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A PARTITIONED METHOD WITH DIFFERENT TIME STEPS FOR COUPLED STOKES AND DARCY FLOWS WITH TRANSPORT

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Abstract. A decoupled finite element algorithm with different time steps on different physical variables for a Stokes-Darcy interface system coupled with the solution transport is studied. The viscosity of the Stokes equation is assumed to depend on the concentration of the transported solution. The numerical algorithm consists of two steps. In the first step, the system is decoupled on the interface. In the second step, the time derivatives are discretized with different step sizes for different partial differential equations in the system. An careful error analysis provides a guidance on the ratio of the step sizes with respect to the ratio of the physical parameters. Numerical examples are presented to verify the theoretical results and illustrate the effectiveness of the decoupled algorithm of using different time steps.

Key words. Coupled Stokes and Darcy flows, solute transport, decoupled algorithm, different time steps, error estimates.

1. Introduction

Recently there have been growing interests in building suitable mathematical and numerical models for the coupling of fluid flows in a porous medium domain and a free flow domain. In the porous medium domain, the fluid flow can be modeled by a Darcy equation while in the free flow domain the fluid flow can be modeled by a Stokes equation. The Darcy equation and the Stokes equation are coupled through conditions on the interface which connects the porous domain and the free flow domain. Modeling through the Stokes-Darcy system has a wide arrange of applications such as hydrology[5], environment science[12], and biofluid dynamics[15].

A number of numerical methods have been developed for the coupled Stokes-Darcy flow system, including the domain decomposition method [2, 39, 9], the mixed finite element method [1, 23, 38], the non-conforming finite element methods [33], the Mortar multiscale finite element methods [20], the Lagrange multiplier and mixed element methods [4, 24, 27, 16, 17], the mixed finite element method combining with the DG method [31, 32], the DG method combining with mimetic finite difference method [25], the pseudospectral least squares method [21] and spectral method [41], and many other numerical methods [18, 10, 26, 29].

The aim of this paper is to construct an efficient numerical algorithm for the Stokes-Darcy flow system coupled with an advection-diffusion equation that models, for example, the transport of a chemical. In [8], Cesmelioglu and Riviere study the existence and stability of the weak solution with the fluid viscosity depending on the concentration for this model. For numerical methods, in [40] the flow equations are solved through the domain decomposition method using classical finite element methods in the Stokes region and mixed finite element methods in the Darcy region, and the transport equation is solved by a local discontinuous Galerkin (LDG) method, while in [34] the authors proposed a mixed weak formulation and use the

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nonconforming piecewise Crouzeix- Raviart finite element, piecewise constant and conforming piecewise linear finite element to approximate velocity, pressure and concentration, respectively.

In this paper, we study the finite element approximation of the Stokes-Darcy-Transport system with different time steps on different physical variables. As a multi-phyiscs problem, each of the equations in the the Stokes-Darcy-Transport system has a different time scale reflected by the corresponding partial differential equation and the related physical parameters. Thus it is natural to use larger time step in the region with slower velocity. The multiple-time-step technique for the non-stationary Stokes-Darcy model was presented in [36, 37]. There the viscosity of the free flow, the hydraulic coefficient of the flow in porous medium and the diffusion coefficient of the transport are assumed to be constants. In this study, we assume that viscosity and the hydraulic conductivity depend on the concentration of the transport and the diffusion coefficient depends on the velocity of flow in the porous medium. Under a modest time step restriction in relation to physical parameters, we obtain the stability of the method and a priori error estimates. With the help of such error analysis we derive criteria of choosing the time step for each physical variables in accordance to the ratios of the physical parameters. In particular, we show that the ratio between the time steps should be proportional to the ratios between these physical parameters. For spatial discretization, we adopt the decoupling method of [28] (see also [6, 22, 37]).

The rest of the article is organized as follows. In Section 2, we introduce the model problem, and present the mixed weak formulation. Coupling and decoupling schemes, and the stability of the decoupling scheme with different time steps on different subdomains are given in Section 3. The error estimates for fluid velocity, kinematic pressure, piezometric head and concentration are presented in Section 4. Finally in Section 5, we present some numerical examples to demonstrate our theoretical results.

Through out this paper we use K and C, with or without subscription, to denote a generic constant, which may have different values in different appearances.

2. Model problem and weak formulation

The model under consideration is a flow in a bounded domain $\Omega \subset R^N$ (N=2 or 3), consisting of an fluid flow region Ω_f, where the flow is governed by the Stokes equation, and a porous medium region $\Omega_p = \Omega \backslash \overline{\Omega}_f$, where the flow is governed by the Darcy's law. Here $\Omega_l (l = f, p)$ are bounded domains with outward unit normal vectors $\boldsymbol{n}_l (l = f, p)$. The whole domain Ω is occupied by a mixture of two miscible fluids which before mixing are each incompressible, therefore the Boussinesq approximation is valid and the mean-volume velocity is thought to be equal to the mean mass velocity. The two regions are separated by an interface $\Gamma_I = \partial\Omega_f \cap \partial\Omega_p$, and $\tau_j, j = 1, \cdots, N-1$ denote an orthonormal system of tangent vector on Γ_I. On the interface, we have $\boldsymbol{n}_f = -\boldsymbol{n}_p$. Let $\Gamma_l = \partial\Omega_l \backslash \Gamma_I (l = f, p)$. Each interface and boundary is assumed to be polygonal or polyhedral. Figure 1 gives a schematic representation of the geometry with N=2.

The equations of motion, continuity and mass transport for the fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$, kinematic pressure $p(\boldsymbol{x}, t)$ and concentration $c(\boldsymbol{x}, t)$ in Ω_f can be written as

$$\partial_t \boldsymbol{u} - \nabla \cdot (2\mu(c)\boldsymbol{S}(\boldsymbol{u})) + \nabla p = \boldsymbol{f}(c), \quad \boldsymbol{x} \in \Omega_f, t \in J, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \boldsymbol{x} \in \Omega_f, t \in J, \tag{2}$$

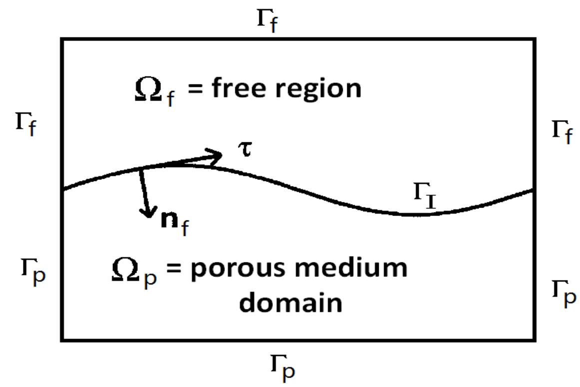


FIGURE 1. The model problem.

$$\partial_t c - \nabla \cdot (d \nabla c) + \boldsymbol{u} \cdot \nabla c = 0, \quad \boldsymbol{x} \in \Omega_f, t \in J, \tag{3}$$

where $J = [0, T]$ denotes the time interval, $\partial_t = \dfrac{\partial}{\partial t}$, $\mu = \mu(c)$ is the concentration (denoted by c) dependent fluid viscosity, $\boldsymbol{S}$ is the deformation rate tensor defined by $\boldsymbol{S}(\boldsymbol{v}) = \dfrac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$. $\boldsymbol{f} \in (L^2(\Omega))^N$ is a term related to body forces. The equations of motion, continuity and mass transport for the piezometric head $\varphi(\boldsymbol{x}, t)$ and concentration $c(\boldsymbol{x}, t)$ in Ω_p can be written as

$$\boldsymbol{u}_p = -\lambda(c) \nabla \varphi, \quad \boldsymbol{x} \in \Omega_p, t \in J, \tag{4}$$

$$S_0 \partial_t \varphi + \nabla \cdot \boldsymbol{u}_p = q^I - q^P, \quad \boldsymbol{x} \in \Omega_p, t \in J, \tag{5}$$

$$\phi \partial_t c - \nabla \cdot (\boldsymbol{D}(\boldsymbol{u}_p) \nabla c) + \boldsymbol{u}_p \cdot \nabla c = (c^I - c) q^I, \quad \boldsymbol{x} \in \Omega_p, t \in J, \tag{6}$$

where $\boldsymbol{u}_p$ is the fluid velocity in Ω_p, S_0 is the specific mass storativity coefficient, which is assumed to be a constant, d is the molecular diffusion coefficient, $\lambda(c) = \dfrac{K(\boldsymbol{x})}{\mu(c)}$ with $K = \operatorname{diag} k_j$ being the permeability of the medium, q^I and q^P represent the source term and sink term, respectively, ϕ is the porosity of the medium, $\boldsymbol{D}$ is the diffusion and dispersion coefficient, c^I is the injected concentration.

From Darcy's law (4), Equation (5) can be rewritten in the parabolic form

$$S_0 \partial_t \varphi - \nabla \cdot (\lambda(c) \nabla \varphi) = q^I - q^P, \quad \boldsymbol{x} \in \Omega_p, t \in J. \tag{7}$$

On the interface Γ_I, the following interface conditions are imposed:

$$\boldsymbol{u} \cdot \boldsymbol{n}_f + \boldsymbol{u}_p \cdot \boldsymbol{n}_p = 0, \tag{8}$$

$$p - \boldsymbol{n}_f \cdot 2\mu(c_f) \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{n}_f = g\varphi, \tag{9}$$

$$2\boldsymbol{n}_f \cdot \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{\tau}_j + \gamma_j \boldsymbol{u} \cdot \boldsymbol{\tau}_j = 0, \ j = 1, \cdots, N-1, \tag{10}$$

$$c_f = c_p, \tag{11}$$

$$d \nabla c_f \cdot \boldsymbol{n}_f + \boldsymbol{D}(\boldsymbol{u}_p) \nabla c_p \cdot \boldsymbol{n}_p = 0. \tag{12}$$

Here g is the gravitational acceleration, which is assumed to be a constant, $\gamma_j = \alpha_1 / \sqrt{k_j}$, α_1 is a parameter determined by experimental evidence. Equations (8) represents continuity of mass flux, (11) represents continuity of concentration, (12) represents the balance of concentration flux, (9) represents the balance of normal forces, and (10) is the Beavers-Joseph-Saffman condition.

From Darcy's law (4), the condition (8) can be rewritten as

$$\boldsymbol{u}\cdot\boldsymbol{n}_f=\lambda(c_p)\frac{\partial\varphi}{\partial\boldsymbol{n}_p}. \tag{13}$$

To complete the system (1)− (13), we assume the following boundary conditions

$$\boldsymbol{u}=0,\quad \boldsymbol{x}\in\Gamma_f, t\in J, \tag{14}$$

$$\varphi=0,\quad \boldsymbol{x}\in\Gamma_p, t\in J, \tag{15}$$

$$\overline{\boldsymbol{D}}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n}=0,\quad \boldsymbol{x}\in\partial\Omega, t\in J, \tag{16}$$

and initial conditions

$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega_f, \tag{17}$$

$$\varphi(\boldsymbol{x},0)=\boldsymbol{\varphi}_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega_p, \tag{18}$$

$$c(\boldsymbol{x},0)=c_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega. \tag{19}$$

Here

$$\overline{\boldsymbol{D}}(\boldsymbol{u})=\begin{cases} d\boldsymbol{u}, & \boldsymbol{x}\in\Omega_f,\\ \boldsymbol{D}(\boldsymbol{u}_p)=\boldsymbol{D}(\lambda(c)\nabla\varphi), & \boldsymbol{x}\in\Omega_p.\end{cases}$$

Equations (1)− (19) consist of the coupled Stokes and Darcy flows system with an advection-diffusion equation that models the transport of a solute in which the viscosity depends on the concentration c.

We now make several assumptions on the coefficients in the system.

(i) $\phi\in L^\infty(\Omega_p)$, $K\in L^\infty(\Omega_p)^{N\times N}$ is uniformly bounded and positive defined in Ω_p. There exists positive constants $\phi_{\min}$, $\phi_{\max}$, $k_{\min}$ and $k_{\max}$ such that

$$\phi_{\min}\le\phi(\boldsymbol{x})\le\phi_{\max},\ \forall\boldsymbol{x}\in\Omega_p, \tag{20}$$

$$k_{\min}|\boldsymbol{x}|^2\le K\boldsymbol{x}\cdot\boldsymbol{x}\le k_{\max}|\boldsymbol{x}|^2,\ \forall\boldsymbol{x}\in\Omega_p. \tag{21}$$

(ii) The form of μ is assumed to follow the quarter-power rule

$$\mu(c)=\mu(0)\left[\Big(\frac{\mu(0)}{\mu(1)}\Big)^{\frac14}c+(1-c)\right]^{-4},\quad \textit{where } c\in[0,1]. \tag{22}$$

From the form of μ, we can see that $\mu(c)$ is bounded and monotone for concentration $c\in[0,1]$

$$\mu_{\min}\le\mu(c)\le\mu_{\max},\quad \forall c\in[0,1], \tag{23}$$

where $\mu_{\min}=\min\{\mu(0),\mu(1)\}$, $\mu_{\max}=\max\{\mu(0),\mu(1)\}$. It is also easy to see that μ is also Lipschitz continuous for $c\in[0,1]$. We assume that μ_L is the Lipschitz constant.

From the assumptions above, we know that λ is also bounded, monotone and Lipschitz continuous for concentration $c\in[0,1]$, and

$$\frac{k_{\min}}{\mu_{\max}}|\boldsymbol{x}|^2\le\lambda(c)\boldsymbol{x}\cdot\boldsymbol{x}\le\frac{k_{\max}}{\mu_{\min}}|\boldsymbol{x}|^2,\quad \forall c\in[0,1],\boldsymbol{x}\in\Omega_p. \tag{24}$$

(iii) The diffusion-dispersion coefficient is taken to be

$$\boldsymbol{D}(\boldsymbol{u})=\phi d\widetilde{\tau}\boldsymbol{I}+|\boldsymbol{u}|(d_l\boldsymbol{E}(\boldsymbol{u})+d_t\boldsymbol{E}^\perp),$$

where $\widetilde{\tau}$ is a positive constant in $(0,1)$, and represents the tortuosity of the porous medium, d_l and d_t are the longitudinal and transverse dispersion coefficients, respectively. For $\boldsymbol{u}=(u_1,\cdots,u^N)$, $|\boldsymbol{u}|=\sqrt{u_1^2+\cdots+u_N^2}$ and the matrices $\boldsymbol{E},\boldsymbol{E}^\perp$ are given by

$$\boldsymbol{E}(\boldsymbol{u})=\Big(\frac{u_iu_j}{|\boldsymbol{u}|^2}\Big)_{N\times N},\qquad \boldsymbol{E}^\perp=\boldsymbol{I}-\boldsymbol{E}.$$

Usually d_l is considerably larger than d_t, hence we assume $d_l > d_t$.

(iv) $\boldsymbol{f} = \boldsymbol{f}(c)$ is a Lipschitz continuous function for concentration $c \in [0, 1]$ with Lipschitz constant f_L.

By the definition of $\boldsymbol{D}$ and the analysis of [14], we know that for $\boldsymbol{u}, \boldsymbol{v} \in C(\overline{\Omega_p})^N$, there holds

$$D_{\min}|\boldsymbol{\xi}|^2 \leq \overline{\boldsymbol{D}}(\boldsymbol{u})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq D_{\max}|\boldsymbol{\xi}|^2, \quad \boldsymbol{\xi} \in R^N, \tag{25}$$

$$(\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{v}))(i, j) \leq (3d_l - 2d_t)|\boldsymbol{u} - \boldsymbol{v}|, \quad 1 \leq i, j \leq 2, \tag{26}$$

where $D_{\min}$ and $D_{\max}$ are constants depending only on $\phi_{\min}$, $\phi_{\max}$, $\widetilde{\tau}$, d_l, d_t and d.

Before giving the suitable weak formulations of the problem (1)− (19), we introduce some useful notations. The Sobolev space $W^{m,n}(\Omega)$ is defined in the usual way with the usual norm $\|\cdot\|_{W^{m,n}(\Omega)}$, where $0 \leq m < \infty, 0 < n$. When $n = 2$, we simply substitute $H^m(\Omega)$ for $W^{m,2}(\Omega)$ with $\|\cdot\|_{m,\Omega} = \|\cdot\|_{W^{m,2}(\Omega)}$, $|\cdot|_{m,\Omega} = |\cdot|_{W^{m,2}(\Omega)}$. In particular, when $m = 0$, we have $L^2(\Omega) = H^0(\Omega)$ with $\|\cdot\|$ for $\|\cdot\|_{0,\Omega}$. Let $(\cdot,\cdot)_\Omega$ denote $L^2(\Omega)$, $L^2(\Omega)^N$ or $L^2(\Omega)^{N\times N}$ inner product or duality pairing. Also, $\|\cdot\|_l, |\cdot|_l$ *and* $(\cdot,\cdot)_l, l = f, p$, will be the same with Ω replaced by $\Omega_l, l = f, p$.

To present a variational form of the coupled problem we define the following spaces H_f H_p and Q for the fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$, piezometric head $\varphi(\boldsymbol{x}, t)$ and kinematic pressure $p(\boldsymbol{x}, t)$, respectively

$$H_f = \{\boldsymbol{v} \in H^1(\Omega_f)^N : v = 0 \ on \ \Gamma_f\}, \tag{27}$$

$$H_p = \{\boldsymbol{v} \in H^1(\Omega_p) : \psi = 0 \ on \ \Gamma_p\}, \tag{28}$$

$$Q = L^2(\Omega_f). \tag{29}$$

The norm of Q is $\|\cdot\|_f$ and the norm of H_f and H_p are

$$\|\boldsymbol{v}\|_{H_f} = \|\nabla\boldsymbol{v}\|_f = \sqrt{(\nabla\boldsymbol{v}, \nabla\boldsymbol{v})_f}, \ \forall\boldsymbol{v} \in H_f, \tag{30}$$

$$\|\psi\|_{H_p} = \|\nabla\psi\|_p = \sqrt{(\nabla\psi, \nabla\psi)_p}, \ \forall\psi \in H_p. \tag{31}$$

The space for concentration is $W = H^1(\Omega)$ with norm $\|\cdot\|_W = \|\cdot\|_1$.

The following inequalities are useful for our analysis.

Poincaré inequalities: there exist constants C_P and $\tilde{C}_P$ which only depend on Ω_f and Ω_p respectively, such that

$$\|\boldsymbol{v}\|_f \leq C_P\|\boldsymbol{v}\|_{H_f}, \quad \forall\boldsymbol{v} \in H_f, \tag{32}$$

$$\|\psi\|_p \leq \tilde{C}_P\|\psi\|_{H_p}, \quad \forall\psi \in H_p. \tag{33}$$

Trace inequalities: there exist constants C_T and $\tilde{C}_T$ which only depend on Ω_f and Ω_p respectively, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma)} \leq C_T\|\boldsymbol{v}\|_f^{1/2}\|\boldsymbol{v}\|_{H_f}^{1/2}, \quad \forall\boldsymbol{v} \in H_f, \tag{34}$$

$$\|\psi\|_{L^2(\Gamma)} \leq \tilde{C}_T\|\psi\|_p^{1/2}\|\psi\|_{H_p}^{1/2}, \quad \forall\psi \in H_p. \tag{35}$$

We now propose the following weak formulation of the coupled problem (1)− (19): find $\boldsymbol{u} \in H_f$, $\varphi \in H_p$, $p \in Q$ and $c \in W$ such that $t \in J$

$$(\partial_t\boldsymbol{u}, \boldsymbol{v})_f + gS_0(\partial_t\varphi, \psi)_p + a_f(c; \boldsymbol{u}, \boldsymbol{v}) + a_p(c; \varphi, \psi) + a_\Gamma(\boldsymbol{v}, \varphi) - a_\Gamma(\boldsymbol{u}, \psi)$$

$$+ b(\boldsymbol{v}, p) = (\boldsymbol{f}(c), \boldsymbol{v})_f + g(q^I - q^P, \psi)_p, \quad \forall\boldsymbol{v} \in H_f, \psi \in H_p, \tag{36}$$

$$b(\boldsymbol{u}, q) = 0, \quad \forall q \in Q, \tag{37}$$

$$(\partial_t c, z)_{\overline{\phi}} + d(\boldsymbol{u}; c, z) = (q^I c^I, z), \quad \forall z \in W, \tag{38}$$

$$(\boldsymbol{u}(0), \boldsymbol{v})_f = (\boldsymbol{u}_0, \boldsymbol{v})_f, \quad \forall\boldsymbol{v} \in H_f, \tag{39}$$

$$(\varphi(0), \psi)_p = (\varphi_0, \psi)_p, \quad \forall \varphi \in H_p, \tag{40}$$

$$(c(0), z)_{\overline{\phi}} = (c_0, z)_{\overline{\phi}}, \quad z \in W, \tag{41}$$

where

$$a_f(c; \boldsymbol{u}, \boldsymbol{v}) = (2\mu(c)\boldsymbol{S}(\boldsymbol{u}), \boldsymbol{S}(\boldsymbol{v}))_f + \sum_{j=1}^{N-1} \langle \gamma_j \mu(c) \boldsymbol{u} \cdot \boldsymbol{\tau}_j, \boldsymbol{v} \cdot \boldsymbol{\tau}_j \rangle_{\Gamma_I}, \tag{42}$$

$$a_p(c; \varphi, \psi) = g(\lambda(c)\nabla\varphi, \nabla\psi)_p, \tag{43}$$

$$a_\Gamma(\boldsymbol{v}, \varphi) = g\langle \varphi, \boldsymbol{v} \cdot \boldsymbol{n}_f \rangle_{\Gamma_I}, \tag{44}$$

$$b(\boldsymbol{v}, p) = -(p, \nabla \cdot \boldsymbol{v})_f, \tag{45}$$

$$(c, z)_{\overline{\phi}} = (\overline{\phi} c, z)_\Omega, \tag{46}$$

$$d(\boldsymbol{u}; c, z) = (\overline{\boldsymbol{D}}(\boldsymbol{u})\nabla c, \nabla z)_\Omega + (\boldsymbol{u} \cdot \nabla c, z) + (q^I c, z)_p, \tag{47}$$

$$\overline{\phi} = \begin{cases} 1, & in \quad \Omega_f, \\ \phi, & in \quad \Omega_p. \end{cases}$$

The fluid velocity $\boldsymbol{u}_p$ in (38) is identified by the Darcy's law (4). Interface conditions (8)−(10) and (12) are posed weakly in the above variational form, while (11) is treated as an essential condition. Due to (20), $(c, z)_{\overline{\phi}} = (\overline{\phi} c, z)_\Omega$ is an equivalent scalar product on $L^2(\Omega)$ and $\|c\|_{\overline{\phi}} = (c, c)_{\overline{\phi}}^{1/2}$ defines an equivalent norm on $L^2(\Omega)$, we have

$$\phi_{\min}\|c\|_{L^2(\Omega)} \leq \|c\|_{\overline{\phi}} \leq \phi_{\max}\|c\|_{L^2(\Omega)}. \tag{48}$$

The existence of the weak solution of Equations (36)-(41) can be found in [8].

3. Finite element discretization

In this section, we construct the finite element discretization of the coupled problem and propose a coupled and a decoupled scheme with different time step length. The stability of the decoupling scheme with different time step sizes will also be proved in this section.

Let $\mathscr{T}_h$ be a family of triangulations of Ω with nondegenerate triangular (N=2) or tetrahedron (N=3) elements. Assume that each element $T \in \mathscr{T}_h$ is in either Ω_f or Ω_p. For any $T \in \mathscr{T}_h$, denote by h_T the diameter of T. Let $h = \max_{T \in \mathscr{T}_h} h_T$ and denote by ρ_T the diameter of the sphere inscribed in $T \in \mathscr{T}_h$. We assume that the triangulation is regular [11]: there exists a constant σ independent of h such that

$$\sigma_T = \frac{h_T}{\rho_T} \leq \sigma, \quad \forall T \in \mathscr{T}_h.$$

We select finite element subspaces $H_{f,h} \subset H_f$, $H_{p,h} \subset H_p$, $Q_h \subset Q$, $W_h \subset W$ and use continuous piecewise polynomials of degrees $j+1,\ j+1,\ j,\ j+1,\ j = 1, 2, \ldots$ for the spaces $H_{f,h}$, $H_{p,h}$, Q_h and W_h. The finite element spaces $H_{f,h}$ and Q_h approximating velocity and pressure, respectively, in Ω_f are assumed to satisfy the well-known discrete inf-sup condition: there exists a positive constant β, independent of h, such that $\forall q_h \in Q_h$, $\exists \boldsymbol{v}_h \in H_{f,h}$, $\boldsymbol{v}_h \neq 0$,

$$b(\boldsymbol{v}_h, q_h) \geq \beta \|\boldsymbol{v}_h\|_{H_f} \|q_h\|_p. \tag{49}$$

We also assume that the inverse properties hold:

$$\|\boldsymbol{v}_h\|_{L^\infty(\Omega_f)} \leq C_I h^{-1} \|\boldsymbol{v}_h\|_{L^2(\Omega_f)}, \forall \boldsymbol{v}_h \in H_{f,h}, \tag{50}$$

$$\|\psi_h\|_{L^\infty(\Omega_p)} \leq \tilde{C}_I h^{-1} \|\psi_h\|_{L^2(\Omega_p)}, \forall \psi_h \in H_{p,h}. \tag{51}$$

Following [28], we define a projection operator P_h from $(\boldsymbol{u}(t), \varphi(t), p(t)) \in (H_f, H_p, Q)$ to
$(P_h^{\boldsymbol{u}}\boldsymbol{u}(t), P_h^{\varphi}\varphi(t), P_h^{p}p(t)) \in (H_{f,h}, H_{p,h}, Q_h), \forall t \in J$, such that

$$\begin{aligned}
&a_f(c; \boldsymbol{u}(t), \boldsymbol{v}_h) + a_p(c; \varphi(t), \psi_h) + a_\Gamma(\boldsymbol{v}_h, \varphi(t)) - a_\Gamma(\boldsymbol{u}(t), \psi_h) + b(\boldsymbol{v}_h, p(t)) \\
&= a_f(c; P_h^{\boldsymbol{u}}\boldsymbol{u}(t), \boldsymbol{v}_h) + a_p(c; P_h^{\varphi}\varphi(t), \psi_h) + a_\Gamma(\boldsymbol{v}_h, P_h^{\varphi}\varphi(t)) - a_\Gamma(P_h^{\boldsymbol{u}}\boldsymbol{u}(t), \psi_h) \\
&\quad + b(\boldsymbol{v}_h, P_h^{p}p(t)), \ \forall \boldsymbol{v}_h \in H_{f,h}, \psi_h \in H_{p,h}, \qquad (52)
\end{aligned}$$

$$b(P_h^{\boldsymbol{u}}\boldsymbol{u}(t), q_h) = 0, \ \forall q_h \in Q_h. \qquad (53)$$

Assume that the exact solution of Stokes-Darcy-transport problem (1)-(19) satisfy $(\boldsymbol{u}(t), \varphi(t)) \in (H^{j+1}(\Omega_f)^N, H^{j+1}(\Omega_p))$ and $p(t) \in H^j(\Omega_f)$. Then we have the following approximation properties for the projection operator P_h:

$$\|P_h^{\boldsymbol{u}}\boldsymbol{u}(t) - \boldsymbol{u}(t)\|_f + h\|P_h^{\boldsymbol{u}}\boldsymbol{u}(t) - \boldsymbol{u}(t)\|_{H_f} \le Ch^{j+1}\|\boldsymbol{u}(t)\|_{H^{j+1}(\Omega_f)}, \qquad (54)$$

$$\|P_h^{\varphi}\varphi(t) - \varphi(t)\|_p + h\|P_h^{\varphi}\varphi(t) - \varphi(t)\|_{H_p} \le Ch^{j+1}\|\varphi(t)\|_{H^{j+1}(\Omega_p)}, \qquad (55)$$

$$\|P_h^{p}p(t) - p(t)\|_f \le Ch^{j+1}\|p(t)\|_{H^j(\Omega_f)}, \qquad (56)$$

where the constants C are independent of h.

Using the inversive assumption (50) and (51), we have that [19, 30, 35], for sufficiently small h,

$$\|P_h^{\boldsymbol{u}}\boldsymbol{u}(t)\|_{L^\infty(\Omega_f)} \le C_{\boldsymbol{u}}, \qquad (57)$$

$$\|P_h^{\varphi}\varphi(t)\|_{L^\infty(\Omega_p)} \le C_{\boldsymbol{\varphi}}, \qquad (58)$$

where the constants $C_{\boldsymbol{u}}$ and $C_{\boldsymbol{\varphi}}$ are independent of h.

Next, we define the elliptic projection operator $\Xi_h : W \to W_h$,[42] (see also [13, 14, 30]) such that for $t \in J$

$$\begin{aligned}
&(\overline{\boldsymbol{D}}(\boldsymbol{u})\nabla(c(t) - \Xi_h c(t)), \nabla z) + (\boldsymbol{u} \cdot \nabla(c(t) - \Xi_h c(t)), z) \\
&+ ((\xi + q^I)(c(t) - \Xi_h c(t)), z)_p = 0, \quad \forall z \in W_h. \qquad (59)
\end{aligned}$$

Here ξ is a fixed positive constant to ensure the uniqueness of the projection solution. From the theory of finite element methods for elliptic problems, when c is sufficiently smooth $c(t) \in H^{j+1}(\Omega)$ there hold

$$\|c(t) - \Xi_h c(t)\|_{L^2(\Omega)} + h\|\nabla(c(t) - \Xi_h c(t))\|_{L^2(\Omega)} \le Ch^{j+1}\|c(t)\|_{H^{j+1}(\Omega)}, \qquad (60)$$

$$\|\partial_t(c(t) - \Xi_h c(t))\|_{L^2(\Omega)} \le Ch^{j+1}\left(\|c(t)\|_{H^{j+1}(\Omega)} + \|\partial_t c(t)\|_{H^{j+1}(\Omega)}\right), \qquad (61)$$

$$\|\Xi_h c(t)\|_{W^{1,\infty}(\Omega)} \le C_c, \qquad (62)$$

where the constants C and C_c are independent of h.

The main purpose of this paper is the use different time step sizes in different subdomains in temporal discretization. But first we consider a uniform partition on $J = [0, T]$ with $t_n = n\Delta t, m = 0, 1, 2, \ldots, N$ for $\Delta t = \dfrac{T}{N}$. The standard finite element approximation to the solution of Stokes-Darcy-transport problem (1)-(19) is to find $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in (H_{f,h}, Q_h)$, $\varphi_h^{n+1} \in H_{p,h}$ and $c_h^{n+1} \in W_h$, such that

$$\boldsymbol{u}_h^0 = P_h^{\boldsymbol{u}}\boldsymbol{u}_0, \ \varphi_h^0 = P_h^{\varphi}\varphi_0, \ c_h^0 = \Xi_h c_0,$$

and for $n = 0, 1, \cdots, N-1$,

$$\begin{aligned}
&(d_t\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + gS_0(d_t\varphi_h^{n+1}, \psi_h) + a_f(c_h^n; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + +a_p(c_h^n; \varphi_h^{n+1}, \psi_h) \\
&+ b(\boldsymbol{v}_h, p_h^{n+1}) + a_\Gamma(\boldsymbol{v}_h, \varphi_h^{n+1}) - a_\Gamma(\boldsymbol{u}_h^{n+1}, \psi_h) \\
&= (\boldsymbol{f}(c_h^n), \boldsymbol{v}_h) + g((q^I - q^P)^{n+1}, \psi_h), \quad \forall \boldsymbol{v}_h \in H_{f,h}, \ \psi_h \in H_{p,h},
\end{aligned}$$

$$b(\boldsymbol{u}_h^{n+1}, q_h) = 0, \quad \forall q_h \in Q_h,$$

$$(d_t c_h^{n+1}, z_h)_{\overline{\phi}} + d(\boldsymbol{u}_h^{n+1}; c_h^{n+1}, z_h) = (q^I c^I, z_h), \quad \forall z_h \in W_h, \tag{63}$$

where

$$d_t \boldsymbol{u}_h^n = \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\Delta t},\ d_t c_h^n = \frac{c_h^n - c_h^{n-1}}{\Delta t},\ d_t \varphi_h^n = \frac{\varphi_h^n - \varphi_h^{n-1}}{\Delta t}.$$

Next we consider the decoupling scheme with different time steps. Let

$$t_k^p = k\Delta t^p,\ k = 0,1,2,\ldots,N_p,$$
$$t_m^c = m\Delta t^c,\ m = 0,1,2,\ldots,N_c,$$
$$t_n^f = n\Delta t^f,\ n = 0,1,2,\ldots,N_f$$

be the temporal partitions for the Darcy equation, the transport equation, and the Darcy equation, respectively, where $\Delta t^p = \dfrac{T}{N_p}$ $\Delta t^c = \frac{T}{N_c}$ and , $\Delta t^f = \frac{T}{N_f}$. Assume that

$$\Delta t^p = r_{pc}\Delta t^c = r_{pc} r_{cf} \Delta t^f = r_{pf}\Delta t^f,$$

where r_{pc}, r_{cf}, and r_{pf} are positive integers. Obviously r_{pc}, r_{cf} and r_{pf} are the ratios of the different time steps and $N_f = r_{cf} N_c = r_{pf} N_p = r_{cf} r_{pc} N_p$. Denote $n_{m_k} = m_k r_{cf} = k r_{pc} r_{cf} = k r_{pf}$. The relationships of different time steps are depicted in Figure 2. Let

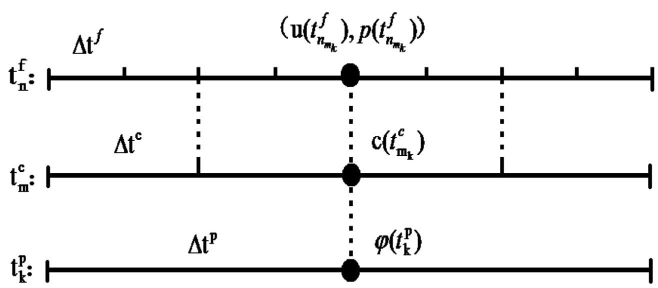


FIGURE 2. Relationship of the time steps.

$$d_{t^f} \boldsymbol{u}_h^n = \frac{(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1})}{\Delta t^f},\ d_{t^c} c_h^{n_m} = \frac{(c_h^{n_m} - c_h^{n_{m-1}})}{\Delta t^c},\ d_{t^p} \varphi_h^{n_{m_k}} = \frac{(\varphi_h^{n_{m_k}} - \varphi_h^{n_{m_{k-1}}})}{\Delta t^p}.$$

The numerical algorithm of the decoupled scheme (see [28]) with different time steps is outline as follows

$Algorithm:\ Take\ \boldsymbol{u}_h^0 = P_h^{\boldsymbol{u}} \boldsymbol{u}_0,\ \varphi_h^{n_{m_0}} = P_h^{\varphi}\varphi_0,\ c_h^{n_0} = \Xi_h c_0$

$for\ k = 0 : N_p - 1$

$for\ m = m_k : m_{k+1} - 1$

$for\ n = n_m : n_{m+1} - 1$

$Find\ (\boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in (H_{f,h}, Q_h),\ such\ that$

$$\begin{aligned}&(d_{t^f} \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + a_f(\overline{c}_h^{n_m}; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h^{n+1})\\ &= (\boldsymbol{f}(\overline{c}_h^{n_m}), \boldsymbol{v}_h) - a_\Gamma(\boldsymbol{v}_h, \varphi_h^{n_{m_k}}),\ \forall \boldsymbol{v}_h \in H_{f,h},\end{aligned} \tag{64}$$

$$b(\boldsymbol{u}_h^{n+1}, q_h) = 0, \ \forall q_h \in Q_h \tag{65}$$

end

$$Take\ S_h^{n_m} = \frac{1}{r_{cf}} \sum_{i=n_m}^{n_{m+1}-1} \boldsymbol{u}_h^i$$

$$Find\ c_h^{n_{m+1}} \in W_h,\ such\ that$$

$$(d_{t^c} c_h^{n_{m+1}}, z_h)_{\overline{\phi}} + d(\boldsymbol{U}_h^{n_m}; c_h^{n_{m+1}}, z_h) = (q^I c^I, z_h), \ \forall z_h \in W_h \tag{66}$$

end

$$Take\ T_h^{n_{m_k}} = \frac{1}{r_{pf}} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \boldsymbol{u}_h^i, \ R_h^{n_{m_k}} = \frac{1}{r_{pc}} \sum_{i=m_k}^{m_{k+1}-1} c_h^{n_i}$$

$$Find\ \varphi_h^{n_{m_{k+1}}} \in H_{p,h},\ such\ that$$

$$\begin{aligned} &gS_0(d_{t^p} \varphi_h^{n_{m_{k+1}}}, \psi_h) + a_p(\overline{R}_h^{n_{m_k}}; \varphi_h^{n_{m_{k+1}}}, \psi_h) \\ &= g((q^I - q^P)^{n_{m_{k+1}}}, \psi_h) + a_\Gamma(T_h^{n_{m_k}}, \psi_h), \ \forall \psi_h \in H_{p,h} \end{aligned} \tag{67}$$

end

Where

$$\overline{c}_h^{n_m} = \min\{1, \max\{0, c_h^{n_m}\}\} \in [0,1], \quad \overline{R}_h^{n_{m_k}} = \min\{1, \max\{0, R_h^{n_{m_k}}\}\} \in [0,1],$$

$$\boldsymbol{U}_h^{n_m} = \begin{cases} S_h^{n_m}, & \boldsymbol{x} \in \Omega_f, \\ u_{p,h}^{n_m}, & \boldsymbol{x} \in \Omega_p, \end{cases} \quad u_{p,h}^{n_m} = u_p(c_h^{n_m}, \varphi_h^{n_{m_k}}).$$

Since $c \in [0,1]$, it is easy to see that

$$|\overline{c}_h^m - c^m| \le |c_h^m - c^m|. \tag{68}$$

In the rest of this section, we study the stability property of the above algorithm. For this purpose we need the coercivity properties of the bilinear forms a_f and a_p. Define the discrete divergence free finite dimensional subspace $V_{f,h} \subset H_{f,h}$ as follows.

$$V_{f,h} = \{\boldsymbol{v}_h \in H_{f,h} : b(\boldsymbol{v}_h, q_h) = 0, \quad \forall q_h \in Q_h\}.$$

Lemma 3.1. *$a_f(\cdot;\cdot,\cdot)$ and $a_p(\cdot;\cdot,\cdot)$ are coercive on $V_{f,h}$ and $H_{p,h}$, respectively: there are positive constants α_f and α_p such that*

$$a_f(c_h; \boldsymbol{v}_h, \boldsymbol{v}_h) \ge \alpha_f \|\boldsymbol{v}_h\|_{H_f}^2, \quad \forall \boldsymbol{v}_h \in V_{f,h}, \tag{69}$$

$$a_p(c_h; \psi_h, \psi_h) \ge \alpha_p \|K^{1/2} \nabla \psi\|_p^2, \quad \forall \psi_h \in H_{p,h}, \tag{70}$$

where $\alpha_f \propto \frac{\mu_{\min}}{k_{\max}}$, $\alpha_p = g/\mu_{\max}$, α_f and α_p are independent of h, Δt^f, Δt^c and Δt^p.

Proof. From (23), we have

$$(2\mu(c_h)\boldsymbol{S}(\boldsymbol{v}_h), \boldsymbol{S}(\boldsymbol{v}_h))_f \ge 2\mu_*(\boldsymbol{S}(\boldsymbol{v}_h), \boldsymbol{S}(\boldsymbol{v}_h))_f.$$

By the Korn's inequality for piecewise H^1 vector field [3], we have

$$(2\mu(c_h)\boldsymbol{S}(\boldsymbol{v}_h), \boldsymbol{S}(\boldsymbol{v}_h))_f + \sum_{j=1}^{N-1} \langle \gamma_j \mu(c_h) \boldsymbol{v}_h \cdot \boldsymbol{\tau}_j, \boldsymbol{v}_h \cdot \boldsymbol{\tau}_j \rangle_{\Gamma_I} \ge \alpha_f \|\boldsymbol{v}_h\|_{H_f}^2.$$

Since $\gamma_j = \alpha_1/\sqrt{k_j}$, $\alpha_f \propto \frac{\mu_{\min}}{k_{\max}}$.

From the definition of $a_p(\cdot;\cdot,\cdot)$ (see (43)), we have

$$g(\lambda(c_h)\nabla\psi_h,\psi_h)_p \geq \frac{g}{\mu_{\max}}\|K^{1/2}\nabla\psi_h\|_p^2.$$

□

For the bilinear form a_Γ defined on the interface, we have the following estimates.

Lemma 3.2. *There exist constants $C_1 = C_T^2\tilde{C}_T^2$, and $C_2 = C_P\tilde{C}_P$, such that $\forall\varepsilon > 0$,*

$$\begin{aligned}|a_\Gamma(\boldsymbol{v},\varphi)| \leq{}& \frac{\varepsilon}{2}(\frac{g}{\mu_{\max}}\|K^{1/2}\nabla\varphi\|_p^2 + \alpha_f\|\boldsymbol{v}\|_{H_f}^2)\\ &+ \frac{C_1 g\sqrt{\mu_{\max}}}{8\varepsilon\sqrt{\alpha_f k_{\min} S_0}}(gS_0\|\varphi\|_p^2 + \|\boldsymbol{v}\|_f^2), \quad \forall\boldsymbol{v}\in H_f,\varphi\in H_p,\end{aligned} \tag{71}$$

$$|a_\Gamma(\boldsymbol{v},\varphi)| \leq \varepsilon\alpha_f\|\boldsymbol{v}\|_{H_f}^2 + \frac{C_1C_2 g}{4\varepsilon k_{\min}\alpha_f}g\|K^{1/2}\nabla\varphi\|_p^2, \quad \forall\boldsymbol{v}\in H_f,\varphi\in H_p. \tag{72}$$

Proof. Using Hölder inequality, trace inequalities (34), (35) and Young inequality, we have

$$\begin{aligned}|a_\Gamma(\boldsymbol{v},\varphi)| ={}& g|\langle\varphi,\boldsymbol{v}\cdot\boldsymbol{n}_f\rangle_{\Gamma_I}| \leq g\|\varphi\|_{L^2(\Gamma_I)}\|\boldsymbol{v}\cdot\boldsymbol{n}_f\|_{L^2(\Gamma_I)}\\ \leq{}& gC_T\tilde{C}_T\|\varphi\|_p^{\frac12}\|\varphi\|_{H_p}^{\frac12}\|\boldsymbol{v}\|_f^{\frac12}\|\boldsymbol{v}\|_{H_f}^{\frac12}\\ \leq{}& gC_T\tilde{C}_T(k_{\min})^{-1/4}\|\varphi\|_p^{\frac12}\|\nabla\varphi\|_p^{\frac12}\|\boldsymbol{v}\|_f^{\frac12}\|\boldsymbol{v}\|_{H_f}^{\frac12}\\ \leq{}& \varepsilon\sqrt{\frac{g}{\mu_{\max}}}\sqrt{\alpha_f}\|K^{1/2}\nabla\varphi\|_p\|\boldsymbol{v}\|_{H_f} + \frac{gC_T^2\tilde{C}_T^2\sqrt{\mu_{\max}}}{4\varepsilon\sqrt{\alpha_f k_{\min}S_0}}\sqrt{gS_0}\|\varphi\|_p\|\boldsymbol{v}\|_f\\ \leq{}& \frac{\varepsilon}{2}(\frac{g}{\mu_{\max}}\|K^{1/2}\nabla\varphi\|_p^2 + \alpha_f\|\boldsymbol{v}\|_{H_f}^2) + \frac{C_T^2\tilde{C}_T^2 g\sqrt{\mu_{\max}}}{8\varepsilon\sqrt{\alpha_f k_{\min}S_0}}(gS_0\|\varphi\|_p^2 + \|\boldsymbol{v}\|_f^2),\end{aligned}$$

take $C_1 = C_T^2\tilde{C}_T^2$, we get (71).

Using Hölder inequality, trace inequalities (34), (35), Poincaré inequalities (32), (33) and Young inequality, we have

$$\begin{aligned}|a_\Gamma(\boldsymbol{v},\varphi)| ={}& g|\langle\varphi,\boldsymbol{v}\cdot\boldsymbol{n}_f\rangle_{\Gamma_I}| \leq g\|\varphi\|_{L^2(\Gamma_I)}\|\boldsymbol{v}\cdot\boldsymbol{n}_f\|_{L^2(\Gamma_I)}\\ \leq{}& gC_T\tilde{C}_T\|\varphi\|_p^{\frac12}\|\varphi\|_{H_p}^{\frac12}\|\boldsymbol{v}\|_f^{\frac12}\|\boldsymbol{v}\|_{H_f}^{\frac12}\\ \leq{}& gC_T\tilde{C}_TC_P^{1/2}\tilde{C}_P^{1/2}\|\varphi\|_{H_p}\|\boldsymbol{v}\|_{H_f}\\ \leq{}& gC_T\tilde{C}_TC_P^{1/2}\tilde{C}_P^{1/2}(k_{\min})^{-1/2}\|K^{1/2}\nabla\varphi\|_p\|\boldsymbol{v}\|_{H_f}\\ \leq{}& \varepsilon\alpha_f\|\boldsymbol{v}\|_{H_f}^2 + \frac{C_T^2\tilde{C}_T^2C_P\tilde{C}_P g}{4\varepsilon k_{\min}\alpha_f}g\|K^{1/2}\nabla\varphi\|_p^2,\end{aligned}$$

which leads to (72) with $C_1 = C_T^2\tilde{C}_T^2$ and $C_2 = C_P\tilde{C}_P$. □

We are now ready to prove the stability for the proposed numerical algorithm.

Theorem 3.3. *Assume that Δt^f satisfies*

$$K\Delta t^f \leq 1, \ \textit{with}\ K := \frac{4C_1 g\sqrt{\mu_{\max}}}{\sqrt{\alpha_f k_{\min}S_0}}. \tag{73}$$

Then for all $0 \leq l \leq N_p - 2$, $0 \leq s \leq r_{pf} - 1$,

$$\|\boldsymbol{u}_h^{n_{m_{l+1}}+s+1}\|_f^2 + \alpha_f\Delta t^f\sum_{i=0}^{n_{m_{l+1}}+s}\|\boldsymbol{u}_h^{i+1}\|_{H_f}^2$$

$$
\begin{aligned}
&+ gS_0\|\varphi_h^{n_{m_{l+1}}}\|_p^2 + \frac{g}{\mu_{\max}}\Delta t^p \sum_{k=0}^{l} \|K^{1/2}\nabla\varphi_h^{n_{m_{k+1}}}\|_p^2 \\
\le& C(T)\Big(\frac{2C_P^2\Delta t^f}{\alpha_f} \sum_{m=0}^{m_{l+1}+[s/r_{cf}]} \|\boldsymbol{f}(\bar{c}_h^{n_m})\|_f^2 + \frac{2g\tilde{C}_P^2\mu_{\max}\Delta t^p}{k_{\min}} \sum_{k=0}^{l} \|(q^I - q^P)^{n_{m_{k+1}}}\|_p^2 \\
&+ \|\boldsymbol{u}_0\|_f^2 + gS_0\|\varphi_0\|_p^2 + \frac{\alpha_f\Delta t^f}{4}\|\boldsymbol{u}_0\|_{H_f}^2 + \frac{g\Delta t^f}{4\mu_{\max}}\|K^{1/2}\nabla\varphi_0\|_p^2\Big)
\end{aligned} \tag{74}
$$

with $C(T) = \exp(\Delta t^f \sum_{i=0}^{n_{m_{l+1}}+s+1} \frac{K}{1-K\Delta t^f})$. *Here* [ ] *denote the integral part of a real number.*

In particular, choosing $l = 0$ *in the above, we have the stability for the fluid velocity* $\boldsymbol{u}_h$ *on* $[0, t_1^p]$*: for any* $0 \le s \le r_{pf} - 2$,

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{s+1}\|_f^2 + \alpha_f\Delta t^f \sum_{i=0}^{s} \|\boldsymbol{u}_h^{i+1}\|_{H_f}^2 \\
\le& C(t_1^p)\Big(\frac{4C_P^2\Delta t^f}{3\alpha_f} \sum_{m=0}^{[s/r_{cf}]} \|\boldsymbol{f}(\bar{c}_h^{n_m})\|_f^2 + \|\boldsymbol{u}_0\|_f^2 + gS_0\|\varphi_0\|_p^2 + \frac{g\Delta t^f}{4\mu_{\max}}\|K^{1/2}\nabla\varphi_0\|_p^2\Big)
\end{aligned} \tag{75}
$$

with $C(t_1^p) \approx \exp(\Delta t^f \sum_{i=0}^{s+1} \frac{K}{1-K\Delta t^f})$.

Proof. Taking $\boldsymbol{v}_h = 2\Delta t^f \boldsymbol{u}_h^{n+1}$ in (64) and (65), we have

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{n+1}\|_f^2 + \|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_f^2 - \|\boldsymbol{u}_h^n\|_f^2 + 2\Delta t^f a_f(\bar{c}_h^{n_m}; \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}) \\
&= 2\Delta t^f(\boldsymbol{f}(\bar{c}_h^{n_m}), \boldsymbol{u}_h^{n+1}) - 2\Delta t^f a_\Gamma(\boldsymbol{u}_h^{n+1}.\varphi_h^{n_{m_k}}),
\end{aligned} \tag{76}
$$

Summing up the above with $n = n_m, n_m + 1, \ldots, n_{m+1} - 1$ and $m = m_k, m_k + 1, \ldots, m_{k+1} - 1$ gives

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{n_{m_{k+1}}}\|_f^2 + \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_f^2 - \|\boldsymbol{u}_h^{n_{m_k}}\|_f^2 \\
&+ 2\Delta t^f \sum_{m=m_k}^{m_{k+1}-1} \sum_{i=n_m}^{n_{m+1}-1} a_f(\bar{c}_h^{n_m}; \boldsymbol{u}_h^{i+1}, \boldsymbol{u}_h^{i+1}) \\
=& 2\Delta t^f \sum_{m=m_k}^{m_{k+1}-1} (\boldsymbol{f}(\bar{c}_h^{n_m}), \sum_{i=n_m}^{n_{m+1}-1} \boldsymbol{u}_h^{i+1}) - 2\Delta t^f a_\Gamma(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \boldsymbol{u}_h^{i+1}, \varphi_h^{n_{m_k}}).
\end{aligned} \tag{77}
$$

Taking $\psi_h = 2\Delta t^p \varphi_h^{n_{m_{k+1}}}$ in (67), from the definition of $T_h^{n_{m_k}}$, we have

$$
\begin{aligned}
&gS_0(\|\varphi_h^{n_{m_{k+1}}}\|_p^2 + \|\varphi_h^{n_{m_{k+1}}} - \varphi_h^{n_{m_k}}\|_p^2 - \|\varphi_h^{n_{m_k}}\|_p^2) \\
&+ 2\Delta t^p a_p(\overline{R}_h^{n_{m_k}}; \varphi_h^{n_{m_{k+1}}}, \varphi_h^{n_{m_{k+1}}}) \\
=& 2\Delta t^p g((q^I - q^P)^{n_{m_{k+1}}}, \varphi_h^{n_{m_{k+1}}}) + \frac{2\Delta t^p}{r_{pf}} a_\Gamma(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \boldsymbol{u}_h^i, \varphi_h^{n_{m_{k+1}}}).
\end{aligned} \tag{78}
$$

Combining (77) and (78), we obtain

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{n_{m_{k+1}}}\|_f^2 + \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_f^2 - \|\boldsymbol{u}_h^{n_{m_k}}\|_f^2 \\
&+ 2\Delta t^f \sum_{m=m_k}^{m_{k+1}-1} \sum_{i=n_m}^{n_{m+1}-1} a_f(\overline{c}_h^{n_m}; \boldsymbol{u}_h^{i+1}, \boldsymbol{u}_h^{i+1}) \\
&+ gS_0(\|\varphi_h^{n_{m_{k+1}}}\|_p^2 + \|\varphi_h^{n_{m_{k+1}}} - \varphi_h^{n_{m_k}}\|_p^2 - \|\varphi_h^{n_{m_k}}\|_p^2) \\
&+ 2\Delta t^p a_p(\overline{R}_h^{n_{m_k}}; \varphi_h^{n_{m_{k+1}}}, \varphi_h^{n_{m_{k+1}}}) \\
=&2\Delta t^f \sum_{m=m_k}^{m_{k+1}-1} (\boldsymbol{f}(\overline{c}_h^{n_m}), \sum_{i=n_m}^{n_{m+1}-1} \boldsymbol{u}_h^{i+1}) + 2\Delta t^p g((q^I - q^P)^{n_{m_{k+1}}}, \varphi_h^{n_{m_{k+1}}}) \\
&+ 2\Delta t^f a_\Gamma(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \boldsymbol{u}_h^i, \varphi_h^{n_{m_{k+1}}}) - 2\Delta t^f a_\Gamma(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \boldsymbol{u}_h^{i+1}, \varphi_h^{n_{m_k}}).
\end{aligned} \tag{79}
$$

Following (69) and (70), we have

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{n_{m_{k+1}}}\|_f^2 + \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_f^2 - \|\boldsymbol{u}_h^{n_{m_k}}\|_f^2 + 2\alpha_f \Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|\boldsymbol{u}_h^{i+1}\|_{H_f}^2 \\
&+ gS_0(\|\varphi_h^{n_{m_{k+1}}}\|_p^2 + \|\varphi_h^{n_{m_{k+1}}} - \varphi_h^{n_{m_k}}\|_p^2 - \|\varphi_h^{n_{m_k}}\|_p^2) + \frac{2g}{\mu_{\max}} \Delta t^p \|K^{1/2} \nabla \varphi_h^{n_{m_{k+1}}}\|_p^2 \\
\le&2\Delta t^f \sum_{m=m_k}^{m_{k+1}-1} (\boldsymbol{f}(\overline{c}_h^{n_m}), \sum_{i=n_m}^{n_{m+1}-1} \boldsymbol{u}_h^{i+1}) + 2\Delta t^p g((q^I - q^P)^{n_{m_{k+1}}}, \varphi_h^{n_{m_{k+1}}}) \\
&+ 2\Delta t^f a_\Gamma(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \boldsymbol{u}_h^i, \varphi_h^{n_{m_{k+1}}}) - 2\Delta t^f a_\Gamma(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \boldsymbol{u}_h^{i+1}, \varphi_h^{n_{m_k}}).
\end{aligned} \tag{80}
$$

For the first two terms on the right hand side of the above, we use the Hölder inequality and Poincaré inequalities together with (71), (72) and Young inequality to obtain

$$
\begin{aligned}
&2\Delta t^f \sum_{m=m_k}^{m_{k+1}-1} (\boldsymbol{f}(\overline{c}_h^{n_m}), \sum_{i=n_m}^{n_{m+1}-1} \boldsymbol{u}_h^{i+1}) + 2\Delta t^p g((q^I - q^P)^{n_{m_{k+1}}}, \varphi_h^{n_{m_{k+1}}}) \\
\le&2C_P \Delta t^f \sum_{m=m_k}^{m_{k+1}-1} \|\boldsymbol{f}(\overline{c}_h^{n_m})\|_f \|\sum_{i=n_m}^{n_{m+1}-1} \boldsymbol{u}_h^{i+1}\|_{H_f} \\
&+ \frac{2g\tilde{C}_P \Delta t^p}{\sqrt{k_{\min}}} \|(q^I - q^P)^{n_{m_{k+1}}}\|_p \|K^{1/2} \nabla \varphi_h^{n_{m_{k+1}}}\|_p \\
\le&\frac{2C_P^2 \Delta t^f}{\alpha_f} \sum_{m=m_k}^{m_{k+1}-1} \|\boldsymbol{f}(\overline{c}_h^{n_m})\|_f^2 + \frac{2g\tilde{C}_P^2 \mu_{\max} \Delta t^p}{k_{\min}} \|(q^I - q^P)^{n_{m_{k+1}}}\|_p^2 \\
&+ \frac{\alpha_f \Delta t^f}{2} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|\boldsymbol{u}_h^{i+1}\|_{H_f}^2 + \frac{g\Delta t^p}{2\mu_{\max}} \|K^{1/2} \nabla \varphi_h^{n_{m_{k+1}}}\|_p^2.
\end{aligned} \tag{81}
$$

For the last two terms of right-hand side in (80) we choose $\varepsilon = 1/8$ in (71) to obtain

$$
\begin{aligned}
&2\Delta t^f\Big[a_\Gamma(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\boldsymbol{u}_h^i,\varphi_h^{n_{m_{k+1}}}) - a_\Gamma(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\boldsymbol{u}_h^{i+1},\varphi_h^{n_{m_k}})\Big]\\
\leq&2\Delta t^f\Big[\frac{1}{16}(\frac{g}{\mu_{\max}}\|K^{1/2}\nabla\varphi_h^{n_{m_{k+1}}}\|_p^2 + \alpha_f\|\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\boldsymbol{u}_h^i\|_{H_f}^2\\
&+\frac{g}{\mu_{\max}}\|K^{1/2}\nabla\varphi_h^{n_{m_k}}\|_p^2 + \alpha_f\|\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\boldsymbol{u}_h^{i+1}\|_{H_f}^2)\\
&+\frac{C_1 g\sqrt{\mu_{\max}}}{\sqrt{\alpha_f k_{\min} S_0}}\big(gS_0\|\varphi_h^{n_{m_{k+1}}}\|_p^2 + \|\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\boldsymbol{u}_h^i\|_f^2\\
&+gS_0\|\varphi_h^{n_{m_k}}\|_p^2 + \|\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\boldsymbol{u}_h^{i+1}\|_f^2\big)\Big]\\
\leq&\frac{\Delta t^f}{4}\Big(\alpha_f\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|\boldsymbol{u}_h^{i+1}\|_{H_f}^2 + \alpha_f\|\boldsymbol{u}_h^{n_{m_k}}\|_{H_f}^2\\
&+\frac{g}{\mu_{\max}}\|K^{1/2}\nabla\varphi_h^{n_{m_{k+1}}}\|_p^2 + \frac{g}{\mu_{\max}}\|K^{1/2}\nabla\varphi_h^{n_{m_k}}\|_p^2\Big)\\
&+\frac{4C_1 g\sqrt{\mu_{\max}}\Delta t^f}{\sqrt{\alpha_f k_{\min} S_0}}\Big(\sum_{i=n_{m_k}}^{n_{m_{k+1}}}\|\boldsymbol{u}_h^i\|_f^2 + gS_0\|\varphi_h^{n_{m_{k+1}}}\|_p^2 + gS_0\|\varphi_h^{n_{m_k}}\|_p^2\Big).
\end{aligned}
\tag{82}
$$

It then follows from (80) that

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{n_{m_{k+1}}}\|_f^2 + \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^i\|_f^2 - \|\boldsymbol{u}_h^{n_{m_k}}\|_f^2 + \frac{5}{4}\alpha_f\Delta t^f\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|\boldsymbol{u}_h^{i+1}\|_{H_f}^2\\
&-\frac{1}{4}\alpha_f\Delta t^f\|\boldsymbol{u}_h^{n_{m_k}}\|_{H_f}^2 + gS_0(\|\varphi_h^{n_{m_{k+1}}}\|_p^2 + \|\varphi_h^{n_{m_{k+1}}}-\varphi_h^{n_{m_k}}\|_p^2 - \|\varphi_h^{n_{m_k}}\|_p^2)\\
&+\frac{5}{4}\frac{g}{\mu_{\max}}\Delta t^p\|K^{1/2}\nabla\varphi_h^{n_{m_{k+1}}}\|_p^2 - \frac{1}{4}\frac{g}{\mu_{\max}}\Delta t^f\|K^{1/2}\nabla\varphi_h^{n_{m_k}}\|_p^2\\
\leq&\frac{2C_P^2\Delta t^f}{\alpha_f}\sum_{m=m_k}^{m_{k+1}-1}\|\boldsymbol{f}(\overline{c}_h^{n_m})\|_f^2 + \frac{2g\tilde{C}_P^2\mu_{\max}\Delta t^p}{k_{\min}}\|(q^I-q^P)^{n_{m_{k+1}}}\|_p^2\\
&+\frac{4C_1 g\sqrt{\mu_{\max}}\Delta t^f}{\sqrt{\alpha_f k_{\min} S_0}}\Big(\sum_{i=n_{m_k}}^{n_{m_{k+1}}}\|\boldsymbol{u}_h^i\|_f^2 + gS_0\|\varphi_h^{n_{m_{k+1}}}\|_p^2 + gS_0\|\varphi_h^{n_{m_k}}\|_p^2\Big).
\end{aligned}
\tag{83}
$$

Summing over (83) with $k = 0, 1, 2, \ldots, l$, $0 \leq l \leq N_p - 2$, and using the fact that $\Delta t^p \geq \Delta t^f$, we have

$$
\|\boldsymbol{u}_h^{n_{m_{l+1}}}\|_f^2 + \alpha_f\Delta t^f\sum_{k=0}^{l}\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|\boldsymbol{u}_h^{i+1}\|_{H_f}^2
$$

$$
\begin{aligned}
&+ gS_0\|\varphi_h^{n_{m_{l+1}}}\|_p^2 + \frac{g}{\mu_{\max}}\Delta t^p \sum_{k=0}^{l} \|K^{1/2}\nabla\varphi_h^{n_{m_{k+1}}}\|_p^2 \\
\leq & \frac{2C_P^2\Delta t^f}{\alpha_f}\sum_{k=0}^{l}\sum_{m=m_k}^{m_{k+1}-1}\|\boldsymbol{f}(\overline{c}_h^{n_m})\|_f^2 + \frac{2g\tilde{C}_P^2\mu_{\max}\Delta t^p}{k_{\min}}\sum_{k=0}^{l}\|(q^I - q^P)^{n_{m_{k+1}}}\|_p^2 \\
&+ \frac{4C_1 g\sqrt{\mu_{\max}}\Delta t^f}{\sqrt{\alpha_f k_{\min} S_0}}\sum_{k=0}^{l}\Big(\sum_{i=n_{m_k}}^{n_{m_{k+1}}}\|\boldsymbol{u}_h^i\|_f^2 + gS_0\|\varphi_h^{n_{m_{k+1}}}\|_p^2 + gS_0\|\varphi_h^{n_{m_k}}\|_p^2\Big) \\
&+ \|\boldsymbol{u}_0\|_f^2 + gS_0\|\varphi_0\|_p^2 + \frac{\alpha_f\Delta t^f}{4}\|\boldsymbol{u}_0\|_{H_f}^2 + \frac{g\Delta t^f}{4\mu_{\max}}\|K^{1/2}\nabla\varphi_0\|_p^2. \qquad (84)
\end{aligned}
$$

Next we analyse the stability on time interval $[t_k^p, t_{k+1}^p]$. Taking $\boldsymbol{v}_h = 2\Delta t^f \boldsymbol{u}_h^{n+1}$ in (64) and (65) and summing over with $n = n_{m_l}, n_{m_l}+1, \ldots, n_{m_l}+s$, $0 \leq l \leq N_p - 1$, $0 \leq s \leq r_{pf} - 2$, we have

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{n_{m_l}+s+1}\|_f^2 + \sum_{i=n_{m_l}}^{n_{m_l}+s}\|\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i\|_f^2 - \|\boldsymbol{u}_h^{n_{m_l}}\|_f^2 + 2\alpha_f\Delta t^f\sum_{i=n_{m_l}}^{n_{m_l}+s}\|\boldsymbol{u}_h^{i+1}\|_{H_f}^2 \\
\leq & \frac{4C_P^2\Delta t^f}{3\alpha_f}\sum_{m=m_l}^{m_l+[s/r_{cf}]}\|\boldsymbol{f}(\overline{c}_h^{n_m})\|_f^2 + \frac{3\alpha_f\Delta t^f}{4}\sum_{i=n_{m_l}}^{n_{m_l}+s}\|\boldsymbol{u}_h^{i+1}\|_{H_f}^2 \\
&+ \frac{C_1 g\sqrt{\mu_{\max}}\Delta t^f}{\sqrt{\alpha_f k_{\min} S_0}}\Big(\sum_{i=n_{m_l}}^{n_{m_l}+s}\|\boldsymbol{u}_h^{i+1}\|_f^2 + gS_0\|\varphi_h^{n_{m_l}}\|_p^2\Big) \\
&+ \frac{\Delta t^f}{4}\Big(\alpha_f\sum_{i=n_{m_l}}^{n_{m_l}+s}\|\boldsymbol{u}_h^{i+1}\|_{H_f}^2 + \frac{g}{\mu_{\max}}\|K^{1/2}\nabla\varphi_h^{n_{m_{l+1}}}\|_p^2\Big). \qquad (85)
\end{aligned}
$$

Combining (84) and (85) and using the condition (73) and applying the Gronwall inequality, we obtain (74) and (75). □

We see that the numerical scheme (66) is linear, since we get the stabilities of $\boldsymbol{v}_h$ and φ_h in Theorem 3.3, we can get the stability of c_h directly.

4. Error estimates

In this section, we derive the error estimates for the proposed numerical algorithm in the previous section. First we split the errors between the approximate solutions and the exact solutions as follow.

$$
\begin{aligned}
&\boldsymbol{u} - \boldsymbol{u}_h = \boldsymbol{u} - P_h^{\boldsymbol{u}}\boldsymbol{u} + P_h^{\boldsymbol{u}}\boldsymbol{u} - \boldsymbol{u}_h = \epsilon_{\boldsymbol{u}} + e_{\boldsymbol{u},h}, \\
&p - p_h = p - P_h^p p + P_h^p p - p_h = \epsilon_p + e_{p,h}, \\
&\varphi - \varphi_h = \varphi - P_h^{\varphi}\varphi + P_h^{\varphi}\varphi - \varphi_h = \epsilon_{\varphi} + e_{\varphi,h}, \\
&c - c_h = c - \Xi_h c + \Xi_h c - c_h = \epsilon_c + e_{c,h}. \qquad (86)
\end{aligned}
$$

From the definition of P_h (see (52) and (53)), we know that, for time step size Δt^f, there holds

$$
\begin{aligned}
&(d_{t^f} P_h^{\boldsymbol{u}}\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) + a_f(c^{n+1}; P_h^{\boldsymbol{u}}\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, P_h^p p^{n+1}) \\
=&(d_{t^f}\boldsymbol{u}^{n+1} - \partial_{t^f}\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) \\
&- (d_{t^f}\epsilon_{\boldsymbol{u}}^{n+1}, \boldsymbol{v}_h) + (\boldsymbol{f}(c^{n+1}), \boldsymbol{v}_h) - a_\Gamma(\boldsymbol{v}_h, P_h^{\varphi}\varphi^{n+1}),\ \forall \boldsymbol{v}_h \in H_{f,h}, \qquad (87)
\end{aligned}
$$

$$
b(P_h^{\boldsymbol{u}}\boldsymbol{u}^{n+1}, q_h) = 0,\ \forall q_h \in Q_h, \qquad (88)
$$

and for time step size Δt^p, there holds

$$\begin{aligned}&gS_0(d_{t^p}P_h^{\varphi}\varphi^{n_{m_{k+1}}},\psi_h)+a_p(c^{n_{m_{k+1}}};P_h^{\varphi}\varphi^{n_{m_{k+1}}},\psi_h)\\ =&gS_0(d_{t^p}\varphi^{n_{m_{k+1}}}-\partial_{t^p}\varphi^{n_{m_{k+1}}},\psi_h)-gS_0(d_{t^p}\epsilon_{\varphi}^{n_{m_{k+1}}},\psi_h)\\ &+g((q^I-q^P)^{n_{m_{k+1}}},\psi_h)+a_\Gamma(P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_{k+1}}},\psi_h),\ \forall\psi_h\in H_{p,h}.\end{aligned}\tag{89}$$

Subtracting (87) and (88) from (64) and (65), respectively, we have that for $\boldsymbol{v}_h\in H_{f,h}$, $q_h\in Q_h$

$$\begin{aligned}&(d_{t^f}e_{\boldsymbol{u},h}^{n+1},\boldsymbol{v}_h)+a_f(\overline{c}_h^{n_m};e_{\boldsymbol{u},h}^{n+1},\boldsymbol{v}_h)+b(\boldsymbol{v}_h,e_{p,h}^{n+1})\\ =&(d_{t^f}\boldsymbol{u}^{n+1}-\partial_{t^f}\boldsymbol{u}^{n+1},\boldsymbol{v}_h)-(d_{t^f}\epsilon_{\boldsymbol{u}}^{n+1},\boldsymbol{v}_h)\\ &-\big(a_f(c^{n+1};P_h^{\boldsymbol{u}}\boldsymbol{u}^{n+1},\boldsymbol{v}_h)-a_f(\overline{c}_h^{n_m};P_h^{\boldsymbol{u}}\boldsymbol{u}^{n+1},\boldsymbol{v}_h)\big)\\ &+\big((\boldsymbol{f}(c^{n+1}),\boldsymbol{v}_h)-(\boldsymbol{f}(\overline{c}_h^{n_m}),\boldsymbol{v}_h)\big)\\ &-\big(a_\Gamma(\boldsymbol{v}_h,P_h^{\varphi}\varphi^{n+1}-P_h^{\varphi}\varphi^{n})+a_\Gamma(\boldsymbol{v}_h,P_h^{\varphi}\varphi^{n}-\varphi_h^{n_{m_k}})\big),\end{aligned}\tag{90}$$

$$b(e_{\boldsymbol{u},h}^{n+1},q_h)=0.\tag{91}$$

Subtracting (89) from (67), we have that for $\psi_h\in H_{p,h}$

$$\begin{aligned}&gS_0(d_{t^p}e_{\varphi,h}^{n_{m_{k+1}}},\psi_h)+a_p(\overline{R}_h^{n_{m_k}};e_{\varphi,h}^{n_{m_{k+1}}},\psi_h)\\ =&gS_0(d_{t^p}\varphi^{n_{m_{k+1}}}-\partial_{t^p}\varphi^{n_{m_{k+1}}},\psi_h)-gS_0(d_{t^p}\epsilon_{\varphi}^{n_{m_{k+1}}},\psi_h)\\ &-\big(a_p(c^{n_{m_{k+1}}};P_h^{\varphi}\varphi^{n_{m_{k+1}}},\psi_h)-a_p(\overline{R}_h^{n_{m_k}};P_h^{\varphi}\varphi^{n_{m_{k+1}}},\psi_h)\big)\\ &+\big(a_\Gamma(P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_{k+1}}}-P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}},\psi_h)+a_\Gamma(P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}}-T_h^{n_{m_k}},\psi_h)\big).\end{aligned}\tag{92}$$

The next theorem is about the error estimate for the velocity and pressure.

Theorem 4.1. *Assume that*

$$\begin{aligned}&c\in L^\infty(J;H^{j+1}(\Omega))\cap H^1(J;H^{j+1}(\Omega))L^2(J;W^{1,\infty}(\Omega))\cap H^2(J;L^2(\Omega)),\\ &\boldsymbol{u}\in W^{1,\infty}(J;H^{j+1}(\Omega_f)^N)\cap H^2(J;L^2(\Omega_f)^N)\cap H^1(J;H^{j+1}(\Omega_f)^N),\\ &\varphi\in L^\infty(J;H^{j+1}(\Omega_p))\cap H^1(J;H^{j+1}(\Omega_p)).\end{aligned}\tag{93}$$

Also assume that the condition (73) holds. Then for sufficiently small h, there exists a positive constant $\hat{C}_1$ independent of h, Δt^f, Δt^c and Δt^p, and exists a positive constant $\hat{C}_2$ independent of h, Δt^f, Δt^c and Δt^p, and independent of physical parameters $\mu_{\max}$, $\mu_{\min}$, $k_{\max}$ and $k_{\min}$, such that for $0\le l\le N_p-1$

$$\begin{aligned}&\|e_{\boldsymbol{u},h}^{n_{m_{l+1}}}\|_f^2+\alpha_f\Delta t^f\sum_{k=0}^{l}\sum_{i=n_{m_k}}^{n_{m_{k+1}}}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2+gS_0\|e_{\varphi,h}^{n_{m_{l+1}}}\|_p^2\\ &+\frac{g}{\mu_{max}}\Delta t^p\sum_{k=0}^{l}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2\\ \le&C(T)\bigg(\hat{C}_1\Big(h^{2(j+1)}+\Delta t^c\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}(\|c^{n_i}-c_h^{n_i}\|^2\Big)\\ &+\hat{C}_2\Big(\frac{1}{\alpha_f}(\Delta t^f)^2+\frac{\mu_{\max}}{k_{\min}}(\Delta t^p)^2+(\frac{1}{\alpha_f}+\frac{\mu_{\max}k_{\max}}{k_{\min}\mu_{\min}})(\Delta t^c)^2\Big)\bigg).\end{aligned}\tag{94}$$

Proof. Taking $\boldsymbol{v}_h=2\Delta t^f e_{\boldsymbol{u},h}^{n+1}$ in (90) and (91), we have

$$\|e_{\boldsymbol{u},h}^{n+1}\|_f^2+\|e_{\boldsymbol{u},h}^{n+1}-e_{\boldsymbol{u},h}^{n}\|_f^2-\|e_{\boldsymbol{u},h}^{n}\|_f^2+2\Delta t^f a_f(\overline{c}_h^{n_m};e_{\boldsymbol{u},h}^{n+1},e_{\boldsymbol{u},h}^{n+1})$$

$$
\begin{aligned}
=&2\Delta t^f(d_{t^f}\boldsymbol{u}^{n+1}-\partial_{t^f}\boldsymbol{u}^{n+1},e_{\boldsymbol{u},h}^{n+1})-2\Delta t^f(d_{t^f}\epsilon_{\boldsymbol{u}}^{n+1},e_{\boldsymbol{u},h}^{n+1})\\
&-2\Delta t^f\big(a_f(c^{n+1};P_h^{\boldsymbol{u}}\boldsymbol{u}^{n+1},e_{\boldsymbol{u},h}^{n+1})-a_f(\overline{c}_h^{n_m};P_h^{\boldsymbol{u}}\boldsymbol{u}^{n+1},e_{\boldsymbol{u},h}^{n+1})\big)\\
&+2\Delta t^f\big((\boldsymbol{f}(c^{n+1}),e_{\boldsymbol{u},h}^{n+1})-(\boldsymbol{f}(\overline{c}_h^{n_m}),e_{\boldsymbol{u},h}^{n+1})\big)\\
&-2\Delta t^f\big(a_\Gamma(e_{\boldsymbol{u},h}^{n+1},P_h^{\varphi}\varphi^{n+1}-P_h^{\varphi}\varphi^{n})+a_\Gamma(e_{\boldsymbol{u},h}^{n+1},P_h^{\varphi}\varphi^{n}-\varphi_h^{n_{m_k}})\big).
\end{aligned}\tag{95}
$$

Summing over (95) with $n=n_m,n_m+1,\ldots,n_{m+1}-1$ and $m=m_k,m_k+1,\ldots,m_{k+1}-1$ we obtain

$$
\begin{aligned}
&\|e_{\boldsymbol{u},h}^{n_{m_{k+1}}}\|_f^2+\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|e_{\boldsymbol{u},h}^{i+1}-e_{\boldsymbol{u},h}^{i}\|_f^2-\|e_{\boldsymbol{u},h}^{n_{m_k}}\|_f^2\\
&+2\Delta t^f\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}a_f(\overline{c}_h^{n_m};e_{\boldsymbol{u},h}^{i+1},e_{\boldsymbol{u},h}^{i+1})\\
=&2\Delta t^f\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}(d_{t^f}\boldsymbol{u}^{i+1}-\partial_{t^f}\boldsymbol{u}^{i+1},e_{\boldsymbol{u},h}^{i+1})-2\Delta t^f\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}(d_{t^f}\epsilon_{\boldsymbol{u}}^{i+1},e_{\boldsymbol{u},h}^{i+1})\\
&-2\Delta t^f\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big(a_f(c^{i+1};P_h^{\boldsymbol{u}}\boldsymbol{u}^{i+1},e_{\boldsymbol{u},h}^{i+1})-a_f(\overline{c}_h^{n_m};P_h^{\boldsymbol{u}}\boldsymbol{u}^{i+1},e_{\boldsymbol{u},h}^{i+1})\big)\\
&+2\Delta t^f\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big((\boldsymbol{f}(c^{i+1}),e_{\boldsymbol{u},h}^{i+1})-(\boldsymbol{f}(\overline{c}_h^{n_m}),e_{\boldsymbol{u},h}^{i+1})\big)\\
&-2\Delta t^f\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\big(a_\Gamma(e_{\boldsymbol{u},h}^{i+1},P_h^{\varphi}\varphi^{i+1}-P_h^{\varphi}\varphi^{i})+a_\Gamma(e_{\boldsymbol{u},h}^{i+1},P_h^{\varphi}\varphi^{i}-\varphi_h^{n_{m_k}})\big).
\end{aligned}\tag{96}
$$

Taking $\psi_h=2\Delta t^p e_{\varphi,h}^{n_{m_{k+1}}}$ in (92), we have

$$
\begin{aligned}
&gS_0\|e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2+gS_0\|e_{\varphi,h}^{n_{m_{k+1}}}-e_{\varphi,h}^{n_{m_k}}\|_p^2\\
&-gS_0\|e_{\varphi,h}^{n_{m_k}}\|_p^2+2\Delta t^p a_p(\overline{R}_h^{n_{m_k}};e_{\varphi,h}^{n_{m_{k+1}}},e_{\varphi,h}^{n_{m_{k+1}}})\\
=&2\Delta t^p gS_0(d_{t^p}\varphi^{n_{m_{k+1}}}-\partial_{t^p}\varphi^{n_{m_{k+1}}},e_{\varphi,h}^{n_{m_{k+1}}})-2\Delta t^p gS_0(d_{t^p}\epsilon_{\varphi}^{n_{m_{k+1}}},e_{\varphi,h}^{n_{m_{k+1}}})\\
&-2\Delta t^p\big(a_p(c^{n_{m_{k+1}}};P_h^{\varphi}\varphi^{n_{m_{k+1}}},e_{\varphi,h}^{n_{m_{k+1}}})-a_p(\overline{R}_h^{n_{m_k}};P_h^{\varphi}\varphi^{n_{m_{k+1}}},e_{\varphi,h}^{n_{m_{k+1}}})\big)\\
&+2\Delta t^p\big(a_\Gamma(P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_{k+1}}}-P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}},e_{\varphi,h}^{n_{m_{k+1}}})+a_\Gamma(P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}}-T_h^{n_{m_k}},e_{\varphi,h}^{n_{m_{k+1}}})\big).
\end{aligned}\tag{97}
$$

Combining (96) and (97), using (69) and (70), and following the definition of $T_h^{n_{m_k}}$, we have that

$$
\begin{aligned}
&\|e_{\boldsymbol{u},h}^{n_{m_{k+1}}}\|_f^2-\|e_{\boldsymbol{u},h}^{n_{m_k}}\|_f^2+2\alpha_f\Delta t^f\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2\\
&+gS_0\|e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2-gS_0\|e_{\varphi,h}^{n_{m_k}}\|_p^2+\frac{2g}{\mu_{max}}\Delta t^p\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2\\
\le&2\Delta t^f\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}(d_{t^f}\boldsymbol{u}^{i+1}-\partial_{t^f}\boldsymbol{u}^{i+1},e_{\boldsymbol{u},h}^{i+1})\\
&+2\Delta t^p gS_0(d_{t^p}\varphi^{n_{m_{k+1}}}-\partial_{t^p}\varphi^{n_{m_{k+1}}},e_{\varphi,h}^{n_{m_{k+1}}})
\end{aligned}
$$

$$
\begin{aligned}
& - 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} (d_{t^f} \epsilon_{\boldsymbol{u}}^{i+1}, e_{\boldsymbol{u},h}^{i+1}) - 2\Delta t^p g S_0 (d_{t^p} \epsilon_\varphi^{n_{m_{k+1}}}, e_{\varphi,h}^{n_{m_{k+1}}}) \\
& - 2\Delta t^f \sum_{m=m_k}^{m_{k+1}-1} \sum_{i=n_m}^{n_{m+1}-1} \big(a_f(c^{i+1}; P_h^{\boldsymbol{u}} \boldsymbol{u}^{i+1}, e_{\boldsymbol{u},h}^{i+1}) - a_f(\overline{c}_h^{n_m}; P_h^{\boldsymbol{u}} \boldsymbol{u}^{i+1}, e_{\boldsymbol{u},h}^{i+1}) \big) \\
& - 2\Delta t^p \big(a_p(c^{n_{m_{k+1}}}; P_h^\varphi \varphi^{n_{m_{k+1}}}, e_{\varphi,h}^{n_{m_{k+1}}}) - a_p(\overline{R}_h^{n_{m_k}}; P_h^\varphi \varphi^{n_{m_{k+1}}}, e_{\varphi,h}^{n_{m_{k+1}}}) \big) \\
& + 2\Delta t^f \sum_{m=m_k}^{m_{k+1}-1} \sum_{i=n_m}^{n_{m+1}-1} \big((\boldsymbol{f}(c^{i+1}), e_{\boldsymbol{u},h}^{i+1}) - (\boldsymbol{f}(\overline{c}_h^{n_m}), e_{\boldsymbol{u},h}^{i+1}) \big) \\
& - 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(e_{\boldsymbol{u},h}^{i+1}, P_h^\varphi \varphi^{i+1} - P_h^\varphi \varphi^i) \\
& + 2\Delta t^p a_\Gamma(P_h^{\boldsymbol{u}} \boldsymbol{u}^{n_{m_{k+1}}} - P_h^{\boldsymbol{u}} \boldsymbol{u}^{n_{m_k}}, e_{\varphi,h}^{n_{m_{k+1}}}) \\
& - 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(e_{\boldsymbol{u},h}^{i+1}, P_h^\varphi \varphi^i - P_h^\varphi \varphi^{n_{m_k}}) \\
& + 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(P_h^{\boldsymbol{u}} \boldsymbol{u}^{n_{m_k}} - P_h^{\boldsymbol{u}} \boldsymbol{u}^i, e_{\varphi,h}^{n_{m_{k+1}}}) \\
& - 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(e_{\boldsymbol{u},h}^{i+1}, e_{\varphi,h}^{n_{m_k}}) + 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(e_{\boldsymbol{u},h}^{i}, e_{\varphi,h}^{n_{m_{k+1}}}) \\
& = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8. \qquad (98)
\end{aligned}
$$

Now we estimate the terms T_1 through T_8 term by term.

Using Hölder inequality, trace inequalities (34), (35) and Young inequality, we have

$$
\begin{aligned}
|T_1| =& \Big| 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} (d_{t^f} \boldsymbol{u}^{i+1} - \partial_{t^f} \boldsymbol{u}^{i+1}, e_{\boldsymbol{u},h}^{i+1}) \\
& + 2\Delta t^p g S_0 (d_{t^p} \varphi^{n_{m_{k+1}}} - \partial_{t^p} \varphi^{n_{m_{k+1}}}, e_{\varphi,h}^{n_{m_{k+1}}}) \Big| \\
\le & \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \| e_{\boldsymbol{u},h}^{i+1} \|_{H_f}^2 + \frac{12 C_P^2 \Delta t^f}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \| d_{t^f} \boldsymbol{u}^{i+1} - \partial_{t^f} \boldsymbol{u}^{i+1} \|_f^2 \\
& + \frac{g \Delta t^p}{10 \mu_{max}} \| K^{1/2} \nabla e_{\varphi,h}^{n_{m_{k+1}}} \|_p^2 + \frac{10 g \tilde{C}_p^2 S_0^2 \mu_{\max} \Delta t^p}{k_{\min}} \| d_{t^p} \varphi^{n_{m_{k+1}}} - \partial_{t^p} \varphi^{n_{m_{k+1}}} \|_p^2 \\
\le & \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \| e_{\boldsymbol{u},h}^{i+1} \|_{H_f}^2 + \frac{g \Delta t^p}{10 \mu_{max}} \| K^{1/2} \nabla e_{\varphi,h}^{n_{m_{k+1}}} \|_p^2 \\
& + \frac{12 C_P^2 \Delta t^f}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \frac{1}{(\Delta t^f)^2} \int_\Omega \Big(\int_{t_i^f}^{t_{i+1}^f} (t - t_i^f) \partial_{t^f}^2 \boldsymbol{u}(t) dt^f \Big)^2 dx \\
& + \frac{10 g \tilde{C}_p^2 S_0^2 \mu_{\max} \Delta t^p}{k_{\min}} \frac{1}{(\Delta t^p)^2} \int_\Omega \Big(\int_{t_k^p}^{t_{k+1}^p} (t - t_k^p) \partial_{t^p}^2 \varphi(t) dt^p \Big)^2 dx
\end{aligned}
$$

$$
\begin{aligned}
\le & \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{g\Delta t^p}{10\mu_{max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 \\
& + \frac{12C_P^2 \Delta t^f}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \frac{1}{(\Delta t^f)^2} \int_\Omega \int_{t_i^f}^{t_{i+1}^f} \left(\partial_{t^f}^2 \boldsymbol{u}(t)\right)^2 dt^f \int_{t_i^f}^{t_{i+1}^f} \left(t - t_i^f\right)^2 dt^f dx \\
& + \frac{10g\tilde{C}_p^2 S_0^2 \mu_{\max} \Delta t^p}{k_{\min}} \frac{1}{(\Delta t^p)^2} \int_\Omega \int_{t_k^p}^{t_{k+1}^p} \left(\partial_{t^p}^2 \varphi(t)\right)^2 dt^p \int_{t_k^p}^{t_{k+1}^p} \left(t - t_k^p\right)^2 dt^p dx \\
= & \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{g\Delta t^p}{10\mu_{max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 \\
& + \frac{4C_P^2 (\Delta t^f)^2}{\alpha_f} \int_{t_{n_{m_k}}^f}^{t_{n_{m_{k+1}}}^f} \|\partial_{t^f}^2 \boldsymbol{u}(t)\|_f^2 dt^f \\
& + \frac{10g\tilde{C}_p^2 S_0^2 \mu_{\max} (\Delta t^p)^2}{3k_{\min}} \int_{t_k^p}^{t_{k+1}^p} \|\partial_{t^p}^2 \varphi(t)\|_p^2 dt^p. \qquad (99)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
|T_2| = & |2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} (d_{t^f} \epsilon_{\boldsymbol{u}}^{i+1}, e_{\boldsymbol{u},h}^{i+1}) + 2\Delta t^p g S_0 (d_{t^p} \epsilon_\varphi^{n_{m_{k+1}}}, e_{\varphi,h}^{n_{m_{k+1}}})| \\
\le & \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{12C_P^2 \Delta t^f}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|d_{t^f} \epsilon_{\boldsymbol{u}}^{i+1}\|_f^2 \\
& + \frac{g\Delta t^p}{10\mu_{\max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 + \frac{10g\tilde{C}_P^2 S_0^2 \mu_{\max} \Delta t^p}{k_{\min}} \|d_{t^p} \epsilon_\varphi^{n_{m_{k+1}}}\|_p^2 \\
\le & \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{g\Delta t^p}{10\mu_{\max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 \\
& + \frac{12C_P^2 \Delta t^f}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \frac{1}{(\Delta t^f)^2} \int_\Omega \Big(\int_{t_i^f}^{t_{i+1}^f} d_{t^f} \epsilon_{\boldsymbol{u}}(t) dt^f \Big)^2 dx \\
& + \frac{10g\tilde{C}_P^2 S_0^2 \mu_{\max} \Delta t^p}{k_{\min}} \frac{1}{(\Delta t^p)^2} \int_\Omega \Big(\int_{t_k^p}^{t_{k+1}^p} d_{t^p} \epsilon_\varphi(t) dt^p \Big)^2 dx \\
\le & \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{g\Delta t^p}{10\mu_{\max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 \\
& + \frac{12C_P^2 \Delta t^f}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \frac{1}{(\Delta t^f)^2} \int_\Omega \Big(\int_{t_i^f}^{t_{i+1}^f} (d_{t^f} \epsilon_{\boldsymbol{u}}(t))^2 dt^f \int_{t_i^f}^{t_{i+1}^f} 1^2 dt^f \Big) dx \\
& + \frac{10g\tilde{C}_P^2 S_0^2 \mu_{\max} \Delta t^p}{k_{\min}} \frac{1}{(\Delta t^p)^2} \int_\Omega \Big(\int_{t_k^p}^{t_{k+1}^p} (d_{t^p} \epsilon_\varphi(t))^2 dt^p \int_{t_k^p}^{t_{k+1}^p} 1^2 dt^p \Big) dx \\
= & \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{g\Delta t^p}{10\mu_{\max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2
\end{aligned}
$$

$$+\frac{12C_P^2}{\alpha_f}\int_{t^f_{n_{m_k}}}^{t^f_{n_{m_{k+1}}}}\|d_{t^f}\epsilon_{\boldsymbol{u}}(t)\|_f^2dt^f+\frac{10g\tilde{C}_P^2S_0^2\mu_{\max}}{k_{\min}}\int_{t_k^p}^{t_{k+1}^p}\|d_{t^p}\epsilon_{\varphi}(t)\|_p^2dt^p. \tag{100}$$

From (57), (68) and the assumption that μ is Lipschitz continuous, we have

$$\begin{aligned}|T_3| =&|2\Delta t^f\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big(a_f(c^{i+1};P_h^{\boldsymbol{u}}\boldsymbol{u}^{i+1},e_{\boldsymbol{u},h}^{i+1})-a_f(\overline{c}_h^{n_m};P_h^{\boldsymbol{u}}\boldsymbol{u}^{i+1},e_{\boldsymbol{u},h}^{i+1})\big)|\\ \le&\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big(\frac{12\Delta t^fC_P^2C_{\boldsymbol{u}}^2}{\alpha_f}\|\mu(c^{i+1})-\mu(\overline{c}_h^{n_m})\|_f^2+\frac{\alpha_f\Delta t^f}{12}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2\big)\\ \le&\frac{12\Delta t^fC_P^2C_{\boldsymbol{u}}^2\mu_L^2}{\alpha_f}\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big(\|c^{n_m}-c_h^{i+1}\|_f^2\big)+\frac{\alpha_f\Delta t^f}{12}\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2.\end{aligned} \tag{101}$$

Since

$$\begin{aligned}\sum_{i=n_m}^{n_{m+1}-1}\|c^{i+1}-c^{n_m}\|^2\le&\sum_{i=n_m}^{n_{m+1}-1}\big(\|c^{i+1}-c^i\|^2+\|c^i-c^{i-1}\|^2+\cdots+\|c^{n_m+1}-c^{n_m}\|^2\big)\\ \le&\, r_{cf}\sum_{i=n_m}^{n_{m+1}-1}\|c^{i+1}-c^i\|^2,\end{aligned}$$

we can estimate the first term of (101) as follows

$$\begin{aligned}&\frac{12\Delta t^fC_P^2C_{\boldsymbol{u}}^2\mu_L^2}{\alpha_f}\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big(\|c^{n_m}-c_h^{i+1}\|_f^2\big)\\ \le&\frac{12\Delta t^fC_P^2C_{\boldsymbol{u}}^2\mu_L^2}{\alpha_f}\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big(\|c^{n_m}-c_h^{n_m}\|_f^2+\|c^{i+1}-c^{n_m}\|_f^2\big)\\ \le&\frac{12\Delta t^fC_P^2C_{\boldsymbol{u}}^2\mu_L^2}{\alpha_f}\big(r_{cf}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i}-c_h^{n_i}\|_f^2+r_{cf}\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|c^{i+1}-c^i\|_f^2\big)\\ \le&\frac{12\Delta t^fr_{cf}C_P^2C_{\boldsymbol{u}}^2\mu_L^2}{\alpha_f}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i}-c_h^{n_i}\|_f^2\\ &+\frac{12\Delta t^c\Delta t^fC_P^2C_{\boldsymbol{u}}^2\mu_L^2}{\alpha_f}\int_{t^f_{n_{m_k}}}^{t^f_{n_{m_{k+1}}-1}}\|\partial_{t^f}c\|_f^2dt^f.\end{aligned}$$

Thus

$$\begin{aligned}|T_3|\le&\frac{12\Delta t^cC_P^2C_{\boldsymbol{u}}^2\mu_L^2}{\alpha_f}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i}-c_h^{n_i}\|_f^2\\ &+6C_P^2C_{\boldsymbol{u}}^2\mu_L^2\frac{\big((\Delta t^c)^2+(\Delta t^f)^2\big)}{\alpha_f}\int_{t^f_{n_{m_k}}}^{t^f_{n_{m_{k+1}}-1}}\|\partial_{t^f}c\|_f^2dt^f\\ &+\frac{\alpha_f\Delta t^f}{12}\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2.\end{aligned} \tag{102}$$

From (58), (68) the definition of λ, and the assumption that μ is Lipschitz continuous, we have

$$\begin{aligned}|T_4| =&|2\Delta t^p\big(a_p(c^{n_{m_{k+1}}}; P_h^\varphi\varphi^{n_{m_{k+1}}}, e_{\varphi,h}^{n_{m_{k+1}}}) - a_p(\overline{R}_h^{n_{m_k}}; P_h^\varphi\varphi^{n_{m_{k+1}}}, e_{\varphi,h}^{n_{m_{k+1}}})\big)| \\ \leq& \frac{10g\tilde{C}_P^2\mu_{\max}\Delta t^p C_\varphi^2}{k_{\min}}\|\lambda(c^{n_{m_{k+1}}}) - \lambda(\overline{R}_h^{n_{m_k}})\|_p^2 + \frac{g\Delta t^p}{10\mu_{\max}}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 \\ \leq& \frac{10g\tilde{C}_P^2\mu_{\max}\Delta t^p C_\varphi^2\mu_L k_{\max}}{k_{\min}\mu_{\min}}\|c^{n_{m_{k+1}}} - R_h^{n_{m_k}}\|_p^2 + \frac{g\Delta t^p}{10\mu_{\max}}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2. \end{aligned} \tag{103}$$

Here, the term $\|c^{n_{m_{k+1}}} - R_h^{n_{m_k}}\|_p^2$ on the right hand side above can be estimated as follows

$$\begin{aligned}\|c^{n_{m_{k+1}}} - R_h^{n_{m_k}}\|_p^2 =& \|c^{n_{m_{k+1}}} - \frac{1}{r_{pc}}\sum_{i=m_k}^{m_{k+1}-1} c_h^{n_i}\|_p^2 \leq \frac{1}{r_{pc}^2}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_{m_{k+1}}} - c_h^{n_i}\|_p^2 \\ \leq& \frac{1}{r_{pc}^2}\sum_{i=m_k}^{m_{k+1}-1}\big(\|c^{n_i} - c_h^{n_i}\|_p^2 + \|c^{n_{m_{k+1}}} - c^{n_i}\|_p^2\big) \\ \leq& \frac{1}{r_{pc}^2}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i} - c_h^{n_i}\|_p^2 + \frac{1}{r_{pc}^2}r_{pc}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_{i+1}} - c^{n_i}\|_p^2 \\ \leq& \frac{1}{r_{pc}^2}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i} - c_h^{n_i}\|_p^2 + \frac{1}{r_{pc}}\Delta t^c\int_{t_{m_k}^c}^{t_{m_{k+1}-1}^c}\|\partial_{t^c}c\|_p^2 dt^c. \end{aligned} \tag{104}$$

Applying the above to (103), we obtain the estimate of T_4:

$$\begin{aligned}|T_4| \leq& \frac{10g\tilde{C}_P^2\mu_{\max}\Delta t^c C_\varphi^2\mu_L k_{\max}}{k_{\min}\mu_{\min}r_{pc}}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i} - c_h^{n_i}\|_p^2 \\ &+ \frac{10g\tilde{C}_P^2\mu_{\max}(\Delta t^c)^2 C_\varphi^2\mu_L k_{\max}}{k_{\min}\mu_{\min}}\int_{t_{m_k}^c}^{t_{m_{k+1}-1}^c}\|\partial_{t^c}c\|_p^2 dt^c \\ &+ \frac{g\Delta t^p}{10\mu_{\max}}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2. \end{aligned} \tag{105}$$

T_5 can be estimated the same way as that for T_3:

$$\begin{aligned}|T_5| =&|2\Delta t^f\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}(\boldsymbol{f}(c^{i+1}) - \boldsymbol{f}(\overline{c}_h^{n_m}), e_{\boldsymbol{u},h}^{i+1})| \\ \leq& \sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big(\frac{12\Delta t^f C_P^2 f_L^2}{\alpha_f}\|c^{i+1} - c_h^{n_m}\|_f^2 + \frac{\alpha_f\Delta t^f}{12}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2\big) \\ \leq& \frac{12\Delta t^c C_P^2 f_L^2}{\alpha_f}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i} - c_h^{n_i}\|_f^2 \\ &+ 6C_P^2 f_L^2\frac{\big((\Delta t^c)^2 + (\Delta t^f)^2\big)}{\alpha_f}\int_{t_{n_{m_k}}^f}^{t_{n_{m_{k+1}}-1}^f}\|\partial_{t^f}c\|_f^2 dt^f \\ &+ \frac{\alpha_f\Delta t^f}{12}\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2. \end{aligned} \tag{106}$$

Following the proof of (72), we have

$$
\begin{aligned}
|T_6| =& \Big| -2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(e_{\boldsymbol{u},h}^{i+1}, P_h^{\varphi}\varphi^{i+1} - P_h^{\varphi}\varphi^{i}) \\
&+ 2\Delta t^p a_\Gamma(P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_{k+1}}} - P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}}, e_{\varphi,h}^{n_{m_{k+1}}})\Big| \\
\leq& \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{12 g^2 C_1 C_2 \Delta t^f}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|P_h^{\varphi}\varphi^{i+1} - P_h^{\varphi}\varphi^{i}\|_{H_p}^2 \\
&+ \frac{g\Delta t^p}{10\mu_{\max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 + \frac{10 g C_1 C_2 \Delta t^p \mu_{\max}}{k_{\min}} \|P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_{k+1}}} - P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}}\|_{H_f}^2. \qquad (107)
\end{aligned}
$$

By the equivalence between $\|\boldsymbol{u}\|_{H_f}$ and $\|\nabla \boldsymbol{u}\|_f$, $\|\varphi\|_{H_p}$ and $\|\nabla\varphi\|_p$, we have

$$
\begin{aligned}
&\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|P_h^{\varphi}\varphi^{i+1} - P_h^{\varphi}\varphi^{i}\|_{H_p}^2 \leq C_P \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|\varphi(t_{i+1}^f) - \varphi(t_i^f)\|_{H_p}^2 \\
&\leq C_P \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \int_{\Omega_p} \big(\nabla(\varphi(t_{i+1}^f) - \varphi(t_i^f))\big)^2 dx \\
&\leq C_P \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \int_{\Omega_p} \Big(\int_{t_i^f}^{t_{i+1}^f} \nabla(\partial_{t^f}\varphi) dt^f\Big)^2 dx \\
&\leq C_P \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \int_{\Omega_p} \int_{t_i^f}^{t_{i+1}^f} |\nabla(\partial_{t^f}\varphi)|^2 dt^f dx \int_{t_i^f}^{t_{i+1}^f} 1^2 dt^f dx \\
&= C_P \Delta t^f \int_{t_{n_{m_k}}^f}^{t_{n_{m_{k+1}}}^f} \|\partial_{t^f}\varphi\|_{H_p}^2 dt^f. \qquad (108)
\end{aligned}
$$

Similarly, we have

$$
\|P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_{k+1}}} - P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}}\|_{H_f}^2 \leq C_P \Delta t^p \int_{t_k^p}^{t_{k+1}^p} \|\partial_{t^p}\boldsymbol{u}\|_{H_f}^2 dt^p. \qquad (109)
$$

Taking (108) and (109) back to (107), we have

$$
\begin{aligned}
|T_6| \leq& \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{g\Delta t^p}{10\mu_{\max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 \\
&+ \frac{12 g^2 C_1 C_2 C_P (\Delta t^f)^2}{\alpha_f} \int_{t_{n_{m_k}}^f}^{t_{n_{m_{k+1}}}^f} \|\partial_{t^f}\varphi\|_{H_p}^2 dt^f \\
&+ \frac{10 g C_1 C_2 C_P (\Delta t^p)^2 \mu_{\max}}{k_{\min}} \int_{t_k^p}^{t_{k+1}^p} \|\partial_{t^p}\boldsymbol{u}\|_{H_f}^2 dt^p. \qquad (110)
\end{aligned}
$$

Similar to the estimate of of T_6 we have that

$$
|T_7| = \Big| -2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(e_{\boldsymbol{u},h}^{i+1}, P_h^{\varphi}\varphi^{i} - P_h^{\varphi}\varphi^{n_{m_k}})
$$

$$
\begin{aligned}
&+ 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}} - P_h^{\boldsymbol{u}}\boldsymbol{u}^i, e_{\varphi,h}^{n_{m_{k+1}}})\Big| \\
\le& \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{12g^2C_1C_2\Delta t^f}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|P_h^{\varphi}\varphi^i - P_h^{\varphi}\varphi^{n_{m_k}}\|_{H_p}^2 \\
&+ \frac{g\Delta t^f}{10\mu_{\max}} \|K^{1/2} e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 + \frac{10C_1C_2\Delta t^f \mu_{\max}}{k_{\min}} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|P_h^{\boldsymbol{u}}\boldsymbol{u}^{n_{m_k}} - P_h^{\boldsymbol{u}}\boldsymbol{u}^i\|_{H_f}^2 \\
\le& \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{g\Delta t^f}{10\mu_{\max}} \|K^{1/2} e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 \\
&+ \frac{12g^2C_1C_2\Delta t^p}{\alpha_f} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|P_h^{\varphi}\varphi^{i+1} - P_h^{\varphi}\varphi^i\|_{H_p}^2 \\
&+ \frac{10C_1C_2\Delta t^p \mu_{\max}}{k_{\min}} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|P_h^{\boldsymbol{u}}\boldsymbol{u}^{i+1} - P_h^{\boldsymbol{u}}\boldsymbol{u}^i\|_{H_f}^2.
\end{aligned} \tag{111}
$$

Similar to (108) and (109), we have

$$
\begin{aligned}
&\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|P_h^{\boldsymbol{u}}\boldsymbol{u}^{i+1} - P_h^{\boldsymbol{u}}\boldsymbol{u}^i\|_{H_f}^2 \le C_P \Delta t^f \int_{t_{n_{m_k}}^f}^{t_{n_{m_{k+1}}}^f} \|\partial_{t^f}\boldsymbol{u}\|_{H_f}^2 dt^f \\
&= C_P \Delta t^f r_{pf} \int_{t_k^p}^{t_{n_{k+1}}^p} \|\partial_{t^p}\boldsymbol{u}\|_{H_f}^2 dt^p = C_P \Delta t^p \int_{t_k^p}^{t_{n_{k+1}}^p} \|\partial_{t^p}\boldsymbol{u}\|_{H_f}^2 dt^p.
\end{aligned} \tag{112}
$$

It follows from (108) and (112) to (111) that

$$
\begin{aligned}
|T_7| \le& \frac{\alpha_f \Delta t^f}{12} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + \frac{g\Delta t^f}{10\mu_{\max}} \|K^{1/2} e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 \\
&+ 6g^2C_1C_2C_P \frac{\big((\Delta t^f)^2 + (\Delta t^p)^2\big)}{\alpha_f} \int_{t_{n_{m_k}}^f}^{t_{n_{m_{k+1}}}^f} \|\partial_{t^f}\varphi\|_{H_p}^2 dt^f \\
&+ \frac{10C_1C_2C_P(\Delta t^p)^2 \mu_{\max}}{k_{\min}} \int_{t_k^p}^{t_{n_{k+1}}^p} \|\partial_{t^p}\boldsymbol{u}\|_{H_f}^2 dt^p.
\end{aligned} \tag{113}
$$

Similar to the estimate (82), we have that

$$
\begin{aligned}
|T_8| =& | - 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(e_{\boldsymbol{u},h}^{i+1}, e_{\varphi,h}^{n_{m_k}}) + 2\Delta t^f \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} a_\Gamma(e_{\boldsymbol{u},h}^{i}, e_{\varphi,h}^{n_{m_{k+1}}})| \\
\le& 2\Delta t^f \Big[\frac{1}{16}\big(\frac{g}{\mu_{\max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_k}}\|_p^2 + \alpha_f \| \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 \\
&+ \frac{g}{\mu_{\max}} \|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2 + \alpha \| \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} e_{\boldsymbol{u},h}^{i}\|_{H_f}^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{gC_1\sqrt{\mu_{\max}}}{\sqrt{\alpha_f k_{\min} S_0}}\big(gS_0\|e_{\varphi,h}^{n_{m_k}}\|_p^2+\|\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} e_{\boldsymbol{u},h}^{i+1}\|_f^2\\
&+gS_0\|e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2+\|\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} e_{\boldsymbol{u},h}^{i}\|_f^2\big)\Big]\\
\le&\frac{\Delta t^f}{4}\big(\alpha_f\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2+\alpha_f\|e_{\boldsymbol{u},h}^{n_{m_k}}\|_{H_f}^2\\
&+\frac{g}{\mu_{\max}}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2+\frac{g}{\mu_{\max}}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_k}}\|_p^2\big)\\
&+\frac{4gC_1\sqrt{\mu_{\max}}\Delta t^f}{\sqrt{\alpha_f k_{\min} S_0}}\big(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|e_{\boldsymbol{u},h}^{i}\|_f^2+gS_0\|e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2+gS_0\|e_{\varphi,h}^{n_{m_k}}\|_p^2\big).
\end{aligned}
\tag{114}
$$

Replacing T_1 thorough T_8 in (98) with the above estimates and summing over with $k=0,1,\dots,l$ and using the fact that $e_{\boldsymbol{u},h}^{n_{m_0}}=0$, $e_{\varphi,h}^{n_{m_0}}=0$, we have that

$$
\begin{aligned}
&\|e_{\boldsymbol{u},h}^{n_{m_{l+1}}}\|_f^2+\alpha_f\Delta t^f\sum_{k=0}^{l}\sum_{i=n_{m_k}}^{n_{m_{k+1}}}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2\\
&+gS_0\|e_{\varphi,h}^{n_{m_{l+1}}}\|_p^2+\frac{g}{\mu_{max}}\Delta t^p\sum_{k=0}^{l}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2\\
\le&\frac{4C_P^2(\Delta t^f)^2}{\alpha_f}\int_0^T\|\partial_{t^f}^2\boldsymbol{u}(t)\|_f^2dt^f+\frac{10g\tilde{C}_p^2S_0^2\mu_{\max}(\Delta t^p)^2}{3k_{\min}}\int_0^T\|\partial_{t^p}^2\varphi(t)\|_p^2dt^p\\
&+\frac{12C_P^2}{\alpha_f}\int_0^T\|d_{t^f}\epsilon_{\boldsymbol{u}}(t)\|_f^2dt^f+\frac{10g\tilde{C}_P^2S_0^2\mu_{\max}}{k_{\min}}\int_0^T\|d_{t^p}\epsilon_{\varphi}(t)\|_p^2dt^p\\
&+\frac{12\Delta t^cC_P^2C_{\boldsymbol{u}}^2\mu_L^2}{\alpha_f}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i}-c_h^{n_i}\|_f^2\\
&+6C_P^2C_{\boldsymbol{u}}^2\mu_L^2\frac{\big((\Delta t^c)^2+(\Delta t^f)^2\big)}{\alpha_f}\int_0^T\|\partial_{t^f}c\|_f^2dt^f\\
&+\frac{10g\tilde{C}_P^2\mu_{\max}\Delta t^cC_\varphi^2\mu_Lk_{\max}}{k_{\min}\mu_{\min}r_{pc}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i}-c_h^{n_i}\|_p^2\\
&+\frac{10g\tilde{C}_P^2\mu_{\max}(\Delta t^c)^2C_\varphi^2\mu_Lk_{\max}}{k_{\min}\mu_{\min}}\int_0^T\|\partial t^c c\|_p^2dt^c\\
&+\frac{12\Delta t^cC_P^2f_L^2}{\alpha_f}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i}-c_h^{n_i}\|_f^2+6C_P^2f_L^2\frac{\big((\Delta t^c)^2+(\Delta t^f)^2\big)}{\alpha_f}\int_0^T\|\partial_{t^f}c\|_f^2dt^f\\
&+\frac{12g^2C_1C_2C_P(\Delta t^f)^2}{\alpha_f}\int_0^T\|\partial_{t^f}\varphi\|_{H_p}^2dt^f\\
&+\frac{10gC_1C_2C_P(\Delta t^p)^2\mu_{\max}}{k_{\min}}\int_0^T\|\partial_{t^p}\boldsymbol{u}\|_{H_f}^2dt^p\\
&+6g^2C_1C_2C_P\frac{\big((\Delta t^f)^2+(\Delta t^p)^2\big)}{\alpha_f}\int_0^T\|\partial_{t^f}\varphi\|_{H_p}^2dt^f
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{10C_1C_2C_P(\Delta t^p)^2\mu_{\max}}{k_{\min}}\int_0^T\|\partial t^p\boldsymbol{u}\|_{H_f}^2dt^p\\
&+\frac{4gC_1\sqrt{\mu_{\max}}\Delta t^f}{\sqrt{\alpha_f k_{\min}S_0}}\sum_{k=0}^{l}\Big(\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|e_{\boldsymbol{u},h}^i\|_f^2+gS_0\|e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2+gS_0\|e_{\varphi,h}^{n_{m_k}}\|_p^2\Big).
\end{aligned}
\tag{115}
$$

Using the approximation properties of the projection operator P_h given by (54) and (55) and the Gronwall inequality, we obtain (94). □

Now we consider the error estimate for the concentration. From (38), (59) and (66), we have that for $z_h\in W_h$,

$$
\begin{aligned}
&(d_{t^c}e_{c,h}^{n_{m+1}},z_h)_{\overline{\phi}}+(\overline{D}(\boldsymbol{U}_h^{n_m})\nabla e_{c,h}^{n_{m+1}},\nabla z_h)\\
=&(d_{t^c}c^{n_{m+1}}-\partial_{t^c}c^{n_{m+1}},z_h)_{\overline{\phi}}-(d_{t^c}\epsilon_c^{n_{m+1}},z_h)_{\overline{\phi}}\\
&-((\overline{D}(\boldsymbol{u}^{n_{m+1}})-\overline{D}(\boldsymbol{U}_h^{n_m}))\nabla\Xi_h c^{n_{m+1}},\nabla z_h)\\
&-((\boldsymbol{u}^{n_{m+1}}-\boldsymbol{U}_h^{n_m})\cdot\Xi_h c^{n_{m+1}},z_h)-(\boldsymbol{U}_h^{n_m}\cdot\nabla e_{c,h}^{n_{m+1}},z_h)\\
&+(\xi\epsilon_c^{n_{m+1}}-q^I e_{c,h}^{n_{m+1}},z_h)_p.
\end{aligned}
\tag{116}
$$

Theorem 4.2. *Suppose that the assumptions of Theorem 4.1 satisfied, and if Δt^f, Δt^c and Δt^p are sufficiently small, then there exists a positive constant $\hat{C}_3$ independent of h, Δt^f, Δt^c and Δt^p, and exists a positive constant $\hat{C}_4$ independent of h, Δt^f, Δt^c and Δt^p, and independent of physical parameters $\mu_{\max}$, $\mu_{\min}$, $k_{\max}$, $k_{\min}$ and $\phi_{\max}$, such that for $0\le l\le N_p-1$*

$$
\begin{aligned}
&\|e_{c,h}^{n_{m_{l+1}}}\|^2+D_{\min}\Delta t^c\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla e_{c,h}^{n_{i+1}}\|^2\\
\le&\hat{C}_3h^{2(j+1)}+\hat{C}_4\Big((\frac{k_{\max}}{\mu_{\min}}+\phi_{\max})(\Delta t^c)^2+\frac{k_{\max}}{\mu_{\min}}(\Delta t^p)^2\Big).
\end{aligned}
\tag{117}
$$

Proof. Taking $z_h=2\Delta t^c e_{c,h}^{n_{m+1}}$ in (116), we have

$$
\begin{aligned}
&\|e_{c,h}^{n_{m+1}}\|_{\overline{\phi}}^2-\|e_{c,h}^{n_m}\|_{\overline{\phi}}^2+2\Delta t^c\overline{D}(\boldsymbol{U}_h^{n_m})\|\nabla e_{c,h}^{n_{m+1}}\|^2\\
=&2\Delta t^c(d_{t^c}c^{n_{m+1}}-\partial_{t^c}c^{n_{m+1}},e_{c,h}^{n_{m+1}})_{\overline{\phi}}-2\Delta t^c(d_{t^c}\epsilon_c^{n_{m+1}},e_{c,h}^{n_{m+1}})_{\overline{\phi}}\\
&-2\Delta t^c((\overline{D}(\boldsymbol{u}^{n_{m+1}})-\overline{D}(\boldsymbol{U}_h^{n_m}))\nabla\Xi_h c^{n_{m+1}},\nabla e_{c,h}^{n_{m+1}})\\
&-2\Delta t^c((\boldsymbol{u}^{n_{m+1}}-\boldsymbol{U}_h^{n_m})\cdot\Xi_h c^{n_{m+1}},e_{c,h}^{n_{m+1}})-2\Delta t^c(\boldsymbol{U}_h^{n_m}\cdot\nabla e_{c,h}^{n_{m+1}},e_{c,h}^{n_{m+1}})\\
&+2\Delta t^c(\xi\epsilon_c^{n_{m+1}}-q^I e_{c,h}^{n_{m+1}},e_{c,h}^{n_{m+1}})_p.
\end{aligned}
\tag{118}
$$

Summing over (118) with $m=m_k,m_k+1,\ldots,m_{k+1}-1$, and following the property (25), we have

$$
\begin{aligned}
&\|e_{c,h}^{n_{m_{k+1}}}\|_{\overline{\phi}}^2-\|e_{c,h}^{n_{m_k}}\|_{\overline{\phi}}^2+2\Delta t^c D_{\min}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla e_{c,h}^{n_{m+1}}\|^2\\
\le&2\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}(d_{t^c}c^{n_{i+1}}-\partial_{t^c}c^{n_{i+1}},e_{c,h}^{n_{i+1}})_{\overline{\phi}}-2\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}(d_{t^c}\epsilon_c^{n_{i+1}},e_{c,h}^{n_{i+1}})_{\overline{\phi}}\\
&-2\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}((\overline{D}(\boldsymbol{u}^{n_{i+1}})-\overline{D}(\boldsymbol{U}_h^{n_i}))\nabla\Xi_h c^{n_{i+1}},\nabla e_{c,h}^{n_{i+1}})
\end{aligned}
$$

$$
\begin{aligned}
&- 2\Delta t^c \sum_{i=m_k}^{m_{k+1}-1} ((\boldsymbol{u}^{n_{i+1}} - \boldsymbol{U}_h^{n_i}) \cdot \Xi_h c^{n_{i+1}}, e_{c,h}^{n_{i+1}}) \\
&- 2\Delta t^c \sum_{i=m_k}^{m_{k+1}-1} (\boldsymbol{U}_h^{n_i} \cdot \nabla e_{c,h}^{n_{i+1}}, e_{c,h}^{n_{i+1}}) \\
&+ 2\Delta t^c \sum_{i=m_k}^{m_{k+1}-1} (\xi \epsilon_c^{n_{i+1}} - q^I e_{c,h}^{n_{i+1}}, e_{c,h}^{n_{i+1}})_p \\
&= H_1 + H_2 + H_3 + H_4 + H_5 + H_6. \qquad (119)
\end{aligned}
$$

Now we estimate the terms $H_1, H_2, \dots, H_8$ term by term.

Similarly as the estimates for T_1 and T_2, we have

$$
\begin{aligned}
|H_1| =& \Big| 2\Delta t^c \sum_{i=m_k}^{m_{k+1}-1} (d_{t^c} c^{n_{i+1}} - \partial_{t^c} c^{n_{i+1}}, e_{c,h}^{n_{i+1}})_{\overline{\phi}} \Big| \\
\leq & \frac{\Delta t^c}{K_1} \sum_{i=m_k}^{m_{k+1}-1} \| d_{t^c} c^{n_{i+1}} - \partial_{t^c} c^{n_{i+1}} \|_{\overline{\phi}}^2 + K_1 \Delta t^c \sum_{i=m_k}^{m_{k+1}-1} \| e_{c,h}^{n_{i+1}} \|_{\overline{\phi}}^2 \\
\leq & \frac{\phi_{\max} \Delta t^c}{K_1} \sum_{i=m_k}^{m_{k+1}-1} \frac{1}{(\Delta t^c)^2} \int_\Omega \Big(\int_{t_i^c}^{t_{i+1}^c} (t - t_i^c) \partial_{t^c}^2 c(t) dt^c \Big)^2 dx \\
& + K_1 \Delta t^c \sum_{i=m_k}^{m_{k+1}-1} \| e_{c,h}^{n_{i+1}} \|_{\overline{\phi}}^2 \\
\leq & \frac{\phi_{\max} \Delta t^c}{K_1} \sum_{i=m_k}^{m_{k+1}-1} \frac{1}{(\Delta t^c)^2} \int_\Omega \Big(\int_{t_i^c}^{t_{i+1}^c} (\partial_{t^c}^2 c(t))^2 dt^c \int_{t_i^c}^{t_{i+1}^c} (t - t_i^c)^2 dt^c \Big) dx \\
& + K_1 \Delta t^c \sum_{i=m_k}^{m_{k+1}-1} \| e_{c,h}^{n_{i+1}} \|_{\overline{\phi}}^2 \\
= & \frac{\phi_{\max} (\Delta t^c)^2}{3K_1} \int_{t_{m_k}^c}^{t_{m_{k+1}}^c} \| \partial_{t^c} c \|^2 dt^c + K_1 \Delta t^c \sum_{i=m_k}^{m_{k+1}-1} \| e_{c,h}^{n_{i+1}} \|_{\overline{\phi}}^2, \qquad (120)
\end{aligned}
$$

$$
\begin{aligned}
|H_2| =& | 2\Delta t^c \sum_{i=m_k}^{m_{k+1}-1} (d_{t^c} \epsilon_c^{n_{i+1}}, e_{c,h}^{n_{i+1}})_{\overline{\phi}} | \\
\leq & \frac{\Delta t^c}{K_2} \sum_{i=m_k}^{m_{k+1}-1} \| d_{t^c} \epsilon_c^{n_{i+1}} \|_{\overline{\phi}}^2 + K_2 \Delta t^c \sum_{i=m_k}^{m_{k+1}-1} \| e_{c,h}^{n_{i+1}} \|_{\overline{\phi}}^2 \\
\leq & \frac{\phi_{\max} \Delta t^c}{K_2} \sum_{i=m_k}^{m_{k+1}-1} \frac{1}{(\Delta t^c)^2} \int_\Omega \Big(\int_{t_i^c}^{t_{i+1}^c} d_{t^c} \epsilon_c(t) dt^c \Big)^2 dx + K_2 \Delta t^c \sum_{i=m_k}^{m_{k+1}-1} \| e_{c,h}^{n_{i+1}} \|_{\overline{\phi}}^2 \\
\leq & \frac{\phi_{\max} \Delta t^c}{K_2} \sum_{i=m_k}^{m_{k+1}-1} \frac{1}{(\Delta t^c)^2} \int_\Omega \Big(\int_{t_i^c}^{t_{i+1}^c} (d_{t^c} \epsilon_c(t))^2 dt^c \int_{t_i^c}^{t_{i+1}^c} 1^2 dt^c \Big)^2 dx \\
& + K_2 \Delta t^c \sum_{i=m_k}^{m_{k+1}-1} \| e_{c,h}^{n_{i+1}} \|_{\overline{\phi}}^2
\end{aligned}
$$

(121)

$$=\frac{\phi_{\max}}{K_2}\int_{t^c_{m_k}}^{t^c_{m_{k+1}}}\|d_{t^c}\epsilon_c(t)\|^2dt^c+K_2\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}\|e_{c,h}^{n_{i+1}}\|^2_{\phi},$$

where the positive constants K_1 and K_2 are independent of h, Δt^f, Δt^c and Δt^p.

Using Hölder inequality, Young inequality, (26) and (62), we have

$$\begin{aligned}|H_3| =&|2\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}((\overline{D}(\boldsymbol{u}^{n_{i+1}})-\overline{D}(\boldsymbol{U}_h^{n_i}))\nabla\Xi_hc^{n_{i+1}},\nabla e_{c,h}^{n_{i+1}})|\\ \le&2\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}C_c\|\overline{D}(\boldsymbol{u}^{n_{i+1}})-\overline{D}(\boldsymbol{U}_h^{n_i})\|\cdot\|\nabla e_{c,h}^{n_{i+1}}\|\\ \le&\frac{2C_c^2\Delta t^c}{D_{\min}}(3d_l-2d_t)\sum_{i=m_k}^{m_{k+1}-1}\|\boldsymbol{u}^{n_{i+1}}-\boldsymbol{U}_h^{n_i}\|^2+\frac{D_{\min}\Delta t^c}{2}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla e_{c,h}^{n_{i+1}}\|^2,\end{aligned}\tag{122}$$

where $\sum_{i=m_k}^{m_{k+1}-1}\|\boldsymbol{u}^{n_{i+1}}-\boldsymbol{U}_h^{n_i}\|^2$ can be split into three parts:

$$\begin{aligned}&\sum_{i=m_k}^{m_{k+1}-1}\|\boldsymbol{u}^{n_{i+1}}-\boldsymbol{U}_h^{n_i}\|^2\\ =&\sum_{i=m_k}^{m_{k+1}-1}\big(\|\boldsymbol{u}^{n_{i+1}}-S_h^{n_i}\|_f^2+\|(\lambda(c^{n_{i+1}})-\lambda(c_h^{n_i}))\nabla\varphi^{n_{i+1}}\|_p^2\\ &+\|\lambda(c_h^{n_i})(\nabla\varphi^{n_{i+1}}-\nabla\varphi_h^{n_{m_k}})\|_p^2\big).\end{aligned}\tag{123}$$

Now we estimate each term of (123). By the definition of $S_h^{n_i}$, we have

$$\begin{aligned}\sum_{i=m_k}^{m_{k+1}-1}\|\boldsymbol{u}^{n_{i+1}}-S_h^{n_i}\|_f^2=&\sum_{m=m_k}^{m_{k+1}-1}\|\boldsymbol{u}^{n_{m+1}}-\frac{1}{r_{cf}}\sum_{i=n_m}^{n_{m+1}-1}\boldsymbol{u}_h^i\|_f^2\\ \le&\frac{1}{r_{cf}^2}\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\|\boldsymbol{u}^{n_{m+1}}-\boldsymbol{u}_h^i\|_f^2\\ \le&\frac{1}{r_{cf}^2}\sum_{m=m_k}^{m_{k+1}-1}\sum_{i=n_m}^{n_{m+1}-1}\big(\|\boldsymbol{u}^i-\boldsymbol{u}_h^i\|_f^2+\|\boldsymbol{u}^{n_{m+1}}-\boldsymbol{u}^i\|_f^2\big)\\ \le&\frac{1}{r_{cf}^2}\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|\boldsymbol{u}^i-\boldsymbol{u}_h^i\|_f^2+\frac{1}{r_{cf}}\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|\boldsymbol{u}^{i+1}-\boldsymbol{u}^i\|_f^2.\\ \le&\frac{1}{r_{cf}^2}\sum_{i=n_{m_k}}^{n_{m_{k+1}}-1}\|\boldsymbol{u}^i-\boldsymbol{u}_h^i\|_f^2+\frac{\Delta t^f}{r_{cf}}\int_{t^f_{n_{m_k}}}^{t^f_{n_{m_{k+1}}}}\|\partial_{t^f}\boldsymbol{u}\|_f^2dt^f.\end{aligned}\tag{124}$$

By the definition of λ and the assumption that μ is Lipschitz continuous, we have

$$\sum_{i=m_k}^{m_{k+1}-1}\|(\lambda(c^{n_{i+1}})-\lambda(c_h^{n_i}))\nabla\varphi^{n_{i+1}}\|_p^2\le\frac{k_{\max}\mu_L}{\mu_{\min}}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_{i+1}}-c_h^{n_i}\|_p^2$$

$$\leq \frac{k_{\max}\mu_L}{\mu_{\min}}\Big(\sum_{i=m_k}^{m_{k+1}-1} \|c^{n_i} - c_h^{n_i}\|_p^2 + \sum_{i=m_k}^{m_{k+1}-1} \|c^{n_i} - c^{n_{i+1}}\|_p^2\Big)$$

$$\leq \frac{k_{\max}\mu_L}{\mu_{\min}}\Big(\sum_{i=m_k}^{m_{k+1}-1} \|c^{n_i} - c_h^{n_i}\|_p^2 + \Delta t^c \int_{t^c_{m_k}}^{t^c_{m_{k+1}}} \|\partial_{t^c} c\|_p^2 dt^c\Big), \tag{125}$$

$$\sum_{i=m_k}^{m_{k+1}-1} \|\lambda(c_h^{n_i})(\nabla\varphi^{n_{i+1}} - \nabla\varphi_h^{n_{m_k}})\|_p^2 \leq \frac{k_{\max}}{\mu_{\min}} \sum_{i=m_k}^{m_{k+1}-1} \|\nabla\varphi^{n_{i+1}} - \nabla\varphi_h^{n_{m_k}}\|_p^2$$

$$\leq \frac{k_{\max}}{\mu_{\min}}\Big(\sum_{i=m_k}^{m_{k+1}-1} \|\nabla\varphi^{n_{m_k}} - \nabla\varphi_h^{n_{m_k}}\|_p^2 + \sum_{i=m_k}^{m_{k+1}-1} \|\nabla\varphi^{n_{m_k}} - \nabla\varphi^{n_{i+1}}\|_p^2\Big)$$

$$\leq \frac{k_{\max}}{\mu_{\min}}\big(r_{pc}\|\nabla\varphi^{n_{m_k}} - \nabla\varphi_h^{n_{m_k}}\|_p^2 + \sum_{i=m_k}^{m_{k+1}-1} r_{pc}\|\nabla\varphi^{n_{i+1}} - \nabla\varphi^{n_i}\|_p^2\big)$$

$$\leq \frac{k_{\max}}{\mu_{\min}}\big(r_{pc}\|\nabla\varphi^{n_{m_k}} - \nabla\varphi_h^{n_{m_k}}\|_p^2 + \Delta t^c r_{pc} \int_{t^c_{m_k}}^{t^c_{m_{k+1}}} \|\partial_{t^c}\nabla\varphi\|_p^2 dt^c\big). \tag{126}$$

Taking (124)-(126) to (122), we get the estimate of H_3:

$$\begin{aligned}
|H_3| \leq& \frac{2C_c^2 \Delta t^f}{r_{cf} D_{\min}}(3d_l - 2d_t) \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|\boldsymbol{u}^i - \boldsymbol{u}_h^i\|_f^2 \\
&+ \frac{2C_c^2 (\Delta t^f)^2}{D_{\min}}(3d_l - 2d_t) \int_{t^f_{n_{m_k}}}^{t^f_{n_{m_{k+1}}}} \|\partial_{t^f}\boldsymbol{u}\|_f^2 dt^f \\
&+ \frac{2C_c^2 k_{\max}\mu_L \Delta t^c}{\mu_{\min} D_{\min}}(3d_l - 2d_t) \sum_{i=m_k}^{m_{k+1}-1} \|c^{n_i} - c_h^{n_i}\|_p^2 \\
&+ \frac{2C_c^2 k_{\max}\mu_L (\Delta t^c)^2}{\mu_{\min} D_{\min}}(3d_l - 2d_t) \int_{t^c_{m_k}}^{t^c_{m_{k+1}}} \|\partial_{t^c} c\|_p^2 dt^c \\
&+ \frac{2C_c^2 k_{\max} \Delta t^p}{\mu_{\min} D_{\min}}(3d_l - 2d_t)\|\nabla\varphi^{n_{m_k}} - \nabla\varphi_h^{n_{m_k}}\|_p^2 \\
&+ C_c^2 \frac{k_{\max}\big((\Delta t^c)^2 + (\Delta t^p)^2\big)}{\mu_{\min}} \frac{(3d_l - 2d_t)}{D_{\min}} \int_{t^c_{m_k}}^{t^c_{m_{k+1}}} \|\partial_{t^c}\nabla\varphi\|_p^2 dt^c \\
&+ \frac{D_{\min}\Delta t^c}{2} \sum_{i=m_k}^{m_{k+1}-1} \|\nabla e_{c,h}^{n_{i+1}}\|^2.
\end{aligned} \tag{127}$$

Similarly, we can get the estimate of H_4:

$$\begin{aligned}
|H_4| =& |2\Delta t^c \sum_{i=m_k}^{m_{k+1}-1} ((\boldsymbol{u}^{n_{i+1}} - \boldsymbol{U}_h^{n_i}) \cdot \Xi_h c^{n_{i+1}}, e_{c,h}^{n_{i+1}})| \\
\leq& \frac{C_c^2 \Delta t^c}{K_4} \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{u}^{n_{i+1}} - \boldsymbol{U}_h^{n_i}\|^2 + K_4 \Delta t^c \sum_{i=m_k}^{m_{k+1}-1} \|e_{c,h}^{n_{i+1}}\|^2 \\
\leq& \frac{C_c^2 \Delta t^f}{K_4 r_{cf}} \sum_{i=n_{m_k}}^{n_{m_{k+1}}-1} \|\boldsymbol{u}^i - \boldsymbol{u}_h^i\|_f^2 + \frac{C_c^2 (\Delta t^f)^2}{K_4} \int_{t^f_{n_{m_k}}}^{t^f_{n_{m_{k+1}}}} \|\partial_{t^f}\boldsymbol{u}\|_f^2 dt^f
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C_c^2 k_{\max}\mu_L \Delta t^c}{\mu_{\min}K_4}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i}-c_h^{n_i}\|_p^2+\frac{C_c^2 k_{\max}\mu_L(\Delta t^c)^2}{\mu_{\min}K_4}\int_{t^c_{m_k}}^{t^c_{m_{k+1}}}\|\partial_{t^c}c\|_p^2dt^c\\
&+\frac{C_c^2 k_{\max}\Delta t^p}{\mu_{\min}K_4}\|\nabla\varphi^{n_{m_k}}-\nabla\varphi_h^{n_{m_k}}\|_p^2\\
&+\frac{C_c^2}{2K_4}\frac{k_{\max}\big((\Delta t^c)^2+(\Delta t^p)^2\big)}{\mu_{\min}}\int_{t^c_{m_k}}^{t^c_{m_{k+1}}}\|\partial_{t^c}\nabla\varphi\|_p^2dt^c\\
&+K_4\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}\|e_{c,h}^{n_{i+1}}\|^2.
\end{aligned}\tag{128}
$$

Here the positive constant K_4 is independent of h, Δt^f, Δt^c and Δt^p.

To estimate H_5, we make the following induction hypothesis.

$$
\|\boldsymbol{u}_h^i\|_{L^\infty(\Omega_f)}\le K_{\boldsymbol{u}},\quad \|\nabla\varphi_h^{n_{m_k}}\|_{L^\infty(\Omega_p)}\le K_\varphi.\tag{129}
$$

Then

$$
\begin{aligned}
|H_5|=&|2\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}(\boldsymbol{U}_h^{n_i}\cdot\nabla e_{c,h}^{n_{i+1}},e_{c,h}^{n_{i+1}})|\\
\le&\frac{D_{\min}\Delta t^c}{2}\|\nabla e_{c,h}^{n_{i+1}}\|^2+\frac{2K_5^2\Delta t^c}{D_{\min}}\|e_{c,h}^{n_{i+1}}\|^2,
\end{aligned}\tag{130}
$$

where $K_5=\max\{K_{\boldsymbol{u}},K_\varphi\}$.

Using Hölder inequality and Young inequality, we have

$$
\begin{aligned}
&|H_6|=|2\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}(\xi\epsilon_c^{n_{i+1}}-q^Ie_{c,h}^{n_{i+1}},e_{c,h}^{n_{i+1}})_p|\\
&\quad\le\frac{\xi^2\Delta t^c}{K_6}\sum_{i=m_k}^{m_{k+1}-1}\|\epsilon_c^{n_{i+1}}\|^2+K_6\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}\|e_{c,h}^{n_{i+1}}\|^2+2q^I_{\max}\Delta t^c\sum_{i=m_k}^{m_{k+1}-1}\|e_{c,h}^{n_{i+1}}\|^2.
\end{aligned}\tag{131}
$$

Taking the estimates of $H_1,H_2,\dots,H_6$ back to (119), summing over with $k=0,1,\dots,l$ and using the fact that $e_{c,h}^{n_{m_0}}=0$ and $(\cdot,\cdot)_{\overline{\phi}}$ is an equivalent scalar product on $L^2(\Omega)$, we have

$$
\begin{aligned}
&\|e_{c,h}^{n_{m_{l+1}}}\|^2+D_{\min}\Delta t^c\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla e_{c,h}^{n_{i+1}}\|^2\\
\le&C\Big(h^{2(j+1)}\Big)+\Delta t^c\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|e_{c,h}^{n_{i+1}}\|^2+\Delta t^f\sum_{i=n_{m_l}}^{n_{m_{l+1}}-1}\|\boldsymbol{u}^i-\boldsymbol{u}_h^i\|_f^2\\
&+\Delta t^c\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|c^{n_i}-c_h^{n_i}\|_p^2+\sum_{k=0}^{l}\Delta t^p\|\nabla\varphi^{n_{m_k}}-\nabla\varphi_h^{n_{m_k}}\|_p^2\Big)\\
&+\hat{C}_4\big(\frac{k_{\max}}{\mu_{\min}}+\phi_{\max})(\Delta t^c)^2+\frac{k_{\max}}{\mu_{\min}}(\Delta t^p)^2\big).
\end{aligned}\tag{132}
$$

If Δt^f, Δt^c and Δt^p are sufficiently small, we can get (117) by using Gronwalll inequality (60) and (94) in Theorem 4.1.

It remains to check the induction hypothesis (129). Note that

$$\begin{aligned}\|\boldsymbol{u}_h^i\|_{L^\infty(\Omega_f)} &\le \|P_h^{\boldsymbol{u}}\boldsymbol{u}^i\|_{L^\infty(\Omega_f)} + Kh^{-1}(\|\boldsymbol{u}_h^i - \boldsymbol{u}^i\|_{L^2(\Omega_f)} + \|\boldsymbol{u}^i - P_h^{\boldsymbol{u}}\boldsymbol{u}^i\|_{L^2(\Omega_f)})\\ &\le \|P_h^{\boldsymbol{u}}\boldsymbol{u}^i\|_{L^\infty(\Omega_f)} + Kh^{-1}(h^{(j+1)} + \Delta t^f).\end{aligned}$$

Using the same method we can derive the estimate for $\|\nabla\varphi_h^{n_{m_k}}\|_{L^\infty(\Omega_p)}$.This completes the proof. □

With the above two theorems, we conclude this section with an error estimate solely in terms of Δt_f under appropriate choices of γ_{pf} and γ_{cf}, which determine time steps for each equation.

Theorem 4.3. *Suppose that the assumptions of Theorem 4.2 hold. Let*

$$r_{pf} = \hat{C}_5\Big(\frac{k_{\max}k_{\min}}{\mu_{\max}\mu_{\min} + k_{\max}k_{\min}}\Big)^{\frac{1}{2}}, \tag{133}$$

$$r_{cf} = \hat{C}_5\Big(\frac{k_{\max}k_{\min}}{k_{\max}k_{\min} + \mu_{\max}k_{\max} + \phi_{\max}\mu_{\min}k_{\min}}\Big)^{\frac{1}{2}}, \tag{134}$$

where the positive constant $\hat{C}_5$ is independent of $\mu_{\max}$, $\mu_{\min}$, $k_{\max}$, $k_{\min}$ and $\phi_{\max}$. Then there $\hat{C}_7$ independent of h, Δt^f, Δt^c and Δt^p and independent of $\mu_{\max}$, $\mu_{\min}$, $k_{\max}$, $k_{\min}$ and $\phi_{\max}$, such that for $0 \le l \le N_p - 1$

$$\begin{aligned}&\|e_{\boldsymbol{u},h}^{n_{m_{l+1}}}\|_f^2 + \alpha_f\Delta t^f\sum_{k=0}^{l}\sum_{i=n_{m_k}}^{n_{m_{k+1}}}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2\\ &+ gS_0\|e_{\varphi,h}^{n_{m_{l+1}}}\|_p^2 + \frac{g}{\mu_{max}}\Delta t^p\sum_{k=0}^{l}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2\\ &+ \|e_{c,h}^{n_{m_{l+1}}}\|^2 + D_{\min}\Delta t^c\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla e_{c,h}^{n_{i+1}}\|^2\\ \le&\hat{C}_7\frac{k_{\max}}{\mu_{\min}}(\Delta t^f)^2 + \hat{C}_6h^{2(j+1)}.\end{aligned} \tag{135}$$

Proof. From Theorem 4.1 and Theorem 4.2, and the fact that $\alpha_f \propto \dfrac{\mu_{\min}}{k_{\max}}$ in Lemma 3.1, we have

$$\begin{aligned}&\|e_{\boldsymbol{u},h}^{n_{m_{l+1}}}\|_f^2 + \alpha_f\Delta t^f\sum_{k=0}^{l}\sum_{i=n_{m_k}}^{n_{m_{k+1}}}\|e_{\boldsymbol{u},h}^{i+1}\|_{H_f}^2 + gS_0\|e_{\varphi,h}^{n_{m_{l+1}}}\|_p^2\\ &+ \frac{g}{\mu_{max}}\Delta t^p\sum_{k=0}^{l}\|K^{1/2}\nabla e_{\varphi,h}^{n_{m_{k+1}}}\|_p^2\\ &+ \|e_{c,h}^{n_{m_{l+1}}}\|^2 + D_{\min}\Delta t^c\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla e_{c,h}^{n_{i+1}}\|^2\\ \le&\hat{C}_8\Big(\frac{k_{\max}}{\mu_{\min}}(\Delta t^f)^2 + (\frac{\mu_{\max}}{k_{\min}} + \frac{k_{\max}}{\mu_{\min}})(\Delta t^p)^2 + (\frac{k_{\max}}{\mu_{\min}} + \frac{\mu_{\max}k_{\max}}{k_{\min}\mu_{\min}} + \phi_{\max})(\Delta t^c)^2\Big)\\ &+ (\hat{C}_1 + \hat{C}_3)h^{2(j+1)}.\\ =&\hat{C}_8\Big(\frac{k_{\max}}{\mu_{\min}} + (\frac{\mu_{\max}}{k_{\min}} + \frac{k_{\max}}{\mu_{\min}})(r_{pf})^2 + (\frac{k_{\max}}{\mu_{\min}} + \frac{\mu_{\max}k_{\max}}{k_{\min}\mu_{\min}} + \phi_{\max})(r_{cf})^2\Big)(\Delta t^f)^2\\ &+ (\hat{C}_1 + \hat{C}_3)h^{2(j+1)},\end{aligned}$$

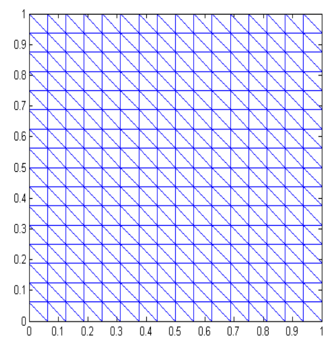


FIGURE 3. Finite element triangulation.

where $\hat{C}_8 = \max\{C(T)\hat{C}_2, \hat{C}_4\}$. Taking $\hat{C}_7 = \hat{C}_8(1 + 2(\hat{C}_5)^2)$ and $\hat{C}_6 = \hat{C}_1 + \hat{C}_3$, we complete the proof. □

5. Numerical experiments

In this section, we present some numerical experiments to demonstrate the error estimates results obtained in the previous section. For all the numerical experiments, we choose $\Omega_f = [0,1] \times [0,1]$ and $\Omega_p = [1,2] \times [0,1]$ with the interface $\Gamma_I = \{1\} \times (0,1)$, and the time interval is $J = [0,1]$. unless specified otherwise, the values of the parameters are assigned as $\phi = 1$, $g = \gamma = d = q^I = S_0 = 1$, $D = I$, $K = \frac{1}{2}I$. A typical triangulation $\mathscr{T}_h$ is depicted in Figure 3. The upper and lower bounds of μ in (22) are $\mu(1) = 37/21$ and $\mu(0) = 1/50$. Since it is difficult to construct the exact solutions that satisfy the entire coupled Stokes and Darcy flows with mass transport (1)− (19), especially because of the the interface conditions, we generalize the interface conditions (9), (10) and (13) (as in [33]) to include nonhomogeneous terms:

$$p - \boldsymbol{n}_f \cdot 2\mu(c_f)\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{n}_f = g\varphi + \eta_1, \tag{136}$$

$$2\boldsymbol{n}_f \cdot \mu(c_f)\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{\tau} + \mu(c_f)\gamma\boldsymbol{u} \cdot \boldsymbol{\tau} = \eta_2, \tag{137}$$

$$\boldsymbol{u} \cdot \boldsymbol{n}_f = \lambda(c_p)\frac{\partial\varphi}{\partial\boldsymbol{n}_p} + \eta_3, \tag{138}$$

where η_1, η_2 and η_3 are given functions on Γ_I according to the analytical solutions. The variational form for this modified system will include two additional terms $-\langle\eta_1, \boldsymbol{v}_h \cdot \boldsymbol{n}_f\rangle_{\Gamma_I} + \langle\eta_2, \boldsymbol{v}_h \cdot \boldsymbol{\tau}\rangle_{\Gamma_I}$ to the right-hand side of (64), and one additional term $-\langle\eta_3, \psi_h\rangle_{\Gamma_I}$ to the right-hand side of (67). We select all the right-hand terms and boundary conditions according to the analytical solution. Notation wise, we use the following symbols to represent the computational errors.

$$\begin{aligned}
&\|E^h_{\boldsymbol{u}}\|_{l^\infty(L^2)} := \max_n \|\boldsymbol{u}^n - \boldsymbol{u}^n_h\|, \quad \|E^h_{\boldsymbol{u}}\|_{l^\infty(H_f)} := \max_n \|\boldsymbol{u}^n - \boldsymbol{u}^n_h\|_{H_f},\\
&\|E^h_p\|_{l^\infty(L^2)} := \max_n \|p^n - p^n_h\|,\\
&\|E^h_\varphi\|_{l^\infty(L^2)} := \max_n \|\varphi^n - \varphi^n_h\|, \quad \|E^h_\varphi\|_{l^\infty(H_p)} := \max_n \|\varphi^n - \varphi^n_h\|_{H_p},\\
&\|E^h_c\|_{l^\infty(L^2)} := \max_n \|c^n - c^n_h\|, \quad \|E^h_c\|_{l^\infty(H^1)} := \max_n |c^n - c^n_h|_1.
\end{aligned} \tag{139}$$

With these notations in hand, we compute the convergence rate (degenerate rate) of the approximate solutions by

$$rate = \frac{\log(\|E^{h_1}\|/\|E^{h_2}\|)}{\log(h_1/h_2)}, \tag{140}$$

where h_1 and h_2 are the meshsizes and $E^{h_i}, i = 1, 2$ are any of the errors described in (139). The finite element spaces are constructed using the well-known MINI elements (P1b-P1) for the Stokes problem and the linear Lagrangian elements (P1) for the Darcy flow and concentration. We use two numerical examples to verify the convergence rates of the decoupled scheme (64)− (67) and illustrate the theoretical conclusion of the stability (73). In each example we also compare the errors, convergence rates and CPU times (unit: second) for the coupled scheme (CS for short) (63) and decoupled scheme (DS for short) (64)− (67) with different time steps.

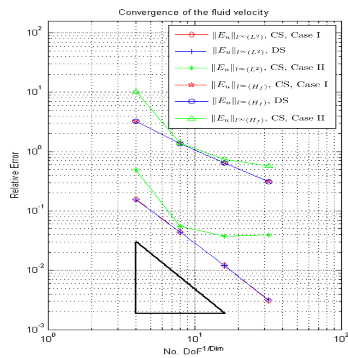
Table 1. The convergence performance and CPU time of coupling scheme in Example 1.

h	$\|E_{\boldsymbol{u}}\|_{l^\infty(L^2)}$	$\|E_{\boldsymbol{u}}\|_{l^\infty(H_f)}$	$\|E_p\|_{l^\infty(L^2)}$	$\|E_\varphi\|_{l^\infty(L^2)}$
1/4	1.57e-1	3.22e+0	1.96e+0	1.49e-1
1/8	4.41e-2	1.36e+0	6.20e-0	9.57e-3
1/16	1.21e-2	6.41e-1	2.35e-1	2.40e-3
1/32	3.07e-3	3.12e-1	1.04e-1	6.71e-4
rate	1.97	1.04	1.18	1.84
h	$\|E_\varphi\|_{l^\infty(H_p)}$	$\|E_c\|_{l^\infty(L^2)}$	$\|E_c\|_{l^\infty(H^1)}$	CPU time (s)
1/4	5.21e-1	1.41e-1	4.76e-1	0.31
1/8	2.52e-1	2.08e-2	1.66e-1	6.61
1/16	1.25e-1	4.97e-3	8.09e-2	378.52
1/32	6.21e-2	1.46e-3	4.02e-2	21766.00
rate	1.00	1.77	1.01	-

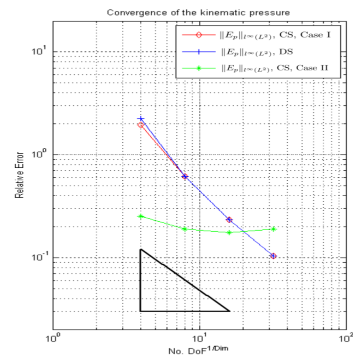
Table 2. The convergence performance and CPU time of decoupling scheme with different time steps in Example 1.

h	$\|E_{\boldsymbol{u}}\|_{l^\infty(L^2)}$	$\|E_{\boldsymbol{u}}\|_{l^\infty(H_f)}$	$\|E_p\|_{l^\infty(L^2)}$	$\|E_\varphi\|_{l^\infty(L^2)}$
1/4	1.55e-1	3.22e+0	2.25e+0	2.07e-2
1/8	4.37e-2	1.36e+0	6.16e-0	5.14e-3
1/16	1.21e-2	6.41e-1	2.35e-1	1.35e-3
1/32	3.14e-3	3.12e-1	1.04e-1	4.19e-4
rate	1.94	1.04	1.18	1.88
h	$\|E_\varphi\|_{l^\infty(H_p)}$	$\|E_c\|_{l^\infty(L^2)}$	$\|E_c\|_{l^\infty(H^1)}$	CPU time (s)
1/4	4.99e-1	2.02e-1	6.33e-1	0.09
1/8	2.50e-1	4.70e-2	1.72e-1	0.53
1/16	1.24e-1	9.65e-3	8.16e-2	27.80
1/32	6.21e-2	2.78e-3	4.04e-2	1628.20
rate	1.00	1.80	1.01	-

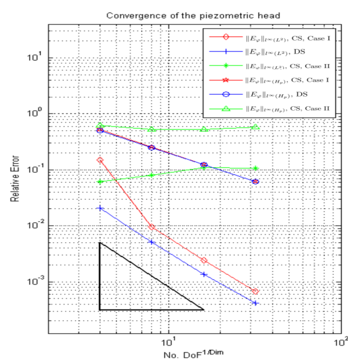
We listed and plot all the errors in (139) and the corresponding convergence rate for $h = 2^{-m}, m = 2, 3, 4, 5$, and the time step $\Delta t = 1/100$ in (63) and $\Delta t^f = 1/100$ (Case I) in (64)− (67), and to illustrate (73)we also plot the errors for $h = 2^{-m}, m =$



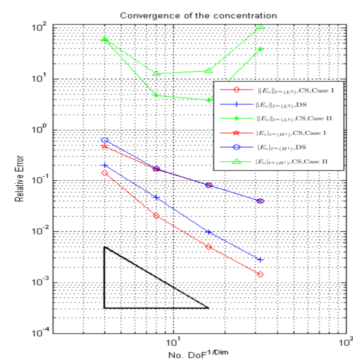
(a) Fluid velocity



(b) Kinematic pressure



(c) Piezometric



(d) Concentration

FIGURE 4. Convergence rates of Example 1. The tangent of the triangle in (b) is 1 and those in (a), (c) and (d) are 2.

$2, 3, 4, 5$ and $\Delta t^f = 1/8$ (Case II) in (64)– (67). To make sure that r_{cf} and r_{pc} are integers, we take $\hat{C}_5 = \sqrt{\frac{9584}{525}} \approx 4.27$ so that $r_{cf} = 2$, $r_{pc} = 2$ in the decoupled scheme (64)– (67).

Example 1 In this example, we choose the nonhomogenenous functions in (5.1)-(5.3) so that The exact solutions for the fluid velocity, kinematic pressure, piezometric head, and concentration are given by

$$\begin{cases} \boldsymbol{u} = (\sin(y^2 + 6x + t), \cos(4x^2 y)e^t)^T, & p = 2(y-1)\cos^2 x e^t, \\ \varphi = 1/6\cos(y^2 + 6x + t), & c = x(2-x)y(1-y)e^{-t}. \end{cases}$$

The numerical results and the convergence rates of Example 1 are listed in Table 1 and Table 2. The errors and convergence rates are plotted with respect of nodes on each direction in Figure 4.

Example 2 In this example, the analytical fluid velocity, kinematic pressure, piezometric head and concentration are given by

$$\begin{cases} \boldsymbol{u} = (e^{-(x+2y+t)}, e^{xy}\sin(t))^T, & p = 12x^2 e^y \sin(t), \\ \varphi = e^{-(x+2y+t)}, & c = \sin(\pi x)\sin(2\pi y)e^t. \end{cases}$$

The errors and the convergence rates of Example 2 are listed in Table 3 and Table 4. The errors and convergence rates are plotted with respect of nodes on each direction in Figure 5.

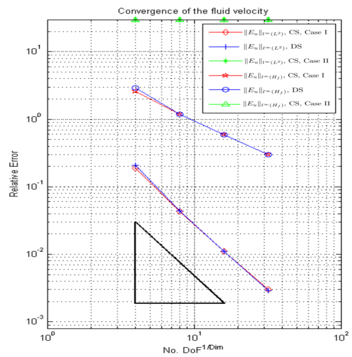
Table 3. The convergence performance and CPU time of coupling scheme in Example 2.

h	$\|E_{\boldsymbol{u}}\|_{l^\infty(L^2)}$	$\|E_{\boldsymbol{u}}\|_{l^\infty(H_f)}$	$\|E_p\|_{l^\infty(L^2)}$	$\|E_\varphi\|_{l^\infty(L^2)}$
1/4	1.88e-1	2.59e+0	1.66e+0	2.80e-2
1/8	4.32e-2	1.18e+0	8.73e-0	4.21e-3
1/16	1.11e-2	5.95e-1	4.58e-1	7.16e-4
1/32	3.05e-3	3.02e-1	2.35e-1	2.01e-4
rate	1.86	0.98	0.96	1.83
h	$\|E_\varphi\|_{l^\infty(H_p)}$	$\|E_c\|_{l^\infty(L^2)}$	$\|E_c\|_{l^\infty(H^1)}$	CPU time (s)
1/4	1.55e-1	1.25e+0	1.77e+1	0.31
1/8	3.28e-2	2.43e-1	6.76e+0	7.67
1/16	1.39e-2	5.88e-2	3.18e+0	431.02
1/32	7.03e-3	1.43e-2	1.57e+0	25370.00
rate	0.98	2.03	1.02	-

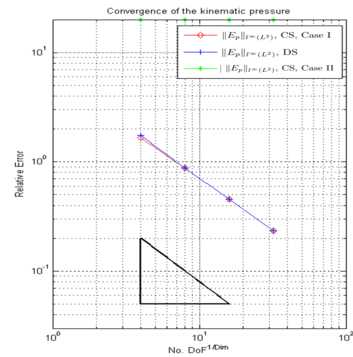
Table 4. The convergence performance and CPU time of decoupling scheme with different time steps in Example 2.

h	$\|E_{\boldsymbol{u}}\|_{l^\infty(L^2)}$	$\|E_{\boldsymbol{u}}\|_{l^\infty(H_f)}$	$\|E_p\|_{l^\infty(L^2)}$	$\|E_\varphi\|_{l^\infty(L^2)}$
1/4	2.11e-1	2.91e+0	1.74e+0	2.29e-2
1/8	4.36e-2	1.19e+0	8.77e-0	3.82e-3
1/16	1.11e-2	5.96e-1	4.58e-1	6.12e-4
1/32	2.93e-3	3.02e-1	2.35e-1	1.60e-4
rate	1.92	0.98	0.96	1.94
h	$\|E_\varphi\|_{l^\infty(H_p)}$	$\|E_c\|_{l^\infty(L^2)}$	$\|E_c\|_{l^\infty(H^1)}$	CPU time (s)
1/4	1.39e-1	1.60e+0	2.09e+1	0.06
1/8	3.22e-2	2.73e-1	6.93e+0	0.53
1/16	1.37e-2	6.44e-2	3.20e+0	33.42
1/32	6.86e-3	1.55e-2	1.57e+0	2149.90
rate	0.99	2.05	1.03	-

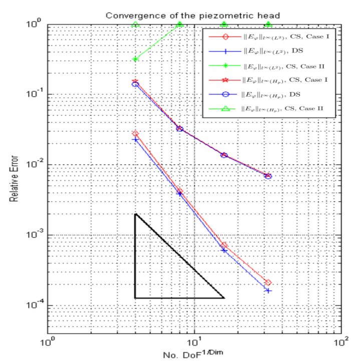
From Tables 1-4 and Figures 4-5 we find that the numerical results are consistent with our theoretical analysis. In particular, the convergence rates for fluid velocity in $l^\infty(H_f)$ norm, kinematic pressure in $l^\infty(L^2)$ norm, piezometric head in $l^\infty(H_p)$ norm and concentration in semi-$l^\infty(H^1)$ norm are first-order, and the convergence rates for fluid velocity in $l^\infty(L^2)$ norm, piezometric head in $l^\infty(L^2)$ norm and concentration in $l^\infty(L^2)$ norm are second-order. All the convergence rates are optimal. And we find that the errors are basically non-convergence when taking a large time step $\Delta t^f = 1/8$. It means that Δt^f should be relatively small to ensure the stability, which illustrate the theoretical conclusion of the stability (73). What's more, from the comparison of coupled scheme and decoupled scheme, we find that, the errors are similar in every discretization parameter, and the convergence rates are the same, but the decoupled scheme costs less CPU time than coupled scheme. The comparison verifies that our method with different subdomain time steps is useful in increasing computational efficiency.



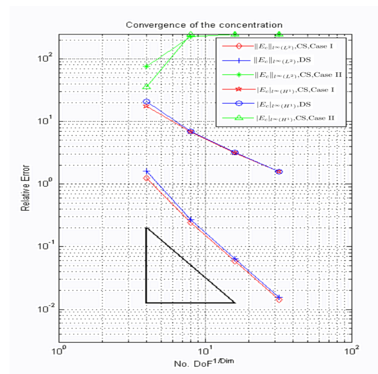
(a) Fluid velocity



(b) Kinematic pressure



(c) Piezometric



(d) Concentration

FIGURE 5. Convergence rates of Example 2. The tangent of the triangle in (b) is 1 and those in (a), (c) and (d) are 2.

Acknowledgments

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