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ERROR ANALYSIS OF AN IMMERSED FINITE ELEMENT METHOD FOR TIME-DEPENDENT BEAM INTERFACE PROBLEMS

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Abstract. This article presents an error analysis of a Hermite cubic immersed finite element (IFE) method for solving certain initial-boundary value problems (IBVP) modeling a time-dependent Euler-Bernoulli beam formed by multiple materials together with suitable jump conditions at material interfaces. The optimal convergence of this IFE method is shown by both theoretical proof and numerical simulations.

Key words. Interface problem, time-dependent beam model, IFE method, fully discrete, error analysis.

1. Introduction

In this paper, we present an error analysis of a Hermite cubic immersed finite element (IFE) method for solving interface problems related to a mathematical model for a time-dependent Euler-Bernoulli beam formed with multiple materials. Without loss of generality, we consider a beam of length 1 formed with two materials, and we assume its dynamics is modeled by the following initial-boundary value problem (IBVP) [23]:

$$\rho(x)u_{tt}(x,t) + \big(\beta(x)u_{xx}(x,t)\big)_{xx} = f(x,t),\ x \in (0,1)\backslash\{\alpha\},\ t \in (0,T], \tag{1a}$$

$$u(0,t) = b_1(t),\ u_x(0,t) = b_2(t),\ u(1,t) = b_3(t),\ u_x(1,t) = b_4(t), \tag{1b}$$

$$u(x,0) = g_1(x),\ u_t(x,0) = g_2(x), \tag{1c}$$

and the rigid connection condition across the material interface α as follows:

$$\begin{cases} \big[u(x,t)\big]_{x=\alpha} = 0, & \text{(continuity in the deflection)}, \\ \big[\dfrac{\partial u(x,t)}{\partial x}\big]_{x=\alpha} = 0, & \text{(continuity in the bending angle)}, \\ \big[\beta(x)\dfrac{\partial^2 u(x,t)}{\partial x^2}\big]_{x=\alpha} = 0, & \text{(continuity of the bending moment)}, \\ \big[\dfrac{\partial(\beta(x)\frac{\partial^2 u(x,t)}{\partial x^2})}{\partial x}\big]_{x=\alpha} = 0, & \text{(continuity of the shear)}, \end{cases} \tag{1d}$$

where $u(x,t)$ is the transverse displacement of the beam at time t and longitudinal coordinate x, $\rho(x)$ is the mass density, $\beta(x)$ is the bending modulus or stiffness parameter, and $f(x,t)$ is the distributed loading force. Note that $[w(x,t)]_{x=\alpha} := \lim_{x\to\alpha^+} w(x,t) - \lim_{x\to\alpha^-} w(x,t)$. For simplicity, we assume that the material parameters

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$\rho(x)$ and $\beta(x)$ are both piecewise positive constant functions:

$$\rho(x) = \begin{cases} \rho^-, & x \in \Omega^-, \\ \rho^+, & x \in \Omega^+, \end{cases} \tag{1e}$$

$$\beta(x) = \begin{cases} \beta^-, & x \in \Omega^-, \\ \beta^+, & x \in \Omega^+, \end{cases} \tag{1f}$$

where $\Omega = (0,1)$, $\Omega^- = (0,\alpha)$, $\Omega^+ = (\alpha,1)$ and $\alpha \in \Omega$ is the interface position of the two materials. In the discussion from now on, we let $\rho_{min} := min\{\rho^-, \rho^+\}, \rho_{max} := max\{\rho^-, \rho^+\}$ and $\beta_{max} := max\{\beta^-, \beta^+\}$, $\beta_{min} := min\{\beta^-, \beta^+\}$.

IFE methods are desirable for solving interface problems with a mesh independent of the discontinuity of the coefficients associated with the material interfaces in the differential equations. The author of [8] introduced an IFE method for solving an interface problem of a two point boundary value problem. Afterwards, authors of [21, 11, 9, 2, 6, 15, 22, 7, 18, 14, 16, 1, 5, 10, 17, 19] developed IFE methods for solving elliptic interface problems, some time-dependent interface problems, Stokes interface problems as well as elasticity interface problems and so on. In particular, a Hermite cubic IFE space was developed in [13, 23] for solving interface problems of the 4-th order differential equations modeling a static Euler-Bernoulli beam and numerical examples were provided in those articles to show the optimal convergence of the related IFE method. A recent followup article [12] carried out an error analysis proving the optimal approximation capability for the Hermite cubic IFE space developed in [13, 23] and the optimal convergence of the numerical solution for the static Euler-Bernoulli beam produced in this IFE space by the usual Galerkin finite element scheme. However, so far there has been no error analysis for the IFE method developed in [23] to solve the time-dependent Euler-Bernoulli Beam interface problem, and this promotes us in this article to extend the error analysis reported in [12] to this fully discrete IFE method.

In the error analysis to be presented later, the standard Sobolev space defined on an open set $D \subseteq \Omega$ will be used: for every integer $m \geq 0$,

$$H^m(D) = \{w(x) \mid w^{(j)} \in L^2(D),\ j = 0, 1, \cdots, m\}, \tag{2}$$

on which we have the following norm and semi-norm:

$$\|w\|_{H^m(D)} = \sqrt{\sum_{j=0}^{m} \left\|w^{(j)}\right\|^2_{L^2(D)}},\ |w|_{H^m(D)} = \left\|w^{(m)}\right\|_{L^2(D)},\ \forall w \in H^m(D). \tag{3}$$

Also, we will use the following related Sobolev space: for every integer $m \geq 1$,

$$H^m_0(D) = \{w(x) \in H^m(D) \mid w^{(j)}|_{\partial D} = 0,\ j = 0, 1, \cdots, m-1\}. \tag{4}$$

In the case when $\alpha \in D$, we let $D^\pm = D \cap \Omega^\pm$ and we will consider the following space:

$$\tilde{H}^m(D) = \{w(x) \mid w|_{D^\pm} \in H^m(D^\pm)\}, \tag{5}$$

which is endowed with the following norm and semi-norm:

$$\begin{cases} \|w(x)\|_{\tilde{H}^m(D)} & = \sqrt{\|w\|^2_{H^m(D^-)} + \|w\|^2_{H^m(D^+)}} \\ |w(x)|_{\tilde{H}^m(D)} & = \sqrt{|w|^2_{H^m(D^-)} + |w|^2_{H^m(D^+)}} \end{cases} \quad \forall w \in \tilde{H}^m(D). \tag{6}$$

For a time-dependent function $w(x,t)$ defined on $D\times(0,T)$, we will use the Sobolev space $L^p(0,T;H^m(D))$ with $p\geq 1$ and the related norms are defined by:

(7)
$$\|w(x,t)\|_{L^p(0,T;\tilde{H}^m(D))} = \left(\int_0^T \|w(\cdot,t)\|^p_{\tilde{H}^m(D)}\,dt\right)^{1/p}, \ \forall w\in L^p(0,T;\tilde{H}^m(D)),$$

(8)
$$\|w(x,t)\|_{L^\infty(0,T;\tilde{H}^m(D))} = \underset{t\in(0,T]}{ess\ sup}\ \|w(\cdot,t)\|_{\tilde{H}^m(D)}, \ \forall w\in L^\infty(0,T;\tilde{H}^m(D)).$$

By the usual procedure based on the integration by parts, we can obtain the following weak form for the IBVP (1): for any $t\in(0,T]$, find $u(x,t)\in H^2(\Omega)$ such that

$$\langle \rho u_{tt}, v\rangle + \mathcal{A}(u,v) = \langle f,v\rangle,\ \forall v\in H_0^2(\Omega),\ t\in(0,T], \tag{9a}$$

$$u(0,t)=b_1(t),\ u_x(0,t)=b_2(t),\ u(1,t)=b_3(t),\ u_x(1,t)=b_4(t), \tag{9b}$$

$$\mathcal{A}(u(x,0),v)=\mathcal{A}(g_1,v),\ \mathcal{A}(u_t(x,0),v)=\mathcal{A}(g_2,v),\ \forall v\in H_0^2(\Omega), \tag{9c}$$

where $\langle w,v\rangle$ and $\mathcal{A}(w,v)$ are such that

$$\langle w,v\rangle = \int_0^1 w(x)v(x)dx, \tag{10}$$

$$\mathcal{A}(w,v) = \langle \beta w'', v''\rangle. \tag{11}$$

2. A Hermite cubic IFE Space and a fully discrete IFE method

Finite element methods can be derived from the weak form (9) by following the Galerkin framework which requires a H^2 finite element space. Since we would like to solve the interface problem of a time-dependent Euler-Bernoulli beam with an interface-independent mesh, we choose the Hermite cubic immersed finite element developed in [13, 23]. To describe this IFE space, let $\mathcal{T}_h$ be a quasi-uniform mesh of the solution domain $\Omega=(0,1)=\Omega^-\cup\{\alpha\}\cup\Omega^+$ with following nodes independent of the interface point α:

$$0 = x_1 < x_2 < x_3 < \cdots < x_{N_s} = 1 \tag{12}$$

with

$$e_k=[x_k,x_{k+1}],\ h_k=x_{k+1}-x_k,\ k=1,2,\ldots,N_s-1,\ h=\max_{1\leq k\leq N_s-1} h_k.$$

On each of the non-interface elements such that $\alpha\notin e_k, k=1,2,\ldots,N_s-1$, we use the following standard Hermitian cubic local shape functions:

$$\psi_{k,j}(x) = \begin{cases} N_j\big(F_k(x)\big),\ j=1,3,\\ h_k N_j\big(F_k(x)\big),\ j=2,4,\end{cases} \tag{13}$$

where $N_j(\xi), j=1,2,3,4$ are the Hermite cubic shape functions on reference element $[0,1]$ defined by

$$\begin{cases} N_1(\xi)=2\xi^3-3\xi^2+1, & N_2(\xi)=\xi^3-2\xi^2+\xi,\\ N_3(\xi)=-2\xi^3+3\xi^2, & N_4(\xi)=\xi^3-\xi^2,\end{cases}\quad 0\leq\xi\leq 1, \tag{14}$$

and $F_k(x)=\dfrac{x-x_k}{h_k}$ is the affine mapping from element e_k to the reference element. On the interface element such that $\alpha\in e_k$, the following immersed Hermite cubic local shape functions are used:

$$\widetilde{\psi}_{k,j}(x) = \begin{cases} \widetilde{N}_j\big(F_k(x)\big), \ j = 1,3, \\ h_k \widetilde{N}_j\big(F_k(x)\big), \ j = 2,4, \end{cases} \tag{15}$$

where $\widetilde{N}_j(\xi), j = 1,2,3,4$ are the immersed Hermite cubic shape functions on the reference element $[0,1]$ [13, 23] such that

$$\begin{aligned} &\widetilde{N}_1(\xi) = \begin{cases} 1 + \xi^2(a_1 + b_1(\xi - \hat{\alpha})) & \text{if } 0 \le \xi \le \hat{\alpha}, \\ (\xi - 1)^2(c_1 + d_1(\xi - \hat{\alpha})) & \text{if } \hat{\alpha} \le \xi \le 1, \end{cases} \\ &\widetilde{N}_2(\xi) = \begin{cases} \xi + \xi^2(a_2 + b_2(\xi - \hat{\alpha})) & \text{if } 0 \le \xi \le \hat{\alpha}, \\ (\xi - 1)^2(c_2 + d_2(\xi - \hat{\alpha})) & \text{if } \hat{\alpha} \le \xi \le 1, \end{cases} \\ &\widetilde{N}_3(\xi) = \begin{cases} \xi^2(a_3 + b_3(\xi - \hat{\alpha})) & \text{if } 0 \le \xi \le \hat{\alpha}, \\ 1 + (\xi - 1)^2(c_3 + d_3(\xi - \hat{\alpha})) & \text{if } \hat{\alpha} \le \xi \le 1, \end{cases} \\ &\widetilde{N}_4(\xi) = \begin{cases} \xi^2(a_4 + b_4(\xi - \hat{\alpha})) & \text{if } 0 \le \xi \le \hat{\alpha}, \\ (\xi - 1) + (\xi - 1)^2(c_4 + d_4(\xi - \hat{\alpha})) & \text{if } \hat{\alpha} \le \xi \le 1, \end{cases} \end{aligned} \tag{16}$$

with $\hat{\alpha} = F_k(\alpha)$ and the coefficients in (16) are determined by the following linear systems:

$$\begin{bmatrix} \hat{\alpha}^2 & 0 & -(\hat{\alpha}-1)^2 & 0 \\ 2\hat{\alpha} & \hat{\alpha}^2 & -2(\hat{\alpha}-1) & -(\hat{\alpha}-1)^2 \\ 2\beta^- & 4\hat{\alpha}\beta^- & -2\beta^+ & -4(\hat{\alpha}-1)\beta^+ \\ 0 & 6\beta^- & 0 & -6\beta^+ \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{bmatrix} = \vec{q}_i, \ i = 1,2,3,4, \tag{17}$$

$$\text{with } \vec{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{q}_2 = \begin{bmatrix} -\hat{\alpha} \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{q}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{q}_4 = \begin{bmatrix} \hat{\alpha}-1 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{18}$$

With these Hermite cubic local shape functions, we can define a function $\phi_{k,i}(x), i = 1,2,3,4$ on each element as follows:

$$\phi_{k,i}(x) = \begin{cases} \psi_{k,i}(x), & \text{if } e_k \in \mathcal{T}_h \text{ is a non-interface element}, \\ \widetilde{\psi}_{k,i}(x), & \text{if } e_k \in \mathcal{T}_h \text{ is an interface element}. \end{cases} \tag{19}$$

By their design, it is easy to verify that $\phi_{k,i}(x), i = 1,2,3,4$ satisfy the following Hermite interpolation conditions:

$$\begin{cases} \phi_{k,1}(x_k) = 1, \phi'_{k,1}(x_k) = 0, \phi_{k,1}(x_{k+1}) = 0, \phi'_{k,1}(x_{k+1}) = 0, \\ \phi_{k,2}(x_k) = 0, \phi'_{k,2}(x_k) = 1, \phi_{k,2}(x_{k+1}) = 0, \phi'_{k,2}(x_{k+1}) = 0, \\ \phi_{k,3}(x_k) = 0, \phi'_{k,3}(x_k) = 0, \phi_{k,3}(x_{k+1}) = 1, \phi'_{k,3}(x_{k+1}) = 0, \\ \phi_{k,4}(x_k) = 0, \phi'_{k,4}(x_k) = 0, \phi_{k,4}(x_{k+1}) = 0, \phi'_{k,4}(x_{k+1}) = 1, \end{cases} \tag{20}$$

and the rigid connection condition (1d) in the interface problem, i.e., across the interface element, they satisfy

$$\big[\phi_{k,j}\big]_\alpha = 0, \ \big[\phi'_{k,j}\big]_\alpha = 0, \ \big[\beta\phi''_{k,j}\big]_\alpha = 0, \ \big[\beta\phi''_{k,j}\big]'_\alpha = 0, \ j = 1,2,3,4. \tag{21}$$

We can continue to define the global IFE basis functions $\widetilde{\phi}_i(x), i = 1,2,\cdots,2N_s - 1, 2N_s$ over Ω as follows:

At node x_1, we define

$$\widetilde{\phi}_1(x) = \begin{cases} \phi_{1,1}(x), & \text{if } x \in [x_1, x_2], \\ 0, & \text{if } x \notin [x_1, x_2], \end{cases} \tag{22}$$

and

$$\widetilde{\phi}_2(x) = \begin{cases} \phi_{1,2}(x), & \text{if } x \in [x_1, x_2], \\ 0, & \text{if } x \notin [x_1, x_2]. \end{cases} \tag{23}$$

At node $x_k, k = 2, 3 \cdots, N_s - 1$, we define

$$\widetilde{\phi}_{2k-1}(x) = \begin{cases} \phi_{k-1,3}(x), & \text{if } x \in [x_{k-1}, x_k], \\ \phi_{k,1}(x), & \text{if } x \in [x_k, x_{k+1}], \\ 0, & \text{if } x \notin [x_{k-1}, x_{k+1}], \end{cases} \tag{24}$$

and

$$\widetilde{\phi}_{2k}(x) = \begin{cases} \phi_{k-1,4}(x), & \text{if } x \in [x_{k-1}, x_k], \\ \phi_{k,2}(x), & \text{if } x \in [x_k, x_{k+1}], \\ 0, & \text{if } x \notin [x_{k-1}, x_{k+1}]. \end{cases} \tag{25}$$

At node x_{N_s}, we define

$$\widetilde{\phi}_{2N_s-1}(x) = \begin{cases} \phi_{N_s-1,3}(x), & \text{if } x \in [x_{N_s-1}, x_{N_s}], \\ 0, & \text{if } x \notin [x_{N_s-1}, x_{N_s}], \end{cases} \tag{26}$$

and

$$\widetilde{\phi}_{2N_s}(x) = \begin{cases} \phi_{N_s-1,4}(x), & \text{if } x \in [x_{N_s-1}, x_{N_s}], \\ 0, & \text{if } x \notin [x_{N_s-1}, x_{N_s}]. \end{cases} \tag{27}$$

Then, the Hermitian cubic IFE space is constructed as follows:

$$S_h(\Omega) = span\{\widetilde{\phi}_1, \widetilde{\phi}_2, \cdots, \widetilde{\phi}_{2N_s-1}, \widetilde{\phi}_{2N_s}\}, \tag{28}$$

$$S_{h,0}(\Omega) = S_h(\Omega) \cap H_0^2(\Omega). \tag{29}$$

We consider a fully discretized method for solving the time-dependent beam with the Hermite cubic IFE space. We introduce a uniform partition for the time interval $[0, T]$:

$$0 = t^0 < t^1 < t^2 < \cdots < t^M = T,$$

and let

$$\tau = t^n - t^{n-1}, \ n = 1, 2, \cdots, M,$$

the time step size. For simplicity, we use $w^n(x) := w(x, t^n)$ to denote a function $w(x, t)$ restricted at time level t^n and we adopt the following notations:

$$w^{n,1/4} := \frac{w^{n+1} + 2w^n + w^{n-1}}{4}, \tag{30}$$

$$\partial_t w^n := \frac{w^{n+1} - w^{n-1}}{2\tau}, \tag{31}$$

$$\partial_{tt} w^n := \frac{w^{n+1} - 2w^n + w^{n-1}}{\tau^2}, \tag{32}$$

$$w^{n+1/2} := \frac{w^{n+1} + w^n}{2}, \tag{33}$$

$$\partial_t w^{n+1/2} := \frac{w^{n+1} - w^n}{\tau}. \tag{34}$$

We now define the IFE solution to the weak problem (9) at the time level t^n as the function $u_h^n(x) \in S_h(\Omega)$ such that:

$$\begin{aligned} &\langle \rho \partial_{tt} u_h^n, v_h \rangle + \mathcal{A}(u_h^{n,1/4}, v_h) = \langle f^n, v_h \rangle, \ \forall v_h \in S_{h,0}(\Omega), \\ &\qquad\qquad n = 1, 2, \cdots, M-1, \end{aligned} \tag{35a}$$

$$\begin{aligned} &u_h^n(0) = b_1(t^n), \ u_{h,x}^n(0) = b_2(t^n), \ u_h^n(1) = b_3(t^n), \ u_{h,x}^n(1) = b_4(t^n), \\ &\qquad\qquad n = 1, 2, \cdots, M, \end{aligned} \tag{35b}$$

$$\mathcal{A}(u_h^0, v_h) = \mathcal{A}(g_1, v_h), \ \mathcal{A}(u_h^1, v_h) = \mathcal{A}(u^*, v_h), \ \forall v_h \in S_{h,0}(\Omega), \tag{35c}$$

with

$$u^*(x) := g_1(x) + \tau g_2(x) + \frac{\tau^2}{2} u_{tt}^0(x), \tag{35d}$$

where we suppose that $u_{tt}^0(x)$ is provided by (1a).

In (35a), we employ $u_h^{n,1/4}$ to approximate the unknown at time level t^n using approximations at three time levels around t^n. This scheme can be traced back to [4] where a similar fully discrete scheme was discussed for the second order hyperbolic equations. For the interface problem (1) of a time dependent beam with multiple materials, our error analysis to be presented later in this article shows that the IFE method described by (35) is not only an optimal descritization of the weak problem (9) but also a stable scheme for any choice of the step size τ in the time variable.

3. Error Bounds for the IFE solution

In this section, we derive error bounds for the IFE solution of the IBVP (1). As usual in error analysis and without loss of generality, we assume that this IBVP has a homogeneous boundary condition, i.e., $b_1(t) = b_2(t) = b_3(t) = b_4(t) = 0$. Proceeding to the error bound estimation, we introduce two auxiliary error functions:

$$\eta(x, t) := P_h u(x, t) - u(x, t), \tag{36}$$

$$\xi^n(x) := u_h^n(x) - P_h u^n(x), \tag{37}$$

where P_h is the Ritz projection such that for every $w(x) \in H_0^2(\Omega)$, $P_h w$ is the IFE function $S_{h,0}(\Omega)$ uniquely determined by

$$\mathcal{A}(P_h w, v) = \mathcal{A}(w, v), \ \forall v \in S_{h,0}(\Omega). \tag{38}$$

Then, we have the following error decomposition for the IFE solution $u_h^n(x)$:

$$\begin{aligned} e^n(x) &:= u_h^n(x) - u^n(x) \\ &= \big(u_h^n(x) - P_h u^n(x)\big) + \big(P_h u^n(x) - u^n(x)\big) \\ &= \xi^n(x) + \eta^n(x). \end{aligned} \tag{39}$$

We start from the estimation for $\eta(x,t)$ which is the error in the Ritz projection $P_h u(x,t)$ of $u(x,t)$ associated with the bilinear form $\mathcal{A}(\cdot,\cdot)$. This Ritz projection of a function can be considered as the IFE solution to an interface problem of the related static beam equation described by the fourth order spatial differential operator in (1a), therefore, following the same arguments of [12], we can obtain error bounds for $\eta(x,t)$ given in the following Lemma.

Lemma 3.1. *Assume that the exact solution $u(x,t)$ to the interface problem described by (9) is such that $u \in W^{2,\infty}(0,T;\tilde{H}^4(\Omega))$, $u_{ttt} \in L^2(0,T;\tilde{H}^4(\Omega))$. Then, there exists a constant C independent of the interface location such that $\eta(x,t)$ has the following error bounds with $m = 0,1,2,3$:*

$$\|\eta(\cdot,t)\|_{\tilde{H}^m(\Omega)} \le Ch^{4-m}\left(\|u(\cdot,t)\|_{H^4(\Omega^-)} + \|u(\cdot,t)\|_{H^4(\Omega^+)}\right), \tag{40}$$

$$\|\eta_{tt}\|_{L^\infty(0,T;\tilde{H}^m(\Omega))} \le Ch^{4-m}\left(\|u_{tt}\|_{L^\infty(0,T;H^4(\Omega^-))} + \|u_{tt}\|_{L\infty(0,T;H^4(\Omega^+))}\right), \tag{41}$$

$$\|\eta_{ttt}\|_{L^2(0,T;\tilde{H}^m(\Omega))} \le Ch^{4-m}\left(\|u_{ttt}\|_{L^2(0,T;H^4(\Omega^-))} + \|u_{ttt}\|_{L^2(0,T;H^4(\Omega^+))}\right). \tag{42}$$

proof. By Theorems 4.1 and 4.2 in [12], we have

$$\begin{aligned} \|\eta(\cdot,t)\|_{\tilde{H}^m(\Omega)} &= \sqrt{\sum_{j=0}^{m} |\eta(\cdot,t)|^2_{\tilde{H}^j(\Omega)}} \\ &\le \sqrt{\sum_{j=0}^{m}\left[Ch^{4-j}\big(|u(\cdot,t)|_{H^4(\Omega^-)} + |u(\cdot,t)|_{H^4(\Omega^+)}\big)\right]^2} \\ &\le Ch^{4-m}\left(|u(\cdot,t)|_{H^4(\Omega^-)} + |u(\cdot,t)|_{H^4(\Omega^+)}\right) \\ &\le Ch^{4-m}\left(\|u(\cdot,t)\|_{H^4(\Omega^-)} + \|u(\cdot,t)\|_{H^4(\Omega^+)}\right) \end{aligned}$$

which establishes the estimate in (40). For (41),

$$\begin{aligned} \|\eta_{tt}\|_{L^\infty(0,T;\tilde{H}^m(\Omega))} &= \underset{t\in(0,T]}{ess\ sup}\ \|\eta_{tt}(\cdot,t)\|_{\tilde{H}^m(\Omega)} \\ &\le Ch^{4-m}\left(\underset{t\in(0,T]}{ess\ sup}\ \|u_{tt}(\cdot,t)\|_{H^4(\Omega^-)} + \underset{t\in(0,T]}{ess\ sup}\ \|u_{tt}(\cdot,t)\|_{H^4(\Omega^+)}\right) \\ &= Ch^{4-m}\left(\|u_{tt}\|_{L^\infty(0,T;H^4(\Omega^-))} + \|u_{tt}\|_{L\infty(0,T;H^4(\Omega^+))}\right). \end{aligned}$$

For (42),

$$\begin{aligned}
&\|\eta_{ttt}\|_{L^2(0,T;\tilde{H}^m(\Omega))} = \left(\int_0^T \|\eta_{ttt}(\cdot,t)\|^2_{\tilde{H}^m(\Omega)}\,dt\right)^{1/2} \\
&\le Ch^{4-m}\left[\int_0^T \left(\|u_{ttt}(\cdot,t)\|_{H^4(\Omega^-)} + \|u_{ttt}(\cdot,t)\|_{H^4(\Omega^+)}\right)^2 dt\right]^{1/2} \\
&\le Ch^{4-m}\left[\left(\int_0^T \|u_{ttt}(\cdot,t)\|^2_{H^4(\Omega^-)}\,dt\right)^{1/2} + \left(\int_0^T \|u_{ttt}(\cdot,t)\|^2_{H^4(\Omega^+)}\,dt\right)^{1/2}\right] \\
&= Ch^{4-m}\left(\|u_{ttt}\|_{L^2(0,T;H^4(\Omega^-))} + \|u_{ttt}\|_{L^2(0,T;H^4(\Omega^+))}\right).
\end{aligned}$$

∎

Correspondingly, we have the following results for the finite differences of $\eta(x,t)$ with respect to the time variable.

Lemma 3.2. *Suppose $u(x,t)$ has the same regularity stated in Lemmas 3.1. Then, there exists a constant C independent of the interface location such that the time variable differences of $\eta(x,t)$ have the following estimates:*

$$\begin{aligned}
\|\partial_{tt}\eta^n\|_{\tilde{H}^0(\Omega)} \le Ch^4 \Big(&\|u_{tt}\|_{L^\infty(0,T;H^4(\Omega^-))} \\
&+ \|u_{tt}\|_{L^\infty(0,T;H^4(\Omega^+))}\Big),\ n = 1,2,\cdots,M-1,
\end{aligned} \tag{43}$$

$$\begin{aligned}
\tau\sum_{n=1}^{N-1}\left\|\partial_{tt}(\partial_t\eta^{n+1/2})\right\|^2_{\tilde{H}^0(\Omega)} \le Ch^8 \Big(&\|u_{ttt}\|^2_{L^2(0,T;H^4(\Omega^-))} \\
&+ \|u_{ttt}\|^2_{L^2(0,T;H^4(\Omega^+))}\Big),\ N = 2,\cdots,M-1.
\end{aligned} \tag{44}$$

Proof. By the definition of the difference operator (32), we have

$$\|\partial_{tt}\eta^n\|_{\tilde{H}^0(\Omega)} = \frac{1}{\tau^2}\left\|\eta^{n+1} - 2\eta^n + \eta^{n-1}\right\|_{\tilde{H}^0(\Omega)}. \tag{45}$$

With Taylor expansions:

$$\begin{aligned}
\eta^{n+1} &= \eta^n + \tau\eta_t^n + R_1,\ R_1 = \int_{t^n}^{t^{n+1}} (t^{n+1}-t)\eta_{tt}(x,t)\ dt, \\
\eta^{n-1} &= \eta^n - \tau\eta_t^n + R_2,\ R_2 = \int_{t^{n-1}}^{t^n} (t-t^{n-1})\eta_{tt}(x,t)\ dt,
\end{aligned}$$

we obtain

$$\left\|\eta^{n+1} - 2\eta^n + \eta^{n-1}\right\|_{\tilde{H}^0(\Omega)} = \|R_1 + R_2\|_{\tilde{H}^0(\Omega)} \le \|R_1\|_{\tilde{H}^0(\Omega)} + \|R_2\|_{\tilde{H}^0(\Omega)}.$$

Now, we continue to estimate the norms of R_1 and R_2. First, applying Cauchy-Schward inequality to R_1, we have

$$R_1^2 \le \int_{t^n}^{t^{n+1}} (t^{n+1}-t)^2\ dt \cdot \int_{t^n}^{t^{n+1}} \eta_{tt}^2\ dt \le \tau^3 \int_{t^n}^{t^{n+1}} \eta_{tt}^2\ dt$$

Then, applying the result above as well as the integral mean value theorem , we have

$$\begin{aligned}
\|R_1\|_{\tilde{H}^0(\Omega)} &= \left(\int_{\Omega^-} R_1^2 \ dx + \int_{\Omega^+} R_1^2 \ dx\right)^{\frac{1}{2}} \\
&\le \tau^{\frac{3}{2}} \left(\int_{\Omega^-}\int_{t^n}^{t^{n+1}} \eta_{tt}^2 \ dtdx + \int_{\Omega^+}\int_{t^n}^{t^{n+1}} \eta_{tt}^2 \ dtdx\right)^{\frac{1}{2}} \\
&= \tau^{\frac{3}{2}} \left[\int_{t^n}^{t^{n+1}} \left(\int_{\Omega^-} \eta_{tt}^2 \ dx + \int_{\Omega^+} \eta_{tt}^2 \ dx\right) dt\right]^{\frac{1}{2}} \\
&= \tau^{\frac{3}{2}} \left(\int_{t^n}^{t^{n+1}} \|\eta_{tt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt\right)^{\frac{1}{2}} \\
&= \tau^2 \|\eta_{tt}(\cdot,t^*)\|_{\tilde{H}^0(\Omega)}, \ t^* \in [t^n, t^{n+1}], \\
&\le \tau^2 \|\eta_{tt}\|_{L^\infty(t^n,t^{n+1};\tilde{H}^0(\Omega))},
\end{aligned}$$

similarly,

$$\|R_2\|_{\tilde{H}^0(\Omega)} = \left(\int_{\Omega^-} R_2^2 \ dx + \int_{\Omega^+} R_2^2 \ dx\right)^{\frac{1}{2}} \le \tau^2 \|\eta_{tt}\|_{L^\infty(t^{n-1},t^n;\tilde{H}^0(\Omega))}.$$

Accordingly, we have

$$\left\|\eta^{n+1} - 2\eta^n + \eta^{n-1}\right\|_{\tilde{H}^0(\Omega)} \le 2\tau^2 \|\eta_{tt}\|_{L^\infty(0,T;\tilde{H}^0(\Omega))}. \tag{46}$$

Hence, combining (45), (46) and (41) yields (43).

As for (44), we notice that

$$\begin{aligned}
\partial_{tt}(\partial_t \eta^{n+1/2}) &= \partial_{tt}\left(\frac{\eta^{n+1}-\eta^n}{\tau}\right) = \frac{1}{\tau}(\partial_{tt}\eta^{n+1} - \partial_{tt}\eta^n) \\
&= \frac{1}{\tau}\left(\frac{\eta^{n+2}-2\eta^{n+1}+\eta^n}{\tau^2} - \frac{\eta^{n+1}-2\eta^n+\eta^{n-1}}{\tau^2}\right) \\
&= \frac{1}{\tau^3}(\eta^{n+2} - 3\eta^{n+1} + 3\eta^n - \eta^{n-1})
\end{aligned} \tag{47}$$

By Taylor expansions:

$$\begin{aligned}
\eta^{n+2} &= \eta^n + 2\tau\eta_t^n + 2\tau^2\eta_{tt}^n + \frac{1}{2}\tilde{R}_1, \ \tilde{R}_1 = \int_{t^n}^{t^{n+2}} (t^{n+2}-t)^2 \eta_{ttt}(x,t) \ dt, \\
\eta^{n+1} &= \eta^n + \tau\eta_t^n + \frac{\tau^2}{2}\eta_{tt}^n + \frac{1}{2}\tilde{R}_2, \ \tilde{R}_2 = \int_{t^n}^{t^{n+1}} (t^{n+1}-t)^2 \eta_{ttt}(x,t) \ dt, \\
\eta^{n-1} &= \eta^n - \tau\eta_t^n + \frac{\tau^2}{2}\eta_{tt}^n - \frac{1}{2}\tilde{R}_3, \ \tilde{R}_3 = \int_{t^{n-1}}^{t^n} (t-t^{n-1})^2 \eta_{ttt}(x,t) \ dt,
\end{aligned}$$

we have

$$\left\|\eta^{n+2} - 3\eta^{n+1} + 3\eta^n - \eta^{n-1}\right\|_{\tilde{H}^0(\Omega)}^2 \le \frac{1}{4}\left(\left\|\tilde{R}_1\right\|_{\tilde{H}^0(\Omega)}^2 + 9\left\|\tilde{R}_2\right\|_{\tilde{H}^0(\Omega)}^2 + \left\|\tilde{R}_3\right\|_{\tilde{H}^0(\Omega)}^2\right).$$

We continue to estimate the norms of $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3$. First, applying Cauchy-Schward inequality to $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3$, we obtain

$$\tilde{R}_1^2 \le \int_{t^n}^{t^{n+2}} (t^{n+2}-t)^4 \ dt \cdot \int_{t^n}^{t^{n+2}} \eta_{ttt}^2 \ dt \le (2\tau)^5 \int_{t^n}^{t^{n+2}} \eta_{ttt}^2 \ dt,$$
$$\tilde{R}_2^2 \le \int_{t^n}^{t^{n+1}} (t^{n+1}-t)^4 \ dt \cdot \int_{t^n}^{t^{n+1}} \eta_{ttt}^2 \ dt \le \tau^5 \int_{t^n}^{t^{n+1}} \eta_{ttt}^2 \ dt,$$
$$\tilde{R}_3^2 \le \int_{t^{n-1}}^{t^n} (t-t^{n-1})^4 \ dt \cdot \int_{t^{n-1}}^{t^n} \eta_{ttt}^2 \ dt \le \tau^5 \int_{t^{n-1}}^{t^n} \eta_{ttt}^2 \ dt.$$

Then,

$$\begin{aligned}\left\|\tilde{R}_1\right\|_{\tilde{H}^0(\Omega)}^2 &= \int_{\Omega^-} \tilde{R}_1^2 \ dx + \int_{\Omega^+} \tilde{R}_1^2 \ dx \\ &\le (2\tau)^5 \left(\int_{\Omega^-} \int_{t^n}^{t^{n+2}} \eta_{ttt}^2 \ dtdx + \int_{\Omega^+} \int_{t^n}^{t^{n+2}} \eta_{ttt}^2 \ dtdx \right) \\ &= (2\tau)^5 \int_{t^n}^{t^{n+2}} \left(\int_{\Omega^-} \eta_{ttt}^2 \ dx + \int_{\Omega^+} \eta_{ttt}^2 \ dx \right) dt \\ &= (2\tau)^5 \int_{t^n}^{t^{n+2}} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt,\end{aligned}$$

similarly,

$$\left\|\tilde{R}_2\right\|_{\tilde{H}^0(\Omega)}^2 \le \tau^5 \int_{t^n}^{t^{n+1}} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt,$$
$$\left\|\tilde{R}_3\right\|_{\tilde{H}^0(\Omega)}^2 \le \tau^5 \int_{t^{n-1}}^{t^n} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt.$$

Accordingly, we obtain

$$\begin{aligned}&\left\|\eta^{n+2}-3\eta^{n+2}+3\eta^n-\eta^{n-1}\right\|_{\tilde{H}^0(\Omega)}^2 \\ &\le \frac{\tau^5}{4}\left(32\int_{t^n}^{t^{n+2}} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt + 9\int_{t^n}^{t^{n+1}} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt \right. \\ &\qquad \left. + \int_{t^{n-1}}^{t^n} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt\right)\end{aligned} \tag{48}$$

Combining (47) and (48) yields

$$\begin{aligned}&\tau\sum_{n=1}^{N-1}\left\|\partial_{tt}(\partial_t\eta^{n-1/2})\right\|_{\tilde{H}^0(\Omega)}^2 \\ &\le \frac{1}{4}\sum_{n=1}^{N-1}\left(32\int_{t^n}^{t^{n+2}} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt + 9\int_{t^n}^{t^{n+1}} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt \right. \\ &\qquad \left. + \int_{t^{n-1}}^{t^n} \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt\right) \\ &\le C\int_0^T \|\eta_{ttt}(\cdot,t)\|_{\tilde{H}^0(\Omega)}^2 \ dt = C\left\|\eta_{ttt}\right\|_{L^2(0,T;\tilde{H}^0(\Omega))}^2.\end{aligned}$$

Finally, (44) follows from applying (42) to the estimate above.

■

Now, let us consider the estimates for the term $\xi^n(x)$.

Lemma 3.3. *Let $u(x,t)$ be the exact solution described by (9) satisfying $u_{ttt} \in L^\infty(0,T;H^2(\Omega))$. Then, there exists a constant C independent of the interface location such that the following inequalities hold:*

$$\left\|\xi^1\right\|_{H^2(\Omega)} \leq C\tau^3 \left(\|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega^-))} + \|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega^+))}\right), \tag{49}$$

$$\left\|\xi^{1/2}\right\|_{H^2(\Omega)} \leq C\tau^3 \left(\|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega^-))} + \|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega^+))}\right), \tag{50}$$

$$\left\|\partial_t\xi^{1/2}\right\|_{H^2(\Omega)} \leq C\tau^2 \left(\|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega^-))} + \|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega^+))}\right). \tag{51}$$

Proof. By the definitions of ξ^n and the Ritz Projection, we have

$$\begin{aligned}\left\|\xi^1\right\|^2_{H^2(\Omega)} &\leq C\mathcal{A}(\xi^1,\xi^1) = C\mathcal{A}(u_h^1 - P_hu^1, \xi^1)\\ &= C\mathcal{A}(u^* - u^1, \xi^1) \leq C\left\|u^* - u^1\right\|_{H^2(\Omega)}\left\|\xi^1\right\|_{H^2(\Omega)},\end{aligned}$$

which, along with (35d), implies

$$\begin{aligned}\left\|\xi^1\right\|_{H^2(\Omega)} &\leq C\left\|\int_0^\tau \frac{(\tau-t)^2}{2}u_{ttt}(\cdot,t)\,dt\right\|_{H^2(\Omega)} \leq C\tau^3\|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega))}\\ &\leq C\tau^3\left(\|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega^-))} + \|u_{ttt}\|_{L^\infty(0,T;H^2(\Omega^+))}\right).\end{aligned}$$

It is obvious that $\xi^{1/2} = \frac{\xi^1}{2}$, $\partial_t\xi^{1/2} = \frac{\xi^1}{\tau}$ due to the fact that $\xi^0 = 0$. Hence, (50) and (51) can be easily obtained from (49).

■

Using similar arguments, we can obtain estimates about u in the following lemma.

Lemma 3.4. *Assume that the exact solution $u(x,t)$ to the interface problem described by (9) has the regularity specified in Lemma 3.1 and $u_{tttt} \in L^2(0,T;\tilde{H}^0(\Omega))$. Then, there exists a constant C independent of the interface location such that*

$$\left\|u^{n,1/4} - u^n\right\|_{H^2(\Omega)} \leq C\tau^2\|u_{tt}\|_{L^\infty(0,T;H^4(\Omega))},\ n = 1,2,\cdots,M-1, \tag{52}$$

$$\tau\sum_{n=1}^N\|\partial_{tt}u^n - u_{tt}^n\|^2_{\tilde{H}^0(\Omega)} \leq C\tau^4\|u_{tttt}\|^2_{L^2(0,T;\tilde{H}^0(\Omega))}, N = 1,2,\cdots,M-1, \tag{53}$$

$$\begin{aligned}\frac{\tau}{2}\sum_{n=1}^N\left\|\partial_t(u^{n+1/2,1/4} - u^{n+1/2})\right\|^2_{H^2(\Omega)} &\leq C\tau^4\left(\|u_{ttt}\|^2_{L^2(0,T;H^4(\Omega^-))}\right.\\ &\left.+\|u_{ttt}\|^2_{L^2(0,T;H^4(\Omega^+))}\right),\ N = 1,2,\cdots,M-2,\end{aligned} \tag{54}$$

Proof. We only provide a proof for (53) and the proofs for the other estimates are omitted because of the similarities. By definition,

$$\partial_{tt}u^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2},$$

and by Taylor expansions:

$$u^{n+1} = u^n + \tau u_t^n + \frac{\tau^2}{2} u_{tt}^n + \frac{\tau^3}{6} u_{ttt}^n + \int_{t^n}^{t^{n+1}} \frac{(t^{n+1}-t)^3}{6} u_{tttt}\, dt,$$
$$u^{n-1} = u^n - \tau u_t^n + \frac{\tau^2}{2} u_{tt}^n - \frac{\tau^3}{6} u_{ttt}^n + \int_{t^n}^{t^{n-1}} \frac{(t^{n-1}-t)^3}{6} u_{tttt}\, dt,$$

we have

$$\begin{aligned}
&\|\partial_{tt} u^n - u_{tt}^n\|_{\tilde{H}^0(\Omega)}^2 \\
&= \frac{1}{\tau^4} \left\| \int_{t^n}^{t^{n+1}} \frac{(t^{n+1}-t)^3}{6} u_{tttt}\, dt + \int_{t^n}^{t^{n-1}} \frac{(t^{n-1}-t)^3}{6} u_{tttt}\, dt \right\|_{\tilde{H}^0(\Omega)}^2 \\
&\le C\tau^3 \left(\|u_{tttt}\|_{L^2(t^n,t^{n+1};\tilde{H}^0(\Omega))}^2 + \|u_{tttt}\|_{L^2(t^{n-1},t^n;\tilde{H}^0(\Omega))}^2 \right),
\end{aligned}$$

from which we straightforwardly obtain (53)

■

Lemma 3.5. *Assume that $u(x,t)$ satisfies the regularity requirements of Lemmas 3.1 and 3.3. Then the following estimate holds:*

$$\text{(55)} \qquad \begin{aligned}
\tau \left| \sum_{n=1}^{N} \left\langle \rho \partial_{tt} \eta^n, \partial_t \xi^n \right\rangle \right| &\le C(h^8 + \tau^6) + \delta \left\| \xi^{N+1/2} \right\|_{H^2(\Omega)}^2 \\
&\quad + \frac{\tau}{2} \sum_{n=0}^{N-1} \left\| \xi^{n+1/2} \right\|_{H^2(\Omega)}^2, \ N = 1, 2, \cdots, M-1
\end{aligned}$$

where δ is an arbitrary positive auxiliary constant.

Proof. When $N = 1$, using the inequality $\left|\left\langle u, v \right\rangle\right| \le \delta \|u\|_{\tilde{H}^0}^2 + \frac{1}{4\delta} \|v\|_{\tilde{H}^0}^2$ for arbitrary $\delta > 0$, we have

$$\begin{aligned}
&\tau \left| \sum_{n=1}^{N} \left\langle \rho \partial_{tt} \eta^n, \partial_t \xi^n \right\rangle \right| = \left| \left\langle \rho \partial_{tt} \eta^1, \xi^{1+1/2} - \xi^{1/2} \right\rangle \right| \\
&\le \left| \left\langle \rho \partial_{tt} \eta^1, \xi^{1+1/2} \right\rangle \right| + \left| \left\langle \rho \partial_{tt} \eta^1, \xi^{1/2} \right\rangle \right| \le \left(\frac{\rho_{max}^2}{4\delta} \left\| \partial_{tt} \eta^1 \right\|_{\tilde{H}^0(\Omega)}^2 + \delta \left\| \xi^{1+1/2} \right\|_{H^2(\Omega)}^2 \right) \\
&\qquad + \left(\frac{\rho_{max}^2}{2} \left\| \partial_{tt} \eta^1 \right\|_{\tilde{H}^0(\Omega)}^2 + \frac{1}{2} \left\| \xi^{1/2} \right\|_{H^2(\Omega)}^2 \right)
\end{aligned}$$

which leads to (55) according to estimates given in (43) and (50). Similarly, for $N > 1$, we have

$$\begin{aligned}
&\tau\left|\sum_{n=1}^{N}\left\langle\rho\partial_{tt}\eta^n,\partial_t\xi^n\right\rangle\right| = \left|\sum_{n=1}^{N}\left\langle\rho\partial_{tt}\eta^n,\xi^{n+1/2}-\xi^{n-1/2}\right\rangle\right| \\
=&\left|\sum_{n=1}^{N}\left\langle\rho\partial_{tt}\eta^n,\xi^{n+1/2}\right\rangle - \sum_{n=0}^{N-1}\left\langle\rho\partial_{tt}\eta^{n+1},\xi^{n+1/2}\right\rangle\right| \\
=&\left|\sum_{n=1}^{N-1}\left\langle\rho(\partial_{tt}\eta^n-\partial_{tt}\eta^{n+1}),\xi^{n+1/2}\right\rangle + \left\langle\rho\partial_{tt}\eta^N,\xi^{N+1/2}\right\rangle - \left\langle\rho\partial_{tt}\eta^1,\xi^{1/2}\right\rangle\right| \\
\le&\left|\tau\sum_{n=1}^{N-1}\left\langle\rho\partial_{tt}(\partial_t\eta^{n+1/2}),\xi^{n+1/2}\right\rangle\right| + \left|\left\langle\rho\partial_{tt}\eta^N,\xi^{N+1/2}\right\rangle\right| + \left|\left\langle\rho\partial_{tt}\eta^1,\xi^{1/2}\right\rangle\right| \\
\le&\left(\frac{\rho_{max}^2}{2}\tau\sum_{n=1}^{N-1}\left\|\partial_{tt}(\partial_t\eta^{n+1/2})\right\|_{\tilde{H}^0(\Omega)}^2 + \frac{\tau}{2}\sum_{n=1}^{N-1}\left\|\xi^{n+1/2}\right\|_{H^2(\Omega)}^2\right) \\
&+\left(\frac{\rho_{max}^2}{4\delta}\left\|\partial_{tt}\eta^N\right\|_{\tilde{H}^0(\Omega)}^2 + \delta\left\|\xi^{N+1/2}\right\|_{H^2(\Omega)}^2\right) \\
&+\left(\frac{\rho_{max}^2}{2}\left\|\partial_{tt}\eta^1\right\|_{\tilde{H}^0(\Omega)}^2 + \frac{1}{2}\left\|\xi^{1/2}\right\|_{H^2(\Omega)}^2\right)
\end{aligned}$$

Then, using the estimates in Lemma 3.2 and Lemma 3.3, we obtain

$$\begin{aligned}
&\tau\left|\sum_{n=1}^{N}\left\langle\rho\partial_{tt}\eta^n,\partial_t\xi^n\right\rangle\right| \\
\le&\left(Ch^8 + \frac{\tau}{2}\sum_{n=1}^{N-1}\left\|\xi^{n+1/2}\right\|_{H^2(\Omega)}^2\right) + \left(Ch^8 + \delta\left\|\xi^{N+1/2}\right\|_{H^2(\Omega)}^2\right) + C(h^8+\tau^6) \\
\le& C(h^8+\tau^6) + \delta\left\|\xi^{N+1/2}\right\|_{H^2(\Omega)}^2 + \frac{\tau}{2}\sum_{n=0}^{N-1}\left\|\xi^{n+1/2}\right\|_{H^2(\Omega)}^2
\end{aligned}$$

which leads to the estimate (55)

■

Lemma 3.6. *Let $u(x,t)$ be the exact solution described by* (9) *with the regularity specified by Lemmas 3.3 and 3.4. Then we have*

$$\begin{aligned}
&\left|\tau\sum_{n=1}^{N}\left\langle\rho(\partial_{tt}u^n-u_{tt}^n),\partial_t\xi^n\right\rangle\right| \le C\tau^4 + \frac{\tau}{2}\sum_{n=0}^{N-1}\left\|\partial_t\xi^{n+1/2}\right\|_{\tilde{H}^0(\Omega)}^2 \\
&\qquad + \frac{\tau\delta}{4}\left\|\partial_t\xi^{N+1/2}\right\|_{\tilde{H}^0(\Omega)}^2,\ N=1,2,\cdots,M-1,
\end{aligned} \tag{56}$$

$$\begin{aligned}
&\left|\tau\sum_{n=1}^{N}\mathcal{A}(u^{n,1/4}-u^n,\partial_t\xi^n)\right| \le C(\tau^4+\tau^6) + \delta\left\|\xi^{N+1/2}\right\|_{H^2(\Omega)}^2 \\
&\qquad + \frac{\tau}{2}\sum_{n=0}^{N-1}\left\|\xi^{n+1/2}\right\|_{H^2(\Omega)}^2,\ N=1,2,\cdots,M-1,
\end{aligned} \tag{57}$$

where $\delta \le 1$ is an arbitrary positive constant.

Proof. By definition, for $n \geq 1$, we have

$$\begin{aligned}\|\partial_t \xi^n\|^2_{\tilde{H}^0(\Omega)} &= \left\| \frac{\xi^{n+1} - \xi^{n-1}}{2\tau} \right\|^2_{\tilde{H}^0(\Omega)} = \frac{1}{4} \left\| \frac{\xi^{n+1} - \xi^n}{\tau} + \frac{\xi^n - \xi^{n-1}}{\tau} \right\|^2_{\tilde{H}^0(\Omega)} \\ &\leq \frac{1}{2} \left(\left\| \partial_t \xi^{n+1/2} \right\|^2_{\tilde{H}^0(\Omega)} + \left\| \partial_t \xi^{n-1/2} \right\|^2_{\tilde{H}^0(\Omega)} \right).\end{aligned}$$

When $N = 1$, we have

$$\begin{aligned}&\left| \tau \sum_{n=1}^{N} \left\langle \rho(\partial_{tt} u^n - u^n_{tt}), \partial_t \xi^n \right\rangle \right| = \left| \tau \left\langle \rho(\partial_{tt} u^1 - u^1_{tt}), \partial_t \xi^1 \right\rangle \right| \\ &\leq \frac{\tau \rho^2_{max}}{2\delta} \left\| \partial_{tt} u^1 - u^1_{tt} \right\|^2_{\tilde{H}^0(\Omega)} + \frac{\tau\delta}{2} \left\| \partial_t \xi^1 \right\|^2_{\tilde{H}^0(\Omega)} \\ &\leq C\tau^4 + \frac{\tau\delta}{4} \left(\left\| \partial_t \xi^{1+1/2} \right\|^2_{\tilde{H}^0(\Omega)} + \left\| \partial_t \xi^{1/2} \right\|^2_{\tilde{H}^0(\Omega)} \right)\end{aligned}$$

which leads to (56) for $\delta \leq 1$. For $N > 1$, using (53), we have:

$$\begin{aligned}&\left| \tau \sum_{n=1}^{N} \left\langle \rho(\partial_{tt} u^n - u^n_{tt}), \partial_t \xi^n \right\rangle \right| \\ &\leq \frac{\tau \rho^2_{max}}{2} \sum_{n=1}^{N-1} \|\partial_{tt} u^n - u^n_{tt}\|^2_{\tilde{H}^0(\Omega)} + \frac{\tau}{2} \sum_{n=1}^{N-1} \|\partial_t \xi^n\|^2_{\tilde{H}^0(\Omega)} \\ &\qquad + \frac{\tau \rho^2_{max}}{2\delta} \left\| \partial_{tt} u^N - u^N_{tt} \right\|^2_{\tilde{H}^0(\Omega)} + \frac{\tau\delta}{2} \left\| \partial_t \xi^N \right\|^2_{\tilde{H}^0(\Omega)} \\ &\leq C\tau^4 \|u_{tttt}\|^2_{L^2(0,T;\tilde{H}^0(\Omega))} + \frac{\tau}{4} \sum_{n=1}^{N-1} \left(\left\| \partial_t \xi^{n+1/2} \right\|^2_{\tilde{H}^0(\Omega)} + \left\| \partial_t \xi^{n-1/2} \right\|^2_{\tilde{H}^0(\Omega)} \right) \\ &\qquad + \frac{\tau\delta}{4} \left(\left\| \partial_t \xi^{N+1/2} \right\|^2_{\tilde{H}^0(\Omega)} + \left\| \partial_t \xi^{N-1/2} \right\|^2_{\tilde{H}^0(\Omega)} \right) \\ &\leq C\tau^4 \|u_{tttt}\|^2_{L^2(0,T;\tilde{H}^0(\Omega))} + \frac{\tau}{2} \sum_{n=0}^{N-1} \left\| \partial_t \xi^{n+1/2} \right\|^2_{\tilde{H}^0(\Omega)} + \frac{\tau\delta}{4} \left\| \partial_t \xi^{N+1/2} \right\|^2_{\tilde{H}^0(\Omega)}\end{aligned}$$

which yields (56) again. For the second estimate in the lemma, we first consider the case for $N = 1$:

$$\begin{aligned}&\left| \tau \sum_{n=1}^{N} \mathcal{A}(u^{n,1/4} - u^n, \partial_t \xi^n) \right| = \left| \tau \mathcal{A}(u^{1,1/4} - u^1, \partial_t \xi^1) \right| \\ &\leq \left| \mathcal{A}(u^{1,1/4} - u^1, \xi^{1+1/2}) \right| + \left| \mathcal{A}(u^{1,1/4} - u^1, \xi^{1/2}) \right| \\ &\leq \beta_{max} \left\| u^{1,1/4} - u^1 \right\|_{H^2(\Omega)} \left\| \xi^{1+1/2} \right\|_{H^2(\Omega)} + \beta_{max} \left\| u^{1,1/4} - u^1 \right\|_{H^2(\Omega)} \left\| \xi^{1/2} \right\|_{H^2(\Omega)} \\ &\leq \frac{1}{4\delta} \beta^2_{max} \left\| u^{1,1/4} - u^1 \right\|^2_{H^2(\Omega)} + \delta \left\| \xi^{1+1/2} \right\|^2_{H^2(\Omega)} \\ &\qquad + \frac{1}{2} \beta^2_{max} \left\| u^{1,1/4} - u^1 \right\|^2_{H^2(\Omega)} + \frac{1}{2} \left\| \xi^{1/2} \right\|^2_{H^2(\Omega)}\end{aligned}$$

which leads to (57) according to estimates given in (52) and (50). Then, for $N > 1$, we have

$$\begin{aligned}
&\left|\tau \sum_{n=1}^{N} \mathcal{A}(u^{n,1/4} - u^n, \partial_t \xi^n)\right| \\
=&\left|\sum_{n=1}^{N} \mathcal{A}(u^{n,1/4} - u^n, \xi^{n+1/2}) - \sum_{n=0}^{N-1} \mathcal{A}(u^{n+1,1/4} - u^{n+1}, \xi^{n+1/2})\right| \\
\leq& \tau \sum_{n=1}^{N-1} \left|\mathcal{A}(\partial_t(u^{n+1/2,1/4} - u^{n+1/2}), \xi^{n+1/2})\right| + \left|\mathcal{A}(u^{N,1/4} - u^N, \xi^{N+1/2})\right| \\
&\quad + \left|\mathcal{A}(u^{1,1/4} - u^1, \xi^{1/2})\right| \\
\leq& \tau \beta_{max} \sum_{n=1}^{N-1} \left\|\partial_t(u^{n+1/2,1/4} - u^{n+1/2})\right\|_{H^2(\Omega)} \left\|\xi^{n+1/2}\right\|_{H^2(\Omega)} \\
&+ \beta_{max} \left\|u^{N,1/4} - u^N\right\|_{H^2(\Omega)} \left\|\xi^{N+1/2}\right\|_{H^2(\Omega)} + \beta_{max} \left\|u^{1,1/4} - u^1\right\|_{H^2(\Omega)} \left\|\xi^{1/2}\right\|_{H^2(\Omega)} \\
\leq& \frac{\tau}{2} \beta_{max}^2 \sum_{n=1}^{N-1} \left\|\partial_t(u^{n+1/2,1/4} - u^{n+1/2})\right\|^2_{H^2(\Omega)} + \frac{\tau}{2} \sum_{n=1}^{N-1} \left\|\xi^{n+1/2}\right\|^2_{H^2(\Omega)} \\
&+ \frac{1}{4\delta} \beta_{max}^2 \left\|u^{N,1/4} - u^N\right\|^2_{H^2(\Omega)} + \delta \left\|\xi^{N+1/2}\right\|^2_{H^2(\Omega)} \\
&\quad + \frac{1}{2} \beta_{max}^2 \left\|u^{1,1/4} - u^1\right\|^2_{H^2(\Omega)} + \frac{1}{2} \left\|\xi^{1/2}\right\|^2_{H^2(\Omega)}
\end{aligned}$$

where δ is an arbitrary positive constant. Then, estimate (57) following from applying (50), (52), and (54) to the above.

■

With these preparations, we can derive an estimate for $\xi^{n+1/2}, n = 1, 2, \cdots, M-1$ in the following theorem.

Theorem 3.1. *Let $u(x,t)$ be the exact solution described by* (9) *and suppose $u \in W^{2,\infty}(0,T;\tilde{H}^4(\Omega))$, $u_{ttt} \in L^2(0,T;\tilde{H}^4(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$, $u_{tttt} \in L^2(0,T;\tilde{H}^0(\Omega))$. Then we have the following estimate:*

$$\left\|\partial_t \xi^{n+1/2}\right\|_{\tilde{H}^0(\Omega)} + \left\|\xi^{n+1/2}\right\|_{H^2(\Omega)} \leq C(h^4 + \tau^2), \ n = 1, \cdots, M-1. \tag{58}$$

Proof. We note that

$$\left\langle \rho \partial_{tt} u_h^n, v_h \right\rangle + \mathcal{A}(u_h^{n,1/4}, v_h) = \left\langle f^n, v_h \right\rangle, \ \forall v_h \in S_{h,0}(\Omega), \tag{59}$$

$$\left\langle \rho u_{tt}^n, v_h \right\rangle + \mathcal{A}(u^{n,1/4}, v_h) = \left\langle f^n, v_h \right\rangle + \mathcal{A}(u^{n,1/4} - u^n, v_h), \ \forall v_h \in S_{h,0}(\Omega), \tag{60}$$

Subtracting (60) from (59), we have

$$\left\langle \rho(\partial_{tt} u_h^n - \partial_{tt} u^n + \partial_{tt} u^n - u_{tt}^n), v_h \right\rangle + \mathcal{A}(e^{n,1/4}, v_h) = -\mathcal{A}(u^{n,1/4} - u^n, v_h),$$

which leads to

$$\left\langle \rho \partial_{tt} e^n, v_h \right\rangle + \mathcal{A}(e^{n,1/4}, v_h) = -\mathcal{A}(u^{n,1/4} - u^n, v_h) - \left\langle \rho(\partial_{tt} u^n - u_{tt}^n), v_h \right\rangle.$$

Because $e^n = \xi^n + \eta^n$ and $\mathcal{A}(\eta^{n,1/4}, v_h) = 0$, we obtain

$$\begin{aligned}
\left\langle \rho \partial_{tt} \xi^n, v_h \right\rangle + \mathcal{A}(\xi^{n,1/4}, v_h) = &-\mathcal{A}(u^{n,1/4} - u^n, v_h) - \left\langle \rho(\partial_{tt} u^n - u_{tt}^n), v_h \right\rangle \\
&- \left\langle \rho \partial_{tt} \eta^n, v_h \right\rangle.
\end{aligned}$$

Choosing $v_h = \partial_t \xi^n$ based on the fact that $\partial_t \xi^n \in S_{h,0}(\Omega)$ and combining the definitions given by (32)-(34) yields

$$\begin{aligned}
&\frac{1}{2\tau}\Big[\big\langle \rho\partial_t \xi^{n+1/2}, \partial_t \xi^{n+1/2}\big\rangle - \big\langle \rho\partial_t \xi^{n-1/2}, \partial_t \xi^{n-1/2}\big\rangle \\
&\qquad + \mathcal{A}(\xi^{n+1/2}, \xi^{n+1/2}) - \mathcal{A}(\xi^{n-1/2}, \xi^{n-1/2})\Big] \\
=& - \mathcal{A}(u^{n,1/4} - u^n, \partial_t \xi^n) - \big\langle \rho(\partial_{tt} u^n - u^n_{tt}), \partial_t \xi^n\big\rangle - \big\langle \rho\partial_{tt}\eta^n, \partial_t \xi^n\big\rangle.
\end{aligned}$$

Furthermore, we sum the above from $n = 1$ to $n = N$ with $N \leq M-1$ and multiply by 2τ to have:

$$\begin{aligned}
&\big\langle \rho\partial_t \xi^{N+1/2}, \partial_t \xi^{N+1/2}\big\rangle + \mathcal{A}(\xi^{N+1/2}, \xi^{N+1/2}) \\
=& \big\langle \rho\partial_t \xi^{1/2}, \partial_t \xi^{1/2}\big\rangle + \mathcal{A}(\xi^{1/2}, \xi^{1/2}) - 2\tau \sum_{n=1}^{N} \mathcal{A}(u^{n,1/4} - u^n, \partial_t \xi^n) \\
&- 2\tau \sum_{n=1}^{N} \big\langle \rho(\partial_{tt} u^n - u^n_{tt}), \partial_t \xi^n\big\rangle - 2\tau \sum_{n=1}^{N} \big\langle \rho\partial_{tt}\eta^n, \partial_t \xi^n\big\rangle
\end{aligned}$$

Then, applying the coercivity and boundedness of the bilinear form $\mathcal{A}(\cdot,\cdot)$ to the above, we have

$$\begin{aligned}
&\rho_{min} \left\|\partial_t \xi^{N+1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \beta_{min} C_0 \left\|\xi^{N+1/2}\right\|^2_{H^2(\Omega)} \\
\leq& \big\langle \rho\partial_t \xi^{N+1/2}, \partial_t \xi^{N+1/2}\big\rangle + \mathcal{A}(\xi^{N+1/2}, \xi^{N+1/2}) \\
=& \big\langle \rho\partial_t \xi^{1/2}, \partial_t \xi^{1/2}\big\rangle + \mathcal{A}(\xi^{1/2}, \xi^{1/2}) - 2\tau \sum_{n=1}^{N} \mathcal{A}(u^{n,1/4} - u^n, \partial_t \xi^n) \\
&- 2\tau \sum_{n=1}^{N} \big\langle \rho(\partial_{tt} u^n - u^n_{tt}), \partial_t \xi^n\big\rangle - 2\tau \sum_{n=1}^{N} \big\langle \rho\partial_{tt}\eta^n, \partial_t \xi^n\big\rangle \\
\leq& \rho_{max} \left\|\partial_t \xi^{1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \beta_{max} \left\|\xi^{1/2}\right\|^2_{H^2(\Omega)} + 2\left|\tau \sum_{n=1}^{N} \mathcal{A}(u^{n,1/4} - u^n, \partial_t \xi^n)\right| \\
&+ 2\left|\tau \sum_{n=1}^{N} \big\langle \rho(\partial_{tt} u^n - u^n_{tt}), \partial_t \xi^n\big\rangle\right| + 2\tau \left|\sum_{n=1}^{N} \big\langle \rho\partial_{tt}\eta^n, \partial_t \xi^n\big\rangle\right|,
\end{aligned}$$

where C_0 is the constant such that $C_0 \|v\|^2_{H^2(\Omega)} \le |v|^2_{H^2(\Omega)}$ for all $v \in H^2_0(\Omega)$. Then, applying the estimates in Lemmas 3.3, 3.5, and 3.6 to the above, we have

$$\begin{aligned}
&\rho_{min} \left\|\partial_t \xi^{N+1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \beta_{min} C_0 \left\|\xi^{N+1/2}\right\|^2_{H^2(\Omega)} \\
\le & C\tau^4 + C\tau^6 + \left(C(\tau^4+\tau^6) + 2\delta \left\|\xi^{N+1/2}\right\|^2_{H^2(\Omega)} + \tau \sum_{n=0}^{N-1} \left\|\xi^{n+1/2}\right\|^2_{H^2(\Omega)} \right) \\
& + \left(C\tau^4 + \tau \sum_{n=0}^{N-1} \left\|\partial_t \xi^{n+1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \frac{\tau\delta}{2} \left\|\partial_t \xi^{N+1/2}\right\|^2_{\tilde{H}^0(\Omega)} \right) \\
& + \left(C(h^8+\tau^6) + 2\delta \left\|\xi^{N+1/2}\right\|^2_{H^2(\Omega)} + \tau \sum_{n=0}^{N-1} \left\|\xi^{n+1/2}\right\|^2_{H^2(\Omega)} \right) \\
\le & C(\tau^4+h^8) + \frac{\tau\delta}{2} \left\|\partial_t \xi^{N+1/2}\right\|^2_{\tilde{H}^0(\Omega)} + 4\delta \left\|\xi^{N+1/2}\right\|^2_{H^2(\Omega)} \\
& + 2\tau \sum_{n=0}^{N-1} \left(\left\|\partial_t \xi^{n+1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \left\|\xi^{n+1/2}\right\|^2_{H^2(\Omega)} \right)
\end{aligned}$$

Because δ is arbitrary, we can let it be small enough such that for a certain constant C there holds

$$\begin{aligned}
&\left\|\partial_t \xi^{N+1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \left\|\xi^{N+1/2}\right\|^2_{H^2(\Omega)} \\
\le & C(\tau^4+h^8) + C\tau \sum_{n=0}^{N-1} \left(\left\|\partial_t \xi^{n+1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \left\|\xi^{n+1/2}\right\|^2_{H^2(\Omega)} \right).
\end{aligned}$$

Hence, by the standard discrete Gronwall-Bellman's inequality [3, 20] and Lemma 3.3, we have

$$\begin{aligned}
&\left\|\partial_t \xi^{N+1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \left\|\xi^{N+1/2}\right\|^2_{H^2(\Omega)} \\
&\le C \left[h^8 + \tau^4 + \left(\left\|\partial_t \xi^{1/2}\right\|^2_{\tilde{H}^0(\Omega)} + \left\|\xi^{1/2}\right\|^2_{H^2(\Omega)} \right) \right] \le C(h^8+\tau^4),
\end{aligned}$$

for $1 \le N \le M-1$ which proves (58).

■

Finally, we can obtain an estimate for the IFE solution u_h^n in the following theorem.

Theorem 3.2. *Assume that $u(x,t)$ satisfies the same regularity as required in Theorem 3.1. Then*

$$\left\|u_h^{n+1/2} - u^{n+1/2}\right\|_{H^2(\Omega)} \le C(h^2+\tau^2), \ n = 1, \cdots, M-1. \tag{61}$$

proof. Estimate in (61) follows easily from the results of Lemma 3.1 and Theorem 3.1:

$$\begin{aligned}
\left\|u_h^{n+1/2} - u^{n+1/2}\right\|_{H^2(\Omega)} &= \left\|\xi^{n+1/2} + \eta^{n+1/2}\right\|_{H^2(\Omega)} \\
&\le \left\|\xi^{n+1/2}\right\|_{H^2(\Omega)} + \left\|\eta^{n+1/2}\right\|_{H^2(\Omega)} \\
&\le C(h^4+\tau^2) + Ch^2 \le C(h^2+\tau^2).
\end{aligned}$$

■

4. Numerical examples

In this section, we numerically examine the order of accuracy of the Hermite cubic IFE solution for the time dependent interface beam problem by applying the IFE method described by (35) to the initial boundary value problem defined by (1) whose exact solution is as follows:

$$u(x,t) = \begin{cases} e^{0.02t}\big(a\,cos(2x) + b\,sin(x) + 1\big), & \text{if } 0 \le x < \alpha, t \ge 0, \\ e^{0.02t}\big(c\,cos(2x) + d\,sin(x)\big), & \text{if } \alpha \le x \le 1, t \ge 0, \end{cases} \tag{62}$$

in which the coefficients a, b, c, d are chosen so that $u(x,t)$ satisfies the interface jump conditions (1d). The force function $f(x,t)$, the boundary condition functions $b_i(t), i = 1,2,3,4$, as well as the initial condition functions $g_i(x), i = 1,2$ are derived by $u(x,t)$. For simplicity, we let the time interval be $t \in (0,1]$, mass density $\rho^- = \rho^+ = 1$, interface $\alpha = \pi/6$ and the following three configurations of $\beta(x)$ are considered:

Case 1: $\beta^- = 2$, $\beta^+ = 3$,
Case 2: $\beta^- = 2$, $\beta^+ = 30$,
Case 3: $\beta^- = 2$, $\beta^+ = 3000$,

which represent small, moderate, and large discontinuities of the coefficient at the interface respectively.

We examine the error of the IFE solution at two representative times: one is in the middle and the last one is in the end of the time interval $(0,1]$. To investigate the order of accuracy numerically, we use a uniform partition for the spacial solution domain $\Omega = (0,1)$ and time interval $(0,1]$ to generate IFE solution $u_h(x,t^n) = u_h^n(x) \in S_h(\Omega), n = 1,2,\cdots,M$. In the numerical simulation, we use mesh size $h = \frac{1}{5\times i}, i = 1,2,\cdots,10$ and a time step size $\tau = \frac{h}{100}$.

Data Tables 1-3 list H^2 norm errors of the cubic Hermite IFE solutions $u_h^{n+1/2}$ at time level $t = t^{300}, t = t^{M-1}$ for various values of mesh size h. Applying linear regression to these data, we note that these numerical results obey the following relationships:

For **Case 1**:

$$\left\|u_h^{300+1/2} - u^{300+1/2}\right\|_{H^2(\Omega)} \approx 0.7722h^{2.0017},$$
$$\left\|u_h^{(M-1)+1/2} - u^{(M-1)+1/2}\right\|_{H^2(\Omega)} \approx 0.7745h^{1.9974}.$$

For **Case 2**:

$$\left\|u_h^{300+1/2} - u^{300+1/2}\right\|_{H^2(\Omega)} \approx 0.2561h^{1.9874},$$
$$\left\|u_h^{(M-1)+1/2} - u^{(M-1)+1/2}\right\|_{H^2(\Omega)} \approx 0.2569h^{1.9831}.$$

For **Case 3**:

$$\left\|u_h^{300+1/2} - u^{300+1/2}\right\|_{H^2(\Omega)} \approx 0.2370h^{1.9851},$$
$$\left\|u_h^{(M-1)+1/2} - u^{(M-1)+1/2}\right\|_{H^2(\Omega)} \approx 0.2378h^{1.9808}.$$

These numerical results obviously corroborate the following error estimate derived in Theorem 3.2:

$$\left\|u_h^{n+1/2} - u^{n+1/2}\right\|_{H^2(\Omega)} \le Ch^2,\ n = 1, \cdots, M-1. \tag{63}$$

TABLE 1. H^2 norm errors of IFE solution for the beam interface problem with $\alpha = \pi/6$, $\beta^- = 2$, $\beta^+ = 3$.

h	$\left\|u_h^{300+1/2} - u^{300+1/2}\right\|_{H^2(\Omega)}$	$\left\|u_h^{(M-1)+1/2} - u^{(M-1)+1/2}\right\|_{H^2(\Omega)}$
1/5	3.080931326602829e-02	3.105553405826669e-02
1/10	7.703511870235500e-03	7.811963275214073e-03
1/15	3.407157315246679e-03	3.462064122715313e-03
1/20	1.920814998973799e-03	1.953728454806770e-03
1/25	1.230544341605216e-03	1.252383613721194e-03
1/30	8.526766070336298e-04	8.681579633540466e-04
1/35	6.266057066929438e-04	6.381653928362991e-04
1/40	4.793927381786260e-04	4.883416064008635e-04
1/45	3.789901082250086e-04	3.861293161711144e-04
1/50	3.070514901767621e-04	3.128774111665804e-04

TABLE 2. H^2 norm errors of IFE solution for the beam interface problem with $\alpha = \pi/6$, $\beta^- = 2$, $\beta^+ = 30$.

h	$\left\|u_h^{300+1/2} - u^{300+1/2}\right\|_{H^2(\Omega)}$	$\left\|u_h^{(M-1)+1/2} - u^{(M-1)+1/2}\right\|_{H^2(\Omega)}$
1/5	1.030202357504485e-02	1.038435492714779e-02
1/10	2.678087182434285e-03	2.715789768272097e-03
1/15	1.181374043129116e-03	1.200412047823178e-03
1/20	6.674660591807658e-04	6.789031907859465e-04
1/25	4.293145532068404e-04	4.369338779112072e-04
1/30	2.962391337062861e-04	3.016177071739973e-04
1/35	2.186421970937499e-04	2.226757307067679e-04
1/40	1.673958506215580e-04	1.705206443226521e-04
1/45	1.320070058112771e-04	1.344936811962275e-04
1/50	1.072021298525442e-04	1.092361569468198e-04

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TABLE 3. H^2 norm errors of IFE solution for the beam interface problem with $\alpha = \pi/6$, $\beta^- = 2$, $\beta^+ = 3000$.

h	$\left\|u_h^{300+1/2} - u^{300+1/2}\right\|_{H^2(\Omega)}$	$\left\|u_h^{(M-1)+1/2} - u^{(M-1)+1/2}\right\|_{H^2(\Omega)}$
1/5	9.554467686977073e-03	9.630824747690184e-03
1/10	2.493511847976944e-03	2.528615949557647e-03
1/15	1.102873666343827e-03	1.120646626695687e-03
1/20	6.214883012612048e-04	6.321375971040403e-04
1/25	4.007901425003836e-04	4.079032259258491e-04
1/30	2.764878460235710e-04	2.815078115158901e-04
1/35	2.038487489048730e-04	2.076093714067065e-04
1/40	1.563032432088006e-04	1.592209702068999e-04
1/45	1.231570948296918e-04	1.254770604054279e-04
1/50	1.000676164110465e-04	1.019662750383065e-04

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