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IMPROVED ERROR ESTIMATION FOR THE PARTIALLY PENALIZED IMMERSED FINITE ELEMENT METHODS FOR ELLIPTIC INTERFACE PROBLEMS

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Abstract. This paper is for proving that the partially penalized immersed finite element (PPIFE) methods developed in [25] converge optimally under the standard piecewise H^2 regularity assumption for the exact solution. In energy norms, the error estimates given in this paper are better than those in [25] where a stronger piecewise H^3 regularity was assumed. Furthermore, with the standard piecewise H^2 regularity assumption, this paper proves that these PPIFE methods also converge optimally in the L^2 norm which could not be proved in [25] because of the excessive H^3 regularity requirement.

Key words. Interface problems, immersed finite element methods, optimal convergence, discontinuous coefficients, finite element spaces, interface independent mesh, regularity.

1. Introduction

In this article, we establish better error estimates for the numerical solutions generated by the partially penalized immersed finite element (PPIFE) methods [25] for the interface problem governed by the second-order elliptic equation:

$$-\nabla \cdot (\beta \nabla u) = f, \quad \text{in } \Omega^- \cup \Omega^+, \tag{1a}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1b}$$

where, without loss of generality, the domain $\Omega \subseteq \mathbb{R}^2$ is divided by an interface curve Γ into two subdomains Ω^- and Ω^+, and the coefficient β is a piecewise positive constant function such that

$$\beta(X) = \begin{cases} \beta^- & \text{for } X \in \Omega^-, \\ \beta^+ & \text{for } X \in \Omega^+. \end{cases}$$

In addition, the exact solution u satisfies the following jump conditions across the interface

$$[u]_\Gamma := u^-|_\Gamma - u^+|_\Gamma = 0, \tag{2}$$

$$\left[\beta \nabla u \cdot \mathbf{n}\right]_\Gamma := \beta^- \nabla u^- \cdot \mathbf{n}|_\Gamma - \beta^+ \nabla u^+ \cdot \mathbf{n}|_\Gamma = 0, \tag{3}$$

where $\mathbf{n}$ is the unit normal vector to the interface Γ. For the sake of simplicity, as in [25], we assume the interface Γ is a C^2-curve and does not intersect $\partial\Omega$.

The immersed finite element (IFE) method is developed to solve the interface problem (1)-(3) on an interface independent mesh, if desirable, a simple structured (Cartesian) mesh can be used. The key idea of this method is to utilize Hsieh-Clough-Tocher type macro polynomials [3, 6], i.e., the piecewise polynomials constructed according to the jump conditions on interface elements to capture the jump behaviors of the exact solutions [2, 9, 16, 22], while standard polynomials are used on all the non-interface elements. The global IFE functions such as those used in [16, 22, 25] are, in general, not continuous across the interface edges, even though the continuity at the mesh nodes is imposed. The partially penalized IFE (PPIFE)

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methods developed in [25] employed the interior penalties [8] on interface edges to control the adverse effects from those discontinuities so that these PPIFE methods converge optimally in a certain energy norm. Penalties are also used in Cut-FEMs [4, 14] mainly for enhancing jump conditions across the interface. IFE methods for interface problems associated with other types of PDEs or jump conditions as well as the applications can be found in [1, 5, 10, 11, 17, 18, 19, 20, 21, 23, 24, 26, 28], to name just a few.

The authors in [25] employed a piecewise H^3 regularity assumption for the exact solution to the interface problem to prove the optimal convergence of the PPIFE solutions. However, given the body force term $f \in L^2(\Omega)$, the exact solution to (1)-(3) only has the piecewise H^2 regularity [7] in general. This motivates us to investigate whether the PPIFE methods developed in [25] can converge optimally under the standard piecewise H^2 regularity assumption instead of the excessive piecewise H^3 regularity. Towards this goal, we introduce a new energy norm that is stronger than the one used in [25]. Inspired by [13], we derive an estimate for the IFE interpolation error gauged by this energy norm on a patch of each interface element. Furthermore, the bilinear form in the PPIFE method has both the continuity and coercivity in this energy norm. These properties enable us to derive an error bound for the PPIFE solution in the energy norm under the standard piecewise H^2 regularity assumption. As an important consequence, the improved estimation further enables us to show the optimal convergence in the L^2 norm, which, to our best knowledge, has not been established in the literature for the PPIFE methods.

This article consists of four additional sections. The next section reviews some notations from [25] which will be also used in this article. In Section 3, we introduce the patches for the interface elements and recover the approximation capabilities of IFE spaces on these patches. In Section 4 we show the optimal convergence of the PPIFE solutions. Finally, we make some conclusions in Section 5.

2. Notations and IFE Spaces

We herein adopt some notations from [25]. For every measurable open set $\tilde{\Omega} \subseteq \Omega$, we let $\tilde{\Omega}^s := \tilde{\Omega} \cap \Omega^s, s = \pm$, and we let $W^{k,p}(\tilde{\Omega})$ be the standard Sobolev space on $\tilde{\Omega}$ with the standard norm $\|\cdot\|_{k,p,\tilde{\Omega}}$ and the semi-norm $|v|_{k,p,\tilde{\Omega}}$. When $\tilde{\Omega}^s \neq \emptyset$, $s = \pm$, we let the related Sobolev norms and semi-norms be

$$\|\cdot\|^2_{k,p,\tilde{\Omega}} = \|\cdot\|^2_{k,p,\tilde{\Omega}^-} + \|\cdot\|^2_{k,p,\tilde{\Omega}^+}, \qquad |\cdot|^2_{k,p,\tilde{\Omega}} = |\cdot|^2_{k,p,\tilde{\Omega}^-} + |\cdot|^2_{k,p,\tilde{\Omega}^+}.$$

Furthermore, we introduce the following spaces on $\tilde{\Omega}$ in the case $\tilde{\Omega}^s \neq \emptyset$, $s = \pm$:

$$PW^{k,p}(\tilde{\Omega}) = \{u : u|_{\tilde{\Omega}^s} \in W^{k,p}(\tilde{\Omega}^s),\ s = \pm;\ [u] = 0,\ [\beta \nabla u \cdot \mathbf{n}_\Gamma] = 0 \text{ on } \Gamma \cap \tilde{\Omega}\},$$

for suitable k and p such that involved qualities on $\Gamma \cap \tilde{\Omega}$ are well defined. As usual, we will drop p from the pertinent norms and semi-norms for $H^k(\tilde{\Omega}) = W^{k,2}(\tilde{\Omega})$ and $PH^k(\tilde{\Omega}) = PW^{k,2}(\tilde{\Omega})$.

We let $\mathcal{T}_h$ be a triangular or a rectangular mesh for the domain $\Omega \subset \mathbb{R}^2$ and let $\mathcal{N}_h$ be the collection of the nodes in the mesh $\mathcal{T}_h$. We denote the sets of interface elements and non-interface elements by $\mathcal{T}_h^i$ and $\mathcal{T}_h^n$. Also, we denote the set of interior edges by $\mathring{\mathcal{E}}_h$, the interior interface edges by $\mathring{\mathcal{E}}_h^i$ and the interior non-interface edges by $\mathring{\mathcal{E}}_h^n$, respectively. For each element $T \in \mathcal{T}_h$, we define its index set as $\mathcal{I}_T = \{1, 2, 3\}$ when T is triangular, but $\mathcal{I}_T = \{1, 2, 3, 4\}$ when T is rectangular. Given each T, let A_i, $i \in \mathcal{I}_T$ be the vertices of T, and the interface partitions the index set $\mathcal{I}$ into $\mathcal{I}_T^- = \{A_i : A_i \in T^-\}$ and $\mathcal{I}_T^+ = \{A_i : A_i \in T^+\}$. As usual [16, 15, 17], we make the following assumptions on the mesh $\mathcal{T}_h$:

(H1) The interface Γ cannot intersect an edge of any element at more than two points unless the edge is part of Γ.

(H2) If Γ intersects the boundary of an element at two points, these intersection points must be on different edges of this element.

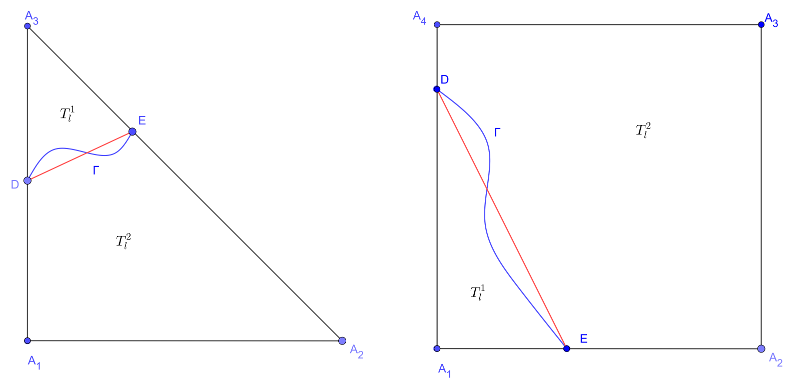

FIGURE 1. left: triangular interface element; right: rectangular interface element.

Let $\psi_{j,T}$, $j \in \mathcal{I}$ be the standard linear or bilinear Lagrangian shape functions on T, i.e.,

$$\psi_{j,T}(A_i) = \delta_{ij}, \ \forall i, j \in \mathcal{I}_T. \tag{4}$$

Then we use the polynomial space

$$\mathbb{P}(T) = \text{Span}\{\psi_{j,T}, j \in \mathcal{I}_T\} \text{ and } \mathbb{Q}(T) = \text{Span}\{\psi_{j,T}, j \in \mathcal{I}_T\} \tag{5}$$

as the local IFE space on the non-interface triangular element and rectangular elements, respectively. But on the interface elements, we will use the linear and bilinear IFE shape functions [15, 16, 22]. To be specific, let the interface Γ intersect the edges of an interface element at the points D and E, as shown in Figure 1. Let l be the line passing through D, E with the normal vector $\bar{\mathbf{n}} = (\bar{n}_x, \bar{n}_y)$, and this line partitions T into two subelements $T_l^{\pm}$. Then, on each triangular interface element T, a linear IFE shape function $\phi_T(x, y)$ is a piecewise linear polynomial specified by [22] :

$$\phi_T(x,y) = \begin{cases} \phi_T^-(x,y) = a^- x + b^- y + c^-, \text{ if } (x,y) \in T_l^-, \\ \phi_T^+(x,y) = a^+ x + b^+ y + c^+, \text{ if } (x,y) \in T_l^+, \\ \phi_T^-(D) = \phi_T^+(D), \phi_T^-(E) = \phi_T^+(E), \\ \beta^+ \frac{\partial \phi_T^+}{\partial \bar{\mathbf{n}}} - \beta^- \frac{\partial \phi_T^-}{\partial \bar{\mathbf{n}}} = 0. \end{cases} \tag{6}$$

Similarly, on each rectangular interface element T, a bilinear IFE shape function $\phi_T(x, y)$ is a piecewise bilinear polynomial specified by [16]:

$$\phi_T(x,y) = \begin{cases} \phi_T^-(x,y) = a^- x + b^- y + c^- + d^- xy, \text{ if } (x,y) \in T_l^-, \\ \phi_T^+(x,y) = a^+ x + b^+ y + c^+ + d^+ xy, \text{ if } (x,y) \in T_l^+, \\ \phi_T^-(D) = \phi_T^+(D), \phi_T^-(E) = \phi_T^+(E), d^- = d^+, \\ \int_{\overline{DE}} (\beta^+ \frac{\partial \phi_T^+}{\partial \bar{\mathbf{n}}} - \beta^- \frac{\partial \phi_T^-}{\partial \bar{\mathbf{n}}}) ds = 0. \end{cases} \tag{7}$$

It has been shown [15, 16, 22] that there is a unique IFE shape function $\phi_{i,T}(x,y), i \in \mathcal{I}_T$ in the format of (6) or (7) satisfying the nodal value conditions

$$\phi_{i,T}(A_j) = \delta_{ij},\ \forall i,j \in \mathcal{I}_T. \tag{8}$$

Then the local IFE space on an interface element T is defined as

$$S_h(T) = \text{Span}\{\phi_{i,T}, i \in \mathcal{I}_T\}. \tag{9}$$

By enforcing the continuity on the mesh nodes, we define the global IFE space as

$$\begin{aligned} S_h(\Omega) = \big\{ v \in L^2(\Omega) \ :\ & v|_T \in S_h(T),\ \forall T \in \mathcal{T}_h, \\ & v \text{ is continuous at each } X \in \mathcal{N}_h, v|_{\partial\Omega} = 0 \big\}. \end{aligned} \tag{10}$$

Now, we recall the PPIFE method [25]. First, we recall the following underline function space

$$\begin{aligned} V_h(\Omega) = \big\{ v \in L^2(\Omega) : & v|_T \in H^1(T), \nabla v \cdot \mathbf{n}|_{\partial T} \in L^2(\partial T), \\ & v \text{ is continuous across each}\, e \in \mathcal{E}_h^n,\ v|_{\partial\Omega} = 0 \big\}. \end{aligned} \tag{11}$$

Clearly, we have $S_h(\Omega) \subset V_h(\Omega)$, and functions in eigher of these two spaces can be discontinuous on the interface edges. For functions in $V_h(\Omega)$, the following operators on each $e \in \mathcal{E}_h^i$ are adopted for the penalties:

$$[v]_e = v|_{T_1^e} - v|_{T_2^e},\ \text{and } \{v\}_e = \frac{1}{2}\left(v|_{T_1^e} + v|_{T_2^e}\right), \tag{12}$$

where T_1^e and T_2^e are the two elements sharing the edge e. Then the bilinear form $a_h(\cdot,\cdot) : V_h(\Omega) \times V_h(\Omega) \to \mathbb{R}$ and the linear form $L_f : V_h(\Omega) \to \mathbb{R}$ for the PPIFE method are given by

$$\begin{aligned} a_h(u,v) = & \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla v dX - \sum_{e \in \mathring{\mathcal{E}}_h^i} \int_e \{\beta \nabla u \cdot \mathbf{n}_e\}_e [v]_e ds \\ & + \epsilon \sum_{e \in \mathring{\mathcal{E}}_h^i} \int_e \{\beta \nabla v \cdot \mathbf{n}_e\}_e [u]_e ds + \sum_{e \in \mathring{\mathcal{E}}_h^i} \frac{\sigma_e^0}{|e|} \int_e [u]_e\, [v]_e ds, \end{aligned} \tag{13}$$

$$L_f(v) = \int_\Omega f v dX. \tag{14}$$

Following the same arguments used in [25], we can see that the exact solution u to the interface problem (1)-(3) with a suitable regularity can satisfy the following weak equation:

$$a_h(u, v_h) = L_f(v_h),\ \forall v \in V_h(\Omega). \tag{15}$$

Accordingly, the PPIFE scheme [25] for the interface problem (1) is to find $u_h \in S_h(\Omega)$ such that

$$a_h(u_h, v_h) = L_f(v_h),\ \forall v_h \in S_h(\Omega). \tag{16}$$

In this article, we follow [25] to consider the PPIFE methods associated with three possible choices $\epsilon = 0, -1, 1$, respectively, and we call the corresponding PPIFE method the incomplete PPIFE (IPPIFE), the symmetric PPIFE (SPPIFE), and the non-symmetric PPIFE (NPPIFE) method, respectively.

3. Approximation Capabilities on a Patch

In this section, following similar ideas in [11, 13], we consider the approximation capability of the IFE spaces locally around an interface element. Specifically, for each interface element $T \in \mathcal{T}_h^i$, we consider a patch around it defined as the union of the neighbor elements:

$$\omega_T = \cup\{T' \in \mathcal{T}_h : \overline{T'} \cap \overline{T} \neq \phi\}, \tag{17}$$

where the notation $\overline{S}$ is the closure of a set S. In the discussions from now on, we make the following assumption on a patch of an interface element:

Patch Assumption: For every interface element T and its patch ω_T, let e be an interface edge of T. Then for $s = \pm$, there exists a triangle $T_e^s \subset \Omega^s \cap \omega_T$ and two constants C_1, C_2 independent of the interface location such that $e \cap T^s$ is one edge of T_e^s and

$$C_1|e \cap T^s|h \leq |T_e^s| \leq C_2|e \cap T^s|h, \ \ s = -, +. \tag{18}$$

For example, for the interface element $T = \triangle A_1A_2A_3$ and the interface edge $e = \overline{A_1A_2}$ in Figure 2, it is easy to see that

$$T_e^+ = \triangle A_1DP, \ T_e^- = \triangle A_2DQ$$

can be used to fulfill the Patch Assumption for this interface element T, here, $D \in e$ is the intersection point of the interface Γ and ∂T, $P \in \omega_T$ and $Q \in \omega_T$ are points whose distance to the line passing A_1 and A_2 is of about h. Basically, the inequality (18) to be satisfied in the patch assumption means that the height of the auxiliary triangle T_e^s corresponding to the edge $e \cap T^s$ has the length $\mathcal{O}(h)$. In general, when the mesh size h is sufficient small so that the interface is locally flat enough, the patch assumption can be easily satisfied.

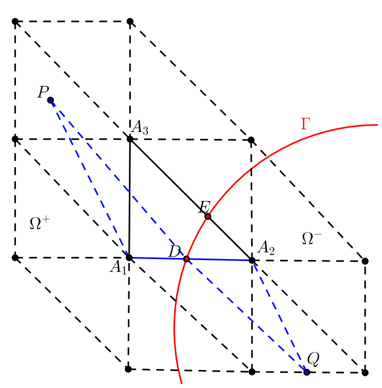

FIGURE 2. The patch of a triangular interface element $T = \triangle A_1A_2A_3$.

We now proceed to investigate the approximation capability of the IFE space on the patch of each interface element. As a preparation, we first consider a few subsets formed according to the interface geometry inside the patch of an interface element. Let T be an interface element. We recall l is the line that passes through the two intersection points of Γ and ∂T. The interface Γ and the line l partition the patch ω_T into the sub-patches ω_T^s and $\hat{\omega}_T^s$ $(s = \pm)$, respectively. Let $\tilde{\omega}_T^s = \omega_T^s \cup \hat{\omega}_T^s$,

$s = \pm$, and we can see that $\tilde{\omega}_T = (\tilde{\omega}_T^+ \cap \omega_T^-) \cup (\tilde{\omega}_T^- \cap \omega_T^+)$ is the sub-patch sandwiched between l and Γ. Following [11, 12], we consider the sub-set

$$\omega_T^{int} = \cup\{l_t \cap \omega_T : l_t \text{ is a tangent line to } \Gamma \cap \omega_T\}. \tag{19}$$

For every vertex A_i of T, $i \in \mathcal{I}_T$, and each point $X \in \omega_T \backslash \omega_T^{int}$, the line segment $\overline{A_i X}$ has either zero or one intersection point with $\Gamma \cap \omega_T$. When there is no intersection point, A_i and X must be on the same side of $\Gamma \cap \omega_T$; while when there is one intersection point, A_i and X are on the different sides of $\Gamma \cap \omega_T$. We further denote $\omega_T^{*,s} = (\hat{\omega}_T^s \cap \omega_T^s) \backslash \omega_T^{int}$, $s = \pm$, and $\omega_T^* = \omega_T \backslash (\omega_T^{*,-} \cup \omega_T^{*,+})$. By Lemma 3.4 of [11], when the mesh size is small enough, there holds

$$|\omega_T^*| \leq Ch^3. \tag{20}$$

For every $X \in \omega_T^{*,s}$, we let $Y_i(t, X) = tA_i + (1-t)X$. When X and A_i are on different sides of Γ, we let $\tilde{t}_i = \tilde{t}_i(X) \in [0,1]$ such that $\tilde{Y}_i = Y_i(\tilde{t}_i, X)$ is on the curve $\Gamma \cap T$. Let $\mathbf{n}(\tilde{X}) = (\tilde{n}_x(\tilde{X}), \tilde{n}_y(\tilde{X}))$ be the normal vector to Γ at every point $\tilde{X} \in \Gamma \cap \omega_T$. Recall $\bar{\mathbf{n}} = (\bar{n}_x, \bar{n}_y)$ be the normal vector to l and denote $\tilde{X}^\perp$ as the projection of a point $\tilde{X}$ onto l. It can be shown, by the similar discussion as the Lemmas 3.1 and 3.2 in [9], that, for any $\tilde{X} \in \Gamma \cap \omega_T$, there holds

$$\|\tilde{X} - \tilde{X}^\perp\| \leq Ch^2, \quad \|\mathbf{n}(\tilde{X}) - \bar{\mathbf{n}}\| \leq Ch. \tag{21}$$

As in [16, 22], for a function $u \in H^2(\Omega^- \cup \Omega^+)$, we let $I_h u \in S_h(\Omega)$ be its IFE interpolation defined by

$$I_h u|_T = I_{h,T} u, \quad \text{with} \quad \begin{cases} I_{h,T} u(X) = \sum_{i \in \mathcal{I}_T} u(A_i) \phi_{i,T}(X), \ \forall X \in T, \ \forall T \in \mathcal{T}_h^i, \\ I_{h,T} u(X) = \sum_{i \in \mathcal{I}_T} u(A_i) \psi_{i,T}(X), \ \forall X \in T, \ \forall T \in \mathcal{T}_h^n. \end{cases}$$

On each interface element T, every IFE shape function $\phi_{i,T}(X), i \in \mathcal{I}_T$ can be naturally considered as a piecewise polynomial defined on the patch ω_T according to the sub-patches $\hat{\omega}_T^s, s = -, +$. Therefore, for a function $u \in H^2(\Omega^- \cup \Omega^+)$, we can consider its local IFE interpolation $I_{h,T} u(X)$ on an interface element T as a piecewise polynomial defined on the patch ω_T according to sub-patches $\hat{\omega}_T^s, s = \pm$, and we proceed to the analysis of its accuracy in the rest of this section. In the discussions below, we denote $s = \pm, s' = \mp$, namely, s and s' take opposite signs whenever a formula have them both. Also, we adopt the following notations: $X = (x, y)$ and $x_1 = x, x_2 = y$.

Following the same arguments in [9], we have the following expansions for $I_h u - u$:

$$\partial_{x_d}(I_{h,T} u(X) - u(X)) = \sum_{i \in \mathcal{I}^{s'}} (E_i^s + F_i^s) \partial_{x_d} \phi_{i,T}(X) + \sum_{i \in \mathcal{I}} R_i^s \partial_{x_d} \phi_{i,T}(X), \ \forall X \in \omega_T^{*,s}, \ s = \pm, \tag{22}$$

$$\partial_{x_d x'_d} I_{h,T} u(X) = \sum_{i \in \mathcal{I}^{s'}} (E_i^s + F_i^s) \partial_{x_d x'_d} \phi_{i,T}(X) + \sum_{i \in \mathcal{I}} R_i^s \partial_{x_d x'_d} \phi_{i,T}(X), \ \forall X \in \omega_T^{*,s}, \ s = \pm, \tag{23}$$

$$\partial_{x_d} I_{h,T} u = \sum_{i \in \mathcal{I}} \tilde{R}_i \partial_{x_d} \phi_{i,T}(X) \ \forall X \in \omega_T^*, \tag{24}$$

$$\partial_{x_d x_{d'}} I_{h,T} u = \sum_{i \in \mathcal{I}} \tilde{R}_i \partial_{x_d x_{d'}} \phi_{i,T}(X) \ \forall X \in \omega_T^*, \tag{25}$$

where $d, d' = 1, 2$ and

$$R_i^s(X) = R_{i1}^s(X) + R_{i2}^s(X) + R_{i3}^s(X), i \in \mathcal{I}^{s'},\ X \in \omega_T^{*,s},\ \text{with} \tag{26}$$

$$\begin{cases} R_{i1}^s(X) = \int_0^{\tilde{t_i}} (1-t) \dfrac{d^2 u^s}{dt^2}(Y_i(t,X))dt, \\ R_{i2}^s(X) = \int_{\tilde{t_i}}^1 (1-t) \dfrac{d^2 u^{s'}}{dt^2}(Y_i(t,X))dt, \\ R_{i3}^s(X) = (1-\tilde{t_i}) \int_0^{\tilde{t_i}} \dfrac{d}{dt}((M^s(\tilde{Y}_i) - I)\nabla u^s(Y_i(t,X)) \cdot (A_i - X))dt, \end{cases} \tag{27}$$

$$E_i^s = ((M^s(\tilde{Y}_i) - \overline{M}^s)\nabla u^s(X))(A_i - \tilde{Y}_i),\ i \in \mathcal{I}^{s'}, \tag{28}$$

$$F_i^s = -((\overline{M}^s - I)\nabla u^s(X))(\tilde{Y}_i - \tilde{Y}_i^{\perp}),\ i \in \mathcal{I}^{s'}, \tag{29}$$

$$\tilde{R}_i(X) = \int_0^1 \frac{d}{dt} u(Y_i(t,X))dt,\ i \in \mathcal{I}, \tag{30}$$

in which $M^- = (N^+)^{-1}N^-$, $M^+ = (N^-)^{-1}N^+$, $\overline{M}^- = (\overline{N}^+)^{-1}\overline{N}^-$, $\overline{M}^+ = (\overline{N}^-)^{-1}\overline{N}^+$, with

$$N^s = N^s(\tilde{X}) = \begin{pmatrix} \tilde{n}_y(\tilde{X}) & -\tilde{n}_x(\tilde{X}) \\ \beta^s \tilde{n}_x(\tilde{X}) & \beta^s \tilde{n}_y(\tilde{X}) \end{pmatrix} \text{ and } \bar{N}^s = \begin{pmatrix} \bar{n}_y & -\bar{n}_x \\ \beta^s \bar{n}_x & \beta^s \bar{n}_y \end{pmatrix},\ s = \pm. \tag{31}$$

Now we show the optimal approximation capabilities for the IFE spaces in terms of the interpolation errors on the patch ω_T for each interface element T. This result is stated in the following theorem and it is complementary to that given in [9, 16, 22].

Theorem 3.1. *Assume that the mesh $\mathcal{T}_h$ is sufficiently fine, then there exists a constant C independent of the interface location such that the following estimate holds on each patch ω_T associated with every interface element T:*

$$\begin{aligned} &\|\nabla(I_{h,T}u - u)\|_{L^2(\omega_T)} + h\|\nabla^2(I_{h,T}u - u)\|_{L^2(\omega_T)} \\ \le& Ch(\|u\|_{PH^2(\omega_T)} + \|u\|_{PW^{1,6}(\omega_T)}),\ \forall u \in PH^2(\omega_T). \end{aligned} \tag{32}$$

Proof. Using Lemma 4.1 in [9] and the fact $\|A_i - X\| \le Ch$ for $i \in \mathcal{I}, X \in \omega_T$, we directly have

$$\begin{aligned} &\|R_i^s\|_{L^2(\omega_T^{*,s})} \\ =&(\int_{\omega_T^{*,s}} (\int_0^1 (1-t)(A_i - X)^T H_u^s(Y_i(t,X))(A_i - X)dt)^2 dX)^{1/2} \\ \le& Ch^2 \int_0^1 (\int_{\omega_T^{*,s}} (1-t)^2 \sum_{k,l=1}^2 |\partial_{x_k}\partial_{x_l} u^s(Y_i,t)|^2 dX)^{1/2} dt \\ \le& Ch^2 \|u\|_{PH^2(\omega_T)}, \end{aligned} \tag{33}$$

where H_u^s is the Hessian matrix given by

$$H_u^s(Y_i(t,X)) = \begin{pmatrix} u_{xx}^s(Y_i(t,X)) & u_{xy}^s(Y_i(t,X)) \\ u_{yx}^s(Y_i(t,X)) & u_{yy}^s(Y_i(t,X))) \end{pmatrix}. \tag{34}$$

Note that (21) implies the $\|M^s(\tilde{Y}_i) - \overline{M}^s\| \le Ch$, $s = \pm$. Then, because of (28), we further have

$$
\begin{aligned}
\|E_i^s\|_{L^2(\omega_T^{*,s})} \le & \|M^s(\tilde{Y}_i) - \overline{M}^s\| \|\nabla u^s\|_{L^2(\omega_T^{*,s})} \|A_i - \tilde{Y}_j\| \\
\le & Ch \|M^s(\tilde{Y}_i) - \overline{M}^s\| \|\nabla u^s\|_{L^2(\omega_T^{*,s})} \\
\le & Ch^2 \|u\|_{PH^2(\omega_T)}.
\end{aligned} \tag{35}
$$

Next, (21) yields

$$
\begin{aligned}
\|F_i^s\|_{L^2(\omega_T^{*,s})} \le & \|\overline{M}^s - I\| \|\nabla u^s\|_{L^2(\omega_T^{*,s})} \|\tilde{Y}_i - \tilde{Y}_i^{\perp}\| \\
\le & Ch^2 \|u\|_{PH^2(\omega_T)}.
\end{aligned} \tag{36}
$$

In addition, using $|\omega_T^*| \le Ch^3$ from (20) and the similar argument as the one used in Lemma 3.2 in [12], we have

$$
\begin{aligned}
\|\tilde{R}_i\|_{L^2(\omega_T^*)} \le Ch^2 \|u\|_{PW^{1,6}(\omega_T)}, \\
\|\partial_{x_d} u\|_{L^2(\omega_T^*)} \le Ch \|u\|_{PW^{1,6}(\omega_T)},
\end{aligned} \tag{37}
$$

where $d = 1, 2$. Note that the IFE shape functions have the following bounds [9, 16, 22]

$$
|\phi_{i,T}|_{W^{k,\infty}(\omega_T)} \le Ch^{-k}, k = 1, 2. \tag{38}
$$

Based on the estimations above, it follows from the expansions (22)-(25) that

$$
\begin{aligned}
& \|\partial_{x_d}(I_{h,T}u - u)\|_{L^2(\omega_T^{*,s})} \\
\le & Ch^{-1} \left(\sum_{i \in \mathcal{I}^{s\prime}} \left(\|E_i^s\|_{L^2(\omega_T^{*,s})} + \|F_i^s\|_{L^2(\omega_T^{*,s})} \right) + \sum_{i \in \mathcal{I}} \|R_i^s\|_{L^2(\omega_T^{*,s})} \right) \\
\le & Ch \|u\|_{PH^2(\omega_T)},
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
& \|\partial_{x_d x_d'}(I_{h,T}u - u)\|_{L^2(\omega_T^{*,s})} \\
\le & Ch^{-2} \left(\sum_{i \in \mathcal{I}^{s\prime}} \left(\|E_i^s\|_{L^2(\omega_T^{*,s})} + \|F_i^s\|_{L^2(\omega_T^{*,s})} \right) + \sum_{i \in \mathcal{I}} \|R_i^s\|_{L^2(\omega_T^{*,s})} \right) \\
\le & C \|u\|_{PH^2(\omega_T)},
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
& \|\partial_{x_d}(I_{h,T}u - u)\|_{L^2(\omega_T^*)} \\
\le & Ch^{-1} \sum_{i \in \mathcal{I}} \|\tilde{R}_i\|_{L^2(w_T^*)} + \|\partial_{x_d} u\|_{L^2(\omega_T^*)} \le Ch \|u\|_{PW^{1,6}(\omega_T)},
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
& \|\partial_{x_d x_d'}(I_{h,T}u - u)\|_{L^2(\omega_T^*)} \\
\le & Ch^{-2} \sum_{i \in \mathcal{I}} \|\tilde{R}_i\|_{L^2(\omega_T^*)} + \|\partial_{x_d x_d'} u\|_{L^2(\omega_T^*)} \le C(\|u\|_{PW^{1,6}(\omega_T)} + \|u\|_{PH^2(\omega_T)}),
\end{aligned} \tag{42}
$$

where $d, d' = 1, 2$. Note that $\omega_T = \omega_T^* \cup \omega_T^{*,-} \cup \omega_T^{*,+}$, thus (32) follows from (39)-(42). □

4. Error estimation for PPIFE method

In this section, we derive optimal estimates for the errors of PPIFE solutions under the usual H^2 regularity assumption for the exact solution. The following quantities will be used to gauge the errors of PPIFE solutions:

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} \int_T \beta\|\nabla v\|^2 dX + \sum_{e\in\mathring{\mathcal{E}}_h^i} \sigma_e^0 \int_e \||e|^{-1/2}[v]\|^2 ds, \ \forall v \in V_h(\Omega), \tag{43}$$

$$|||v|||_h^2 = \|v\|_h^2 + \sum_{e\in\mathring{\mathcal{E}}_h^i} (\sigma_e^0)^{-1} \int_e \||e|^{1/2}\{\beta\nabla v\cdot \mathbf{n}_e\}\|^2 ds, \ \forall v \in V_h(\Omega). \tag{44}$$

In fact, the following Lemma shows the quantities defined in (43) and (44) are indeed energy norms on the space $V_h(\Omega)$.

Lemma 4.1. *$\|\cdot\|_h$ and $|||\cdot|||_h$ are both norms on $V_h(\Omega)$.*

Proof. We only present a proof for $|||\cdot|||_h$ and the argument for $\|\cdot\|_h$ is similar. Suppose $|||v|||_h = 0$ for some $v \in V_h(\Omega)$. By (43), it is easy to see that v is a piecewise constant on each element and sub-elements of interface elements. Besides, since $v|_{\partial\Omega} = 0$ and v is continuous across all the non-interface edges, we have $v = 0$ on $\cup_{T\in\mathcal{T}_h^n} T$. In addition, the second term in (43) vanishing implies that v is actually continuous over all the interface edges, and thus, $v = 0$ on $\cup_{T\in\mathcal{T}_h^i} T$. Hence, $v = 0$ on the whole Ω. Since it is easy to verify that $|||\cdot|||_h$ is a semi-norm, we conclude that $|||\cdot|||_h$ is a norm. □

We note that the energy norm (43) was already used for the analysis in [25]. It is easy to see that

$$\|v\|_h \leqslant |||v|||_h, \ v \in V_h(\Omega). \tag{45}$$

The following lemma shows the norms (43) and (44) are actually equivalent when restricted on the IFE space $S_h(\Omega)$.

Lemma 4.2. *For sufficiently large σ_e^0, there exists a constant C independent of the interface location such that $|||v|||_h \leqslant C\|v\|_h$, $\forall v \in S_h(\Omega)$.*

Proof. For each $e \in \mathring{\mathcal{E}}_h^i$, let T_1^e and T_2^e be the two elements sharing the same edge e. By the trace inequality given by Lemmas 3.2 and 3.5 in [25], there exists a constant C independent of the interface location on both T_1^e and T_2^e, such that

$$\begin{aligned}\int_e \||e|^{1/2}\{\beta\nabla v\cdot\mathbf{n}_e\}\|^2 ds \leqslant& Ch(\|\beta\nabla v|_{T_1^e}\cdot\mathbf{n}_e\|^2_{L^2(T_1^e)} + \|\beta\nabla v|_{T_2^e}\cdot\mathbf{n}_e\|^2_{L^2(T_2^e)})\\ \leqslant& C\|\sqrt{\beta}\nabla v\|^2_{L^2(T_1^e\cup T_2^e)}, \ \forall v\in S_h(\Omega).\end{aligned} \tag{46}$$

Then, adding and subtracting the term $\sum_{e\in\mathring{\mathcal{E}}_h^i}(\sigma_e^0)^{-1}\int_e \||e|^{1/2}\{\beta\nabla v\cdot\mathbf{n}_e\}\|^2 ds$ in $\|\cdot\|_h$ yields

$$\begin{aligned}\|v\|_h^2 \geqslant& \left(1-\frac{2C}{\sigma_e^0}\right)\sum_{T\in\mathcal{T}_h}\int_T \beta\|\nabla v_h\|^2 dX + \sum_{e\in\mathring{\mathcal{E}}_h^i}\sigma_e^0\int_e \||e|^{-1/2}[v_h]\|^2 ds\\ &+ \sum_{e\in\mathring{\mathcal{E}}_h^i}(\sigma_e^0)^{-1}\int_e \||e|^{1/2}\{\beta\nabla v\cdot\mathbf{n}_e\}\|^2 ds,\end{aligned} \tag{47}$$

where the constant C is from (46). It is easy to see that the desired result follows from taking σ_e^0 large enough in (47). □

The following theorem derives an optimal bound for the error in the flux of the IFE interpolation of a piecewise H^2 function on interface edges.

Theorem 4.1. *Assume the mesh $\mathcal{T}_h$ is sufficiently fine and satisfies the Patch Assumption. Then there exists a constant C independent of the interface location such that:*

$$\sum_{e\in\mathring{\mathcal{E}}_h^i} \|\{\beta\nabla(u-I_hu)\cdot\mathbf{n}_e\}\|_{L^2(e)}^2 \leq Ch\|u\|_{PH^2(\Omega)}^2,\ \forall u\in PH^2(\Omega). \tag{48}$$

Proof. For each interface element $T\in\mathcal{T}_h^i$, let $e\in\mathcal{E}_h^i$ be one of its edges and let $e^s=e\cap\Omega^s, s=\pm$. According to the Patch Assumption, there exists an auxiliary triangle $T_e^s\subset\omega_T$, possessing e^s as one of its edges, such that $T_e^s\subset\Omega^s$ and $|e^s|/|T_e^s|\leq Ch^{-1}$, $s=\pm$. Letting $\beta_{max}=\max\{\beta^-,\beta^+\}$, applying the standard trace inequality on T_e^s and using the estimation in (32), we have

$$\begin{aligned}
&\|\beta\nabla(u-I_hu)\cdot\mathbf{n}_e\|_{L^2(e)}\\
\leq&\beta_{max}(\|\nabla(I_{h,T}u-u)\|_{L^2(e^-)}+\|\nabla(I_{h,T}u-u)\|_{L^2(e^+)})\\
\leq&C\sum_{s=-,+}\left(|e^s|/|T_e^s|\right)^{1/2}(\|\nabla(I_{h,T}u-u)\|_{L^2(T_e^s)}+h\|\nabla^2(I_{h,T}u-u)\|_{L^2(T_e^s)})\\
\leq&Ch^{1/2}(\|u\|_{PH^2(\omega_T)}+\|u\|_{PW^{1,6}(\omega_T)}).
\end{aligned} \tag{49}$$

For each interface edge $e\in\mathring{\mathcal{E}}_h^i$, let T_1^e and T_2^e be the two neighbor elements. Then (49) implies

$$\begin{aligned}
&\sum_{e\in\mathring{\mathcal{E}}_h^i}\|\{\beta\nabla(u-I_hu)\cdot\mathbf{n}_e\}\|_{L^2(e)}^2\\
\leqslant&C\sum_{e\in\mathring{\mathcal{E}}_h^i}(\|\beta\nabla(u-I_{h,T_1^e}u)\cdot\mathbf{n}_e\|_{L^2(e)}^2+\|\beta\nabla(u-I_{h,T_2^e}u)\cdot\mathbf{n}_e\|_{L^2(e)}^2)\\
\leqslant&Ch\sum_{T\in\mathcal{T}_h^i}(\|u\|_{PH^2(\omega_T)}^2+\|u\|_{PW^{1,6}(\omega_T)}^2)\\
\leqslant&Ch(\|u\|_{PH^2(\Omega)}^2+\|u\|_{PW^{1,6}(\Omega)}^2),
\end{aligned} \tag{50}$$

where we have used the finite-overlapping property of the patches ω_T, $T\in\mathcal{T}_h^i$. Then (48) is obtained by applying the standard embedding inequality [27] $\|w\|_{1,6,\Omega^s}\leq C\|w\|_{2,\Omega^s}$, $s=\pm$ to (50). □

The following theorem is about the approximation capabilities of the IFE spaces in terms of the energy norms on the whole domain Ω.

Theorem 4.2. *Assume the mesh $\mathcal{T}_h$ is sufficiently fine and satisfies the Patch Assumption. Then there exists a constant C independent of the interface location such that*

$$\|I_hu-u\|_h\leq Ch\|u\|_{PH^2(\Omega)},\ \forall u\in PH^2(\Omega), \tag{51}$$

$$|||I_hu-u|||_h\leq Ch\|u\|_{PH^2(\Omega)},\ \forall u\in PH^2(\Omega). \tag{52}$$

Proof. By (45), estimate (51) follows from (52). Estimate (52) simply comes from the estimate (48) and the definition (44) together with the global optimal approximation capabilities of the linear and bilinear IFE spaces given in [16, 22]. □

Now we show the coercivity and continuity for the bilinear form $a_h(\cdot,\cdot)$ defined in (13) in terms of the energy norm $|||\cdot|||_h$.

Theorem 4.3. *For $a_h(\cdot,\cdot)$ defined in (13), if σ_e^0 is sufficiently large, then there exists a constant κ such that*

$$a_h(v,v) \geq \kappa |||v|||_h^2, \ \forall v \in S_h(\Omega). \tag{53}$$

Proof. The coercivity (53) follows from Lemma 4.1 in [25] together with the norm equivalence given in Lemma 4.2. ☐

Theorem 4.4. *For $a_h(\cdot,\cdot)$ defined in (13), there exists a constant C such that*

$$a_h(w,v) \leq C|||w|||_h |||v|||_h, \ \forall w, v \in V_h(\Omega). \tag{54}$$

Proof. Note that

$$\begin{aligned} |a_h(w,v)| \leq & \big|\sum_{T\in\mathcal{T}_h}\int_T \beta\nabla w \nabla v dX\big| + \big|\sum_{e\in\mathring{\mathcal{E}}_h^i}\int_e \{\beta\nabla w\cdot\mathbf{n}_e\}[v]ds\big| \\ & + \big|\sum_{e\in\mathring{\mathcal{E}}_h^i}\int_e \{\beta\nabla v\cdot\mathbf{n}_e\}[w]\big| + \big|\sum_{e\in\mathring{\mathcal{E}}_h^i}\int_e \frac{\sigma_e^0}{|e|^\alpha}[w][v]ds\big|. \end{aligned} \tag{55}$$

Denote each term on the right in (55) as $Q_i (i = 1, 2, 3, 4)$. Applying Hölder inequality on Q_i, we have

$$|Q_1| \leq C\|w\|_{L^2(T)}\|v\|_{L^2(T)} \leq C|||w|||_h |||v|||_h, \tag{56}$$

$$\begin{aligned} |Q_2| \leq & \sum_{e\in\mathring{\mathcal{E}}_h^i} \|\{\beta\nabla w\cdot\mathbf{n}_e\}\|_{L^2(e)}\|[v]\|_{L^2(e)} \\ \leq & \big(\sum_{e\in\mathring{\mathcal{E}}_h^i}(\sigma_e^0)^{-1}\||e|^{1/2}\{\beta\nabla w\cdot\mathbf{n}_e\}\|_{L^2(e)}^2\big)^{1/2}\big(\sum_{e\in\mathring{\mathcal{E}}_h^i}\sigma_e^0\||e|^{-1/2}[v]\|_{L^2(e)}^2\big)^{1/2} \\ \leq & |||w|||_h |||v|||_h. \end{aligned} \tag{57}$$

Using similar arguments, we obtain

$$|Q_3| \leq |||w|||_h |||v|||_h, \tag{58}$$

$$|Q_4| \leq |||w|||_h |||v|||_h. \tag{59}$$

Thus, (54) follows from applying (56)-(59) to (55). ☐

We note the estimate for $\|I_h u - u\|_h$ given in (51) was also established in [25], but the present article proves it by alternative arguments in which (51) follows from (52) which is the optimal approximation capability of the IFE space in the stronger energy norm $|||\cdot|||_h$. More importantly, adopting the stronger norm $|||\cdot|||_h$ in the error estimation allows us to establish both the coercivity and continuity for the bilinear form $a_h(\cdot,\cdot)$ employed in the PPIFE method, which are critical components in obtaining the optimal error estimates for the PPIFE solution with the standard $PH^2(\Omega)$ regularity in the following theorems.

Theorem 4.5. *Assume that the exact solution u to the interface problem (1)-(3) is in $PH^2(\Omega)$ and u_h is the related PPIFE solution with σ_e^0 in $a_h(\cdot,\cdot)$ large enough on a mesh $\mathcal{T}_h$ fine enough, then there exists a constant C such that*

$$|||u - u_h|||_h \leq Ch\|u\|_{PH^2(\Omega)}. \tag{60}$$

Proof. From (15) and (16) we have

$$a_h(u_h - I_h u, v) = a_h(u - I_h u, v), \quad \forall v \in S_h(\Omega). \tag{61}$$

Letting $v = u_h - I_h u$ and using both the coercivity and the continuity of $a_h(\cdot,\cdot)$, we have

$$\kappa|||u_h - I_h u|||_h^2 \le a_h(u_h - I_h u, u_h - I_h u) = a_h(u - I_h u, u_h - I_h u) \le C|||u - I_h u|||_h |||u_h - I_h u|||_h. \tag{62}$$

Thus, $|||u_h - I_h u|||_h \le C|||u - I_h u|||_h$. Then, by (52), we have

$$|||u - u_h|||_h \le |||u - I_h u|||_h + |||u_h - I_h u|||_h \le (1 + C)|||u - I_h u|||_h \le Ch\|u\|_{PH^2(\Omega)}$$

which proves (60). □

Because of (45), the estimate given by (60) leads to

$$\|u - u_h\|_h \le Ch\|u\|_{PH^2(\Omega)}, \tag{63}$$

which is not only an optimal error estimate for the PPIFE solution u_h in the energy norm $\|\cdot\|_h$ but also a better estimate than the one given in Theorem 4.3 of [25] because (63) follows from the standard regularity assumption for the exact solution u.

Furthermore, using the standard regularity assumption in the error analysis allows us to derive an optimal error estimate in the L^2 norm in the following theorem, which could not be accomplished by the analysis approaches employed in [25] that relied on the excessive $PH^3(\Omega)$ regularity.

Theorem 4.6. *Under the conditions of Theorem 4.5, there exists a constant C such that*

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^2\|u\|_{PH^2(\Omega)}. \tag{64}$$

Proof. The proof is based on the standard duality argument. Let $w \in PH^2(\Omega)$ be the auxiliary function that is the solution to (1)-(3) with f at the right hand side replaced by $u - u_h$. Then, following standard arguments we have

$$\|u - u_h\|_{L^2(\Omega)}^2 = a_h(w, u - u_h). \tag{65}$$

Let $I_h w$ be the interpolent of w in IFE space. Since $I_h w \in S_h(\Omega)$, by (15) and (16) we have $a_h(I_h w, u - u_h) = 0$ which leads to $a_h(w, u - u_h) = a_h(w - I_h w, u - u_h)$. Then, by (65) and the continuity of $a_h(\cdot,\cdot)$, we have

$$\|u - u_h\|_{L^2(\Omega)}^2 = a_h(w - I_h w, u - u_h) \le C|||w - I_h w|||_h |||u - u_h|||_h. \tag{66}$$

According to (52) and the regularity for the elliptic interface problem [7], we have

$$|||w - I_h w|||_h \le Ch\|w\|_{PH^2(\Omega)} \le Ch\|u - u_h\|_{L^2(\Omega)}. \tag{67}$$

Putting (67) to (66) leads to

$$\|u - u_h\|_{L^2(\Omega)} \le Ch|||u - u_h|||_h, \tag{68}$$

which yields (64) by applying (60). □

We present a numerical example that corroborates the optimal error estimates obtained in Theorems 4.5 and 4.6. Consider the domain $\Omega = (-1, 1) \times (-1, 1)$ that is separated by the circular interface $\Gamma : x^2 + y^2 - r_0^2 = 0$, $r_0 = \pi/6.28$ into two subdomains

$$\Omega^- = \{(x, y) : x^2 + y^2 < r_0^2\},\ \Omega^+ = \Omega\backslash\overline{\Omega^-}.$$

On Ω, we choose functions f and g such that the interface problem (1)-(3) has the following exact solution:

$$u(x,y) = \begin{cases} \frac{1}{\beta^-} r^\alpha, & (x,y) \in \Omega^-, \\ \frac{1}{\beta^+} r^\alpha + \left(\frac{1}{\beta^-} - \frac{1}{\beta^+}\right) r_0^\alpha, & (x,y) \in \Omega^+, \end{cases} \tag{69}$$

in which $\alpha = 1.5, r = \sqrt{x^2+y^2}, \beta^- = 1, \beta^+ = 10$. It can be verified that $u \in PH^2(\Omega)\backslash PH^3(\Omega)$. Table 4 presents errors of the PPIFE solution u_h generated on a sequence of uniform triangular meshes $\mathcal{T}_h$ of Ω in which $h = 2/N$ with the integer N listed in the first column in Table 4. The data in this table clearly demonstrate that the PPIFE solutions converge optimally in both the L^2 and H^1 norms to the exact solution u that is a function in the Sobolev space $PH^2(\Omega)$ but not in $PH^3(\Omega)$.

Table 1. Errors of SPPIFE solutions, $\beta^- = 1$, $\beta^+ = 10$, $\alpha = 1.5$.

N	$\|u-u_h\|_{0,\Omega}$	rate	$\|u-u_h\|_{1,\Omega}$	rate
10	2.9428e-03	NA	3.2747e-02	NA
20	8.4280e-04	1.8039	1.5430e-02	1.0856
40	1.9635e-04	2.1018	7.8261e-03	0.9793
80	4.5931e-05	2.0958	3.9244e-03	0.9958
160	1.1242e-05	2.0305	1.9596e-03	1.0019
320	2.9990e-06	1.9064	9.7966e-04	1.0002
640	7.7099e-07	1.9597	4.8967e-04	1.0005
1280	1.9814e-07	1.9602	2.4490e-04	0.9996

5. Conclusions

In this article, we employ a new analysis framework to derive the error bounds for the PPIFE methods developed in [25]. This new framework uses an energy norm $|||\cdot|||_h$ which is stronger than $\|\cdot\|_h$ norm originally used [25]. There are two key-components in this analysis framework. First, it employs a patch technique to show the optimal approximation capability on interface edges for the flux of the IFE interpolation of a function with the standard piecewise H^2 regularity. Second, it shows that the bilinear form $a_h(\cdot,\cdot)$ in the PPIFE methods is both coercive and continuous in terms of the stronger energy norm $|||\cdot|||_h$. Benefitted from these two key-components, not only can we show that the IFE space has the optimal approximation capability gauged by the energy norm $|||\cdot|||_h$, but also we can show the PPIFE solution converges optimally in both $|||\cdot|||_h$ and $\|\cdot\|_h$ with the standard piecewise H^2 regularity for the exact solution. As a very important consequence of the standard piecewise H^2 regularity assumption, we can further show that the PPIFE solution converges optimally in the L^2 norm which the analysis techniques used in [25] could not achieve.

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