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FEM-ANALYSIS ON GRADED MESHES FOR TURNING POINT PROBLEMS EXHIBITING AN INTERIOR LAYER

SIMON BECHER

Abstract. We consider singularly perturbed boundary value problems with a simple interior turning point whose solutions exhibit an interior layer. These problems are discretised using higher order finite elements on layer-adapted graded meshes proposed by Liseikin. We prove ε-uniform error estimates in the energy norm. Furthermore, for linear elements we are able to prove optimal order ε-uniform convergence in the L^2-norm on these graded meshes.

Key words. Singular perturbation, turning point, interior layer, layer-adapted meshes, higher order finite elements.

1. Introduction

We consider singularly perturbed boundary value problems of the type

$$\begin{aligned} -\varepsilon u''(x) + a(x)u'(x) + c(x)u(x) &= f(x) \qquad \text{in } (-1,1), \\ u(-1) = \nu_{-1}, \quad u(1) &= \nu_1, \end{aligned} \tag{1a}$$

where $0 < \varepsilon \ll 1$ is a small parameter and a, c, f are sufficiently smooth with

$$a(x) = -(x - x_0)b(x), \qquad b(x) > 0, \qquad c(x) \geq 0, \qquad c(x_0) > 0 \tag{1b}$$

for a point $x_0 \in (-1, 1)$. Thus, the solution of (1) exhibits an interior layer of "cusp"-type at the simple interior turning point x_0.

In the literature (see e.g. [2], [4, p. 95], [7, Lemma 2.3]) the bounds for such interior layers are well known. We have

$$\left| u^{(i)}(x, \varepsilon) \right| \leq C \left(1 + \left(\varepsilon^{1/2} + |x - x_0| \right)^{\lambda - i} \right) \tag{2}$$

where the parameter λ satisfies $0 < \lambda < \bar{\lambda} := c(x_0)/|a'(x_0)|$. The estimate also holds for $\lambda = \bar{\lambda}$, if $\bar{\lambda}$ is not an integer. Otherwise there is an additional logarithmic factor, see references cited above. For convenience we assume $x_0 = 0$ in the following.

In the last decades a multitude of numerical methods has been developed to solve singularly perturbed problems with turning points and interior layers. For a general review we refer to [6]. Many authors have considered finite difference methods. A selection of possible schemes for problems of the form (1) may be found in [3] and the references therein. Also some layer-adapted meshes have been proposed to handle interior layers of "cusp"-type. As an example Liseikin [4] proved the ε-uniform first order convergence of an upwind finite difference method on special graded meshes. Moreover, Sun and Stynes [7] studied finite elements on a piecewise uniform mesh.

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We shall also analyse the finite element method, but on the graded meshes proposed by Liseikin which are described by the mesh generating function

$$\varphi(\xi,\varepsilon)=\begin{cases}\left(\varepsilon^{\alpha/2}+\xi\left[(1+\varepsilon^{1/2})^{\alpha}-\varepsilon^{\alpha/2}\right]\right)^{1/\alpha}-\varepsilon^{1/2} & \text{for } 0\le\xi\le 1,\\ \varepsilon^{1/2}-\left(\varepsilon^{\alpha/2}-\xi\left[(1+\varepsilon^{1/2})^{\alpha}-\varepsilon^{\alpha/2}\right]\right)^{1/\alpha} & \text{for } 0\ge\xi\ge -1,\end{cases}$$

where $0<\alpha\le\lambda$. In order to handle these meshes, we adapt some basic ideas from [4, pp. 243–244]. While the strategy of Sun and Stynes in [7, Section 5] is restricted to linear finite elements, our approach is more general. Thus, we are able to treat finite elements of higher order as well. Also note that recently, in [1], a similar approach was used to study the streamline diffusion finite element method on the piecewise uniform mesh of Sun and Stynes.

Under certain assumptions, we prove ε-uniform convergence in the energy norm of the form

$$|||u-u_N|||_{\varepsilon}\le CN^{-k}$$

for finite elements of order k, where C may depend on α and k, see Theorem 3.5. On the basis of a supercloseness result we also give an optimal error estimate in the L^2-norm of the form

$$\|u-u_N\|\le CN^{-2}$$

for linear finite elements, see Theorem 3.10. Numerical experiments confirm our theoretical results.

Notation: In this paper C denotes a generic constant independent of ε and the number of mesh points. Furthermore, for an interval I the usual Sobolev spaces $H^1(I)$, $H_0^1(I)$, and $L^2(I)$ are used. The spaces of continuous and k times continuously differentiable functions on I are written as $C(I)$ and $C^k(I)$, respectively. Let $(\cdot,\cdot)_I$ denote the usual $L^2(I)$ inner product and $\|\cdot\|_I$ the $L^2(I)$-norm. We will also use the supremum norm on I given by $\|\cdot\|_{\infty,I}$ and the seminorm in $H^1(I)$ given by $|\cdot|_{1,I}$. If $I=(-1,1)$, the index I in inner products, norms, and seminorms will be omitted. Additionally, for all $v\in H^1((-1,1))$ we define a weighted energy norm by

$$|||v|||_{\varepsilon}:=\left(\varepsilon\,|v|_1^2+\|v\|^2\right)^{1/2}.$$

Further notation will be introduced later at the beginning of the sections where it is needed.

2. The graded meshes proposed by Liseikin

The basic idea of Liseikin is to find a transformation $\varphi(\xi,\varepsilon)$ that eliminates the singularities of the solution when it is studied with respect to ξ. In our case the approach can be condensed to the task to find $\varphi:[0,1]\to[0,1]$ such that

$$\varphi'\left(\varphi+\varepsilon^{1/2}\right)^{\lambda-1}\le C,\qquad \varphi(0)=0,\qquad \varphi(1)=1. \tag{3}$$

The outcome of this approach is the mesh generating function

$$\varphi(\xi,\varepsilon)=\begin{cases}\left(\varepsilon^{\alpha/2}+\xi\left[(1+\varepsilon^{1/2})^{\alpha}-\varepsilon^{\alpha/2}\right]\right)^{1/\alpha}-\varepsilon^{1/2} & \text{for } 0\le\xi\le 1,\\ \varepsilon^{1/2}-\left(\varepsilon^{\alpha/2}-\xi\left[(1+\varepsilon^{1/2})^{\alpha}-\varepsilon^{\alpha/2}\right]\right)^{1/\alpha} & \text{for } 0\ge\xi\ge -1,\end{cases} \tag{4}$$

where $0<\alpha\le\lambda$. By construction we have $\varphi(0,\varepsilon)=0$ and $\varphi(\pm1,\varepsilon)=\pm1$. Note that Liseikin derived the same transformation indirectly. Based on the principle of equidistribution, he used basic majorants of the solution derivatives to find basic layer-damping transformations. This procedure allows to handle also various other types of singularities, see e.g. [4, Chapter 6].

Now, the mesh points are generated by $x_i = \varphi(\frac{i}{N}, \varepsilon)$, $i = -N, \dots, N$. We will denote the lengths of the mesh intervals by $h_i := x_i - x_{i-1}$, $i = -N+1, \dots, N$ and $h := N^{-1}$. Additionally, set $\hbar_i := (h_i + h_{i+1})/2$ for $i = -N+1, \dots, N-1$.

Motivated by Lemma A.2 and A.3, we define and estimate a special constant κ dependent on $\alpha \in (0,1]$ and $\varepsilon \in (0,1]$ by

$$(5) \quad 0 < \ln(2) \le \kappa := \kappa(\alpha, \varepsilon) := \frac{(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}}{\alpha} \le \min\left\{\alpha^{-1}, 1 + \left|\log_2(\varepsilon^{1/2})\right|\right\}.$$

It remains to check whether or not φ defined in (4) satisfies (3). An easy calculation shows

$$\begin{aligned} \frac{\partial \varphi}{\partial \xi}\left(\varphi + \varepsilon^{1/2}\right)^{\lambda-1} &= \frac{1}{\alpha}\left[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\right]\left(\varepsilon^{\alpha/2} + \xi\left[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\right]\right)^{(1-\alpha)/\alpha} \\ &\qquad \left(\varepsilon^{\alpha/2} + \xi\left[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\right]\right)^{(\lambda-1)/\alpha} \\ &= \kappa\left(\varepsilon^{\alpha/2} + \xi\left[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\right]\right)^{(\lambda-\alpha)/\alpha} \le C\kappa \end{aligned}$$

which can be bounded independent of ε due to (5).

Since the arguments are very similar thanks to the symmetry of the mesh, we will consider the case $\xi \ge 0$ only. The next lemmas comprise some basic results concerning the mesh points and mesh intervals. Their proofs are deferred to Appendix B. The argumentation substantially uses the property (3). Here, the derivative of φ comes into play since the mean value theorem guarantees the estimate $h_i \le h \frac{\partial \varphi}{\partial \xi}(\xi_i, \varepsilon)$ for a $\xi_i \in (x_{i-1}, x_i)$.

Lemma 2.1. *Let $\hat\alpha > 0$ and $0 < \alpha \le \min\{\hat\alpha/k, 1\}$ with $k \in \mathbb{N}$, $k \ge 1$ then*

$$h_i^k\left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat\alpha - k} \le \begin{cases} Ch^k & \text{for } 2 \le i \le N, \\ h_1^k \varepsilon^{(\hat\alpha - k)/2} \le Ch^k & \text{for } i = 1, \quad \varepsilon \ge h^{2/\alpha}. \end{cases}$$

If $0 < \alpha \le 1/k$ with $k \in \mathbb{N}$, $k \ge 1$ and $\varepsilon \le h^{2/\alpha}$ then we have

$$x_1 \le Ch^k.$$

In general, we have for $0 < \alpha \le 1$

$$h_i \le Ch \qquad \text{for } 1 \le i \le N.$$

Lemma 2.2. *For $0 < \alpha \le \frac{1}{2}$ the following inequality holds*

$$h_i - h_{i-1} \le Ch^2\left(x_i + \varepsilon^{1/2}\right)^{1-2\alpha} \qquad \text{for } 2 \le i \le N.$$

Let $\hat\alpha > 0$ and $0 < \alpha \le \min\{\hat\alpha/2, 1/2\}$ then

$$(h_i - h_{i-1})\left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat\alpha - 1} \le Ch^2 \qquad \text{for } 2 \le i \le N.$$

Remark 2.3. *Note that an estimate similar to the first one of Lemma 2.2 can also be found in* [8].

Remark 2.4. *In the FEM-analysis a generalised version of* (3)*, i.e.,*

$$(\varphi')^k \left(\varphi + \varepsilon^{1/2}\right)^{\lambda-k} \le C, \qquad \varphi(0) = 0, \qquad \varphi(1) = 1,$$

would be convenient. In fact, this is ensured for $0 < \alpha \le \lambda/k$ *which is already used in the proof of Lemma 2.1.*

3. FEM-analysis on graded meshes

This section follows the paper of Sun and Stynes [7], but while they studied linear finite elements on a layer-adapted piecewise uniform mesh, we shall use the graded mesh proposed by Liseikin instead. Besides our more general approach enables to analyse finite elements of higher order as well. We will only consider homogeneous Dirichlet boundary condition $\nu_{-1} = \nu_1 = 0$. This is no restriction at all since it can be easily ensured by modifying the right hand side f. Furthermore, due to [7, Lemma 2.1] we may assume without loss of generality that

$$\left(c - \tfrac{1}{2}a'\right)(x) \ge \gamma > 0 \qquad \text{for all } x \in [-1,1], \quad \varepsilon \text{ sufficiently small.} \tag{6}$$

For $v, w \in H_0^1((-1,1))$ we set

$$B_\varepsilon(v,w) := (\varepsilon v', w') + (av', w) + (cv, w).$$

The bilinear form $B_\varepsilon(\cdot,\cdot)$ is uniformly coercive over $H_0^1((-1,1)) \times H_0^1((-1,1))$ in terms of the energy norm $|||\cdot|||_\varepsilon$ thanks to (6).

The weak formulation of (1) with $\nu_{-1} = \nu_1 = 0$ reads as follows:

Find $u \in H_0^1((-1,1))$ such that

$$B_\varepsilon(u,v) = (f,v), \qquad \text{for all } v \in H_0^1((-1,1)).$$

Let $k \ge 1$ and let $P_k((x_a,x_b))$ denote the space of polynomial functions of maximal order k over (x_a,x_b). We define the trial and test space V^N by

$$V^N := \left\{ v \in C([-1,1]) : v|_{(x_{i-1},x_i)} \in P_k((x_{i-1},x_i))\, \forall i,\ v(-1) = v(1) = 0 \right\}.$$

Then the discrete problem is given by:

Find $u_N \in V^N$ such that

$$B_\varepsilon(u_N, v_N) = (f, v_N), \qquad \text{for all } v_N \in V^N. \tag{7}$$

Let $\hat\phi_0, \ldots, \hat\phi_k$ denote the Lagrange basis functions on the reference interval $[0,1]$ with respect to the points $0 = \hat x_0 < \hat x_1 < \ldots < \hat x_k = 1$. We shall denote by $u_I \in V^N$ the interpolant of u which is defined on each mesh interval (x_{i-1},x_i) by

$$u_I\big|_{(x_{i-1},x_i)} = \sum_{j=0}^{k} u(x_{i,j})\phi_{i,j},$$

with $x_{i,j} := x_{i-1} + h_i \hat x_j$ and $\phi_{i,j}(x) := \hat\phi_j((x - x_{i-1})/h_i)$.

Assuming $u \in C^{k+1}([x_{i-1},x_i])$, for all $j = 0, \ldots, k+1$ the standard interpolation theory leads to the error estimates:

For $x \in (x_{i-1},x_i)$ there are $\xi_i^j \in (x_{i-1},x_i)$ such that

$$\left|(u - u_I)^{(j)}(x)\right| \le C h_i^{k+1-j} \left|u^{(k+1)}(\xi_i^j)\right| \tag{8}$$

and

$$\big\|(u-u_I)^{(j)}\big\|_{\infty,(x_{i-1},x_i)} \le C\big\|u^{(j)}\big\|_{\infty,(x_{i-1},x_i)}. \tag{9}$$

Furthermore, for all $v_N \in V^N$ the inverse inequality

$$|v_N|_{1,(x_{i-1},x_i)} \le Ch_i^{-1}\,\|v_N\|_{(x_{i-1},x_i)}$$

holds.

3.1. Finite elements of higher order. In the following we shall present the analysis for finite elements of order $k \ge 1$ for problems of the form (1). We assume that $\lambda \in (0, k+1)$ which is the most difficult case. Otherwise all crucial derivatives of the solution could be bounded by a generic constant independent of ε and consequently optimal order ε-uniform estimates could be proven with standard methods on uniform meshes.

Lemma 3.1. *Let u be the solution of* (1) *and u_N the solution of* (7) *on an arbitrary mesh. Then we have*

$$|||u_I-u_N|||_\varepsilon \le C\,|||u_I-u|||_\epsilon + C\left(\sum_{i=-N+1}^{N} h_i^{-2}\,\|x(u_I-u)\|^2_{(x_{i-1},x_i)}\right)^{1/2}.$$

Proof. By the coercivity of $B_\varepsilon(\cdot,\cdot)$ and due to orthogonality, we have

$$C\,|||u_I-u_N|||^2_\varepsilon \le B_\varepsilon(u_I-u_N,u_I-u_N) = B_\varepsilon(u_I-u,u_I-u_N). \tag{10}$$

Integrating by parts, we obtain

$$\begin{aligned}
&B_\varepsilon(u_I-u,u_I-u_N)\\
&\quad= \varepsilon\big((u_I-u)',(u_I-u_N)'\big) + \big(a(u_I-u)',u_I-u_N\big) + \big(c(u_I-u),u_I-u_N\big)\\
&\quad= \varepsilon\big((u_I-u)',(u_I-u_N)'\big) - \big(a(u_I-u),(u_I-u_N)'\big)\\
&\qquad\qquad + \big((c-a')(u_I-u),u_I-u_N\big).
\end{aligned}$$

Hence, triangle inequality and Cauchy-Schwarz inequality yield

$$\begin{aligned}
&|B_\varepsilon(u_I-u,u_I-u_N)|\\
&\qquad\le \varepsilon\left|\big((u_I-u)',(u_I-u_N)'\big)\right| + \left|\big((c-a')(u_I-u),u_I-u_N\big)\right|\\
&\qquad\quad + \sum_{i=-N+1}^{N}\left|\big(a(u_I-u),(u_I-u_N)'\big)_{(x_{i-1},x_i)}\right|\\
&\qquad\le \sqrt{\varepsilon}\,|u_I-u|_1\,\sqrt{\varepsilon}\,|u_I-u_N|_1 + \|c-a'\|_\infty\,\|u_I-u\|\,\|u_I-u_N\|\\
&\qquad\quad + \sum_{i=-N+1}^{N}\|a(u_I-u)\|_{(x_{i-1},x_i)}\,|u_I-u_N|_{1,(x_{i-1},x_i)}\,.
\end{aligned} \tag{11}$$

Now, for $-N+1 \le i \le N$ an inverse inequality and the fact that a is smooth with $a(0)=0$ imply

$$\begin{aligned}
&\|a(u_I-u)\|_{(x_{i-1},x_i)}\,|u_I-u_N|_{1,(x_{i-1},x_i)}\\
&\qquad\le Ch_i^{-1}\,\|x(u_I-u)\|_{(x_{i-1},x_i)}\,\|u_I-u_N\|_{(x_{i-1},x_i)}\,.
\end{aligned}$$

Using this bound to estimate (11), we get by Cauchy-Schwarz' inequality

$$
\begin{aligned}
&|B_\varepsilon(u_I - u, u_I - u_N)| \\
&\quad \le \max\{1\,,\, \|c - a'\|_\infty\}\left(\sqrt{\varepsilon}\, |u_I - u|_1 + \|u_I - u\|\right) |||u_I - u_N|||_\varepsilon \\
&\qquad + \sum\nolimits_i C h_i^{-1} \|x(u_I - u)\|_{(x_{i-1},x_i)} \|u_I - u_N\|_{(x_{i-1},x_i)} \\
&\quad \le \sqrt{2} \max\{1\,,\, \|c - a'\|_\infty\} |||u_I - u|||_\varepsilon \, |||u_I - u_N|||_\varepsilon \\
&\qquad + C \left(\sum\nolimits_i h_i^{-2} \|x(u_I - u)\|^2_{(x_{i-1},x_i)}\right)^{1/2} \left(\sum\nolimits_i \|u_I - u_N\|^2_{(x_{i-1},x_i)}\right)^{1/2} \\
&\quad \le \left(C\, |||u_I - u|||_\varepsilon + C \left(\sum\nolimits_i h_i^{-2} \|x(u_I - u)\|^2_{(x_{i-1},x_i)}\right)^{1/2} \right) |||u_I - u_N|||_\varepsilon \,.
\end{aligned}
$$

Combining this and (10) completes the proof. □

Remark 3.2. *For linear finite elements Sun and Stynes* [7, Lemma 5.2] *proved an estimate of the form*

$$
|||u_I - u_N|||_\varepsilon \le C \left(\|u - u_I\|^{1/2} + \max_i h_i^2 \right),
$$

see also Lemma 3.7. Aside from the fact that their argumentation works for linear elements only, such an estimate would not enable optimal estimates for finite elements of higher order.

The next two lemmas give bounds for the interpolation error on the layer-adapted mesh proposed by Liseikin.

Lemma 3.3. *Let u be the solution of problem* (1)*. Let $u_I \in V^N$ interpolate to u on the mesh generated by* (4) *with $0 < \alpha \le \min\{\lambda/(k+1), 1/(2(k+1))\}$. Then*

$$
\|u - u_I\| \le C N^{-(k+1)} \tag{12}
$$

and

$$
|||u - u_I|||_\varepsilon \le C N^{-k}. \tag{13}
$$

Proof. Thanks to the symmetry of the problem, we shall consider only $x \in [0, 1]$. Furthermore, we use $j \in \{0, 1\}$ to switch between the L^2-norm term and the ε-weighted H^1-seminorm term.

Let $x \in (x_{i-1}, x_i)$ where $2 \le i \le N$. Then for some $\xi_i^j \in (x_{i-1}, x_i)$

$$
\begin{aligned}
\varepsilon^{j/2} \left|(u - u_I)^{(j)}(x)\right| &\le C \varepsilon^{j/2} h_i^{k+1-j} \left|u^{(k+1)}(\xi_i^j)\right| \\
&\le C \varepsilon^{j/2} h_i^{k+1-j} \left(1 + \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-(k+1)}\right) \\
&\le C N^{-(k+1-j)},
\end{aligned}
$$

where we used (8), (2), and Lemma 2.1. Hence,

$$
\varepsilon^j \int_{x_1}^1 \left((u - u_I)^{(j)}(x)\right)^2 dx \le C N^{-2(k+1-j)}.
$$

Now, let $x \in (x_0, x_1)$. We consider two different cases.

First, if $\varepsilon \geq h^{2/\alpha}$ then as above Lemma 2.1 yields

$$\begin{aligned}\varepsilon^{j/2}\left|(u-u_I)^{(j)}(x)\right| &\leq C\varepsilon^{j/2}h_1^{k+1-j}\left(1+\left(x_0+\varepsilon^{1/2}\right)^{\lambda-(k+1)}\right)\\ &\leq Ch_1^{k+1-j}\left(1+\varepsilon^{(\lambda-(k+1-j))/2}\right)\\ &\leq CN^{-(k+1-j)}\end{aligned}$$

and therefore

$$\varepsilon^j\int_0^{x_1}\left((u-u_I)^{(j)}(x)\right)^2dx \leq \left(CN^{-(k+1-j)}\right)^2\int_0^{x_1}1dx \leq Ch_1N^{-2(k+1-j)}.$$

If $\varepsilon \leq h^{2/\alpha}$ we estimate the integral directly. We have

$$\begin{aligned}\varepsilon^j\int_0^{x_1}\left((u-u_I)^{(j)}(x)\right)^2dx &\leq C\varepsilon^j\int_0^{x_1}\left\|u^{(j)}\right\|^2_{\infty,(0,x_1)}dx\\ &\leq C\varepsilon^j\left(1+\varepsilon^{(\lambda-j)/2}\right)^2x_1 \leq CN^{-2(k+1)}\end{aligned}$$

by (9), (2), and Lemma 2.1.

Combining the above estimates for $j=0$ and using symmetry on $[-1,0]$ we get (12). This estimate together with the above estimates for $j=1$ immediately gives (13). $\square$

It remains to estimate the second term in Lemma 3.1.

Lemma 3.4. *Let u be the solution of problem* (1). *Let $u_I \in V^N$ interpolate to u on the mesh generated by* (4) *with $0<\alpha\leq\min\{\lambda/(k+1),1/(2(k+1))\}$. Then*

$$\left(\sum_{i=-N+1}^{N}h_i^{-2}\left\|x(u_I-u)\right\|^2_{(x_{i-1},x_i)}\right)^{1/2} \leq CN^{-k}. \tag{14}$$

Proof. The proof is similar to the proof of Lemma 3.3 but advanced in some way.

Let $2\leq i\leq N$ and $x\in(x_{i-1},x_i)$. Then for some $\xi_i\in(x_{i-1},x_i)$

$$\begin{aligned}x\left|(u_I-u)(x)\right| &\leq Cxh_i^{k+1}\left|u^{(k+1)}(\xi_i)\right|\\ &\leq C(x_{i-1}+h_i)h_i^{k+1}\left(1+\left(x_{i-1}+\varepsilon^{1/2}\right)^{\lambda-(k+1)}\right)\\ &\leq Ch_i\left(N^{-k}+N^{-(k+1)}\right)\leq Ch_iN^{-k},\end{aligned}$$

where we used (8), (2), and Lemma 2.1. Hence, for $2\leq i\leq N$

$$\begin{aligned}h_i^{-2}\left\|x(u_I-u)\right\|^2_{(x_{i-1},x_i)} &= h_i^{-2}\int_{x_{i-1}}^{x_i}\left(x(u_I-u)(x)\right)^2dx\\ &\leq h_i^{-2}\left(Ch_iN^{-k}\right)^2\int_{x_{i-1}}^{x_i}1dx\leq Ch_iN^{-2k}.\end{aligned}$$

Now, let $i=1$ and $x\in(x_0,x_1)$. We consider two different cases.

First, if $\varepsilon \geq h^{2/\alpha}$ then as above Lemma 2.1 yields

$$\begin{aligned} x\,|(u_I - u)(x)| &\leq Cxh_1^{k+1}\left(1 + \left(x_0 + \varepsilon^{1/2}\right)^{\lambda-(k+1)}\right) \\ &\leq Ch_1^{k+2}\left(1 + \varepsilon^{(\lambda-(k+1))/2}\right) \\ &\leq Ch_1 N^{-(k+1)} \end{aligned}$$

and therefore

$$h_1^{-2}\,\|x(u_I - u)\|^2_{(x_0,x_1)} \leq h_1^{-2}\left(Ch_1 N^{-(k+1)}\right)^2 \int_{x_0}^{x_1} 1dx \leq Ch_1 N^{-2(k+1)}.$$

If $\varepsilon \leq h^{2/\alpha}$ we estimate the integral directly. We have

$$\begin{aligned} h_1^{-2}\int_0^{x_1} \left(x(u_I - u)(x)\right)^2 dx &\leq h_1^{-2}\,\|u_I - u\|^2_{\infty,(0,x_1)} \int_0^{x_1} x^2 dx \\ &\leq Ch_1^{-2}\,\|u\|^2_{\infty,(0,x_1)}\, x_1^3 \\ &\leq Cx_1 \leq CN^{-2(k+1)} \end{aligned}$$

by (9), (2), and Lemma 2.1.

Summing up the above estimates gives in the worst case

$$\begin{aligned} \sum_{i=1}^{N} h_i^{-2}\,\|x(u_I - u)\|^2_{(x_{i-1},x_i)} &\leq CN^{-2(k+1)} + \sum_{i=2}^{N} Ch_i N^{-2k} \\ &\leq CN^{-2k}\left(\sum_{i=2}^{N} h_i + N^{-2}\right) \leq CN^{-2k}. \end{aligned}$$

Since, thanks to symmetry the sum for $i = -N+1, \ldots, 0$ can be bounded analogously, the proof is completed. □

Now, we are able to prove the ε-uniform error estimate of P_k-FEM in the energy norm.

Theorem 3.5. *Let u be the solution of* (1) *and u_N the solution of* (7) *on a mesh generated by* (4) *with $0 < \alpha \leq \min\{\lambda/(k+1), 1/(2(k+1))\}$. Then we have*

$$|||u - u_N|||_\varepsilon \leq CN^{-k}.$$

Proof. Based on the splitting $u - u_N = (u - u_I) + (u_I - u_N)$, the bound in the energy norm follows easily from the triangle inequality, Lemma 3.1, (13), and (14). □

3.2. Special features of linear finite elements. In this section we present some special features of linear finite elements. So, we shall assume $k = 1$. The following two lemmas hold for arbitrary meshes and are borrowed from [7]. In particular, they show that for linear finite elements the L^2-norm interpolation error estimate (12) suffices to prove the ε-uniform convergence in the energy norm.

Lemma 3.6 (see [7, Lemma 5.1])**.** *Let u be the solution of problem* (1) *and $k = 1$. Then on an arbitrary mesh we have*

$$|||u - u_I|||^2_\varepsilon \leq C\,\|u - u_I\| \tag{15}$$

and

$$\int_{-1}^{1} \left(x\,(u-u_I)'\,(x)\right)^2 dx \le C\,\|u-u_I\|. \tag{16}$$

Proof. See [7, Lemma 5.1]. □

Lemma 3.7 (similar to [7, Lemma 5.2]). *Let u be the solution of* (1) *and $u_N \in V^N$ $(k=1)$ the solution of* (7) *on an arbitrary mesh. Then we have*

$$|||u_I - u_N|||_\varepsilon \le C\,\|u-u_I\|^{1/2}.$$

Proof. As in the proof of Lemma 3.1 the coercivity of $B_\varepsilon(\cdot,\cdot)$ and orthogonality yield

$$C\,|||u_I-u_N|||_\varepsilon^2 \le B_\varepsilon(u_I-u_N, u_I-u_N) = B_\varepsilon(u_I-u, u_I-u_N).$$

Integrating by parts and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|B_\varepsilon(u_I-u, u_I-u_N)| \\
&\quad\le |\varepsilon(u_I-u, \underbrace{(u_I-u_N)''}_{=0})| + |(a(u_I-u)', u_I-u_N)| + |(c(u_I-u), u_I-u_N)| \\
&\quad\le C\,\|x(u_I-u)'\|\,\|u_I-u_N\| + C\,\|u_I-u\|\,\|u_I-u_N\| \\
&\quad\le C\,\|u_I-u\|^{1/2}\,\|u_I-u_N\|,
\end{aligned}$$

where we used (16) and $\|u_I\|_\infty \le \|u\|_\infty \le C$. □

The next lemma provides an auxiliary inequality that will be needed later. We defer its proof to Appendix C.

Lemma 3.8. *Let $e \in V^N$ $(k=1)$ on an arbitrary mesh and $-N \le L < R \le N$. Then*

$$\sum_{i=L+1}^{R-1} \hbar_i |e_i| + \frac{1}{2}\left(h_{L+1}|e_L| + h_R|e_R|\right) \le C\,\|e\|_{(x_L,x_R)},$$

where $e_i = e(x_i)$.

Proof. See Appendix C. □

Lemma 3.9 (Supercloseness). *Let u be the solution of* (1) *and $u_N \in V^N$ $(k=1)$ the solution of* (7) *on a mesh generated by* (4) *with $0 < \alpha \le \min\{\lambda/2, 1/4\}$. Then we have*

$$|||u_I-u_N|||_\varepsilon \le CN^{-2}.$$

Proof. Following the argument of Lemma 3.7 we have

$$C\,|||u_I-u_N|||_\varepsilon^2 \le |(a(u_I-u)', u_I-u_N)| + C\,\|u_I-u\|\,\|u_I-u_N\|. \tag{17}$$

Integrating by parts and applying Cauchy-Schwarz' inequality yield for $j \in \{0,1\}$

$$\begin{aligned}
&\left|(a(u_I-u)', u_I-u_N)_{(x_j,1)}\right| \\
&\qquad \le \left|(a(u_I-u), (u_I-u_N)')_{(x_j,1)}\right| + C\,\|u_I-u\|_{(x_j,1)}\,\|u_I-u_N\|_{(x_j,1)}.
\end{aligned} \tag{18}$$

Set $e_i = (u_I - u_N)(x_i)$ for $i = -N, \dots, N$. Then we have

$$
\begin{aligned}
&(a(u-u_I),(u_I-u_N)')_{(x_1,1)} \\
&\quad= \sum_{i=2}^{N} \frac{e_i - e_{i-1}}{h_i} \int_{x_{i-1}}^{x_i} a(x)(u-u_I)(x)dx \\
&\quad= \sum_{i=2}^{N-1} e_i a(x_{i-1}) \left\{ \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \right) (u-u_I)(x)dx \right\} \\
&\qquad + \sum_{i=2}^{N-1} e_i \left\{ \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \right) (a(x)-a(x_{i-1}))(u-u_I)(x)dx \right\} \\
&\qquad - e_1 \frac{1}{h_2} \int_{x_1}^{x_2} (a(x)-a(x_0))(u-u_I)(x)dx \\
&\quad= Y_1 + Y_2 + Y_3.
\end{aligned} \tag{19}
$$

Inspecting the proof of Lemma 3.3, we see that

$$
|(u-u_I)(x)| \le CN^{-2} \quad \begin{cases} \text{if } x_1 \le x \le 1, \\ \text{if } x_1 > x \ge 0, \quad \varepsilon \ge h^{2/\alpha}. \end{cases}
$$

Consequently, we have for $i = 2, \dots, N-1$ or if $i = 1$ and $\varepsilon \ge h^{2/\alpha}$

$$
\frac{1}{h_i} \int_{x_{i-1}}^{x_i} (a(x)-a(x_{i-1}))(u-u_I)(x)dx \le C\hbar_i \|u-u_I\|_{\infty,(x_{i-1},x_i)} \le C\hbar_i N^{-2},
$$

and analogously for $i = 1, \dots, N-1$

$$
\frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} (a(x)-a(x_{i-1}))(u-u_I)(x)dx \le C\hbar_i \|u-u_I\|_{\infty,(x_i,x_{i+1})} \le C\hbar_i N^{-2}.
$$

Hence, recalling Lemma 3.8, we conclude

$$
|Y_2| + |Y_3| \le CN^{-2} \sum_{i=1}^{N-1} e_i \hbar_i \le CN^{-2} \|u_I - u_N\|_{(0,1)} \tag{20}
$$

and if $\varepsilon \ge h^{2/\alpha}$ (note that $x_1 = h_1$)

$$
\begin{aligned}
\left|(a(u_I-u),(u_I-u_N)')_{(0,x_1)}\right| &= \left| \frac{e_1 - e_0}{h_1} \int_0^{x_1} a(x)(u-u_I)(x)dx \right| \\
&\le C \frac{|e_1| + |e_0|}{h_1} \|u-u_I\|_{\infty,(0,x_1)} \int_0^{x_1} x\,dx \le CN^{-2} \|u_I - u_N\|_{(0,x_1)}.
\end{aligned} \tag{21}
$$

Next, we bound $|Y_1|$. By an integral transformation, standard interpolation error estimates, the mean value theorem, and (2), we obtain

$$\begin{aligned}
&\left| \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \right) (u - u_I)(x)\, dx \right| \\
&\quad = \left| \int_0^1 (u-u_I)(x_{i-1} + t h_i) - (u - u_I)(x_i + t h_{i+1})\, dt \right| \\
&\quad = \frac{1}{2} \left| \int_0^1 (1-t)t \left[h_i^2 u''(\xi_{i-1}(t)) - h_{i+1}^2 u''(\xi_{i+1}(t)) \right] dt \right| \\
&\quad = \frac{1}{2} \left| \int_0^1 (1-t)t \left[h_i^2 \left(u''(\xi_{i-1}(t)) - u''(\xi_{i+1}(t)) \right) + \left(h_i^2 - h_{i+1}^2 \right) u''(\xi_{i+1}(t)) \right] dt \right| \\
&\quad \le \frac{1}{2} \left(2\hbar_i h_i^2 \max_{x_{i-1} \le \xi \le x_{i+1}} |u'''(\xi)| + 2\hbar_i \left(h_{i+1} - h_i \right) \max_{x_i \le \xi \le x_{i+1}} |u''(\xi)| \right) \int_0^1 (1-t)t\, dt \\
&\quad = \frac{1}{6} \hbar_i \left(h_i^2 \max_{x_{i-1} \le \xi \le x_{i+1}} |u'''(\xi)| + (h_{i+1} - h_i) \max_{x_i \le \xi \le x_{i+1}} |u''(\xi)| \right) \\
&\quad \le C \hbar_i \left(h_i^2 + (h_{i+1} - h_i) + h_i^2 \left(x_{i-1} + \varepsilon^{1/2} \right)^{\lambda - 3} + (h_{i+1} - h_i) \left(x_i + \varepsilon^{1/2} \right)^{\lambda - 2} \right),
\end{aligned}$$

where $x_{i-1} < \xi_{i-1}(t) < x_i$ and $x_i < \xi_{i+1}(t) < x_{i+1}$. Combining this with the estimates of Lemma 2.1 and Lemma 2.2 yields together with Lemma 3.8

$$\begin{aligned}
|Y_1| &\le C \sum_{i=2}^{N-1} e_i x_{i-1} \left\{ \left| \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \right) (u - u_I)(x) dx \right| \right\} \\
&\le C N^{-2} \sum_{i=2}^{N-1} e_i x_{i-1} \hbar_i \left(1 + \left(x_{i-1} + \varepsilon^{1/2} \right)^{-1} + \left(x_i + \varepsilon^{1/2} \right)^{-1} \right) \qquad (22) \\
&\le C N^{-2} \sum_{i=2}^{N-1} e_i \hbar_i \le C N^{-2} \left\| u_I - u_N \right\|_{(x_1, 1)}.
\end{aligned}$$

Altogether (18) – (22), and (12) give

$$\left| (a(u_I - u)', u_I - u_N)_{(x_j, 1)} \right| \le C N^{-2} \left(\| u_I - u_N \|_{(x_j, 1)} + \| u_I - u_N \|_{(0,1)} \right)$$

for $j = 1$ or $j = 0$ if $\varepsilon \ge h^{2/\alpha}$. Because of symmetry, it remains to bound the term $(a(u_I - u)', u_I - u_N)_{(0, x_1)}$ if $\varepsilon \le h^{2/\alpha}$. Applying the Cauchy-Schwarz inequality gives

$$\left| (a(u_I - u)', u_I - u_N)_{(0, x_1)} \right| \le C \left(\int_0^{x_1} \left(x (u_I - u)' \right)^2 dx \right)^{1/2} \| u_I - u_N \|_{(0, x_1)}.$$

Using integration by parts twice, (2), and $\|u_I\|_\infty \le \|u\|_\infty \le C$, we obtain

$$\begin{aligned}
&\int_0^{x_1} \left(x(u_I-u)'\right)^2 dx \\
&\quad = -\int_0^{x_1} 2x\,(u_I-u)'\,(u_I-u)\,dx - \int_0^{x_1} x^2 u''\,(u_I-u)\,dx \\
&\quad = \int_0^{x_1} 2x\,(u_I-u)\,(u_I-u)'\,dx + \int_0^{x_1} 2\,(u_I-u)^2\,dx - \int_0^{x_1} x^2 u''\,(u_I-u)\,dx \\
&\quad = \int_0^{x_1} (u_I-u)^2\,dx - \int_0^{x_1} x^2 u''\,(u_I-u)\,dx \\
&\quad \le \int_0^{x_1} \left((u-u_I)^2(x) + C\,|(u-u_I)(x)|\right) dx \\
&\quad \le Cx_1 \le CN^{-4}
\end{aligned}$$

by Lemma 2.1.

Combining (17), the last three estimates, and (12) completes the proof. □

We now prove the ε-uniform error estimate in the energy and L^2-norm.

Theorem 3.10. *Let u be the solution of* (1) *and $u_N \in V^N$ $(k=1)$ the solution of* (7) *on a mesh generated by* (4) *with $0 < \alpha \le \min\{\lambda/2, 1/4\}$. Then we have*

$$|||u-u_N|||_\varepsilon \le CN^{-1}$$

and

$$\|u-u_N\| \le CN^{-2}.$$

Proof. The bound in the energy norm is already given in Theorem 3.5, but also follows easily from the splitting $u - u_N = (u-u_I) + (u_I - u_N)$, the triangle inequality, Lemma 3.7, (15), and (12). To prove the bound in the L^2-norm only the supercloseness result of Lemma 3.9 and (12) have to be used. □

Remark 3.11. *For linear elements in* [7, Theorem 5.1] *the presumably non-optimal L^2-norm estimate*

$$\|u-u_N\| \le C\left(N^{-1}\ln N\right)^{3/2}$$

is proven for a discrete solution calculated on a piecewise equidistant mesh. The argumentation there is similar to the proof of Lemma 3.9. But, in one of the occurring terms there are problems when $h_i \ne h_{i+1}$ since the difference $|h_i - h_{i+1}|$ is not sufficiently small on the piecewise equidistant mesh. For the graded meshes generated by (4) *we have the estimates of Lemma 2.2. Thus, these problems can be circumvented.*

4. Numerical experiments

Now, we shall present some numerical results to verify the theoretical findings of this paper. Therefore, we study a test problem taken from [7] whose solution exhibits typical interior layer behaviour of "cusp"-type.

All computations where performed using a FEM-code based on $\mathbb{SOFE}$ by Lars Ludwig [5]. In general, the parameter α needed to generate the graded mesh was chosen as

$$\alpha = \alpha_0 \min\{\lambda/(k+1), 1/(2(k+1))\}$$

with $\alpha_0 = 1$. Exceptions are explicitly stated. For given errors $E_{\varepsilon,N}$ we calculate the rates of convergence by $(\ln E_{\varepsilon,N} - \ln E_{\varepsilon,2N}) / \ln 2$.

Example 4.1 (see [7]). *We consider the singularly perturbed turning point problem*

$$-\varepsilon u'' - x(1+x^2)u' + \lambda(1+x^3)u = f, \qquad \textit{for} \quad x \in (-1,1),$$
$$u(-1) = u(1) = 0,$$

where the right hand side $f(x)$ *is chosen such that the solution* $u(x)$ *is given by*

$$u(x) = \left(x^2+\varepsilon\right)^{\lambda/2} + x\left(x^2+\varepsilon\right)^{(\lambda-1)/2} - (1+\varepsilon)^{\lambda/2}\left(1 + x\,(1+\varepsilon)^{-1/2}\right).$$

Note that the problem parameter λ *coincides with the quantity* $\bar{\lambda} = c(0)/|a'(0)|$.

In Figure 1 the energy norm error is plotted for finite elements of order $k = 1, \ldots, 4$ applied to Example 4.1 with $\varepsilon = 10^{-8}$ and $\lambda = 0.005$. The expected convergence behaviour, cf. Theorem 3.5, can be clearly seen. The numerical results suggest that the energy norm error is almost independent of ε. Anyway it stays stable for small ε, see Table 1.

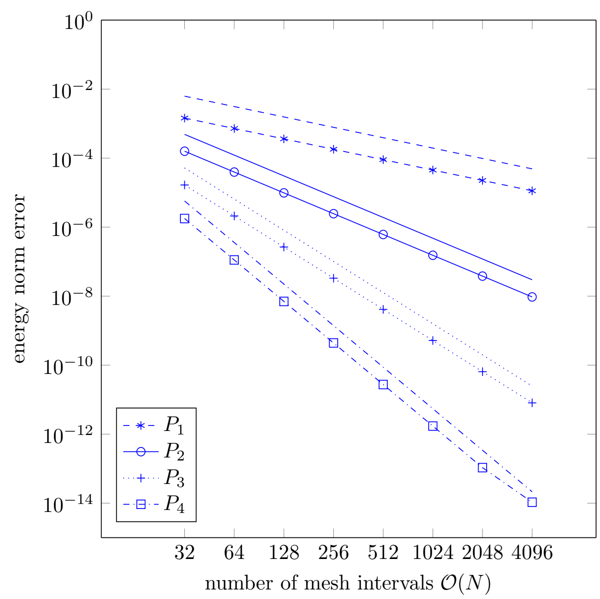

FIGURE 1. Energy norm error for finite elements of order $k = 1, \ldots, 4$ applied to Example 4.1 with $\varepsilon = 10^{-8}$ and $\lambda = 0.005$. Reference curves of the form $\mathcal{O}(N^{-k})$.

Furthermore, we study the influence of varying λ and α_0. Therefore, we consider Example 4.1 with fixed $\varepsilon = 10^{-8}$ on correspondent layer-adapted meshes with $N = 1024$. In Figure 2 the energy norm error is plotted against λ for $\alpha_0 = 1$ (left) and against α_0 for $\lambda = 0.005$ (right), respectively. In both cases the error is almost constant in the studied ranges. Thus, it seems to be plausible to presume the method to be robust in α.

TABLE 1. Energy norm error for finite elements of order $k = 1, \dots, 4$ applied to Example 4.1 with certain ε and $\lambda = 0.005$.

	P_1-elements		P_2-elements		P_3-elements		P_4-elements	
$\varepsilon \setminus N$	512	1024	512	1024	512	1024	512	1024
1	5.89e-04	2.95e-04	2.36e-07	5.91e-08	1.37e-10	1.71e-11	1.49e-13	2.06e-13
10^{-2}	7.64e-04	3.82e-04	1.31e-06	3.28e-07	2.36e-09	2.95e-10	4.07e-12	3.06e-13
10^{-4}	4.61e-04	2.30e-04	1.52e-06	3.81e-07	5.28e-09	6.60e-10	1.75e-11	1.10e-12
10^{-6}	2.16e-04	1.08e-04	1.07e-06	2.68e-07	5.56e-09	6.95e-10	2.76e-11	1.73e-12
10^{-8}	9.04e-05	4.52e-05	6.12e-07	1.53e-07	4.14e-09	5.17e-10	2.75e-11	1.72e-12
10^{-10}	3.54e-05	1.77e-05	3.69e-07	9.17e-08	2.54e-09	3.17e-10	2.17e-11	1.35e-12
10^{-12}	1.33e-05	6.66e-06	3.53e-07	8.79e-08	1.38e-09	1.72e-10	1.80e-11	1.12e-12
10^{-14}	4.95e-06	2.45e-06	4.49e-07	1.12e-07	6.87e-10	8.58e-11	2.31e-11	1.44e-12

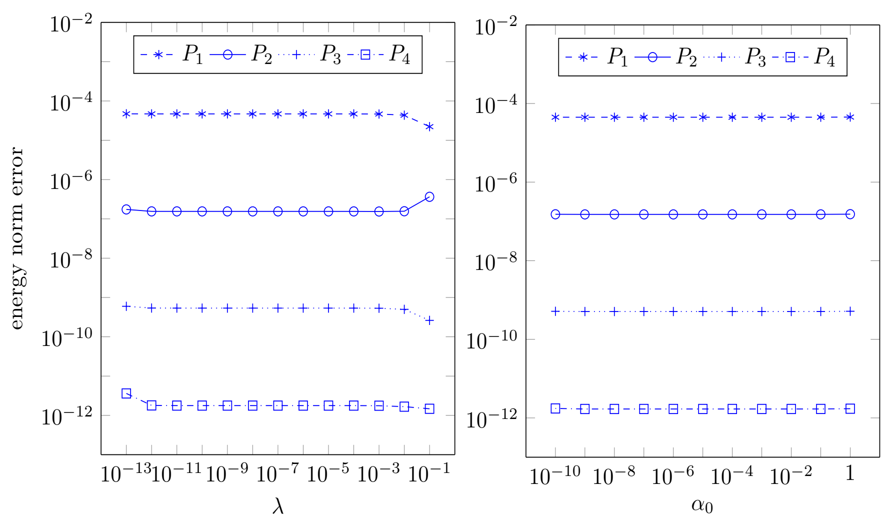

FIGURE 2. Energy norm error for $N = 1024$ and finite elements of order $k = 1, \dots, 4$ applied to Example 4.1 with $\varepsilon = 10^{-8}$, $\lambda = 10^{-13}, \dots, 10^{-1}$, and $\alpha_0 = 1$ (left). Same setting with $\lambda = 0.005$ and $\alpha_0 = 10^{-10}, \dots, 1$ (right).

Finally, in Table 2 we compare the energy norm and the L^2-norm error for linear finite elements. As predicted by theory, cf. Theorem 3.10, the L^2-error is uniformly convergent of second order whereas the error in the $|||\cdot|||_\varepsilon$-norm converges with order one only.

In summary, our numerical experiments confirm the theoretical results of Section 3.

Acknowledgments

The author would like to thank Hans-Görg Roos for helpful comments and discussions.

TABLE 2. Energy norm and L^2-norm error for linear finite elements applied to Example 4.1 with $\varepsilon = 10^{-8}, 10^{-12}$ and $\lambda = 0.005$.

	$\varepsilon = 10^{-8}$				$\varepsilon = 10^{-12}$			
	$\|\|\|u-u_N\|\|\|_\varepsilon$		$\|u-u_N\|$		$\|\|\|u-u_N\|\|\|_\varepsilon$		$\|u-u_N\|$	
N	error	rates	error	rates	error	rates	error	rates
8	7.58e-03	1.402	4.11e-03	2.467	2.22e-03	1.623	2.06e-03	1.846
16	2.87e-03	0.986	7.43e-04	2.108	7.22e-04	1.460	5.72e-04	1.853
32	1.45e-03	1.002	1.72e-04	2.009	2.62e-04	1.203	1.59e-04	1.947
64	7.23e-04	1.001	4.28e-05	2.002	1.14e-04	1.070	4.11e-05	1.986
128	3.62e-04	1.000	1.07e-05	2.000	5.43e-05	1.019	1.04e-05	1.997
256	1.81e-04	1.000	2.67e-06	2.000	2.68e-05	1.005	2.60e-06	1.999
512	9.04e-05	1.000	6.68e-07	2.000	1.33e-05	1.001	6.50e-07	2.000
1024	4.52e-05	1.000	1.67e-07	2.000	6.66e-06	1.000	1.63e-07	2.000
2048	2.26e-05	1.000	4.17e-08	2.000	3.33e-06	1.000	4.07e-08	2.000
theory		1		2		1		2

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Appendix A. Auxiliary lemmas

In this section we provide some auxiliary lemmas and prove some basic inequalities that are needed in the paper.

Lemma A.1. *Let $\alpha \in \mathbb{R}$, $0 < \alpha \le 1$ and let $a, b > 0$. Then*

$$\left.\begin{aligned} (a+b)^{1/\alpha} &\le 2^{1/\alpha-1}\left(a^{1/\alpha}+b^{1/\alpha}\right), \\ (a+b)^{\alpha} &\le a^\alpha + b^\alpha, \end{aligned}\right. \qquad \textit{for } 0<\alpha\le 1.$$

Proof. To prove the first inequality, we use that $x \mapsto x^{1/\alpha}$ is convex on $[0,\infty)$ for $0 < \alpha \le 1$. Hence,

$$(a+b)^{1/\alpha} = 2^{1/\alpha}\left(\frac{1}{2}a+\frac{1}{2}b\right)^{1/\alpha} \le 2^{1/\alpha}\left(\frac{1}{2}a^{1/\alpha}+\frac{1}{2}b^{1/\alpha}\right) = 2^{1/\alpha-1}\left(a^{1/\alpha}+b^{1/\alpha}\right).$$

We gain the second inequality by studying the function $f : (0,\infty) \to \mathbb{R}$ which is defined by $f(x) = (a+x)^\alpha - (a^\alpha + x^\alpha)$. Since $a > 0$, $0 < \alpha \le 1$, and $x \mapsto x^{\alpha-1}$ is monotonically decreasing we have

$$f'(x) = \alpha\big((a+x)^{\alpha-1} - x^{\alpha-1}\big) \le 0 \qquad \text{for } x > 0.$$

Hence, also f is monotonically decreasing and for $x \in (0,b)$

$$(a+b)^\alpha - (a^\alpha + b^\alpha) \le f(x) \le \lim_{x\to 0} f(x) = (a+0)^\alpha - (a^\alpha + 0^\alpha) = a^\alpha - a^\alpha = 0. \square$$

Lemma A.2. *Let $\alpha, c \in [0,1]$. Then $2^\alpha - 1 \le (1+c)^\alpha - c^\alpha \le 1$.*

Proof. Since the case $\alpha = 0$ is easy, we assume $\alpha > 0$. The proof uses the monotonicity properties of the function $x \mapsto (1+x)^\alpha - x^\alpha$. To detect these properties we study the first derivative

$$\frac{\partial}{\partial x}\left[(1+x)^\alpha - x^\alpha\right] = \alpha\left[(1+x)^{\alpha-1} - x^{\alpha-1}\right]$$

for $x \in (0,1)$. By assumption $\alpha - 1 \le 0$. Consequently, $(1+x)^{\alpha-1} \le 1$ and $x^{\alpha-1} \ge 1$ for $x \in (0,1)$. Therefore, the first derivative of $x \mapsto (1+x)^\alpha - x^\alpha$ is negative in $(0,1)$ and the function is monotonically decreasing in this interval. Hence,

$$2^\alpha - 1 = (1+1)^\alpha - 1^\alpha \le (1+c)^\alpha - c^\alpha \le (1+0)^\alpha - 0^\alpha = 1. \qquad \square$$

Lemma A.3. *Let $\alpha, c \in (0,1]$. Then*

$$0 < (1+c)^\alpha - c^\alpha \le \frac{\big(2^\alpha - 1\big)}{\ln(2)}\big(\ln(1+c) - \ln(c)\big).$$

Furthermore, we have

$$\alpha\ln(2) \le \big(2^\alpha - 1\big) \le \alpha.$$

Proof. We study the function

$$f(x) = \frac{\alpha\ln(1+x) - \alpha\ln(x)}{(1+x)^\alpha - x^\alpha} = \frac{\ln\Big(\big(\frac{1+x}{x}\big)^\alpha\Big)}{(1+x)^\alpha - x^\alpha}$$

for $x \in (0,1]$, $\alpha \in (0,1]$. The first derivative of f is calculated to be

$$\begin{aligned}
f'(x) &= \frac{\alpha\left(\frac{1}{1+x} - \frac{1}{x}\right)\big((1+x)^\alpha - x^\alpha\big) - \alpha\ln\Big(\big(\frac{1+x}{x}\big)^\alpha\Big)\big((1+x)^{\alpha-1} - x^{\alpha-1}\big)}{\big((1+x)^\alpha - x^\alpha\big)^2} \\
&= \frac{\alpha}{x(1+x)\big((1+x)^\alpha - x^\alpha\big)^2}\Big(\big(x - (1+x)\big)\big((1+x)^\alpha - x^\alpha\big) \\
&\qquad\qquad - \ln\Big(\big(\tfrac{1+x}{x}\big)^\alpha\Big)\big((1+x)^{\alpha-1} - x^{\alpha-1}\big)x(1+x)\Big) \\
&= \frac{-\alpha}{x(1+x)\big((1+x)^\alpha - x^\alpha\big)^2} \\
&\qquad \underbrace{\Big[\big((1+x)^\alpha - x^\alpha\big) + \ln\Big(\big(\tfrac{1+x}{x}\big)^\alpha\Big)\big(x(1+x)^\alpha - x^\alpha(1+x)\big)\Big]}_{=:g(x)}.
\end{aligned}$$

By Lemma A.1 we have for $x \in (0,1]$, $\alpha \in (0,1]$

$$x(1+x)^\alpha - x^\alpha(1+x) \le x(1^\alpha + x^\alpha) - x^\alpha - x^{1+\alpha} = x - x^\alpha \le 0.$$

Therefore, due to $\ln(x) \le x - 1$ we obtain

$$\begin{aligned} g(x) &\ge \big((1+x)^\alpha - x^\alpha\big) + \Big(\big(\tfrac{1+x}{x}\big)^\alpha - 1\Big)\big(x(1+x)^\alpha - x^\alpha(1+x)\big) \\ &= \big((1+x)-1\big)x^\alpha + \big(1 - x - (1+x)\big)(1+x)^\alpha + x^{1-\alpha}(1+x)^{2\alpha} \\ &= x^{1+\alpha} - 2x(1+x)^\alpha + x^{1-\alpha}(1+x)^{2\alpha} \\ &= \Big(x^{(1+\alpha)/2} - x^{(1-\alpha)/2}(1+x)^\alpha\Big)^2 \\ &\ge 0. \end{aligned}$$

Thus, $f'(x) \le 0$ for $x \in (0,1]$, $\alpha \in (0,1]$ and f is monotonically decreasing. This yields

$$f(x) = \frac{\alpha\ln(1+x) - \alpha\ln(x)}{(1+x)^\alpha - x^\alpha} \ge \frac{\alpha\ln(2) - \alpha\ln(1)}{2^\alpha - 1^\alpha} = \frac{\alpha\ln(2)}{2^\alpha - 1}$$

and for $x = c \in (0,1]$

$$(1+c)^\alpha - c^\alpha \le \frac{(2^\alpha - 1)}{\ln(2)}\big(\ln(1+c) - \ln(c)\big).$$

The lower bound of the second statement follows from $1 + x \le e^x$ since

$$1 + \alpha\ln(2) \le e^{\alpha\ln(2)} = 2^\alpha.$$

The upper bound is verified using the convexity of $x \mapsto 2^x$. We have

$$2^\alpha = 2^{\alpha\cdot 1 + (1-\alpha)\cdot 0} \le \alpha \cdot 2^1 + (1-\alpha)\cdot 2^0 = 2\alpha + (1-\alpha) = 1 + \alpha. \qquad \square$$

Remark A.4. *The bound of the last lemma is exact for $c = 1$ and asymptotically exact for $\alpha \searrow 0$.*

Appendix B. Proof of Lemmas 2.1 and 2.2

For $\xi \ge 0$ we have

(23a)
$$\varphi(\xi,\varepsilon) = \Big(\varepsilon^{\alpha/2} + \xi\Big[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\Big]\Big)^{1/\alpha} - \varepsilon^{1/2},$$

(23b)
$$\frac{\partial}{\partial\xi}\varphi(\xi,\varepsilon) = \frac{1}{\alpha}\Big[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\Big]\Big(\varepsilon^{\alpha/2} + \xi\Big[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\Big]\Big)^{(1-\alpha)/\alpha},$$

(23c)
$$\frac{\partial^2}{\partial\xi^2}\varphi(\xi,\varepsilon) = \frac{1-\alpha}{\alpha^2}\Big[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\Big]^2\Big(\varepsilon^{\alpha/2} + \xi\Big[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\Big]\Big)^{(1-2\alpha)/\alpha}.$$

Using the definition of the mesh points $x_i = \varphi(ih,\varepsilon)$ we obtain

$$x_i + \varepsilon^{1/2} = \Big(\varepsilon^{\alpha/2} + ih\Big[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\Big]\Big)^{1/\alpha}. \tag{24}$$

Proof of Lemma 2.1. Recalling (23) and (24), using the mean value theorem, and the monotony of $\frac{\partial}{\partial\xi}\varphi(\xi,\varepsilon)$ (here $\alpha \le 1$ is needed) we can bound the lengths of the mesh intervals as follows

$$
\begin{aligned}
h_i &= x_i - x_{i-1} = \varphi(ih,\varepsilon) - \varphi((i-1)h,\varepsilon)\\
&\le h\,\frac{\partial\varphi}{\partial\xi}(ih,\varepsilon)\\
&= h\frac{1}{\alpha}\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\left(\varepsilon^{\alpha/2} + ih\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\right)^{(1-\alpha)/\alpha}\\
&= h\frac{1}{\alpha}\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\left(x_i + \varepsilon^{1/2}\right)^{(1-\alpha)}.
\end{aligned} \tag{25}
$$

Consequently, for $0 < \alpha \le 1$ we get the third wanted estimate

$$
h_i \le h\frac{1}{\alpha}\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\left(x_i + \varepsilon^{1/2}\right)^{(1-\alpha)} \le h\kappa 2^{(1-\alpha)} \le 2\kappa h \le Ch.
$$

Now, let $0 < \alpha \le 1/k$ with $k \in \mathbb{N}$, $k \ge 1$ and $\varepsilon \le h^{2/\alpha}$. Using the definition of the mesh (4) and Lemma A.2 we obtain

$$
\begin{aligned}
x_1 &= \left(\varepsilon^{\alpha/2} + h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\right)^{1/\alpha} - \varepsilon^{1/2}\\
&\le \left(h + h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\right)^{1/\alpha}\\
&\le (2h)^{1/\alpha} = \left(2h^{(1-k\alpha)}\right)^{1/\alpha} h^k\\
&\le 2^{1/\alpha}h^k.
\end{aligned}
$$

Note that for $0 < \alpha \le 1/(2k)$ and $h \le 1/4$ an α-independent estimate is guaranteed because of

$$
2h^{(1-k\alpha)} \le 2h^{1/2} \le 2\left(\tfrac{1}{4}\right)^{1/2} = 1.
$$

Finally, let $\hat{\alpha} > 0$ and $0 < \alpha \le \min\{\hat{\alpha}/k, 1\}$ with $k \in \mathbb{N}$, $k \ge 1$. If $\varepsilon \ge h^{2/\alpha}$ we obtain by (25) and Lemma A.2

$$
\begin{aligned}
h_1&\left(x_0 + \varepsilon^{1/2}\right)^{\hat{\alpha}/k-1} = h_1\varepsilon^{(\hat{\alpha}/k-1)/2}\\
&\le h\frac{1}{\alpha}\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\left(\varepsilon^{\alpha/2} + h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\right)^{(1-\alpha)/\alpha}\varepsilon^{(\hat{\alpha}/k-1)/2}\\
&\le h\kappa\left(2\varepsilon^{\alpha/2}\right)^{(1-\alpha)/\alpha}\varepsilon^{(\hat{\alpha}/k-1)/2}\\
&= h\kappa 2^{(1-\alpha)/\alpha}\varepsilon^{(\hat{\alpha}/k-\alpha)/2}\\
&\le Ch.
\end{aligned}
$$

In general, the estimate (25) and the identity (24) yield for $1 \le i \le N$

$$\begin{aligned}
&h_i \left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat{\alpha}/k-1} \\
&\quad \le h\frac{1}{\alpha}\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\left(x_i + \varepsilon^{1/2}\right)^{(1-\alpha)}\left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat{\alpha}/k-1} \\
&\quad = h\frac{1}{\alpha}\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\left(\frac{x_i + \varepsilon^{1/2}}{x_{i-1} + \varepsilon^{1/2}}\right)^{(1-\alpha)}\left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat{\alpha}/k-\alpha} \\
&\quad = h\frac{1}{\alpha}\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right] \\
&\qquad\qquad \left(\frac{\varepsilon^{\alpha/2} + ih\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]}{\varepsilon^{\alpha/2} + (i-1)h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]}\right)^{(1-\alpha)/\alpha}\left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat{\alpha}/k-\alpha}.
\end{aligned}$$

Furthermore, for $2 \le i \le N$ we have

$$\begin{aligned}
\frac{\varepsilon^{\alpha/2} + ih\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]}{\varepsilon^{\alpha/2} + (i-1)h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]} &= 1 + \frac{h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]}{\varepsilon^{\alpha/2} + (i-1)h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]} \\
&\le 1 + \frac{h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]}{(i-1)h\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]} \\
&= 1 + \frac{1}{i-1} = \frac{i}{i-1}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
h_i \left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat{\alpha}/k-1} &\le h\kappa\left(\frac{i}{i-1}\right)^{(1-\alpha)/\alpha}\left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat{\alpha}/k-\alpha} \\
&\le Ch \qquad\qquad \text{for } 2 \le i \le N.
\end{aligned}$$

Raising the inequalities to the k-th power gives the wanted estimate. □

Proof of Lemma 2.2. First of all, recall (23). For $i = 2, \dots, N$ the mean value theorem, the monotony of $\frac{\partial^2}{\partial \xi^2}\varphi(\xi, \varepsilon)$ (here $0 < \alpha \le \frac{1}{2}$ is needed), (24), and (5) enable to estimate the difference $h_i - h_{i-1}$ as follows

$$\begin{aligned}
h_i - h_{i-1} &= \Big(\varphi\big(ih, \varepsilon\big) - \varphi\big((i-1)h, \varepsilon\big)\Big) - \Big(\varphi\big((i-1)h, \varepsilon\big) - \varphi\big((i-2)h, \varepsilon\big)\Big) \\
&= h\left(\frac{\partial \varphi}{\partial \xi}(\xi_i, \varepsilon) - \frac{\partial \varphi}{\partial \xi}(\xi_{i-2}, \varepsilon)\right) \\
&= h\left(\xi_i - \xi_{i-2}\right)\frac{\partial^2 \varphi}{\partial \xi^2}(\xi_{i-1}, \varepsilon) \\
&\le 2h^2\frac{\partial^2 \varphi}{\partial \xi^2}(ih, \varepsilon) \\
&= 2h^2\frac{1-\alpha}{\alpha^2}\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]^2\left(\varepsilon^{\alpha/2} + ih\left[(1+\varepsilon^{1/2})^{\alpha} - \varepsilon^{\alpha/2}\right]\right)^{(1-2\alpha)/\alpha} \\
&= 2h^2(1-\alpha)\kappa^2\left(x_i + \varepsilon^{1/2}\right)^{1-2\alpha} \le Ch^2\left(x_i + \varepsilon^{1/2}\right)^{1-2\alpha}
\end{aligned}$$

where $\xi_i \in ((i-1)h, ih)$, $\xi_{i-2} \in ((i-2)h, (i-1)h)$, and $\xi_{i-1} \in (\xi_{i-2}, \xi_i)$.

Using this bound and applying the same technique as in the proof of Lemma 2.1, we obtain

$$(h_i - h_{i-1})\left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat{\alpha}-1} \le Ch^2 \left(\frac{i}{i-1}\right)^{(1-2\alpha)/\alpha} \left(x_{i-1} + \varepsilon^{1/2}\right)^{\hat{\alpha}-2\alpha} \le Ch^2$$

for $2 \le i \le N$. □

Appendix C. Proof of Lemma 3.8

Proof of Lemma 3.8. An easy calculation shows

$$\begin{aligned}
\|e\|_{(x_L,x_R)} &= \left(\sum_{i=L+1}^{R} \int_{x_{i-1}}^{x_i} \left[e_i - \frac{x_i - x}{h_i}(e_i - e_{i-1})\right]^2 dx\right)^{1/2} \\
&= \left(\sum_{i=L+1}^{R} \int_{x_{i-1}}^{x_i} e_i^2 - 2\frac{x_i - x}{h_i}(e_i - e_{i-1})e_i + \frac{(x_i - x)^2}{h_i^2}(e_i - e_{i-1})^2 dx\right)^{1/2} \\
&= \left(\sum_{i=L+1}^{R} \left(e_i^2 h_i - \frac{(x_i - x_{i-1})^2}{h_i}(e_i - e_{i-1})e_i + \frac{1}{3}\frac{(x_i - x_{i-1})^3}{h_i^2}(e_i - e_{i-1})^2\right)\right)^{1/2} \\
&= \left(\sum_{i=L+1}^{R} h_i \left(e_i^2 - e_i^2 + e_i e_{i-1} + \frac{1}{3}e_i^2 - \frac{2}{3}e_i e_{i-1} + \frac{1}{3}e_{i-1}^2\right)\right)^{1/2} \\
&= \left(\frac{1}{3}\sum_{i=L+1}^{R} h_i \left(e_i^2 + e_i e_{i-1} + e_{i-1}^2\right)\right)^{1/2} \ge \left(\frac{1}{6}\sum_{i=L+1}^{R} h_i \left(e_i^2 + e_{i-1}^2\right)\right)^{1/2}.
\end{aligned}$$

Furthermore, using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\sum_{i=L+1}^{R-1} \hbar_i |e_i| + \frac{1}{2}\left(h_{L+1}|e_L| + h_R|e_R|\right) \\
&\quad = \frac{1}{2}\sum_{i=L+1}^{R-1} \left(h_i|e_i| + h_{i+1}|e_i|\right) + \frac{1}{2}\left(h_{L+1}|e_L| + h_R|e_R|\right) = \frac{1}{2}\sum_{i=L+1}^{R} h_i\left(|e_i| + |e_{i-1}|\right) \\
&\quad \le \left(\frac{1}{2}\sum_{i=L+1}^{R} h_i\right)^{1/2} \left(\frac{1}{2}\sum_{i=L+1}^{R} h_i\left(|e_i| + |e_{i-1}|\right)^2\right)^{1/2} \\
&\quad \le \left(\frac{x_R - x_L}{2}\right)^{1/2} \left(\sum_{i=L+1}^{R} h_i\left(e_i^2 + e_{i-1}^2\right)\right)^{1/2}.
\end{aligned}$$

Hence, we have

$$\sum_{i=L+1}^{R-1} \hbar_i |e_i| + \frac{1}{2}\left(h_{L+1}|e_L| + h_R|e_R|\right) \le \sqrt{3}\left(x_R - x_L\right)^{1/2} \|e\|_{(x_L,x_R)}.$$ □

Institute of Numerical Mathematics, Technical University of Dresden, Dresden 01062, Germany
E-mail: Simon.Becher@tu-dresden.de