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A REVIEW OF THEORETICAL MEASURE APPROACHES IN OPTIMAL SHAPE PROBLEMS

ALIREZA FAKHARZADEH JAHROMI AND HAJAR ALIMORAD

Abstract. Some optimal shape design problems lack classical solutions, or at least, the existence of such solutions is far from being straightforward. In such cases, to obtain an optimal solution, a variety of methods have been employed. In this study, we review the works that used measures which can basically be divided in two groups: using Young measures and embedding process (Shape-measure method). We also survey the advantages and disadvantages of these two methods and investigate their improved version in the presented works and applications.

Key words. Young measure, radon measure, atomic measure, optimal shape, shape-measure, linear programming problem, relaxed problem.

1. Introduction

The field of shape optimization problems has recently attracted the attention of many scholars. These researchers argue for a number of applications in physics and engineering which require a focus on shapes rather than on parameters or functions. The purpose of such applications is to modify admissible shapes so that they can comply with a given cost function which needs to be optimized. In general, the study of optimal shape design (OSD) tries to answer the question of "*What is the best shape for a physical system*". The term OSD is used whenever a function has to be minimized with respect to a particular geometric element (or elements) like curve, domain, or point.

It is well-known that a measurable set, like a shape, can be considered as a measure. On the other hand, a question which comes into mind is when a measure can be considered as a shape. This is the base for special techniques in optimal shape (OS) and OSD problems which try to determine the optimal shape as a measure. The main aim of this paper is to present a complete survey on the OS and OSD theoretical measure based method. As literature show, two measures are used in this application: Young measure and positive Radon measure (we also even see traces of atomic measure in these works and it is necessary to remind that the atomic measure is a part of the methods which are based on these two measures as will be explained later).

We can use Young measures mainly as a tool to organize and comprehend the behavior of sequences of functions with respect to integral functionals; in particular, relaxed optimization problems whose generalized solutions come from a sequence of functions generate the generalized minimizer of the original problem. Young measures can be used to describe these relaxed formulations of different types of optimization problems [68]. There is an extensive literature about the application of Young measures in different subjects; for instance control theory [9], differential equations ([78] and [11]). A key feature of these kinds of measures is their capacity to capture the oscillations of minimizing sequences of non-convex variational problems, and many applications appear in ([10] and [35]); for example in models of

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elastic crystals. Some other properties of Young measures can be found in many studies like [45] and [46].

Using Radon measures for solving optimal control problems based on the idea of L. C. Young (see [81]) was applied for the first time in [80] and the method was theoretically established by Rubio [75]. This approach is based on the definition of a measure on a product space, of which control is just a factor. In this way, one constructs a linear optimization problem corresponding to the control problem. This method was extended and improved by others such as Kamyad et al. [40], [42] and [43] and Farahi et al. [25], [6] and [26] to name some. In 1996, Fakharzadeh [11] used this method to solve some optimal shape design problems governed by elliptic equations in two dimensions. The similar idea for solving OSD problems in a relaxed sense is named Shape-measure. The measures used in Shape-measure method are uniquely defined while those used in Young measure are not defined uniquely [56].

It is necessary to indicate that measures like Occupation measure applied to solve optimal control problems [47] have not yet been used for solving OS and OSD problems. Therefore, this paper is divided into two main sections. In the first one, the OS and OSD problems which are solved by using Young measures are reviewed. In the next section, the application of Radon measures (Shape-measure method) for solving the mentioned problems are surveyed. In each section, to make the readers more familiar with the studies, one of those studies has been explained relatively extensively. Then, an attempt has been made to explain all the studies conducted on the basis of these methods with a view of expanding the method (in terms of the model of the problem, method of utilizing the measures and the way of transferring the main problem into the measures space). It is necessary to remind that since generally used methods for shape optimization have a general framework, applying them for each problem depends on the kind of the problem and the creativity of the person who uses them. In this regard, in each application of these two methods we are faced with some novelties which have to be represented in this review carefully. Finally, the critical and comparative analysis of the two methods for solving OS and OSD problems are presented. Since this paper may not be that comprehensive, we hereby apologize to those authors and readers whose works could not be cited in this study.

2. Young measure based methods

Young measures which were originally conceived as 'generalized curves' by L. C. Young complete sets of ordinary curves in the calculus of variations. It has been proved that a broad class of problems in the calculus of variations has solutions in the form of these generalized curves [2]. Consider an energy functional which lacks the property of lower semi-continuity, in such circumstances, the infimum of energy is achieved only in some generalized sense while a minimizing sequence may develop finer and finer oscillations, reminiscent of a finely twinned microstructure [44]. Young measures are used in optimization problems (and shape optimization) where a local, integral cost functional is to be minimized in a suitable class of functions which often lack optimal solutions because of the presence of some non-convexity. In such cases, a single function is unable to reproduce the optimal behavior due precisely to this lack of optimal solutions, and one must resort to sequences (the so-called minimizing sequences) in order to comprehend the main features of optimality.

Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in $L^{\infty}(U, R^m)$, where U denotes an open bounded subset of R^n, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ and for almost every $x \in U$ a Borel probability measure v_x on R^m such that for each $F \in C(R^m)$ we have

$$F(f_{k_j}) \to \int_{R^m} F\left(y\right) dv(y)$$

in $L^{\infty}(U)$. The measures v_x are called the Young measures generated by the sequence $\{f_k\}_{k=1}^{\infty}$. Let $I : \mathcal{A} \to R \cup \{\infty\}$ be a local, integral cost functional defined on an admissible class of functions $\mathcal{A}$. Typically,

$$I\left(u\right) = \int F\left(x, u\left(x\right), \nabla u\left(x\right), \dots\right) dx,$$

the optimization problem is to comprehend how $\{\inf \quad I(u) \mid \quad u \in \mathcal{A}\}$ is realized. One introduces a generalized optimization problem, intimately connected to the one above, by putting

$$\tilde{I}(v) = \lim_{j \to \infty} \quad I(u_j)$$

when v is the Young measure associated to the sequence $\{u_j\} \subset \mathcal{A}$. If $\widetilde{\mathcal{A}}$ stands for the set of all such Young measures, one would like to understand the optimal behavior for $\left\{\inf \ \tilde{I}\left(v\right) \ \middle| \ v \in \widetilde{\mathcal{A}} \right\}$.

Due to the fundamental property of the Young measure indicated above, the optimal behavior for this new optimization problem can always be described with a single element in $\widetilde{\mathcal{A}}$, which in turn, is generated by minimizing sequences of the original optimization problem. The main point is being able to study the generalized optimization problem by itself, and then interpret that information in terms of minimizing sequences of the initial optimization problem. The significant issue, here, is to find ways of characterizing the admissible set $\widetilde{\mathcal{A}}$ that may allow for an independent treatment of the generalized optimization problem. In particular, understanding how constraints in $\widetilde{\mathcal{A}}$ are determined by constraints in $\mathcal{A}$ is a major challenge [45]. It is not clear how to set up the numerical framework so as to be helpful in approximating minimizing sequences for the original optimization problem [68]. We remind that if the minimizing sequences are gradients, Young measure generated by these sequences of gradients is called gradient Young measures. In 2007, Pedregal introduced div-curl Young measures to examine optimal design problems governed by a linear state law in divergence form [69]. He worked directly with divergence-free vector fields and gradients. The associated Young measures were called div-curl Young measures. To use div-curl Young measures which arose in this kind of optimal design problems, he considered a pair of sequences of feasible designs for the original problem which was a sequence of div-curl pairs. Therefore, he had a div-curl Young measure whose support was restricted to contain the union of those two linear manifolds.

In the next section, we review the optimal shape (OS) and the optimal shape design (OSD) problems which have been solved by using Young measure method; these studies are reviewed in 2 subsections.

2.1. Applications of Young measures in OS and OSD. Nicolaides et al. (1992) investigated a characteristic feature of the energy functional having multiple structures [66]. Typically, each well represented a potential equilibrium state of the crystal, and at a transformation temperature, more than one well was accessible to the crystal as a stable configuration. The variational approach for finding an overall

equilibrium state for the crystal required that the energy functional be minimized in some suitable sense. In attempting such minimizations, one frequently encounters minimizing sequences of rapidly oscillating functions. In computational practice, the minimizing sequences were often constructed using a finite mesh, for example by finite elements. The oscillations referred to above then show up as grid scale oscillations of the (generally nonunique) minimizer [44]. Usually, one wishes to know the values of macroscopic quantities associated with the deformation. Essentially, linear functions of the deformation are obtained as the limits of the same linear functions of the minimizing sequence. On the other hand, nonlinear functions of the deformation (including energy) in general had to be computed as expected values of the probability distribution. In the above mentioned paper, the authors were interested in the opposite case. Although in principle it must be possible to compute the probabilities from the oscillatory minimizing sequence, in practice this could be very difficult if there were a relatively large number of wells.
They considered variational integrals

$$J(u) := \int_{\Omega} F(x, u, \nabla u)dx, \quad u \in W_0^{1,p}(\Omega)^m, \tag{1}$$

where $\Omega \subset R^n$ was a bounded domain and $F(.,.,.)$ was continuous. Inhomogeneous boundary conditions could easily be accommodated if necessary. The case of most interest was when $F(x, u, .)$ was not convex with respect to its last variable. The multiple well property of stored elastic energy functions caused this lack of convexity. In this case, the infimum of $J(.)$ could not be reached in $W_0^{1,p}(\Omega)^m$ and it was necessary to admit generalized solutions. For any bounded sequence in $W_0^{1,p}(\Omega)^m$, $\| u_k \|_{1,p} \leq M$, $\{u_k\}$ contained a subsequence $\{u_{k_j}\}$ such that $u_{k_j} \to u \in L^p(\Omega)^m$. Additionally, a subsequence of $\{u_{k_j}\}$ existed (denoted the same way) with the property that for any continuous g which was reasonably behaved at infinity, and for each $x \in \Omega$ there was a probability measure υ_x such that

$$\begin{aligned} &g(\nabla u_{k_j}) \rightharpoonup G \in L^p(\Omega)^m \\ &G(x) = \textstyle\int_{R^n} g(y) d\upsilon_x(y) \end{aligned} \tag{2}$$

for almost all $x \in \Omega$. Where $\upsilon_{x,\delta}^k$ denoted the probability distribution of the values of $\nabla u_k(z)$, z was chosen uniformly at random from $B(x, \delta)$, the open ball with radius δ and center $x \in \Omega$. Then

$$\lim_{\delta \to 0} \lim_{k \to \infty} \mid \int_{R^n} g(y) d\upsilon_x(y) - \int_{R^n} g(y) d\upsilon_{x,\delta}^k(y) \mid \to 0.$$

They assumed that

$$\begin{aligned} &\upsilon_x = \textstyle\sum_{l=1}^L \lambda_l(x) \delta_{A_l}(x), \\ &\textstyle\sum_{l=1}^L \lambda_l(x) = 1, \quad 0 \leq \lambda_l \leq 1, \end{aligned} \tag{3}$$

where $\delta_{A_l}(x)$ denoted a Dirac mass with pole at $A_l(x)$ and $\lambda_l(x)$ varied measurably with x. They chose g in (2) to be the identity mapping showing that

$$\nabla u := \sum_{l=1}^L \lambda_l(x) A_l(x).$$

These results motivated the following generalized variational problem: minimize

$$I(u) := \int_{\Omega} \langle \upsilon_x, F(x, u(x), .) \rangle dx, \quad u \in W_0^{1,p}(\Omega)^m \tag{4}$$

subject to:

$$\nabla u := \sum_{l=1}^{L} \lambda_l(x) A_l(x)$$

over suitable $A_l \in L^p(\Omega)^{mn}, \lambda_l \in L^\infty(\Omega), l = 1, 2, ..., L$. Solutions to this problem were regarded as generalized solutions to (1).

In the next section of that paper, the authors considered discretizations of the generalized problem. Basically, they used continuous piecewise linear approximations for u, and piecewise constant approximations for A_l but these approximations cannot always be arbitrarily chosen, since the combination on the right of (3) must be a gradient Young measure. The algorithm presented involves several constraints, namely,

$$\nabla u := \sum_{l=1}^{L} \lambda_l A_l, \quad \sum_{l=1}^{L} \lambda_l = 1, \quad and \quad 0 \leq \lambda_l \leq 1$$

(recall that the discrete u was piecewise linear, so its gradient was piecewise constant, as were the discrete A_l and λ_l). In addition to these obvious constraints, when u was vector valued further constraints on the representation of the gradient were required to guarantee that $\upsilon = \sum_{l=1}^{L} \lambda_l \delta_{A_l}$ was a gradient Young measure. They outlined an algorithm that effectively eliminated the constraints on ∇u analytically and began by considering the case with $L = 2$, i.e.

$$\nabla u = \lambda A_0 + (1 - \lambda) A_1.$$

To obtain a solution of the discrete problem, simple relaxation was used in conjunction with the "numerical tricks". The idea behind relaxation was to freeze all but one unknown, ξ (a nodal value of u, or a λ value for an element, etc).

Bonnetier and Conca (1994) introduced Young measures as a means of studying problems of the calculus of variations that did not admit solutions in the classical sense [5]. The sequence of functions that the authors constructed has "rapid variations" and can be interesting in the characterization of minimizing sequences in the calculus of variations problems. As an example, they applied their method to a problem of optimal design of orthotropic plates with parallel stiffeners. They minimized the compliance (the work done by the load) under the constraint of a prescribed volume. However, this minimization problem did not have a solution in the set of admissible thicknesses. They showed that, regarding their approximation, the set of admissible thicknesses and the definition of the compliance could be extended so that the minimization problem had a solution. Given a parameterized measure and a family of continuous φ_n functions, they constructed a sequence of functions $\{u_k\}$ such that $k \to \infty$, function $\varphi_n(u_k)$ converge to the corresponding moments of the measure, in the weak* topology. Using the sequence $\{u_k\}$ corresponding to a dense family of continuous functions, they applied these techniques to an optimal design problem for plates with variable thickness. The relaxation of the compliance functional involves three continuous functions of the thickness. They characterized a set of admissible generalized thicknesses on which the relaxed functional attains its minimum. Let Ω be a smooth domain in R^2. The authors considered a Kirchhoff model for pure bending of symmetric plates with mid-plane Ω. The deflection w satisfied an equation of the form

$$\partial_{\alpha\beta} \left(M_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} \right) = F \qquad in\ \Omega,$$

where the tensor $M_{\alpha\beta\gamma\delta}$ depended on the half-thickness $h(x,y)$ of the plate, F is a load which is sufficiently smooth. They assumed that the plate was clamped, i.e. w satisfies the boundary conditions

$$w = \frac{\partial w}{\partial n} = 0 \quad on\ \partial\Omega,$$

which made w the minimizer of the following energy functional:

$$E\left(w\right) = \frac{1}{2}\int_{\Omega} M_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w\partial_{\gamma\delta}w - \int_{\Omega} Fw.$$

In 1998, Theil and Muller studied the viscoelastically damped wave equation

$$\ddot{u} = \partial_x\left(\sigma\left(\partial_x u\right) + \beta\partial_x\dot{u}\right) - \alpha u, \qquad \alpha \geq 0, \quad \beta > 0, \quad x \in (0,1)$$

with a non-monotone stress-strain relation σ [79]. The interest lies in the case where σ was the derivative of non-convex stored-energy density W which was assumed to satisfy $W\left(z_1\right) = W\left(z_3\right) = 0$ and $W\left(z\right) > 0$ elsewhere; then, the above equation was able to describe the evolution of phase transitions between two stable phases located near z_1 and z_3. The total energy

$$E_\alpha\left(t\right) = E_\alpha\left(u\left(t\right), \dot{u}\left(t\right)\right) = \int_0^1 \left(W\left(\partial_x u\right) + \frac{1}{2}\dot{u}^2 + \frac{\alpha}{2}u^2\right) dx$$

decreases monotonically in time. This system describes the dynamics of phase transitions which is closely related to the creation of microstructures. In order to analyze the dynamic behavior of microstructures, the authors showed that highly oscillatory initial states generated an evolution in a unique fashion in the space of Young measures and they derived the determining equations. Furthermore, they proved a generalized dissipation identity for Young measure solutions.

Pedregal (1999) considered some models of material behavior where the energy functional consisted of two contributions [68]: The first one was local and the second was not in the form of a control problem. An explicit form of the relaxed energy functional in term of Young measure value magnetizations was provided. Then, he considered an optimal design problem for two phase conductors. The aim was to emphasize how a relaxed formulation of the problem based on Young measures, could be considered, and how was the relationship of this formulation with quasiconvexity and the vector variational problems. Another optimal design problem examined there was concerned with plates of variable thickness under a given load. A recent development in continuum mechanics was the introduction of continuum energy nonlinear functional modeling effects of crystal thermo elasticity [1, 2, 54 and 73]; among other things, these functional can be used to study displacive phase transformations and shape memory effects [45]. The first optimal design example was an optimal control problem governed by a second order ordinary differential equation completed by boundary values. It was a one-dimensional simplified version of an important optimal design problem concerned with two phase conductors. In that example $\alpha > 0$ and $f \in L^2(0,1)$ were given, $F(x,u,y)$ was continuous in (u,y) and measurable in x, $\mathcal{U} = \{u \in L^1\left(0,1\right) : u \geq \alpha\}$ was the set of competing functions for optimization problem and the objective function was

$$I\left(u\right) = \int_0^1 F\left(x, u\left(x\right), y\left(x\right)\right) dx$$

where $y \in H_0^1(0,1)$ was the solution of $-(uy^{'})^{'} = f$. The next example in that paper was a shape optimization problem in which set $\Omega \subset R^N$ and function $f \in L^2(\Omega)$ were given. Then, the class of all possible shapes was defined as

$\mathcal{U} = \{A \subset \Omega : A \; is \; regular, \quad |A| \; given\}$ and the cost function was

$$I\left(A\right) = \int_{\Omega} F(x, y_A\left(x\right), \nabla y_A\left(x\right))dx$$

where $y_A \in H_0^1(A)$ was the solution of $-\triangle y_A = f$ in A. The author solved problem $-\triangle y = f$ in Ω, $y \in H_0^1(\Omega)$ and put $u_A = y_A - y$ for each $A \in \mathcal{U}$; since u_A was harmonic in A and $u_A = y$ on ∂A, he set $u_A = y$ outside A and found that u_A consisted in replacing y in A by the harmonic function that agreed with y in ∂A. Then

$$I\left(A\right) = \int_{\Omega} F(x, u_A\left(x\right), \nabla u_A\left(x\right))dx.$$

Since u_A was harmonic when it did not coincide with y:

$$\int_{\Omega} \left|\nabla u_A\left(x\right)\right|^2 dx \leq \int_{\Omega} \left|\nabla y_A\left(x\right)\right|^2 dx$$

for any $A \in \mathcal{U}$. He solved this problem by using Young measure and minimizing sequence $\{A_j\}$. In this way, he established the existence of optimal solutions within the class of Young measures associated with sequences of gradients coming from sequences of subsets of Ω. In order to pursue the analysis in this framework, one needs to characterize all those families of Young measures. This looked like a hard problem even in dimension one [68]. He finally considered an optimal design problem for plates of variable thickness.

In his book, Pedregal (1999) focused on explaining the complexity of vector variational problems from the aspect of existence- nonexistence of equilibrium configurations, with special emphasis on the relevance of structural assumptions [69]. In particular, his point was to communicate how the different notions of convexity arise in vector veariational problems and to explain their significance with respect to the existence issue. The analysis did not go into any deeper examination of polyconvexity, quasi convexity, or rank-one convexity; almost nothing was said about characterizing gradient Young measures (see [67] for the basic properties of these measures). It was also important to point out that applications to real materials had not been included.

Failure of quasi convexity, nonexistence, oscillatory behavior, ultimately, rank-one convexity and laminates, as the main example of microstructures within the context of phase transitions in crystalline solids, were addressed in Chapter 4 of this book. In this chapter, he described a class of materials whose stored-energy densities lack the property of quasi convexity. This situation led to considering generalized variational principles where $W^{1,p}-$Young measures axe allowed to enter the minimization problem. These objects are physically interpreted as microstructures and represent highly oscillatory minimizing sequences on smaller and smaller spatial scales.

In 2002, Bellido and Pedregal studied relaxation for optimal design problems in conductivity in a two-dimensional space [3]. The authors reformulated the optimal design problem in an equivalent way as a genuine vector variational problem. Their main achievement was to explicitly compute the quasiconvexification of the involved density in this problem for some interesting cases. They determined how to mix two conducting materials, with conductivities, or dielectric permittivity, α and β, $0 < \alpha < \beta$, to fill out a domain $\Omega \subset R^2$ in such a way that minimizes the cost functional

$$I(\chi) = \int_\Omega \varphi(x, \alpha\chi(x) + \beta(1-\chi(x)), u(x), \nabla u(x))dx,$$

where the design variable, χ, was the characteristic function of a subset of Ω occupied by the material with conductivity α, and $u \in H^1(\Omega)$ was the unique solution of the state equation

$$\begin{cases} -div((\alpha\chi(x) + \beta(1-\chi(x))\nabla u(x)) = P(x), & in\ \Omega \\ u = u_0\ , & on\ \partial\Omega, \end{cases}$$

which $u_0 \in H^1(\Omega)$ and $P \in H^{-1}(\Omega)$ stand for the exterior charges performing in Ω. General optimal design problems of this kind fail to have optimal solution in the class of characteristic functions. When the cost functional does not depend on the derivatives of the state, the main tool to analyze and understand this lack of existence has been Homogenization Theory [3]. In the mentioned paper, the authors explicitly computed the relaxed density for this variational problem in some interesting cases of objective functions.

Also, Pedregal in 2005 ([70]) solved an optimal design problem with two different conducting materials by using Young measures. He exploited optimality conditions for relaxation problem which were easily handled in a standard way. Indeed, the optimality requirement on these measures providing the polyconvexification had a lot to do with their support being as small as possible. He was looking for some extra information so that he could succeed in the final step of showing that optimal polyconvex measures were indeed laminates all over the domain. In this case, one can often setup a simpler variational problem encoding the behavior of minimizing sequences which can be approximated numerically in a suitable way. The optimal solutions together with the associated optimal measures encode all the ingredients to reconstruct minimizing sequences for the original optimization problem. In that paper, the optimal design problem in [2] was solved with the objective functional

$$J(\chi) = \int_\Omega |\nabla u(x)|^2 dx$$

and a new condition $\int_\Omega \chi(x)\,dx \le t_0|\Omega|$, $t_0 \in (0,1)$. Thus, the following problem was solved

$$\begin{array}{ll} Min: & J(\chi) = \int_\Omega |\nabla u(x)|^2 dx \\ S.to: & \begin{array}{ll} -div((\alpha\chi(x) + \beta(1-\chi(x))\nabla u(x)) = P(x), & in\ \Omega; \\ u = u_0\ , & on\ \partial\Omega; \end{array} \\ & \int_\Omega \chi(x)\,dx \le t_0|\Omega|. \end{array} \tag{5}$$

The author interpreted this design problem as a problem deciding where to place both materials so as to minimize the mean quadratic deviation from the zero vector field.

Next year, he solved the above problem based on the following objective function

$$J(\chi) = \int_\Omega \left|\nabla u(x) - \nabla u^{(0)}(x)\right|^2 dx.$$

His main contribution was to stress that in some cases, such analysis could be carried out even if no explicit form of the underlying quasiconvexification was at disposal. In a sense, the comprehensive knowledge of those relaxed integrands was not needed since his efforts would be directed to single out the optimal Young measures without having to exhaust or describe all possibilities. In particular, he illustrated his strategy by treating the optimal design problem in (two-dimensional) conductivity with a general cost which depends on the underlying electric field satisfying a specific constraint which he called the "joint boundary optimality condition" (JBOC). Under this main structural constraint, he provided a completely explicit relaxation of the problem [71].

Maestre and Pedregal (2006), analyzed a typical 3-D conductivity problem which consists of seeking the optimal layout of two materials in a given design domain $\Omega \subset R^3$ by minimizing the L^2-norm of the electric field under a constraint [48]. They utilized a characterization of the 3-D divergence-free vector fields which was especially appropriate for a variational reformulation. By using gradient Young measures as the main tool, they could give an explicit form of the "*constrained quasiconvexification*" of the cost density. This result is similar to the one in the 2-D situation [70]. However, the characterization of the divergence-free vector fields introduces certain nonlinearity in the problem that needs to be addressed properly. They studied a typical optimal design problem in conductivity which consisted of looking for the optimal distribution of two different conducting materials with isotropic constants α and β $(0 < \alpha < \beta)$ on a domain $\Omega \subset R^3$, such that it minimized a certain functional cost which depended on the underlying electric field of the state equation in the form

$$I\left(\chi\right) = \int_{\Omega} a\left(x, \chi\left(x\right)\right) \left|\nabla u\left(x\right) - F\left(x\right)\right|^2 dx, \tag{6}$$

where u was the unique solution of constraints in problem (5). The reason why they reformulated the problem through relaxation techniques was the lack of classical solutions for this type of problems. Their strategy was directed towards the understanding and computing the constrained quasi-convexification of a certain integrand which was obtained as a result of a suitable variational reformulation of the problem.

In 2006, Munch et al. considered the following damped wave equation:

$$\begin{array}{l} inf\, J(\chi_w) = \frac{1}{2}\int_0^T \int_\Omega (\left|u_t\right|^2 + \left|\nabla u\right|^2) dxdt \\ w \in \Omega_L \end{array} \tag{7}$$

$$\begin{cases} u_{tt} - \triangle u + a\left(x\right)\chi_w u_t = 0 & in \quad (0,T)\times\Omega \\ u = 0 & on \;\; (0,T)\times\partial\Omega \\ u(0,.) = u_0(x), \quad u_t(0,.) = u_1(x) & in\ \Omega \end{cases}$$

where $\Omega \subset C^2\left(R^N\right), \quad N = 1, 2$, is a bounded domain and χ_w denotes the characteristic function of w, a subset of Ω of positive Lebesgue measure and independent of the time $t \in (0,T)$. Moreover, the damping potential $a \in L^\infty(\Omega, R^+)$ is such that $a\left(x\right) > \alpha > 0$ $a.e.$ $x \in w$. Also,

$$\Omega_L = \{w \subset \Omega : |w| = L|\Omega|, \quad 0 < L < 1\},$$

$|w|$ and $|\Omega|$ begin the Lebesgue measures of w and Ω [57]; indeed, optimizing the shape and the position of the damping set for the internal stabilization of the linear wave equation were considered in (3). In the first theoretical part, they reformulated the problem into an equivalent non-convex vector variational one using a characterization of divergence-free vector fields. Then, by means of gradient Young measures, they obtained a relaxed formulation of the problem in which the original cost density was replaced by its constrained quasi-convexification. This implied that the new relaxed problem was well-posed in the sense that there existed a minimizer and the infimum of the original problem coincided with the minimum of the relaxed one. The presented numerical simulations indicated that when a was small, the problem was well-posed; that is, there was a minimizer in the class of characteristic functions, but if a was large, then the problem was ill-posed and it was necessary to relax it.

Maestre et al. (2007) analyzed a spatio-temporal optimal design problem governed by a linear damped 1-D wave equation [49]. The problem consisted of seeking simultaneously the spatio-temporal layout of two isotropic materials and the static position of the damping set in order to minimize a functional depending quadratically on the gradient of the state. By means of gradient Young measures, the authors computed an explicit form of the constrained quasi-convexification of the cost density. Moreover, this quasi-convexification was recovered by first order laminates which gave the optimal distribution of materials and damping set at every point. The problem was the following damped wave equation posed in $(0,T)\times\Omega$

$$\inf_{\chi_{w_1},\chi_{w_2}} \quad I(\chi_{w_1},\chi_{w_2}) = \int_0^T \int_\Omega (u_t^2 + a\,(t,x,\chi_{w_1})\,|u_x|^2)dxdt \tag{8}$$

$$\begin{cases} u_{tt} - \nabla_x\left([\alpha\chi_{w_1} + \beta\,(1-\chi_{w_1})]\,u_x\right) + d\,(x)\,\chi_{w_2}u_t = 0 & in \;\; (0,T)\times\Omega \\ u = 0 & on \;\; (0,T)\times\partial\Omega \\ u\,(0,x) = u_0\,(x)\,, \quad u_t\,(0,x) = u_1\,(x) & in \;\; \Omega \end{cases}$$

For any bounded interval $\Omega\subset R$ and any positive time T, χ_{w_1} and χ_{w_2}designate the characteristic function of two subsets $w_1\subset\Omega\times(0,T)$ and $w_2\subset\Omega$ respectively and $0<\alpha<\beta$. System (4) modeled the stabilization of an elastic string made of two materials α and β located on w_1 and $((0,T)\times\Omega)\backslash w_1$ respectively, by an internal dissipative mechanism located on w_2. The unknown $u(t,x)$ represented the transversal displacement of the string at point x and at time t. By using gradient Young measures as generalized solutions of variational problems, an explicit relaxation of the original problem in the form of a relaxed variational problem has been computed.

In [57], the pure damping case (corresponding to $w_1=\emptyset$; and a minimization with respect to w_2 only) has been studied. In that study, the authors aim at mixing these two cases and minimizing the objective function with respect to w_1 and w_2 simultaneously. In this respect, they derived and analyzed a well-posed relaxation of the problem.

In the next year, they sought the time-dependent optimal layout of two isotropic materials on a one dimensional domain by minimizing a functional depending quadratically on the gradient of the state with coefficients that may depend on space, time and design [50]. Typically, such problems are ill-posed in the sense that there is no optimal design. They examined relaxation using the representation of two-dimensional divergence free vector fields as rotated gradients. By means of gradient Young-measures, they computed an explicit form of the "constrained quasiconvexification" of the cost density. Moreover, this quasiconvexification was recovered by first or second order laminates which gave the optimal microstructure at every point. The perspective was similar to the previous papers for linear elliptic state equations. The novelty here lies in the state equation (the wave equation), and their contribution consists of understanding the differences with respect to elliptic cases.

Bellido et al. (2010) studied the relaxation of an optimal design problem in conductivity under a point-wise constraint on the heat flux [3]. They solved problem (6) with point-wise constraint on the heat flux as follow:

$$|(\alpha\chi(x) + (1-\chi(x))\beta)\nabla u(x)| \leq M \quad a.\ e.\ \ x \in \Omega.$$

The emphasis of their work was on the local constraint on the heat flux which was a new ingredient for these kinds of optimal design problems. In that paper, they proposed an analysis of relaxation for this problem using a different approach based on a suitable variational reformulation of the problem developed by the authors to deal with situations in which dependence on the gradient of the state was admitted in the cost functional. The approach relied on the introduction of a new variable in the form of a potential, or stream function, in order to avoid the non-local character of the state equation.
They explored whether a simpler, more manageable, relaxed formulation of the problem was possible and did this in that particular case following the ideas of [71]. In that paper an strategy was developed to obtain simple, amenable numerical simulation and relaxations for optimal design problems based on the relaxation of the variational reformulation of the problem.

In 2017, Jean-francois et al. were interested in a shape optimization problem for a fluid-structure interaction system composed by an elastic structure immersed in a viscous incompressible fluid. The cost functional to minimize was an energy functional involving together the fluid and the elastic parts of the structure. The shape optimization problem was introduced in the 2-dimensional case. However, the results in that paper were obtained for a simplified free-boundary 1-dimensional problem. They proved that the shape optimization problem was well-posed. The full characterization of the associated material derivatives was given together with the shape derivative of the energy functional.

They considered an elastic structure immersed in a viscous incompressible fluid. Let $\omega \subset\subset \Omega_S^{'} \subset\subset \Omega \subset R^2$ be three bounded domains where $\Omega_S^{'}$ and Ω are simply-connected domains. The deformed elastic body occupied the domain $\Omega_S = \Omega_S^{'} \backslash \bar{\omega} \subset R^2$ and the elastic structure was attached to the inner fixed boundary $\partial\omega$. The fluid filled up a bounded domain $\Omega_F = \Omega \backslash \bar{\Omega_S^{'}} = \Omega \backslash (\bar{\Omega_S} \cup \bar{\omega})$ surrounding the elastic body Ω_S. The boundary between the fluid and the elastic structure was $\Gamma_{FS} = \partial\Omega_F \cap \partial\Omega_S$ and $\partial\Omega_F = \Gamma_{FS} \cup \Sigma$ where $\Sigma = \partial\Omega$. The boundary Σ corresponded also to the outer boundary of the fluid domain Ω_F. The fluid flow was governed by the Stokes equations for the velocity u and the pressure p of the fluid:

$$\begin{array}{ll} -div\ \sigma(u,p) = f & in\ \Omega_F \\ div\ u = 0 & in\ \Omega_F \end{array}$$

where $\sigma(u,p) = 2\nu D(u) - pI_d$ was the Cauchy stress tensor with the symetric strain tensor $D(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$. The fluid was subjected to a given force f and ν is the viscosity of the fluid. At the boundary of the fluid domain, they imposed

$$u = 0 \quad on\ \partial\Omega_F = \Gamma_{FS} \cup \Sigma.$$

The elastic structure Ω_S was a deformation of a given reference bounded domain $\Omega_0 \subset R^2$ by a mapping X i.e. $\Omega_S = X(\Omega_0)$. In summary, the fluid-elasticity system for (u, p, w) read as

$$\begin{array}{ll}
-div\ \sigma(u,p) = f & in\ \Omega_F \\
div\ u = 0 & in\ \Omega_F \\
u = 0 & on\ \partial\Omega_F = \Gamma_{FS} \cup \Sigma \\
-div \prod(w) = g & in\ \Omega_0 \\
w = 0 & on\ \partial\omega \\
\prod(w) n_0 = (\sigma(u,p) o X) cof(\nabla X) n_0 & on\ \Gamma_0
\end{array}$$

where w was the elastic displacement of the structure which satisfied the linearized elasticity equation

$$-div \prod(w) = g \quad in\ \Omega_0$$

where $\prod$ was the second Piola-Kirchhoff stress tensor of the elastic structure given by

$$\prod(w) = \lambda tr(D(w)) I_d + 2\mu D(w)$$

with the Lame coefficients $\lambda > 0, \mu > 0$. The elastic body was subjected to a given external force g [41].

2.2. Div-curl Young measure in OS and OSD. In 2007, Pedregal solved problem (5) with a different objective function as follow:

$$I(\chi) = \int_\Omega (a_\alpha \chi(x) + a_\beta (1 - \chi(x))) |\nabla u(x)|^2 dx.$$

In that paper, he introduced exploit div-curl Young measure to examine optimal design problems governed by a linear state law in a divergence form [72]. The cost was allowed to depend explicitly on the gradient of the state. By means of this family of measures, he formulated a suitable relaxed version of the problem and, in a subsequent step, put it in a similar form as the original optimal design problem with an appropriated set of designs and generalized state laws. By using div-curl Young measures, he made the treatment dimension-independent. His contribution in that note was to develop, in the context of the above optimal design problem, a similar framework as in other previous works ([70]), but without the need to introduce additional potentials. In this way, he treated the problem, regardless of the dimension, and at the same time, simplified some of the computations performed in similar problems in other contexts ([48]).

Munch et al. (2008) considered the heat equation and addressed the nonlinear optimal design problem which consists of finding the distribution of two given isotropic materials in Ω that minimizes a suitable cost functional depending on the heat flux. Both cases of a time-independent and a time-dependent design were analyzed [58]. For solving the first problem, in a standard way, they used the homogenization method which seems to be very suitable for dealing with time-independent designs. In the second case, the relaxation procedure was derived by using the classical tools of non-convex, vector, variational problems: quasi-convexification and div-curl Young measures. The proposed method directly provided the behavior of (some) minimizing sequences of the original problem. They found that this measure was a convex combination of two Dirac masses. In addition, they conjectured that the weights of these Dirac masses (which represented the local volume fraction of the two materials) were time-independent.

After a year, by a suitable reformulation of a typical optimal design problem in conductivity, Boussaid and Pedregal determined the quasiconvexification (to fixed volume fraction) of certain sets of matrices that were the union of two manifolds [7]. They explicitly computed such hulls in a rather straightforward fashion without the need for seeking laminates by hand, and hence avoided some tedious computations. Also, they examined the linear case, both elliptic and hyperbolic, in 2 and higher dimensions using div-curl Young-measure. In non-linear situations, despite various attempts by several researchers, there has not been much progress; hence, they tried to explore the simplest non-linear example which was as close as possible to the linear situation.

At the same time, Maestre and Pedregal analyzed a dynamic optimal design problem in conductivity governed by the two-dimensional wave equation [51]. Under this dynamic perspective, the optimal design problem consisted of seeking the time-dependent optimal layout of two isotropic materials on a 2-D domain. This was done by minimizing a cost functional depending on the square of the gradient of the state function involving coefficients which could depend on time, space and design (using div-curl Young measure). The most important difference with the one treated in [49] and [50] was the non-linear character introduced by the characterization of the divergence-free vector fields in terms of Clebsch potentials.

Following some previous works (see [57] and [72]), Munch and Pedregal (2010) considered the shape design problem consisting of optimizing the distribution of two materials with different conductivities in Ω in order to reduce energy release rate [59]. This work was the first one which attempted to minimize this rate, and therefore to control the crack growth, with respect to the conductivity coefficient. Since this kind of problem is usually ill-posed, they first derived a relaxation by using the classical non-convex variational method. The computation of the quasiconvex envelope of the cost was performed using div-curl Young measures (following the procedure described in [72]) which led to an explicit relaxed formulation of the original problem and exhibited fine microstructure in the form of first order laminates. The numerical experiments suggested that an optimal distribution permits the cost to reduce significantly (with respect to an isotropic one). However, the optimal cost was not arbitrarily small so that the singularities around the crack tip were not canceled in contrast to the cases in which the control variable was an additional boundary load. This phenomenon was highly likely due to the condition (necessary in their context) which imposed the conductivity to be constant around the crack tip.

Also, Pedregal and Zhang (2012) analyzed the typical optimal design problems in conductivity when three or more materials were at disposal [73]. They believed that a more direct approach based on quasiconvexification of sets and quasiconvexification of integrands (see [70]) was possible and interesting. They focused on two different situations and for the first one, concentrated on an optimal design problem where the cost functional depended linearly on the gradient of the state. For the second one, they focused on a cost functional that was quadratic on that gradient. This second situation required a step well beyond the first one. More specifically, the problem was

Minimize χ : $\quad \int_{\Omega} \Psi\left(x, \nabla u\left(x\right)\right) dx$

S. to: $\chi = \{\chi_i\} \in \{0,1\}^n, \quad \chi.\mathbf{1} = 1, \quad \int_\Omega \chi(x)\,dx = |\Omega| r,$

$$div\left[\sum_i A_i \nabla u(x)\right] = 0 \quad in\ \Omega, \qquad u = u_0 \qquad on\ \ \partial\Omega.$$

Their main tool, in that analysis, was an appropriate reformulation of the optimal design problem as a non-convex, vector variational problem for which, due to the underlying structure carried by the conductivity law, a relaxation could be either fully computed or appropriately estimated. This procedure resulted in dealing with div-curl Young measures as introduced in [72]. This program was carried out for two materials in [7]. Their objective was to understand the differences with the more complex situation of more than two materials, and how the various ingredients and computations change for three or more materials. The main advantage, from the relaxation point of view, was that this new version was more amenable to finding a relaxed version of it, or sub-relaxation, because they had to deal with pairs of usual vector fields with special properties instead of characteristic functions.

In 2016, the optimal control problem was reformulated as a system of equations (an optimality system) that consisted of an initial value problem for the underlying (linear or semilinear) wave equation and a terminal value problem for the adjoint wave equation by Steven Hou et al. In that paper, the authors considered an optimal boundary control approach for solving the exact boundary control problem for one-dimensional linear or semilinear wave equations defined on a time interval $(0,T)$ and spatial interval $(0,X)$. The exact boundary control problem they considered was to seek a boundary control $g = (g_L, g_R) \in L^2(0,T) \subset [L^2(0,T)]^2$ and a corresponding state u such that the following system of equations held:

$$\begin{cases} u_{tt} - u_{xx} + f(u) = V \quad in\ \ Q \equiv (0,T) \times (0,X), \\ u|_{t=0} = u_0 \quad and\ \ u_t|_{t=0} = u_1 \quad in\ (0,X), \\ u|_{t=T} = W \quad and\ \ u_t|_{t=T} = Z \quad in\ (0,X), \\ u|_{x=0} = g_L \quad and\ \ u|_{x=1} = g_R \quad in\ (0,T), \end{cases}$$

where u_0 and u_1 were given initial conditions defined on $(0,X)$, $W \in L^2(0,X)$ and $Z \in H^{-1}(0,X)$ were prescribed terminal conditions, V was a given function defined on $(0,T) \times (0,X)$, f was a given function defined on R, and $g = (g_L, g_R) \in [L^2(0,T)]^2$ is the boundary control [77]. In this regard, they attempted to solve the exact controllability problems by an optimal control approach. Precisely, they considered the following optimal control problem: minimize the cost functional

$$J_0(u,g) = \frac{\sigma}{2}\int_0^1 |u(T,x) - W(x)|^2 dx + \frac{\tau}{2}\int_0^1 |u_t(T,x) - Z(x)|^2 dx \\ + \frac{1}{2}\int_0^1 (|g_L|^2 + |g_R|^2) dt$$

$$\begin{cases} u_{tt} - u_{xx} + f(u) = V \quad in\ Q \equiv (0,T) \times (0,X), \\ u|_{t=0} = u_0 \quad and\ \ u_t|_{t=0} = u_1 \quad in\ (0,X), \\ u|_{x=0} = g_L \quad and\ \ u|_{x=1} = g_R \quad in\ (0,T), \end{cases}$$

The optimal control problem was converted into an optimality system of equations and this optimality system of equations was solved by a shooting method. In 2017, Aursand et al. considered a nonlinear variational wave equation that modeled the dynamics of the director field in nematic liquid crystals with high molecular rotational inertia. Being derived from an energy principle, energy stability was an intrinsic property of solutions to that model. For the two-dimensional case, they designed numerical schemes based on the discontinuous Galerkin framework that either conserve or dissipate a discrete version of the energy. They considered the elastic dynamics of the liquid crystal director field in the inertiadominated case (zero viscosity). Associated with the director field n, the classical Oseen-Frank elastic energy density W wa s given by

$$W(n, \nabla n) = \alpha|n \times (\nabla \times n)|^2 + \beta(\nabla .n)^2 + \gamma(n.(\nabla \times n))^2$$

The constants α, β and γ were elastic material constants of the liquid crystal, and were associated with the three basic types of deformations of the medium; bend, splay and twist; respectively. In that paper, the problem was solved using a method different from Young measure method [1].

3. Methods based on Radon measures (Shape-measure)

In Shape-measure method, the problem of optimal design was transferred to an optimal control problem by introducing artificial controls (which has geometry feature). The main idea of the solution is based on replacing the classical problem by a problem defined on a subset of positive Radon measures, to find a pair of measures (or one measure, sometimes), subject to some related linear conditions. Thus, the problem was replaced by a measure-theoretical one in which one seeks to minimize a linear form over a subset of positive Radon measures defined by infinite linear equalities. Hence, the optimal solution can be approximated by a finite combination of atomic measures so that the transferred optimal control problem is approximated by a finite linear programming one. The existence of the optimal solution has been immediately proved by the use of compactness properties of the weak* topology via existence theorems.

In this section, we review optimal shape design problems which have been solved by using positive Radon measures (Shape-measure method). These studies are reviewed in 3 subsections:

3.1) Shape-measure method for elliptic equations;

3.2) Application-oriented problems;

3.3) Some generalization on Shape-measure method.

3.1. Shape-measure method for elliptic equations. In this subsection, different kinds of OSD or OS problems governed by elliptic system, which were solved with respect to Radon measures are reviewed. The involved geometrical element is a pair consisting of a measurable set (in R^2) that can be regarded as a domain and a simple closed curve which was the boundary of the measurable set and passes a given point. In an introductory work, Fakharzadeh and Rubio (1999) introduced a solution procedure for such OS problem defined in polar coordinate in which the unknowns were a set C whose boundary ∂C [12] for the first time. For the readers to have a sufficient identification about the shape-measure method, here we explain this work in more details. The authors of [12] defined $J = [0, 2\pi], J^0 = (0, 2\pi), A = [0, 1], \Omega = J \times A, \tilde{\Omega} = J \times A \times W$ and supposed $r : J \to A$ was an absolutely continuous trajectory function that determines the unknown boundary ∂C, and $\omega : J \to W$ a Lebesgue-measurable control function, where $W \subset R$ was a bounded set. The pair of trajectory and artificial control functions satisfied the following differential equation:

$$\dot{r}(\theta) = \omega(\theta) \equiv g(\theta, r, \omega) \quad for \quad \theta \in J^0. \tag{9}$$

Pair $(C, \partial C)$ was called admissible if equation (9) holds, ∂C contains fixed points $(r(0) = r(2\pi) = r_a)$ and the area of C was fixed. The set of all admissible pairs was denoted by F. Therefore, a classical OS problem was introduced to find the minimizer pair in F for the given functional

$$I\left(C,\partial C\right)=\int_{C}f_{0}dA+\int_{\partial C}\frac{1}{\sqrt{r^{2}+w^{2}}}\ h_{0}ds.$$

First, they transformed this classical problem by introducing the necessary conditions for admissibility of $(C,\ \partial C)$ pair in variational formulation (see [75]) as follow:

$$\begin{aligned}\int_{\partial C}\frac{1}{\sqrt{r^{2}+w^{2}}}\Phi^{g}\left(\theta,r,w\right)ds&\equiv\int_{0}^{2\pi}\left\{\varphi_{r}\left(\theta,r\left(\theta\right)\right)+\varphi_{\theta}\left(\theta,r\left(\theta\right)\right)\right\}d\theta\\&=\varphi(2\pi,r(2\pi))-\varphi(0,r(0))\equiv\delta_{\varphi},\qquad\forall\varphi\in C^{'}(B);\end{aligned}$$

$$\int_{J}\Psi^{g}\left(\theta,r\left(\theta\right),w\left(\theta\right)\right)d\theta\equiv\int_{0}^{2\pi}\left\{r\psi^{'}+w\psi\right\}d\theta=0,\quad\forall\psi\in D(J^{0});$$

$$\int_{J}f\left(\theta,r,w\right)d\theta=a_{f},\qquad\forall f\in C_{1}(\tilde{\Omega})$$

where a_f is the integral of $f\left(\theta,r,w\right)$ over $[0,2\pi]$, independent of r and w, $C^{'}(B)$ is the set of real-valued continuously differentiable function on open ball B in R, $D(J^0)$ is the space of infinitely differentiable real-valued functions with compact support in J^0 and $C_1(\tilde{\Omega})$ is the set of functions in $C(\tilde{\Omega})$ which depends only on variable θ.

To ensure that F was not empty and the problem had a solution, the authors tried to somehow enlarge this set; the basis of this metamorphosis was the fact that an admissible pair could be considered as a pair of positive Radon measures, say (μ,v). Moreover, the transformation $(C,\partial C)\to(\mu,v)$ was an injection and it changed the classical OS problem into a measure-theoretical one. This transformation had an intermediate step; first, the positive linear functional Λ_C and $\Lambda_{\partial C}$ were defined as follow:

$$\begin{aligned}&\Lambda_{C}:C(\Omega)\to R:g\mapsto\textstyle\int_{C}gdA,\\&\Lambda_{\partial C}:C(\tilde{\Omega})\to R:h\mapsto\textstyle\int_{J}hd\theta.\end{aligned}$$

in which each admissible pair $(C,\partial C)$ was changed to $(\Lambda_C,\Lambda_{\partial C})$ monotonically. Then, by using Riesz representation theorem ([76]) each pair $(\Lambda_C,\Lambda_{\partial C})$ was uniquely represented by a pair of positive Radon measure (μ_c,v_c) and therefore the classical problem was transferred into a nonclassic linear measure one whose unknowns were the pair of measures. But, this transformation is one-to-one and the mentioned difficulties were also transferred. To achieve something new and useful, they enlarged the image of the transformation; that is, instead of the induced measures by Riesz representation theorem, they considered all positive Radon measures satisfying the admissibility conditions, not only the induced one. In this regard, the primal nonlinear problem transferred into the following one on the product space of positive Radon measures $M^+(\Omega)\times M^+(\tilde{\Omega})$:

$$\begin{aligned}Min:&\quad I\left(C,\partial C\right)=\mu\left(f_{0}\right)+v(h_{0})\\S.\ to:&\quad v\left(\Phi^{g}\right)=\delta_{\Phi},\qquad\forall\varphi\in C^{'}\left(B\right);\\&\quad v\left(\Psi^{g}\right)=0,\quad\forall\psi\in D\left(J^{0}\right);\\&\quad v\left(f\right)=a_{f},\quad\forall f\in C_{1}(\tilde{\Omega})\end{aligned}\tag{10}$$

where μ and v were introduced as:

$$\mu = \int_C g dA \quad \forall g \in C(\Omega), \qquad v = \int_J h d\theta \quad \forall h \in C(\tilde{\Omega})$$

Next, to prove that there is such a solution, they equipped the problem with weak* topology (see [76]) and proved that the global infimum of problem (10) was guaranteed and could be approximated with the solution of a finite dimensional linear one (like Rubio in [75]).

In the first step of the approximation, by introducing countable total sets $\{\varphi_i; i = 1, 2, ...\}, \{\psi_h; h = 1, 2, ...\}, \{f_s; s = 1, 2, ...\}$ in appropriate spaces, they chose a finite number of functions in each of these sets. Then, problem (10) in a semi-infinite linear programming form was presented as:

$$\begin{aligned} Min: \quad & I(C, \partial C) = \mu(f_0) + v(h_0) \\ S.\ to: \quad & v(\Phi_i{}^g) = \delta_{\Phi_i}, \qquad i = 1, 2, ..., M_1; \\ & v(\Psi_h{}^g) = 0, \qquad h = 1, 2, ..., M_2; \\ & v(f_s) = a_s, \qquad s = 1, 2, ..., M_3. \end{aligned} \tag{11}$$

By regarding Rosenbloom's theorem [74], they showed that the optimal measures μ^* and v^* of (11) had the following forms:

$$\mu^* = \sum_{i=1}^{n} \alpha_i^* \delta(Z_i^*), \qquad v^* = \sum_{j=1}^{m} \beta_j^* \delta(z_j^*),$$

where $\delta(t)$ is an atomic measure with support of the singleton set $\{t\}$ and the coefficients $\alpha_i^* \geq 0, \quad \beta_j^* \geq 0$ and points $Z_i^*, \quad z_j^*$ are the unknown supports. So, in the second step, by using discretization on the appropriate spaces, the problem was approximated by a linear problem with unknowns α_i's and β_j's. This result suggested that the solution of problem (11) could be approximated by the following linear programming problem in which Z_i and z_j for $i = 1, 2, ..., N, j = 1, 2, ..., M$, belong to dense subsets of Ω and $\tilde{\Omega}$ respectively:

$$\begin{aligned} Min: \quad & i(\alpha, \beta) = \textstyle\sum_{i=1}^{N} \alpha_i f_0(Z_i) + \sum_{j=1}^{M} \beta_j h_0(z_j) \\ S.\ to: \quad & \textstyle\sum_{j=1}^{M} \beta_j \phi_k^g(z_j) = \delta_{\phi_k}, \qquad k = 1, 2, ..., M_1; \\ & \textstyle\sum_{j=1}^{M} \beta_j \psi_h(z_j) = 0, \qquad h = 1, 2, ..., M_2; \\ & \textstyle\sum_{j=1}^{M} \beta_j f_s(z_j) = a_s, \qquad s = 1, 2, ..., M_3; \\ & \alpha_i \geq 0, \qquad i = 1, 2, ..., N; \\ & \beta_j \geq 0, \qquad j = 1, 2, ..., M. \end{aligned} \tag{12}$$

To reduce the number of unknowns, they offered a new idea and proved that measure μ could be expressed in terms of the boundary measure v by using the properties of electromagnetic field in each point z_j (the details are stated in the Appendix of [13]). It was also proved that when $M_1, M_2, M_3, N, M \to \infty$, the optimal solution of (12) tends to the optimal solution of (10) (see [75]).

At the end, by using the optimal solution $\alpha_1^*, \alpha_2^*, ..., \alpha_N^*, \beta_1^*, \beta_2^*, ..., \beta_M^*$ of (12), they constructed the artificial control function,$\omega(.)$, as explained in [75] Chapter 5. Then, the trajectory or optimal shape and the corresponding solution of the differential equation (9) were constructed. These pair of trajectory (shape) and control functions, turned out to be the solution to the modified shape design problem. Hence, they showed that *a measure can also be presented by a shape*, something new whose inverse was known before.

By using this method, sometimes, the dimensions of the problem could slightly be enlarged for the sake of very precise computations. Hence, due to the particular choice of f_s functions and the right-hand-side value of the second class of constraints, a large number of coefficients matrix elements of (12) are zero. This reduces the computations and causes the coefficients matrix to be a kind of sparse. Second, the existed methods (such as interior point) for solving linear programming problems with sparse matrices makes the process of solving (12) much easier by decreasing the consumed time as well as the complexity of the computations. In such a case, the number of iterations and the consumed time will be reduced. It is also necessary to remind that it is possible to solve the related semi-infinite linear problem to (12), where the number of constriants are infinite (see [37]). Therefore, an approximation steps will be reduced and the solution can be more accurate.

Based on the concepts of the previous paper, in [13], they solved the related OSD problem which included the solution of the following elliptic equation on C

$$div\left(k\left(\theta,r\right)\nabla u\right)-f\left(\theta,r,u\right)=0,$$

with the Neumann condition $\nabla u.\mathbf{n}|_{\partial C}=v$. The aim of that paper was to find the minimizer of the general functional

$$I\left(C,\partial C\ ,\ u,\ v\right)=\int_C f_0\left(\theta,r,u,\nabla u\right)drd\theta+\int_{\partial C}h_0\left(\theta,r,w,v\right)ds,$$

here (r,u) was the trajectory and (w,v) was the control pair.

In general, it was difficult to identify a classical solution for the elliptic problem; thus, they applied the variational form of the elliptic problem as

$$\int_C\left(k\nabla u\nabla\varphi+f\varphi\right)rdrd\theta-\int_{\partial C}k\varphi vds=0,\qquad\forall\varphi\in H^1\left(C\right),\tag{13}$$

and looked for a bounded weak solution u satisfying the above equation for all φ in $H^1(C)$ (the Sobolev space of order 1 on C). So, the problem was as follows

$$\begin{gathered}I\left(C,\partial C,u,v\right)=\int_C f_0dA+\int_{\partial C}\frac{1}{\sqrt{r^2+w^2}}\ h_0ds\\ S.\ to:\left(C,\partial C,u,v\right)\in F;\\ the\ area\ of\ C=given;\\ \theta_a\ and\ r_a=given;\\ div\left(k\left(\theta,r\right)\nabla u\right)-f\left(\theta,r,u\right)=0;\\ \nabla u.\mathbf{n}|_{\partial C}=v.\end{gathered}$$

This problem was solved by using Radon measures and an admissible $(C,\ \partial C,\ u,\ v)$ defined by two positive Radon measures as

$$\lambda_u\left(F\right)=\int_C F\left(\theta,r,u,\nabla u\right)drd\theta\quad,\quad\sigma_v\left(G\right)=\int_J G(\theta,r,\ w,\ v)d\theta;$$

here $F\in C(\Omega^{'})$ and $G\in C(w^{'})$ where $\Omega^{'}=\Omega\times U\times U^{'}$ and $w^{'}=w\times V$ (where $\nabla u\in U^{'}$). Thus, the problem was changed into a measure-theoretical one. Then, the problem was extended to a bigger space defined by all pairs of measures $(\lambda,\ \sigma)$ satisfying some linear conditions. Considering the variational equality (4), the first

set of conditions was introduced. Another relation was also defined between λ and σ by Stokes's theorem [36]. The last set of conditions was obtained by using Green's formula [36]. Thus, they replaced the problem with a new one which definitely had a minimizer. Then, the minimizer was approximated by a solution of a finite linear programming problem in a similar way as [12] and [75].
In that paper, in comparison with [12], the solution of OS and OSD problems which were governed by elliptic equations and defined in terms of a pair of geometrical elements have been discussed.

In 2001, Fakharzadeh and Rubio solved an OS (or OSD problem with a fixed control) in Cartesian coordinates, which aimed to find the optimal domain like D for a given functional, I, which was incorporated with the solution of a linear or nonlinear elliptic partial differential equation with a boundary condition over D [14]. For a bounded domain $D \subset R^2$ with a piecewise, smooth, closed and simple boundary ∂D, like Haslinger [39] in chapters 4 and 5, they assumed that some parts of ∂D were fixed and the rest, Γ, with the given initial and final points A and B, were not fixed. Domain D was called admissible if the elliptic equation

$$\triangle u\left(X\right)+f\left(X,u\right)=v(X)$$

with the boundary condition $u|_{\partial D}=0$, has a bounded solution on D. The aim of that paper was to solve an optimal shape problem with a fixed control $(v\left(x\right))$, for the functional $I\left(D\right)=\int_D f_0\left(X,u\right)dX$. The problem was solved in two stages: first, for a fixed domain, by using the density property and the idea of approximating a curve by broken line, Γ (and hence∂D) was determined with M number of unknown points (M-representation). Then, D and any integral on D were considered as a function of these M variables. Next, based on the elliptic equations, the weak solution u was determined by the following integral equality

$$\int_D\left(u\triangle\psi+\psi f\right)dX=\int_D\psi v dX;\qquad\forall\psi\in H_0^1\left(D\right). \tag{14}$$

Using the embedding process by defining measure

$$\mu_u\left(F\right)\equiv\int_\Omega F d\mu_u=\int_D F\left(X,u\right)dX,\quad\forall F\in C(\Omega)$$

, $(\Omega\equiv U\times\overline{D}\ \ that\ u\in U)$, they transferred the problem into a measure theoretical one. Then, they enlarged the underlying space to reach an infinite linear system of equations so that the unknown could be a measure in $\mathcal{M}^+(\Omega)$ and the problem could have been approximated by a finite linear system. Hence, the value of objective function $I(D)$ for any given domain D was calculated as a function of M unknown points (variables).
In the second stage, considering the previous one, a vector function $J: D\in D_M\rightarrow I(D)$ was set up such that every admissible domain D would give the value of objective function calculated by the first step. Using a standard minimization algorithm on J, the minimizer domain for J was obtained with an initial guess. The convergence of this method was proved in [14]. In the presented numerical examples, they used Nelder-Mead method [65] as the standard minimization algorithm. In another study, in [18] the method was extended for similar problems with non-fixed control function v and the general performance criterion:

$$I\left(D,v\right)=\int_D f_1\left(X,u\left(X\right)\right)dX+\int_{\partial D}f_2\left(s,v\left(s\right)\right)ds. \tag{15}$$

First, for any given domain D, the method of obtaining the optimal control function v_D^* was explained by using embedding method; then, $I\left(D,v_D^*\right)$ was calculated in terms of the finite number of domain variables and in the same manner as [14], the optimum domain and its related optimal control function were illustrated at the same time.

In 2013, Fakharzadeh considered a different type of elliptic equation by changing the boundary condition into $u|_{\partial D} = v$; that is, v is a boundary control. Hence, he tried to solve an OSD problem with objective function (15). To this aim, he replaced the problem with a measure-theoretical one and followed the concepts of the previous paper for the rest of the process to reach the solution.

Then, he extended the method for obtaining the nearly optimal domain for optimal shape design problems associated with the solution of a control problem involved with nonlinear wave equation [16]. The aim of that paper was to find the optimal domain and its related optimal control for a given wave equation.
This approach also consisted of two steps: first, for a fixed domain, the optimal control was identified by the use of measures in terms of the mentioned M unknowns. Based on the results of the previous step, the second one was similar to [14].
Then, in [20], the authors determined the best standard minimization algorithm for the second step of the solution procedure in [13]. To do this, they examined six related and convenient algorithms, namely, Random search, Nelder-Mead algorithm, Hook and Jeeves algorithm, Simulated annealing algorithm, Genetic and Honey bee mating optimization algorithm. The results showed that Random search and Honey bee mating optimization algorithm were the most appropriate ones.

Also, in [22] Fakharzadeh et al. (2013) solved the mentioned problems in [14] with the presence of obstacles. For this purpose, they imposed some new constraints on the elliptic OSD problem.

3.2. Application-oriented problems.

In 2002 [15], Fakharzadeh and Torabi found the best domain for a *non-linear diffusion equation* in Cartesian coordinates. In that study, the following optimal shape design problem was investigated.

$$I(u,v) = \int_{D\times(0,T)} f_0(u(x,t), \nabla u(x,t), x, t)dxdt + \int_{\partial D\times(0,T)} f_1(v(s,t), s, t)\, dsdt$$

$$\begin{aligned} S.to: \qquad & u_t(x,t) - div\left(k(x,t)\nabla u(x,t)\right) = f(u, \nabla u, x, t) \\ & u(x,0) = 0 \qquad \forall x \in D \times \{0\} \\ & \nabla u.n|_{\partial D\times(0,T)} = v \end{aligned}$$

where v is a control function which gets its values from a specific bounded set. The authors used the approximation domain method (like [14]) by applying suitable changes in Shape-measure for solving this problem. This change was done in the fixed part of the domain. In numerical examples, it was shown that the optimal value of the objective function depended on the fixed part of the unknown domain.
Farahi et al. (2005) presented a new method for designing a nozzle in which they used Radon measure with a completely different viewpoint [27]. In their study, they designed a symmetric nozzle that gave a prescribed velocity u_d in some given bounded open region $D \subset R^2$ near the exit. Their aim was to find a boundary which minimizes the following functional

$$I(\Phi(\mathcal{R})) = \int_D \|\nabla\Phi - u_d\|^2 dx$$

over admissible domains and $\Phi : \mathcal{R} \to R$ satisfies

$$\triangle\Phi = 0 \qquad (16)$$

in an unspecified region $\mathcal{R}$ and

$$\frac{\partial \Phi}{\partial n} = \begin{cases} -1, & on\ \Gamma_1 \\ 0\ , & on\ \Gamma_3 \cup \Gamma \\ \frac{|\Gamma_1|}{|\Gamma_2|}, & on\ \Gamma_2 \end{cases} \tag{17}$$

where Γ_1, Γ_2 and Γ_3 are given boundaries and the boundary Γ is to be found, where $|\Gamma_i|$ denotes the length of Γ_i .

Regarding the generalized solution of (16)- (17), they considered the unknown boundary $\Gamma(x)$ as a control function; then, the OSD problem was converted to an optimal control problem and followed the embedding process. Considering the symmetry property of shape with respect to x-axis, the optimal shape was obtained from rotating the obtained optimal control (optimal curve).

In a similar direction, there are many industrial devices consisting of a 2D slot nozzle; for example in jet wiping process applied in galvanization industry, the liquid film is dragged on the surface of a moving strip and undergoes the effect of air knives created by a 2D slot nozzle. As mentioned above, in [27] optimal shape design for a nozzle with specified velocity in a given region has been investigated. In addition, Mehne et al. (2005) solved a slot nozzle problem considering the sensitivity of the optimal shape with respect to the changes of the following cases:

a) Miss distance: The wall of the nozzle is a trajectory made by using control values and initial condition. Because of some approximations in Shape-measure method, the resulting piecewise linear trajectory may not satisfy the final condition. This miss distance can be reduced by increasing the number of total functions although this may cause difficulties in solving the linear programming problem. To succeed in dealing with this difficulty, they added a weighted functional to the objective function.

b) Minimum length: the piecewise linear trajectory constructed by using control values in Shape-measure method, may have suboptimal length. Reduction in the length of the wall may cause reduction in the material used to construct it. Designing a nozzle with minimum length is important in applications. For improving the method to handle this case, they added a special functional to the objective.

c) Changing in the position of D: They concluded that the pressure changes along the nozzle length; the changes in the size of D along the stream direction changed the optimal shape, but the transversal changes in the size of D did not have any effect on the shape because the pressure remained constant perpendicular to the stream [53].

Later, Farhadinia and Farahi (see [31]) found the optimal shape of a nozzle with respect to some given target flow fields including viscosity effect. That study was different from [29] in which an OSD problem was considered which included the incompressible potential flow equation and Driclet conditions; also, in comparison with [53], Neumann condition was considered. The authors (in [31]) evaluated and solved several case studies in which target flow taken linear and nonlinear profiles. As a nozzle physical behavior point of view, the sensitivity and treatment of the unspecified boundary of the nozzle were considered with respect to target flow and obtained results verified the reliability of the measure theoretical approach.

The problem of determining the shape of a thin wing for minimum drag has been examined by Farahi et al. (2006). This problem is an important industrial problem in which the presence of drag optimization means a great saving on commercial airplanes. In general, the stream function of a flow around an airfoil satisfied the Navier–Stokes equation and the resulting problem was expressed in terms of the optimal control of distributed systems. Then, the authors used the embedding procedure to solve the problem using Radon measures [28].

In 2007, Fahkarzadeh J. and Rostami solved the optimal shape design problem governed by a parabolic control system [17]. In that paper, on the basis of Shape-measure method and [13], the best curve passes a specific point and a supposed objective function was minimized. The classical problem of optimal shape design was as follow:

$$Min: \ J(D, \partial D, u, v) = \int_{D\times(0,T)} f_0(x,t,u,\nabla u)dxdt + \int_{\partial D\times(0,T)} f_1(s,t,\ u,v)\,dsdt$$

$$S.to: \quad u_t(x,t) = \nabla u(x,t) + f(t, u(x,t)) \quad \forall (x,t) \in D\times(0,T)$$

$$u(x,0) = \xi_1 \qquad u(x,T) = \xi_2$$

$$\nabla u.n|_{\partial D\times(0,T)} = H(t,u) + v(s,t)$$

where $D \subset R^2$ was a bounded domain having boundary ∂D and $u : D\times(0,T) \to R$ was a differentiable and bounded function in $C^2(D\times(0,T))$; additionally, $v : \partial D \to R$ was a system control function and H was a known function. The difference between that paper and [13] is that the equations of system are parabolic with specified initial and boundary conditions.

In many industrial applications, scholars are interested in the control of the thermal deformation of an isotopic and homogeneous solid body subjected to a prescribed thermal treatment. Due to temperature changes, the body undergoes a thermoelastic deformation; that is, the induced thermal stress force the body to change its shape in time. As the final shape depends on the initial shape, one is interested in finding the initial shape of the body such that its final shape is in prescribed form as closely as possible. In 2007, Mehne et al. solved this shape optimization problem by the use of Radon measures [54].

In that article, measure theory approach in function space was derived resulting in an effective algorithm for the discretized optimization problem by applying the famous embedding process.

A year later, Mehne formulated the problems of optimization of cylindrical bar cross-sections in variational forms [55]. The shape of the boundary of the cross-section was taken as a design variable. The author considered a cylindrical circular bar which was isotropic and homogeneous with a simply planar connected cross-section. The rigidity properties of this bar depend on the shape of its cross-section. The presented OSD problem in that article was defined as finding the cross-section in order to maximize the bending rigidity of the bar subjected to inequality constraints on torsional rigidity and volume. He considered an isotropic and homogeneous cylindrical circular bar with a simply planar connected cross-section $\Omega \subset R^2$. The class of admissible shapes whose abbreviation is shown as O , defined as the class of simply connected bounded domains with boundary $\partial\Omega \in C^2$, starshaped with respect to a fixed open ball $B_\delta(0)$. Each admissible shape Ω was uniquely determined by a periodic function $r : [0,\ 2\pi] \to [0,\ \delta]$ which represented the boundary $\Gamma = \partial\Omega$, then

$$\Gamma := \{\gamma(\theta) = r(\theta) \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} : \quad \theta \in [0, 2\pi]\}$$

where $r \in C^2_{per}[0, 2\pi]$ with

$$C^2_{per}[0,2\pi] = \left\{ \mathrm{r} \in C^2[0,2\pi] : \quad r^{(i)}(0) = r^{(i)}(2\pi),\ i = 0,1,2 \right\}.$$

Next, he introduced the mathematical formulation of the quantities of interest. The bending rigidity with respect to a fixed barycenter in the origin was given by

$$B(\Omega) = \int_{\Omega} y^2 dx = \frac{1}{4}\int_0^{2\pi} sin^2\theta r^4(\theta)\, d\theta$$

The torsional rigidity was calculated by

$$T(\Omega) = 2\int_{\Omega} u(X)\, dX = 2\int_0^{2\pi}\int_0^{r(\theta)} u(\rho,\theta)\, d\rho d\theta,$$

where $X = [x, y]^T$ and the stress function $u = u(\Omega)$ satisfies

$$\Delta u = -2 \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \Gamma.$$

In this manner, the problem was first expressed as an optimal control problem. Then, by using an embedding method, the optimal shape was approximated from the solution of a finite dimensional linear programming problem.

Fluid flow models associated with the Navier-Stokes equations have become a matter of deep study in the mathematical investigations. Although the range of mathematical issues in the OSD problems is wide, the problems governed by various cases of the Navier-Stokes equations such as compressible or incompressible, viscous or inviscid, are in their beginning and infancy. In 2007, Farhadinia and Farahi considered a particular OSD problem in which minimization of viscous drag was studied through shape modification [30]. Their aim was to investigate the existence of a special drag-minimizing shape which had been solved in [28] by using Shape-measure method.

Nazemi et al. (2008) developed to a new numerical technique for the approximation of the flow problem of incompressible liquid through an inhomogeneous porous medium (say dam) [60]. The stationary flow of fluid through an inhomogeneous porous medium led to a problem posed in a domain with a partially unknown boundary separating the wet and dry part of the dam. In that article, the authors converted the shape optimization problem to an optimal control problem. Then, to each admissible control state, a linear continuous functional was associated. Correspondence between continuous positive linear functional and positive Radon measures led to an optimization problem in measure space.
Also, sensitivity analysis was done for different permeability coefficients without computing complexity.

At the same time, Nazemi et al. ([61]) considered the cross section of a depicted device shape. The electromagnet consists of an iron core w_1 and a coil which penetrates the cross section plane at w_1 and w_2, respectively. A current J flows in the coil, pointing outward on w_1 and inward on w_2. The material-dependent magnetic reluctivities were given by constants v_1 in the iron region, by v_2 in copper and air. The electromagnetic potential A satisfies the equation

$$-div\,(v(x_1,\ x_2)\,\nabla A) = J(x_1,\ x_2)\,, \qquad (x_1,\ x_2) \in w,$$

where

$$v(x_1,\ x_2) = \begin{cases} v_1, & on\ w_1 \\ v_2, & elsewhere \end{cases}$$

and

$$J(x_1,\ x_2) = \begin{cases} j, & on\ w_1 \\ -j, & on\ w_2 \\ 0, & elsewhere \end{cases}$$

while j was the current density. A physically reasonable boundary condition for A was given by

$$A_{\partial w} = 0,$$

By setting $w_2 = w - \overline{w}_1$, they designed the pole such that the electromagnet field was as close as possible to a desired vector $u_d = (u_{d_1},\ \ u_{d_2})^T$ in the given area of D. A cost functional which realizes that this objective was given as

$$\mathcal{I} = \frac{1}{2}\int_D \|\nabla A - u_d\|_2^2 dx_1 dx_2 \quad .$$

The author found the optimal shape of the iron pole which minimizes the above cost functional. Where only a part of the boundary ∂w_1 of the iron core was variable. Then, the transformed weak variational form of the optimal shape design problem was formulated and they used the measure theoretical approach for designing the unknown upper part of the iron core of a magnet.

Meanwhile, Nazemi and Farahi presented a numerical technique in the study of aorto-coronaric bypass anastomoses configurations using unsteady Stokes equations [59 and 61]. The theory of optimal control based on notations of the measure theory was applied in order to optimize the shape of the zone of the incoming branch of the bypass (the toe) into the coronary. The authors considered an idealized two-dimensional bypass bridge configuration where the dotted line represents the geometry of the complete anastomosis; Γ_{w_2} was the section of the original artery, Γ_{in} was the new anastomosis inflow after bypass surgery, Γ_{out} was the anastomosis outflow (see Figure 1 and 2).

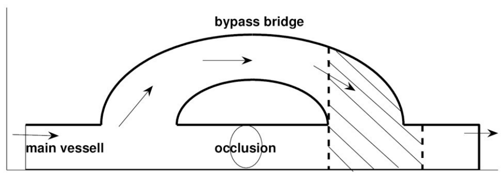

FIGURE 1. Idealized, 2-D bypass bridge configuration.

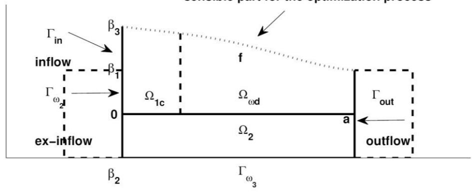

FIGURE 2. The dotted curve f represents the portion of the boundary.

They considered the following boundary-value problem for the Stokes equations which was used to model low Reynolds blood flow.

Find $\underline{v}$, p

$$\text{(18)} \qquad \text{S.to}: \begin{cases} -v\triangle\underline{v} + \nabla p = \underline{F} & in\ \Omega \\ \nabla.\underline{v} = 0 & in\ \Omega \\ \underline{v} = \underline{v}_{in} & on\ \Gamma_{in} \\ \underline{v} = 0 & on\ \ \Gamma_{w_1} \cup \Gamma_{w_3} \\ -p.n + v\frac{\partial v}{\partial n} = \underline{g}_{out} \quad on\ \Gamma_{out} \cup\ \ \Gamma_{w_2} & \end{cases}$$

where $\underline{v} = (u, v)^T$, $\quad n = (n_1, n_2)^T$ is the outward unit normal vector on Γ, v =const > 0 and $\underline{v}_f = \{\underline{v}_{in}\ on\ \ \Gamma_{in};\ \underline{o}\ \ \ on\ \ \Gamma_{w_1} \cup \Gamma_{w_3}\}$ and Γ_{w_1} was the sensible part f of the bypass bridge that must have been determined.

Considering the weak statement of (18), the optimal shape design problem was interpreted as an optimal control problem. So, they introduced the equivalent problem in measures space and by using an embedding method, the class of admissible shapes was replaced by a class of positive Radon measures. Then, optimization problem in measure space was approximated by an LP problem.

In 2009, Nazemi and Farahi studied a two-dimensional model of the orientation of fibres in a paper machine headbox [63]. The goal was to control the fibre orientation distribution at the outlet of contraction by changing its shape. The mathematical formulation led to an optimization problem with control in coefficients of a linear convection-diffusion equation as the state problem. The distribution was controlled by an optimal shape design of the contraction. The mathematical procedure used in that paper was based on three steps:

(1) First, any admissible shape was replaced by exactly one point in a geometry.
(2) Then, any point in this geometry was injected to a functional in a functional space.
(3) Any functional in functional space was embedded by a measure in some measure space.

The geometry of the planar contraction nozzle (one-dimensional headbox) was described by a Lipschitz continuous function α. The model further considered the distribution $\Psi(x, \Phi)$ of the projected angle Φ of the fibre only along the central streamline.

The probability distribution was denoted by $\Psi(x, \Phi)$ and was given as the solution of the linear diffusion-convection type problem in domain $\Omega = (0, 1) \times (\frac{-\pi}{2}, \frac{\pi}{2})$:

$$\text{(19)} \qquad \begin{cases} -\nabla.\left(\mathcal{A}\nabla\Psi\right) + b_\alpha.\nabla\Psi + c_\alpha\Psi = 0 & in\ \Omega, \\ \Psi = \pi^{-1}, & on\ \Gamma_1\ , \\ \left(\mathcal{A}\nabla\Psi\right).v = 0, & on \Gamma_2 \cup \Gamma_3 \end{cases}$$

Here $\mathcal{A}$ was a constant positive definite matrix, v denoted the unit outward normal vector to $\partial\Omega$ (the boundary of Ω) and the coefficients b_α and c_α were given by

$$b_\alpha\left(x, \Phi\right) = \left(u_1,\ -\sin\left(2\Phi\right)\frac{\partial u_1}{\partial x}\right), \qquad c_\alpha\left(x, \Phi\right) =\ -2\cos\left(2\Phi\right)\frac{\partial u_1}{\partial x}\ .$$

Then, they found a function $\alpha(x)$ (see Figure 3) such that the fibre orientation distribution at the outlet was close to a given target distribution $\Psi_0 \in L^2(\Gamma_2)$. Cost functional was given as follows:

$$\mathcal{I}\left(\Psi\left(\alpha\right)\right) = \int_{\Gamma_2}\left(\Psi\left(\alpha\right) - \Psi_0\right)^2 d\Gamma = 0,$$

where $\Psi\left(\alpha\right)$ was a weak solution of (19).

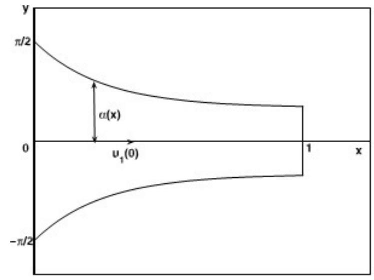

FIGURE 3. Geometry of the planar contraction.

To solve the shape optimization problem, they determined the control function $\alpha(x)$ in this problem. The derivative of the free boundary $\alpha(x)$ was defined as an artificial control function:

$$\frac{d\alpha}{dx} = F\left(\theta\left(x\right)\right), \qquad \alpha\left(0\right) = \alpha_0\ , \qquad \alpha\left(1\right) = \alpha_1,$$

where the artificial control function $\theta\left(x\right)$ was Lebesgue measurable, α_0 and α_1 were known and the shape optimization problem was interpreted as an optimal control problem. Then, they changed the space of optimization problem to measure spaces and the optimization problem in this space was approximated by a linear programming problem.

In 2012, Farhadinia drew attention to the optimal control problem modeled like the one in [63] while his aim was to make a fresh exploration for the solving procedure of the shape optimization problem. The superiorities of his approach over [63] are as follow:
(1) Against the progress made in [63], no additional constraints were made during the present procedure. Hence, a simplified model which clearly illustrated the optimal shape design problem was developed in that study.
(2) He constructed a bijective transformation to convert the varied geometry of the planar contraction into a fixed one and the admissible shape function played role of a state variable. Particularly, the extra constraints and their counterparts in [63] were inessential in his modeling. It had a very simple structure since no extra constraints were imposed on the model. Moreover, [63] was restricted to the differentiable cost functions but that approach dealt with non-differentiable cost functions and furthermore it was self-starting [34].

In 2010, Farhadinia applied Shape-measure method to find an optimal solution of a shape design problem that modeled reducing the amount of noise radiated from aircraft turbofan engines. The proposed method had some advantages comparing to the gradient-based optimization methods ([8] and [52]); for instance, it did not require any information of gradients and the differentiability of cost function. He supposed that the model composition of the aircraft noise source was specified on the source plane Γ_f. The inlet of fan was surrounded by two boundaries: fixed boundary Γ_c and flexible boundary Γ_α which were characterized by the function (x) . It was assumed an acoustic liner exists on the boundary Γ_c. Γ_∞ was far enough from the noise source. This implied that the radiated field treated locally as a plane wave at local incidence [32]. So, the optimal shape design problem was stated as follow:

$$Min: \quad J = A\int_\Omega u^2 d\Omega + B\int_\Omega |\nabla u|^2 d\Omega + \lambda\int_a^b \left(\alpha\left(x_1\right) - \alpha_0\left(x_1\right)\right)^2 dx_1$$

$$\begin{aligned} \triangle u + k^2 u = 0, \quad on\ \Omega, \\ u|_{\Gamma_f} = g(\alpha), \\ \frac{\partial u}{\partial n}\Big|_{\Gamma_\alpha} = 0, \\ (u + x\frac{\partial u}{\partial n})\Big|_{\Gamma_c} = 0, \\ (iku + x\frac{\partial u}{\partial n})\Big|_{\Gamma_\infty} = 0, \end{aligned}$$

where $x > 0$ and the three constants $A,\ B$ and λ satisfied $A^2 + B^2 > 0, \quad \lambda \geq 0$.

Regarding the importance of shape optimization problems within the field of computational fluid dynamics, especially those which were governed by full Navier-Stokes equations, in 2011, Farhadinia found an optimal shape of a wing such that the drag force which was experienced by a body moving through a viscous fluid, was reduced [33]. To this purpose, the airfoil of a symmetric wing was considered in a fixed vertical slit as a bump included in a virtual channel. The boundary of airfoil was separated into two parts: one part was the fixed side Γ_{low} and the other part was the moving side Γ_{wing} which should have been designed (see Figure 4).

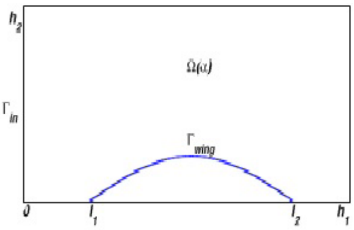

FIGURE 4. The geometry $\Omega(\alpha)$.

The problem was a free boundary problem for a viscous and compressible barotropic fluid. The stationary motion of the fluid in a bounded domain $\Omega(\alpha)$ with a free boundary $\Gamma(\alpha)$ was described by the following equations:

$$\begin{aligned} -\mu\triangle\mathbf{u} - v\nabla div\mathbf{u} + \varrho(p)(\mathbf{u}.\nabla)\mathbf{u} + \nabla p = 0, \qquad \Omega(\alpha), \\ div(\varrho\mathbf{u}) = 0, \qquad \Omega(\alpha), \\ u = u_0, \qquad \Gamma(\alpha), \qquad p = p_0, \qquad \Gamma_{in}(\alpha), \end{aligned}$$

where $\mathbf{u} = (u_1,\ u_2)^T$ was the velocity vector, p was the pressure, $\varrho = \varrho(p)$ was the given density, μ and ν were the coefficients of viscosity which satisfy the thermodynamic restrictions $\mu > 0$ and $\nu+\mu > 0$.

Indeed, due to the difficulty of finding an exact solution of the problem, the procedure was equipped with a measure theoretical approach. Then, a bijective transformation mapped the moving geometry of the problem to a fixed one. For this propose, a weak variational form of the problem was derived from the linearized governing equations.

In [57] the Young measure method was used to solve a damped wave equation problem; but, Fakharzadeh J. et al. solved the similar problem by using Shape-measure method [24]. The measures that they used were uniquely defined while Young measure was not defined uniquely [56]. Their approach for solving the problem was totally different from [57], even from a measure theoretical point of view. Also, in that paper damping coefficient was unknown but in [57] this parameter was fixed. In addition, they found the unknown region and the damping function simultaneously through a three phase optimization procedure which was based on an embedding technique.

3.3. Some generalization on shape-measure.

In Shape-measure method, the optimal shape (and also its related control if exists) and the optimal trajectory as well as the optimal solution of the involved differential system are determined separately. First, the optimal shape (artificial control) is determined and then based on this, the optimal trajectory is calculated. Moreover, there has been no procedure that introduce this optimal trajectory by classical functions like Fourier series. But, Fakharzadeh and Jafarpour (2013) presented a generalized form of the Shape-measure method for solving the controlled systems governed by wave equation in which it would be possible to determine a classical trajectory [18]. Moreover, the method could obtain the optimal trajectory and optimal control simultaneously. For these reasons, an optimal control problem governed by a vibrating shell with the initial and boundary conditions (in general form) on an arbitrary domain D was considered.

First, for a given time interval$[0,T]$, the following optimal control problem was solved:

$$\begin{aligned} Min: \quad & I\left(p\right) = \int_{D'} f_0\left(r,\theta,t,v\right)dA \\ S.to: \quad & u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}); \\ & u\left(r,\theta,0\right) = f\left(r,\theta\right); \\ & u_t\left(r,\theta,0\right) = g\left(r,\theta\right); \\ & u\left(h\left(\theta\right),\theta,t\right) = 0; \\ & u_t\left(r_0,\ \theta_0,t\right) = v\left(t\right); \end{aligned}$$

here, $D^{'} = D\times[0,T]$ and f_0,f and g were the given real valued continuous functions, $v\left(t\right):[0,T] \rightarrow V \subset R$ (a shock on a specified point $(r_0,\ \theta_0)$ of the shell) was a Lebesgue-measurable control function which took its values on a bounded set v and c was a constant. By setting a partition on the domain and approximating each part with a sector of a suitable circle, the system solution was identified as a trigonometric series with unknown coefficients. After taking into account the harmony and smoothness of the solution, the problem was transferred into a new linear one in which it was involved with Radon measures. Existence of the optimal solution for the new problem was proved automatically. Then, by some discretization schemes, they showed how the optimal classical trajectory and control were identified via the results of a finite linear programming simultaneously.

Furthermore, in [22] a comparison between Shape-measure and penalty method had been done. By using the penalty method and considering an initial domain, the athours employed the finite element approach to divide the domain into a finite number of triangles; next, on each of these triangles, they defined two variable polynomials which were uniquely defined by their values at the three vertices of a triangle to replace the unknown domain by a piecewise linear arc. By presenting some examples, they compared the shape-measure method with the penalty approach to dettermine the advantages of these two methods. In penalty method, the preciseness of the method depended on the triangulation and the number of points. In addition, if there were more of these points, the obtained solution was closer to the exact solution of the problem. Also, the necessary number of operations and the performance time were high. In shape-measure method, the number of constraint and the dimension of the problem were finite. They obtained the approximation solution by solving a linear programming problem which was simply solvable. Consuming time and the number of operations were less and the obtained solutions were very close to the solution obtained by finite element method.

Later, Fakharzadeh J. and Alimorad, for the first time, tried to develop a version of Shape-measure method for determining 3-dimensional shapes [23]. They presented a based measure method so that it could solve the optimal shape problems in 3-dimensional space directly. To do this, they found a surface whose image was specific; they illustrated this optimal surface for two cases of presence and absence of obstacles. In addition, several states, such as being symmetric or asymmetric also were considered in which its smoothness was considered by applying outlier detection. This new method did not depend on an initial shape or value. They investigated the advantages of this method by solving numerical examples and comparing results with other studies.

4. A short discussion

The works done in optimal shape using two famous measure theoretical methods have been reviewed in this study: Young-measure and Shape- measure method. We showed how these two methods have been successfully used for solving these kinds of problems in different areas. The advantage of these two methods is that they are suitable for solving problems lacking classical solutions when the existence of such a solution is difficult to characterize. But, both methods have some defects whose removal can be considered in the future studies; the used measures in Young measure method may not be defined uniquely; for instance, for two different sequences which are generated by the same Young measure, the amounts of the related function may be different. Also, it is not clear how to set up the numerical framework such that it can be helpful in approximating the minimizing sequences for original optimization problems. Also, in Shape-measure method, since usually two or three steps of approximation are used to characterize the optimal shape via a finite linear programming problem, we may face unpredictable errors; though finding the solution via a simple finite programming made the shape characterization much easier. A quick overview of the published papers shows that the majority of papers in the first group have been published in Engineering journals while the second group appeared mostly in Applied mathematics journals. Therefore, these studies are empirical and can be used in different applications, specially in the future. We believe that future studies in this area can directly be three-dimensional in design, specially regarding methods, analysis of non-rotation states in optimal shape design, and extending and combining of the used methods.

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Shiraz University of Technology, Department of Mathematics, Faculty of basic Sciences.
E-mail: `a_fakharzadeh@sutech.ac.ir`

Dept. of Mathematics, Jahrom University, P.O. Box: 74135-111, Iran.
E-mail: `hajaralimorad@yahoo.com`