}. \qquad (27)
\end{aligned}$$

Proof. One only need to modify the proof of Theorem 3.2 by estimating the term

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \langle F(z+W_A) - F(z+W_A^{m,n}), |e^{m,n}|^{p-2}e^{m,n}\rangle \mathrm{d}r\Big|\Big].$$

Using mean value theorem and the conditions (4)-(5), and applying Young inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \langle F(z+W_A) - F(z+W_A^{m,n}), |e^{m,n}|^{p-2}e^{m,n}\rangle \mathrm{d}r\Big|\Big] \\
&\le C\int_0^T \|F(z+W_A) - F(z+W_A^{m,n})\|_{L_{\omega,x}^{\frac{p+q-2}{q-1}}}^{\frac{p+q-2}{q-1}} \mathrm{d}r + \frac{L_f}{2}\|e^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{p+q-2} \\
&\le C\Big\|\Big(1+|z|^{q-2}+|W_A|^{q-2}+|W_A^{m,n}|^{q-2}\Big)|W_A - W_A^{m,n}|\Big\|_{L_{t,\omega,x}^{\frac{p+q-2}{q-1}}}^{\frac{p+q-2}{q-1}} \\
&\quad + \frac{L_f}{2}\|e^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{p+q-2}.
\end{aligned}$$

By Young and Hölder inequalities and the estimation (15), we have

$$\begin{aligned}
&\Big\|\Big(1+|z|^{q-2}+|W_A|^{q-2}+|W_A^{m,n}|^{q-2}\Big)|W_A-W_A^{m,n}|\Big\|_{L_{t,\omega,x}^{\frac{p+q-2}{q-1}}}^{\frac{p+q-2}{q-1}}\\
&\le C\big\|W_A-W_A^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{\frac{p+q-2}{q-1}}\Big(1+\|z\|_{L_{t,\omega,x}^{p+q-2}}^{p+q-2}+\|W_A\|_{L_{t,\omega,x}^{p+q-2}}^{p+q-2}+\|W^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{p+q-2}\Big)\\
&\le C\Big(1+\mathbb{E}\Big[\|u_0\|_{L_x^p}^p\Big]\Big)\big\|W_A-W_A^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{\frac{p+q-2}{q-1}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\langle F(z+W_A)-F(z+W_A^{m,n}),|e^{m,n}|^{p-2}e^{m,n}\rangle \mathrm{d}r\Big|\Big]\\
&\le C\Big(1+\mathbb{E}\Big[\|u_0\|_{L_x^p}^p\Big]\Big)\big\|W_A-W_A^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{\frac{p+q-2}{q-1}}+\frac{L_f}{2}\|e^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{p+q-2}.
\end{aligned}$$

Taking $L^1_\omega L^\infty_t$-norm on both sides of (25) and substituting into the above estimation, we have by Grönwall inequality that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|e^{m,n}(t)\|_{L_x^p}^p\Big]+\int_0^T\mathbb{E}\Big[\|e^{m,n}(t)\|_{L_x^{p+q-2}}^{p+q-2}\Big]\mathrm{d}t\\
&\le C\Big(1+\mathbb{E}\Big[\|u_0\|_{L_x^p}^p\Big]\Big)\big\|W_A-W_A^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{\frac{p+q-2}{q-1}}.
\end{aligned}$$

Following the proof of Theorem 3.2, we get

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)-u^{m,n}(t)\|_{L_x^p}^p\Big]+\int_0^T\mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|_{L_x^{p+q-2}}^{p+q-2}\Big]\mathrm{d}t\\
&\le C\Big(1+\mathbb{E}\Big[\|u_0\|_{L_x^p}^p\Big]\Big)\Big(\|W_A-W_A^{m,n}\|_{L_{t,\omega,x}^{p+q-2}}^{\frac{p+q-2}{q-1}}+\|W_A-W_A^{m,n}\|_{L_t^\infty L_{\omega,x}^{p+q-2}}^p\Big).
\end{aligned}$$

Applying Theorem 3.1, we get (27). □

Remark 3.3. *The assumption on the initial datum, to derive a strong convergence rate between u and $u^{m,n}$ under the $L^p_{\omega,x}$-norm, is minimal. However, the convergence rate in Remark 3.1 is far from sharp, since $\frac{p+q-2}{q-1}\le p$ and the equality holds if and only if $q=2$ which reduces to the Lipschitz case.*

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