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GLOBAL STABILITY OF CRITICAL TRAVELING WAVES WITH OSCILLATIONS FOR TIME-DELAYED REACTION-DIFFUSION EQUATIONS

MING MEI, KAIJUN ZHANG*, AND QIFENG ZHANG

Abstract. For a class of non-monotone reaction-diffusion equations with time-delay, the large time-delay usually causes the traveling waves to be oscillatory. In this paper, we are interested in the global stability of these oscillatory traveling waves, in particular, the challenging case of the critical traveling waves with oscillations. We prove that, the critical oscillatory traveling waves are globally stable with the algebraic convergence rate $t^{-1/2}$, and the non-critical traveling waves are globally stable with the exponential convergence rate $t^{-1/2}e^{-\mu t}$ for some positive constant μ, where the initial perturbations around the oscillatory traveling wave in a weighted Sobolev can be arbitrarily large. The approach adopted is the technical weighted energy method with some new development in establishing the boundedness estimate of the oscillating solutions, which, with the help of optimal decay estimates by deriving the fundamental solutions for the linearized equations, can allow us to prove the global stability and to obtain the optimal convergence rates. Finally, numerical simulations in different cases are carried out, which further confirm our theoretical stability for oscillatory traveling waves, where the initial perturbations can be large.

Key words. Nicholson's blowflies equation, time-delayed reaction-diffusion equation, critical traveling waves, oscillation, stability, numerical simulations.

1. Introduction and main result

This is a continuation of the previous studies [4, 25] on the stability of oscillatory traveling waves for a class of non-monotone reaction-diffusion equations with time-delay

$$(1)\qquad \begin{cases} \dfrac{\partial v(t,x)}{\partial t} - D\dfrac{\partial^2 v(t,x)}{\partial x^2} + dv(t,x) = b(v(t-r,x)), & (t,x)\in \mathbb{R}_+\times\mathbb{R}, \\ v(s,x) = v_0(s,x), \quad s\in[-r,0],\ x\in\mathbb{R}, \end{cases}$$

which describes the population dynamics of a single species like the Australian blowflies [13, 14, 29, 30, 33, 43]. Here, $v(t,x)$ represents the mature population at time t and location x, $D>0$ the spatial diffusion rate of the mature species, $d>0$ the death rate, and $r>0$ the maturation delay. As described in [4, 25], $b:[0,\infty)\to(0,\infty)$ is the birth rate function, and is assumed to satisfy the following hypothesis:

- (H_1) Two constant equilibria $v_\pm$: $b(v_\pm)-dv_\pm=0$ for the homogeneous part of (1). We may take $v_-=0$ and thus $b(0)=0$. We further assume that v_- is unstable and v_+ is stable for the homogeneous part of (1). That is, $d-b'(0)<0$ and $d-b'(v_+)>0$.
- (H_2) The uni-modality condition: there is a $v_*\in(0,v_+)$ such that $b(\cdot)$ is increasing on $[0,v_*]$ and decreasing on $[v_*,+\infty)$. In particular, $b'(0)>0$ and $b'(v_+)<0$.
- (H_3) $b\in C^2[0,\infty)$ and $|b'(v)|\le b'(0)$ for $v\in[0,\infty)$.

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*Corresponding author.

Clearly, Hypothesis (H_1) implies that (1) is a mono-stable system, namely, one equilibrium of (1) is stable and the other one is unstable. The typical model for such mono-stable equations is the classic Fisher-KPP equation

$$v_t - v_{xx} = v(1-v).$$

Hypothesis (H_2) means that $b(v)$ is non-monotone for $v \in [0, v_+]$. As we shall see later, this leads to some oscillations for the solutions when the time-delay r is big.

There are also two featured examples for the equation (1) satisfying (H_1)-(H_3). One is the Nicholson's blowflies model by taking the birth rate function as

$$b(v) = pve^{-av}, \quad a > 0,\ p > 0, \tag{2}$$

where the constant equilibria are $v_- = 0$ and $v_+ = \frac{1}{a}\ln\frac{p}{d}$, and $b(v)$ is unimodal on $v \in [0, v_+]$ for $p/d > e$, and satisfies $|b'(v)| \le b'(0)$ for $v \in [0, \infty)$. This model was initially proposed by Gurney, Blythe, and Nisbet [11] based on the experiment data of blowflies by Nicholson [36, 37], see also the follow-up studies on wellposedness and asymptotic behavior of solutions in [13, 14, 19, 29, 30, 33, 43, 44].

The other is the Mackey-Glass model proposed in [27] (see also [12, 23, 29, 30] for further studies) by setting the birth rate function as

$$b(v) = \frac{pv}{1+av^q}, \quad a > 0, \quad p > 0, \quad q > 1, \tag{3}$$

where $v_- = 0$ and $v_+ = \left(\frac{p-d}{da}\right)^{\frac{1}{q}}$. $b(v)$ is unimodal for $v \in [0, v_+]$ for $\frac{p}{d} > \frac{q}{q-1}$, satisfies $|b'(v)| \le b'(0)$ for $v \in [0, \infty)$.

Throughout this paper, naturally we assume that

$$\lim_{x\to\pm\infty} v_0(s,x) = v_\pm \quad \text{uniformly in } s \in [-r, 0]. \tag{4}$$

A traveling wave for (1) is a special solution to (1) of the form $\phi(x+ct) \ge 0$ with $\phi(\pm\infty) = v_\pm$:

$$\begin{cases} c\phi'(\xi) - D\phi''(\xi) + d\phi(\xi) = b(\phi(\xi - cr)), \\ \phi(\pm\infty) = v_\pm, \end{cases} \tag{5}$$

where $\xi = x + ct$, $' = \frac{d}{d\xi}$, and c is the wave speed. As summarized in [4, 25], there exists a number $c_* > 0$, called the minimum wave speed, which is uniquely determined by

$$c_*\lambda_* - D\lambda_*^2 + d = b'(0)e^{-\lambda_* c_* r} \quad \text{and} \quad c_* - 2D\lambda_* = -c_* r b'(0) e^{-\lambda_* c_* r}, \tag{6}$$

and when $c > c_*$, there exist two numbers $\lambda_2 > \lambda_1 > 0$ such that

$$c\lambda_i - D\lambda_i^2 + d = b'(0)e^{-\lambda_i cr}, \qquad \text{for } i = 1, 2, \tag{7}$$

and

$$c\lambda - D\lambda^2 + d > b'(0)e^{-\lambda cr}, \qquad \text{for } \lambda \in (\lambda_1, \lambda_2). \tag{8}$$

As showed in [6, 7, 12, 26, 47, 48], see also the summary in [4, 25], we have the following existence and uniqueness of the traveling waves as well as the property of oscillations:

• When $d \ge |b'(v_+)|$, the traveling wave $\phi(x+ct)$ exists uniquely (up to a shift) for every $c \ge c_* = c_*(r)$, where the time-delay r is allowed to be any number in $[0, \infty)$. If $0 \le r < \underline{r}$, where $\underline{r}$, given by

$$|b'(v_+)|\underline{r}e^{d\underline{r}+1} = 1, \tag{9}$$

is the critical point for the traveling waves to possibly occur oscillations, then these traveling waves are monotone [12]; while, if the time delay $r \geq \underline{r}$, then the traveling waves are still monotone for $(c, r) \in [c_*, c^*] \times [\underline{r}, r_0]$, where $c_* = c_*(r)$ is the minimum wave speed as mentioned before, $c^* = c^*(r)$ is given by the characteristic equation for (5) around v_+, namely, the pair of (c^*, λ^*) is determined by

$$(10) \quad -c^*\lambda^* - D(\lambda^*)^2 + d = b'(v_+)e^{\lambda^* c_* r}, \quad \text{and} \quad -c^* - 2D\lambda^* = b'(v_+)c^* r e^{\lambda^* c^* r},$$

and $r_0(> \underline{r})$ is the unique intersection point of two curves $c_*(r)$ and $c^*(r)$; and the traveling waves are oscillating around v_+ for $(c, r) \notin [c_*, c^*] \times [\underline{r}, r_0]$, namely, either $c > c^*$ or $r > r_0$ (c.f. [12, 25]).

• When $d < |b'(v_+)|$, on the other hand, the traveling wave $\phi(x + ct)$ with $c \geq c_*$ can exist only when $r < \overline{r}$, and no traveling wave can exist for $r \geq \overline{r}$, where $\overline{r}$ is the Hopf-bifurcation point given by

$$(11) \qquad \overline{r} := \frac{\pi - \arctan(\sqrt{|b'(v_+)|^2 - d^2}/d)}{\sqrt{|b'(v_+)|^2 - d^2}}.$$

In the case of $r < \overline{r}$, the waves are monotone for $0 < r < \underline{r}$ and oscillating for $r \in (\underline{r}, \overline{r})$ (c.f. [12, 25]).

Since the traveling waves $\phi(x + ct)$ exist for $c \geq c_*$, and the waves do not exist for $c < c_*$, we call c_* as the critical wave speed, and the corresponding traveling wave $\phi(x + c_* t)$ is called the critical wave, while the waves $\phi(x + ct)$ with $c > c_*$ are said to be non-critical. The study on the critical traveling waves in the biological invasions is particularly interesting but also quite challenging, because the critical wave speed is usually the spreading speed for all solutions with initial data having compact supports [24, 45].

For regular mono-stable reaction-diffusion equations without time-delay ($r = 0$) including the classic Fisher-KPP equation, the existence of traveling waves and their stability have been one of the hot research spots. In 1976, by using the spectral analysis method, Sattinger [41] first proved that, for given non-critical waves with $c > c_*$, when the initial perturbations around the waves are space-exponentially decay at the far field $-\infty$, then these non-critical waves are time-exponentially stable. Since then, the study on stability of non-critical traveling waves has been intensively studied, for example, see [1, 3, 8–10, 15–17, 21, 22, 28, 42] and the references therein, see also the textbook [50] and the survey paper [52]. However, the study on stability of critical traveling waves with $c = c_*$ is very limited, because this is critical case with special difficulty. In 1978, by using the maximum principle method, Uchiyama [49] proved the local stability for the traveling waves including the critical waves, but no convergence rate for the critical waves case was related. Later then, Bramson [2] derived the sufficient and necessary condition for the stability of noncritical and critical waves (no convergence rates issued) by probability method. Lau [22] obtained the same results in a different way. Regarding the convergence rates to the critical traveling waves, Moet [35] first obtained the algebraic convergence rate $O(t^{-1/2})$ by using the Green function method. Kirchgässner [21] then showed the algebraic stability for the critical waves in the form $O(t^{-1/4})$ by the spectral method. Gallay [9] further improved the algebraic rate to $O(t^{-3/2})$ by using the renormalization group method, when the corresponding initial data converges to the critical wave much fast like $O(e^{-x^2/4})$ as $x \to -\infty$. More general case for parabolic equations was investigated by Eckmann and Wayne in [5].

For the mono-stable reaction-diffusion equations with time-delay, in 1987 Schaaf [40] first proved the linear stability for the non-critical traveling waves by the spectral analysis method. This topic was not touched until in 2004 Mei-So-Li-Shen [33] proved the nonlinear stability of fast traveling waves with $c \gg c_*$ by the weighted energy method. Since then, when the equation is monotone (namely, $b(v)$ is increasing), Mei and his collaborators [29–32, 34] further showed that all non-critical traveling waves are exponentially stable and all critical traveling waves are algebraically stable. When $b(v)$ is non-monotone, the equation (1) losses its monotonicity. The solution is usually oscillating as the time-delay r is large, and the traveling waves may occur oscillations around v_+, as theoretically proved in [12, 48] and numerically reported in [4, 25]. Such oscillations of the traveling waves are interesting and important from both physical and mathematical points of view. Recently, by using the weighted energy method with the help of nonlinear Hanalay's inequality, Lin-Lin-Lin-Mei [25] proved that when the initial perturbation is small, then all non-critical oscillatory traveling waves are locally stable with exponential convergence rate. Furthermore, by analyzing the decay rate of the (oscillatory) critical traveling waves at the unstable node $v_- = 0$, and applying the anti-weighted energy method, Chern-Mei-Yang-Zhang [4] obtained the local stability for the critical oscillatory traveling waves. But the convergence rate to the critical waves is still open. The interesting but also challenging questions are whether these oscillatory wavefronts are globally stable, and what will be the optimal convergence rates, particularly the convergence rate for the critical wavefronts. Note that, the existing methods cannot be applied to our case, due to the lack of monotonicity of the equation and the waves, and the bad effect of time-delay. Of course, the critical wave case is always challenging as we know.

In this paper, the main targets are to prove that all critical or non-critical oscillatory traveling waves are globally stable, and further to derive the optimal convergence rates to the critical/non-critical oscillatory traveling wave. That is, for all oscillatory traveling waves, including the critical traveling waves, the original solution of (1) will time-asymptotically converge to the targeted wave, even if the corresponding initial perturbation in a certain weighted space is big. The optimal convergence rates for the critical/non-critical wavefronts are $O(t^{-\frac{1}{2}})$ and $O(t^{-\frac{1}{2}}e^{-\mu t})$, respectively. As mentioned in [4], the usual approaches for deriving the convergence rate are either the monotonic method with the help of the decay estimates for linearized equations [29, 31], or Fourier transform [18, 34], or the multiplier method [39, 46], or the method of approximate Green function [38, 51]. Since our equation is lack of monotonicity and the traveling waves are oscillating, and the bad effect of time-delay, it seems that we could not be able to adopt these methods mentioned before. However, we have some key observation, that is, although the equation and the waves are non-monotone, and we loss the comparison principle, due to the structure of the governing equation, we realize that the absolute value of the oscillating solution can be bounded by the positive solution of linear delayed heat equation with constant coefficients. This can make us possibly to reach our goal by deriving the optimal decay estimate for the linearized solution, where, by using Fourier transform, we derive the fundamental solution for the time-delayed linear heat equation and its optimal convergence rate. Based on the boundedness estimates for the oscillatory solution, we further prove the global stability for the oscillatory traveling waves. To our best knowledge, this is the first frame work to show the global stability for the oscillatory traveling waves with the optimal convergence rates, particularly, for the critical wave case.

Before stating our main results, we first introduce some notations on the solution spaces. Throughout this paper, $C > 0$ denotes for a generic constant, and $C_i > 0$ $(i = 0, 1, 2, \cdots)$ for specific positive constants. $L^2(\mathbb{R})$ is the space of the square integrable functions, $H^k(\mathbb{R})$ is the Sobolev space, $C(\mathbb{R})$ is the space of bounded continuous functions, $C([0,T];\mathbf{B})$ is the space of the $\mathbf{B}$-valued continuous functions on $[0,T]$, where $\mathbf{B}$ is a Banach space, and $T > 0$ is a number. Similarly, $L^2([0,T];\mathbf{B})$ is the space of the $\mathbf{B}$-valued L^2-functions on $[0,T]$.

To handle delay equation with delay r, as denoted in [4, 25], we define the uniformly continuous space $\mathcal{C}_{unif}[-r,T]$, for $0 < T \le \infty$, by

$$\begin{aligned} \mathcal{C}_{unif}[-r,T] := \{ & v(t,x) \in C([-r,T]\times\mathbb{R}) \text{ such that} \\ & \lim_{x\to+\infty} v(t,x) \text{ exists uniformly in } t \in [-r,T], \text{ and} \\ & \lim_{x\to+\infty} v_x(t,x) = \lim_{x\to+\infty} v_{xx}(t,x) = 0 \\ & \text{uniformly with respect to } t \in [-r,T]\}. \end{aligned} \tag{12}$$

For perturbed equation around the traveling waves, we now define the solution space. First, we introduce a weight function. For $c \ge c_*$, we define a weight function

$$w(\xi) := \begin{cases} e^{-2\lambda\xi}, & \xi \in \mathbb{R}, \quad \text{for } c > c_*,\ \lambda \in (\lambda_1, \lambda_2), \\ e^{-2\lambda_*\xi}, & \xi \in \mathbb{R}, \quad \text{for } c = c_*, \end{cases} \tag{13}$$

where $\xi = x + ct$ for $(t,x) \in [-r,\infty)\times\mathbb{R}$, and the numbers λ_1 and λ_2 are specified in (7). Notice that, for $c \ge c_*$, $\lim_{\xi\to-\infty} w(\xi) = +\infty$ and $\lim_{\xi\to+\infty} w(\xi) = 0$, because $\lambda > 0$ and $\lambda_* > 0$. We also denote the weighted Sobolev space $W^{2,1}_w(\mathbb{R})$ by

$$W^{2,1}_w(\mathbb{R}) = \{u | wu \in L^1(\mathbb{R}), w\partial^i_\xi u \in L^1(\mathbb{R}),\ i = 1, 2\}.$$

Our main stability theorems are as follows.

Theorem 1.1 (Global stability with optimal convergence rates)**.** *Assume that $(H_1)-(H_3)$ hold. Let $b'(v_+)$ and r satisfy, either $d \ge |b'(v_+)|$ with arbitrarily given $r > 0$, or $d < |b'(v_+)|$ with $0 < r < \overline{r}$, where $\overline{r}$ is defined in* (11)*. Let $\phi(x+ct) = \phi(\xi)$ be any given traveling wave with $c \ge c_*$, no matter it is oscillatory or not, and the initial perturbation be $v_0(s,x) - \phi(x+cs) \in C_{unif}(-r,0) \cap C([-r,0]; W^{2,1}_w(\mathbb{R}))$ and $\partial_s(v_0 - \phi) \in L^1([-r,0]; L^1_w(\mathbb{R}))$, even if it is arbitrarily large, then the global solution $v(t,x)$ of* (1) *uniquely exists such that $v - \phi \in C_{unif}(-r,0)$ satisfies*

- *if $c > c_*$, then*

$$\sup_{x\in\mathbb{R}} |v(t,x) - \phi(x+ct)| \le Ct^{-\frac{1}{2}} e^{-\mu t} \tag{14}$$

for some positive constant μ satisfying

$$0 < \mu < \min\{d, \quad c\lambda + d - D\lambda^2 - b'(0)e^{-\lambda cr}\}, \tag{15}$$

where $\lambda \in (\lambda_1, \lambda_2)$.

- *if $c = c_*$, then*

$$\sup_{x\in\mathbb{R}} |v(t,x) - \phi(x+c_*t)| \le Ct^{-\frac{1}{2}}. \tag{16}$$

Remark 1.2. (1) *When the equation* (1) *is non-monotone, and the time-delay r is large, the original solution $v(t,x)$ and the traveling wave $\phi(x+ct)$ with $c \ge c_*$ both are oscillatory. As we know, the oscillatory waves are usually expected to be locally stable when the initial perturbations around the waves are sufficiently small. However, in Theorem 1.1, we obtain the* *global stability of these oscillatory traveling waves, where the initial perturbations around the oscillatory waves in the weighted space* $C_{unif}(-r,0)\cap C([-r,0];W^{2,1}_w(\mathbb{R}))$ *can be arbitrarily large. To our best knowledge, this is the first work to show the global stability for the oscillatory waves.*

(2) *When* $c=c^*$*, we obtain the optimally algebraic convergence rate* $O(t^{-\frac{1}{2}})$ *to the critical oscillating waves. This answers the open question left in [4]. The algebraic convergence rate also matches what shown for the critical monotone traveling waves for Fisher-KPP equations with or without time-delay [31, 34, 35].*

(3) *When* $c>c^*$*, the obtained exponential convergence* $O(e^{-\mu}t^{-\frac{1}{2}})$ *improves the the previous study for the non-critical oscillating waves in [25].*

Corollary 1.3 (Stability for Nicholson's blowflies equation)**.** *Let* $b(v)=pve^{-av}$ *for* $p>0$ *and* $a>0$*. For any given traveling wave* $\phi(x+ct)=\phi(\xi)$ *with* $c\ge c_*$*, no matter it is oscillatory or not, when the initial perturbation satisfies* $v_0(s,\xi)-\phi(\xi)\in C_{unif}(-r,0)\cap C([-r,0];W^{2,1}_w(\mathbb{R}))$ *and* $\partial_s(v_0-\phi)\in L^1([-r,0];L^1_w(\mathbb{R}))$*, then the following global stability holds.*

- *When* $e<\frac{p}{d}\le e^2$ *(equivalently to* $d\ge|b'(v_+)|$*), for any time-delay* $r>0$*, then*

$$\begin{cases}\sup_{x\in\mathbb{R}}|v(t,x)-\phi(x+ct)|\le Ct^{-\frac{1}{2}}e^{-\mu t} & \text{for } c>c_*,\\ \sup_{x\in\mathbb{R}}|v(t,x)-\phi(x+c_*t)|\le Ct^{-\frac{1}{2}} & \text{for } c=c_*,\end{cases}\tag{17}$$

with some positive number μ *satisfying* (15).

- *When* $\frac{p}{d}>e^2$ *(equivalently to* $d<|b'(v_+)|$*) but with a small time-delay* $0<r<\overline{r}$*, where*

$$\overline{r}:=\frac{\pi-\arctan\sqrt{\ln\frac{p}{d}(\ln\frac{p}{d}-2)}}{d\sqrt{\ln\frac{p}{d}(\ln\frac{p}{d}-2)}},\tag{18}$$

then the stability (17) *holds.*

Corollary 1.4 (Stability for Mackey-Glass equation)**.** *Let* $b(v)=\frac{pv}{1+av^q}$ *for* $p>0$*,* $q>1$ *and* $a>0$*. For any given traveling wave* $\phi(x+ct)=\phi(\xi)$ *with* $c\ge c_*$*, no matter it is oscillatory or not, when the initial perturbation satisfies* $v_0(s,\xi)-\phi(\xi)\in C_{unif}(-r,0)\cap C([-r,0];W^{2,1}_w(\mathbb{R}))$ *and* $\partial_s(v_0-\phi)\in L^1([-r,0];L^1_w(\mathbb{R}))$*, then the following global stability holds.*

- *When* $\frac{q}{q-1}<\frac{p}{d}\le\frac{q}{q-2}$ *(equivalently to* $d\ge|b'(v_+)|$*), for any time-delay* $r>0$*, then, then, the solution* $v-\phi\in X(0,\infty)$ *satisfies* (15) *with* $c\ge c_*$*;*
- *When* $\frac{p}{d}>\frac{q}{q-2}$ *(equivalently to* $d<|b'(v_+)|$*) but with a small time-delay* $0<r<\overline{r}$*, where*

$$\overline{r}:=\frac{\pi-\arctan\left(d^{-1}\sqrt{[(q-1)\frac{p}{d}-q]^2-d^2}\right)}{\sqrt{[(q-1)\frac{p}{d}-q]^2-d^2}},\tag{19}$$

then, the solution $v-\phi$ *satisfies* (15) *with* $c\ge c_*$*.*

2. Proof of main theorem

Let $\phi(x+ct)=\phi(\xi)$ be any given monotne/non-monotone traveling wave with $c\ge c_*$, and define

$$u(t,\xi):=v(t,x)-\phi(x+ct),\quad u_0(s,\xi):=v_0(s,x)-\phi(x+cs),\quad c\ge c_*.$$

Then, from (1), $u(t,\xi)$ satisfies, for $c \geq c_*$,

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial \xi} - D\dfrac{\partial^2 u}{\partial \xi^2} + du = P(u(t-r,\xi-cr)), & (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(s,\xi) = u_0(s,\xi), \quad s \in [-r,0], \ \xi \in \mathbb{R}, \end{cases} \tag{20}$$

where

$$P(u) := b(\phi + u) - b(\phi) = b'(\tilde{\phi})u, \tag{21}$$

for some $\tilde{\phi}$ between ϕ and $\phi + u$, with $\phi = \phi(\xi - cr)$ and $u = u(t-r,\xi-cr)$ for $c \geq c_*$.

We first prove the existence and uniqueness of solution to the initial value problem (20) in the uniformly continuous space $C_{unif}[-r,\infty)$.

Proposition 2.1 (Existence and Uniqueness)**.** *Assume (H_1)-(H_3) hold. If the initial perturbation $u_0 \in C_{unif}[-r,0]$ for $c \geq c_*$, then the solution $u(t,\xi)$ of the perturbed equation* (20) *is unique and time-globally exits in $C_{unif}[-r,\infty)$.*

Proof. When $t \in [0,r]$, since $t-r \in [-r,0]$ and $u(t-r,\xi-cr) = u_0(t-r,\xi-cr)$, then (20) reduce to the following linear equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial \xi} - D\dfrac{\partial^2 u}{\partial \xi^2} + du = P(u_0(t-r,\xi-cr)), & (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,\xi) = u_0(0,\xi), \quad \xi \in \mathbb{R}, \end{cases} \tag{22}$$

and it possesses a unique solution $u(t,\xi)$ for $t \in [0,r]$ in the integral form of

$$\begin{aligned} u(t,\xi) &= e^{-dt}\int_{-\infty}^{\infty} G(t,\eta)u_0(0,\xi-\eta)d\eta \\ &\quad + \int_0^t e^{-d(t-s)}\int_{-\infty}^{\infty} G(t-s,\eta)P(u_0(s-r,\xi-\eta-cr)d\eta ds, \end{aligned} \tag{23}$$

where $G(t,\eta)$ is the heat kernel

$$G(t,\eta) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(\eta+ct)^2}{4Dt}}, \quad c \geq c_*.$$

Since $u_0 \in C_{unif}[-r,0]$, namely, $\lim_{\xi\to\infty} u_0(t,\xi) = u_0(t,\infty)$ and $\lim_{\xi\to\infty} u_{0,\xi}(t,\xi) = 0$ and $\lim_{\xi\to\infty} u_{0,\xi\xi}(t,\xi) = 0$ uniformly in $t \in [-r,0]$, we immediately prove the following uniform convergence

$$\begin{aligned} \lim_{\xi\to\infty} u(t,\xi) &= e^{-dt}\int_{-\infty}^{\infty} G(t,\eta)\lim_{\xi\to\infty} u_0(0,\xi-\eta)d\eta \\ &\quad + \int_0^t e^{-d(t-s)}\int_{-\infty}^{\infty} G(t-s,\eta)\lim_{\xi\to\infty} P(u_0(s-r,\xi-\eta-cr)d\eta ds \\ &= e^{-dt}u_0(0,\infty)\int_{-\infty}^{\infty} G(t,\eta)d\eta \\ &\quad + \int_0^t e^{-d(t-s)}P(u_0(s-r,\infty)\int_{-\infty}^{\infty} G(t-s,\eta)d\eta ds \\ &= e^{-dt}u_0(0,\infty) + \int_0^t e^{-d(t-s)}P(u_0(s-r,\infty)ds \\ &=: g_1(t), \quad \text{uniformly in } t \in [0,r], \end{aligned} \tag{24}$$

and

$$\lim_{\xi\to\infty} \partial_\xi^k u(t,\xi)$$

$$
\begin{aligned}
&= e^{-dt}\int_{-\infty}^{\infty}\partial_\eta^k G(t,\eta)\lim_{\xi\to\infty}u_0(0,\xi-\eta)d\eta \\
&\quad+\int_0^t e^{-d(t-s)}\int_{-\infty}^{\infty}\partial_\eta^k G(t-s,\eta)\lim_{\xi\to\infty}P(u_0(s-r,\xi-\eta-cr)d\eta ds \\
&= e^{-dt}u_0(0,\infty)\int_{-\infty}^{\infty}\partial_\eta^k G(t,\eta)d\eta \\
&\quad+\int_0^t e^{-d(t-s)}P(u_0(s-r,\infty)\int_{-\infty}^{\infty}\partial_\eta^k G(t-s,\eta)d\eta ds \\
&= 0, \quad \text{for } k=1,2, \quad \text{uniformly in } t\in[0,r]. \qquad (25)
\end{aligned}
$$

Here, we used the facts

$$\int_{-\infty}^{\infty} G(t-s,\eta)d\eta = 1, \quad G(t-s,\pm\infty)=0, \quad \text{and } \partial_\eta G(t-s,\eta)\Big|_{\eta=\pm\infty}=0.$$

Thus, we have proved $u\in C_{unif}[-r,r]$.

Now we consider (20) for $t\in[r,2r]$. Since $t-r\in[0,r]$ and $u(t-r,\xi-cr)$ is solved already in last step for (22), thus $P(u(t-r,\xi-cr))$ is known for (20) with $t\in[0,2r]$, namely, the equation (20) is linear for $t\in[0,2r]$. As showed before, we can similarly prove the existence and uniqueness of the solution u to (20) for $t\in[0,2r]$, and particularly $u\in C_{unif}[-r,2r]$.

By repeating this procedure for $t\in[nr,(n+1)r]$ with $n\in\mathbb{Z}_+$ (the set of all positive integers), we prove that there exists a unique solution $u\in C_{unif}([-r,(n+1)r]$ for (20), and step by step, we finally prove the uniqueness and time-global existence of the solution $u\in C_{unif}[-r,\infty)$ for (20). The proof is complete. □

The most important part of the paper is to prove the following global stability with the optimal convergence rates.

Proposition 2.2 (Stability with optimal convergence rates). *Under the conditions (H$_1$)-(H$_3$), when $b'(v_+)$ and r satisfy, either $d\ge|b'(v_+)|$ with arbitrarily given $r>0$, or $d<|b'(v_+)|$ with $0<r<\bar{r}$, where $\bar{r}$ is defined in* (11)*, and $u_0\in C_{unif}(-r,0)\cap L^1([-r,0];W_w^{2,1}(\mathbb{R}))$ and $\partial_s u_0\in L^1([-r,0];L_w^1(\mathbb{R}))$ for $c\ge c_*$, no matter how large the initial perturbation u_0 is, then*

- *when $c=c_*$, it holds*

$$\sup_{\xi\in\mathbb{R}}|u(t,\xi)|\le Ct^{-\frac{1}{2}}. \qquad (26)$$

- *when $c>c_*$, it holds*

$$\sup_{\xi\in\mathbb{R}}|u(t,\xi)|\le Ct^{-\frac{1}{2}}e^{-\mu t} \qquad (27)$$

for some positive constant μ satisfying (15).

In order to prove Proposition 2.2, we need several lemmas to complete it. Since $b'(0)>d$, namely, $u_-=0$ is the unstable node of (20), heuristically, for a general initial data u_0, we cannot expect the convergence $u\to0$ as $t\to\infty$. But, inspired by [40,41] for the equation (20) with or without time-delay, we expect the solution u to decay to zero when the initial perturbation is only exponentially decay at the far field $\xi=-\infty$. Thus, let us define

$$u(t,\xi)=[w(\xi)]^{-\frac{1}{2}}\tilde{u}(t,\xi), \quad i.e., \quad \tilde{u}(t,\xi)=\sqrt{w(\xi)}u(t,\xi)=e^{-\lambda\xi}u(t,\xi), \qquad (28)$$

where $\lambda \in (\lambda_1, \lambda_2)$ for $c > c_*$ and $\lambda = \lambda_*$ for $c = c_*$, we get the following equations for the new unknown $\tilde{u}(t,\xi)$ with $c \geq c_*$:

(29)
$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - D\dfrac{\partial^2 \tilde{u}}{\partial \xi^2} + a_0(c)\dfrac{\partial \tilde{u}}{\partial \xi} + a_1(c)\tilde{u} = \tilde{P}(\tilde{u}(t-r,\xi-cr)), \quad (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}, \\ \tilde{u}(s,\xi) = \sqrt{w(\xi)}u(s,\xi) =: \tilde{u}_0(s,\xi), \quad s \in [-r,0], \ \xi \in \mathbb{R}, \end{cases}$$

where

$$a_0(c) := c - 2D\lambda, \quad a_1(c) := c\lambda + d - D\lambda^2, \quad \text{for } c \geq c_*, \tag{30}$$

which satisfy (by (6) and (8))

$$a_1(c) = c\lambda + d - D\lambda^2 > b'(0)e^{-\lambda cr}, \qquad \text{for } c > c_*, \lambda \in (\lambda_1, \lambda_2), \tag{31}$$

$$a_1(c_*) = c_*\lambda_* + d - D\lambda_*^2 = b'(0)e^{-\lambda_* c_* r}, \quad \text{for } c = c_*. \tag{32}$$

Here,

$$\tilde{P}(\tilde{u}) = e^{-\lambda\xi}P(u) \tag{33}$$

satisfies (by Taylor's expansion formula)

$$\begin{aligned} \tilde{P}(\tilde{u}(t-r,\xi-cr)) &= e^{-\lambda\xi}P(u(t-r,\xi-cr)) \\ &= e^{-\lambda\xi}b'(\tilde{\phi})u(t-r,\xi-cr) \\ &= e^{-\lambda cr}b'(\tilde{\phi})\tilde{u}(t-r,\xi-cr), \quad c \geq c_*, \end{aligned} \tag{34}$$

for some function $\tilde{\phi}$ between ϕ and $\phi + u$ with $c \geq c_*$ (see (21)). From $(\mathrm{H})_3$ we further have

$$|\tilde{P}(\tilde{u}(t-r,\xi-cr))| \leq e^{-\lambda cr}b'(0)|\tilde{u}(t-r,\xi-cr)|. \tag{35}$$

Since $b'(s)$ can be negative for $s \in (0, v_+)$, then the solution $\tilde{u}$ for the equation (29) with the nonlinear term (34) will be oscillating around v_+ when the time-delay r is large, as numerically reported in [4, 25], and the comparison principle doesn't hold in this case. So the monotonic technique cannot be applied. On the other hand, in (34), since the coefficient $b'(\tilde{\phi})$ is variable, we are unable to derive the decay rate directly by applying Fourier's transform. However, by a deep observation we may establish a crucial boundedness estimate for the oscillating solution $\tilde{u}(t,\xi)$. In order to look for such a boundedness, let us consider the following linear delayed reaction-diffusion equation, for $c \geq c_*$,

$$\begin{cases} \dfrac{\partial u^+}{\partial t} - D\dfrac{\partial^2 u^+}{\partial \xi^2} + a_0(c)\dfrac{\partial u^+}{\partial \xi} + a_1(c)u^+ \\ \qquad = b'(0)e^{-\lambda cr}u^+(t-r,\xi-cr)), \quad (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}, \\ u^+(s,\xi) = u_0^+(s,\xi) \geq 0, \quad s \in [-r,0], \ \xi \in \mathbb{R}. \end{cases} \tag{36}$$

Since $b'(0) > 0$, as showed in [29], we can similarly prove the positiveness of the solution $u^+(t,\xi)$.

Lemma 2.3. *When $u_0^+(s,\xi) \geq 0$ for $(s,\xi) \in [-r,0] \times \mathbb{R}$, then $u^+(t,\xi) \geq 0$ for $(t,\xi) \in [-r,\infty) \times \mathbb{R}$.*

Next we establish the following crucial boundedness estimate for the oscillating solution $\tilde{u}(t,\xi)$ of (20).

Lemma 2.4 (Boundedness estimate)**.** *Let $\tilde{u}(t,\xi)$ and $u^+(t,\xi)$ be the solutions of* (20) *and* (36)*, respectively. When*

$$|\tilde{u}_0(s,\xi)| \leq u_0^+(s,\xi), \quad (s,\xi) \in [-r,0] \times \mathbb{R}, \tag{37}$$

then

$$|\tilde{u}(t,\xi)| \le u^+(t,\xi), \quad (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}. \tag{38}$$

Proof. First of all, we prove $|\tilde{u}(t,\xi)| \le u^+(t,x)$ for $t \in [0,r]$. In fact, when $t \in [0,r]$, namely, $t-r \in [-r,0]$, from (37), we have

$$\begin{aligned} |\tilde{u}(t-r,\xi-cr)| &= |\tilde{u}_0(t-r,\xi-cr)| \\ &\le u_0^+(t-r,\xi-cr) \\ &= u^+(t-r,\xi-cr), \quad \text{for } t\in[0,r] \text{ and } c \ge c_*. \end{aligned} \tag{39}$$

Let

$$U^-(t,\xi) := u^+(t,\xi) - \tilde{u}(t,\xi) \ \text{ and } \ U^+(t,\xi) := u^+(t,\xi) + \tilde{u}(t,\xi),$$

we are going to estimate $U^\pm(t,\xi)$ respectively.

From (29), (34) and (36), then $U^-(t,\xi)$ satisfies, for $c \ge c_*$,

$$\begin{aligned} &\frac{\partial U^-}{\partial t} + a_0(c)\frac{\partial U^-}{\partial \xi} - D\frac{\partial^2 U^-}{\partial \xi^2} + a_1(c)U^- \\ &= b'(0)e^{-\lambda cr}u^+(t-r,\xi-cr) - b'(\tilde{\phi})e^{-\lambda cr}\tilde{u}(t-r,\xi-cr) \\ &\ge b'(0)e^{-\lambda cr}u^+(t-r,\xi-cr) - |b'(\tilde{\phi})|e^{-\lambda cr}|\tilde{u}(t-r,\xi-cr)| \\ &\ge 0, \quad t\in[0,r], \end{aligned} \tag{40}$$

where $\lambda \in (\lambda_1,\lambda_2)$ for $c > c_*$ and $\lambda = \lambda_*$ for $c = c_*$. Here we used (H_3) for $|b'(\tilde{\phi})| \le b'(0)$ and (39) for $|\tilde{u}(t-r,\xi-cr)| \le |u^+(t-r,\xi-cr)|$. Thus, (40) with the initial data $U_0^-(s,\xi) = u_0^+(s,\xi) - \tilde{u}_0(s,\xi) \ge 0$ reduces to

$$\begin{cases} \dfrac{\partial U^-}{\partial t} + a_0(c)\dfrac{\partial U^-}{\partial \xi} - D\dfrac{\partial^2 U^-}{\partial \xi^2} + a_1(c)U^- \ge 0, \\ U_0^-(0,\xi) \ge 0, \end{cases}$$

which, by the regular comparison principle for parabolic equation without delay, implies

$$U^-(t,\xi) = u^+(t,\xi) - \tilde{u}(t,\xi) \ge 0 \ \text{ for } (t,\xi)\in[0,r]\times\mathbb{R} \ \text{ and } c \ge c_*. \tag{41}$$

On the other hand, $U^+(t,\xi)$ satisfies

$$\begin{aligned} &\frac{\partial U^+}{\partial t} + a_0(c)\frac{\partial U^+}{\partial \xi} - D\frac{\partial^2 U^+}{\partial \xi^2} + a_1(c)U^+ \\ &= b'(0)e^{-\lambda cr}u^+(t-r,\xi-cr) + b'(\tilde{\phi})e^{-\lambda cr}\tilde{u}(t-r,\xi-cr) \\ &\ge b'(0)e^{-\lambda cr}u^+(t-r,\xi-cr) - |b'(\tilde{\phi})|e^{-\lambda cr}|\tilde{u}(t-r,\xi-cr)| \\ &\ge 0, \quad t\in[0,r], \end{aligned} \tag{42}$$

because of $|b'(\tilde{\phi})| < b'(0)$ from (H_3) and $|\tilde{u}(t-r,\xi-cr)| \le |u^+(t-r,\xi-cr)|$ from (39). Therefore, we can similarly prove that (42) with the initial data $U_0^+(s,\xi) = u_0^+(s,\xi) + \tilde{u}_0(s,\xi) \ge 0$ implies

$$U^+(t,\xi) = u^+(t,\xi) + \tilde{u}(t,\xi) \ge 0 \ \text{ for } (t,\xi)\in[0,r]\times\mathbb{R} \ \text{ and } c \ge c_*. \tag{43}$$

Combining (41) and (43), we prove

$$|\tilde{u}(t,\xi)| \le u^+(t,\xi) \ \text{ for } (t,\xi)\in[0,r]\times\mathbb{R} \ \text{ and } c \ge c_*. \tag{44}$$

Next, when $t \in [r,2r]$, namely, $t-r \in [0,r]$, based on (44) we can similarly prove

$$\begin{cases} U^-(t,\xi) = u^+(t,\xi) - \tilde{u}(t,\xi) \ge 0, \\ U^+(t,\xi) = u^+(t,\xi) + \tilde{u}(t,\xi) \ge 0, \end{cases} \quad \text{for } (t,\xi)\in[r,2r]\times\mathbb{R} \ \text{ and } c \ge c_*,$$

namely,

$$|\tilde{u}(t,\xi)| \le u^+(t,\xi) \quad \text{for } (t,\xi) \in [r,2r]\times\mathbb{R} \quad \text{and } c \ge c_*. \tag{45}$$

Repeating this procedure, we then further prove

$$|\tilde{u}(t,\xi)| \le u^+(t,\xi) \quad \text{for } (t,\xi) \in [nr,(n+1)r]\times\mathbb{R} \quad \text{and } c \ge c_*, \quad n = 1,2,\cdots, \tag{46}$$

which implies

$$|\tilde{u}(t,\xi)| \le u^+(t,\xi) \quad \text{for } (t,\xi) \in [0,\infty)\times\mathbb{R} \quad \text{and } c \ge c_*. \tag{47}$$

The proof is complete. □

Next we derive the global stability with the optimal convergence rates for the linear equation (36) by using the weighted energy method and by carrying out the crucial estimates on the fundamental solutions. In order to derive the optimal decay rates for the solution of (36) in the cases of $c > c_*$ and $c = c_*$, respectively, we first need to derive the fundamental solution, then show the optimal decay rates of the fundamental solution. Now let us recall the properties of the solutions to the delayed ODE.

Lemma 2.5 ([20]). *Let $z(t)$ be the solution to the following linear time-delayed ODE with time-delay $r > 0$ and two constants k_1 and k_2*

$$\begin{cases} \dfrac{d}{dt}z(t) + k_1 z(t) = k_2 z(t-r), \\ z(s) = z_0(s), \quad s \in [-r,0]. \end{cases} \tag{48}$$

Then

$$z(t) = e^{-k_1(t+r)} e_r^{\bar{k}_2 t} z_0(-r) + \int_{-r}^{0} e^{-k_1(t-s)} e_r^{\bar{k}_2(t-r-s)} [z_0'(s) + k_1 z_0(s)]ds, \tag{49}$$

where

$$\bar{k}_2 := k_2 e^{k_1 r}, \tag{50}$$

and $e_r^{\bar{k}_2 t}$ is the so-called `delayed exponential function` *in the form*

$$e_r^{\bar{k}_2 t} = \begin{cases} 0, & -\infty < t < -r, \\ 1, & -r \le t < 0, \\ 1 + \frac{\bar{k}_2 t}{1!}, & 0 \le t < r, \\ 1 + \frac{\bar{k}_2 t}{1!} + \frac{\bar{k}_2^2 (t-r)^2}{2!}, & r \le t < 2r, \\ \vdots & \vdots \\ 1 + \frac{\bar{k}_2 t}{1!} + \frac{\bar{k}_2^2 (t-r)^2}{2!} + \cdots + \frac{\bar{k}_2^m [t-(m-1)r]^m}{m!}, & (m-1)r \le t < mr, \\ \vdots & \vdots \end{cases} \tag{51}$$

and $e_r^{\bar{k}_2 t}$ is the fundamental solution to

$$\begin{cases} \dfrac{d}{dt}z(t) = \bar{k}_2 z(t-r) \\ z(s) \equiv 1, \quad s \in [-r,0]. \end{cases} \tag{52}$$

The property of the solution to the delayed linear ODE (48) is well-known [34].

Lemma 2.6 ([34]). *Let $k_1 \ge 0$ and $k_2 \ge 0$. Then the solution $z(t)$ to (48) (or equivalently (49)) satisfies*

$$|z(t)| \le C_0 e^{-k_1 t} e_r^{\bar{k}_2 t}, \tag{53}$$

where

$$C_0 := e^{-k_1\tau}|z_0(-r)| + \int_{-r}^{0} e^{k_1 s}|z_0'(s) + k_1 z_0(s)|ds, \tag{54}$$

and the fundamental solution $e_r^{\bar{k}_2 t}$ *with* $\bar{k}_2 > 0$ *to* (52) *satisfies*

$$e_\tau^{\bar{k}_2 t} \le C(1+t)^{-\gamma} e^{\bar{k}_2 t}, \quad t > 0 \tag{55}$$

for arbitrary number $\gamma > 0$.

Furthermore, when $k_1 \ge k_2 \ge 0$, *there exists a constant* $0 < \varepsilon_1 < 1$ *such that*

$$e^{-k_1 t} e_r^{\bar{k}_2 t} \le C e^{-\varepsilon_1 (k_1 - k_2)t}, \quad t > 0 \tag{56}$$

and the solution $z(t)$ *to* (48) *satisfies*

$$|z(t)| \le C e^{-\varepsilon_1 (k_1 - k_2)t}, \quad t > 0. \tag{57}$$

Let us take Fourier transform to (36), and denote the Fourier transform of $u^+(t,\xi)$ by $\hat{u}^+(t,\eta)$, that is,

$$\begin{cases} \dfrac{d}{dt}\hat{u}^+(t,\eta) + A(\eta)\hat{u}^+(t,\eta) = B(\eta)\hat{u}^+(t-r,\eta), \\ \hat{u}^+(s,\eta) = \hat{u}_0^+(s,\eta), \ s \in [-r,0], \quad \eta \in R, \end{cases} \tag{58}$$

where

$$\begin{cases} A(\eta) := D|\eta|^2 + a_1(c) + \mathrm{i}a_0(c)\eta, \\ B(\eta) := b'(0)e^{\mathrm{i}cr\eta}e^{-\lambda cr}, \end{cases} \quad \text{for } c \ge c_*. \tag{59}$$

From (49), the linear time-delayed ordinary differential equation (58) can be solved by

$$\begin{aligned} \hat{u}^+(t,\eta) &= e^{-A(\eta)(t+r)} e_r^{\bar{B}(\eta)t} \hat{u}_0^+(-r,\eta) \\ &\quad + \int_{-r}^{0} e^{-A(\eta)(t-s)} e_r^{\bar{B}(\eta)(t-r-s)} \\ &\quad\quad \times \Big[\frac{d}{ds}\hat{u}_0^+(s,\eta) + A(\eta)\hat{u}_0^+(s,\eta)\Big]ds, \end{aligned} \tag{60}$$

where

$$\bar{B}(\eta) := B(\eta)e^{A(\eta)r}, \tag{61}$$

Taking the inverse Fourier transform to (60), we have

$$\begin{aligned} u^+(t,\xi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\xi\eta} e^{-A(\eta)(t+r)} e_r^{\bar{B}(\eta)t}\hat{u}_0^+(-r,\eta)d\eta \\ &\quad + \frac{1}{2\pi}\int_{-r}^{0}\int_{-\infty}^{\infty} e^{\mathrm{i}\xi\eta} e^{-A(\eta)(t-s)} e_r^{\bar{B}(\eta)(t-r-s)} \\ &\quad\quad \times \Big[\frac{d}{ds}\hat{u}_0^+(s,\eta) + A(\eta)\hat{u}_0^+(s,\eta)\Big]d\eta ds. \end{aligned} \tag{62}$$

Next, we are going to estimate the decay rates for the solution $u^+(t,\xi)$.

Lemma 2.7 (Optimal decay rates for linear delayed equation)**.** *Let the initial data* $u_0^+(s,\xi)$ *be such that* $u_0^+ \in L^1([-r,0]; W^{2,1}(\mathbb{R}))$ *and* $\partial_s u_0^+ \in L^1([-r,0]; L^1(\mathbb{R}))$, *then*

$$\|u^+(t)\|_{L^\infty(\mathbb{R})} \le \begin{cases} C(1+t)^{-\frac{1}{2}}e^{-\mu_1 t}, & \text{for } c > c_*, \\ C(1+t)^{-\frac{1}{2}}, & \text{for } c = c_*, \end{cases} \tag{63}$$

for some $0 < \mu_1 < c\lambda + d - c\lambda^2 - b'(0)e^{-c\lambda r}$ *with* $c > c_*$.

Proof. Using Parseval's inequality, from (60) we have

$$\begin{aligned}
\|u^+(t)\|_{L^\infty(\mathbb{R})} &\le \|\hat{u}^+(t)\|_{L^1(\mathbb{R})} \\
&\le \int_{-\infty}^{\infty} \Big| e^{-A(\eta)(t+r)} e_r^{\bar{B}(\eta)t} \hat{u}_0^+(-r,\eta) \Big| d\eta \\
&\quad + \int_{-r}^{0}\int_{-\infty}^{\infty} \Big| e^{-A(\eta)(t-s)} e_r^{\bar{B}(\eta)(t-r-s)} \\
&\qquad \times \Big[\frac{d}{ds}\hat{u}_0^+(s,\eta) + A(\eta)\hat{u}_0^+(s,\eta) \Big] \Big| d\eta ds \\
&=: I_1(t) + I_2(t). \qquad (64)
\end{aligned}$$

To estimate $I_i(t)$ for $i = 1,2$, from the definitions of $\bar{B}(\eta)$ (see (61) and (59)) and the delayed-exponential function $e_r^{\bar{k}_2 t}$ (see (51)), we first note

$$|e^{-A(\eta)(t+r)}| = e^{-(D\eta^2 + a_1(c))(t+r)} = e^{-k_1(c,\eta)(t+r)}, \quad \text{for } c \ge c_*, \qquad (65)$$

with

$$k_1(c,\eta) := D\eta^2 + a_1(c), \qquad (66)$$

and

$$\begin{aligned}
|\bar{B}(\eta)| &\le |B(\eta)| e^{|A(\eta)|r} \le b'(0)e^{-\lambda c r} e^{k_1(c,\eta)} \\
&=: \bar{k}_2(c,\eta), \quad \text{for } c \ge c_*, \qquad (67)
\end{aligned}$$

where

$$\bar{k}_2(c,\eta) := k_2(c) e^{k_1(c,\eta)}, \quad \text{and } k_2(c) := b'(0)e^{-\lambda c r}, \qquad (68)$$

and

$$|e_r^{\bar{B}(\eta)t}| \le e_r^{|\bar{B}(\eta)|t} = e_r^{\bar{k}_2(c,\eta)t}, \quad \text{for } c \ge c_*. \qquad (69)$$

Noting (6) and (8), we have

$$k_1(c,\eta) = D\eta^2 + a_1(c) = D\eta^2 + c\lambda + d - D\lambda^2 \ge D\eta^2 + b'(0)e^{-\lambda c r} = D\eta^2 + k_2(c),$$

and

$$k_1(c,\eta) - k_2(c) = D\eta^2 + c\lambda + d - D\lambda^2 - b'(0)e^{-\lambda c r} = D\eta^2 + \mu_0, \quad \text{for } c > c_*,$$

where $\mu_0 := c\lambda + d - D\lambda^2 - b'(0)e^{-\lambda c r} > 0$ for $c > c_*$, and

$$k_1(c_*,\eta) - k_2(c_*) = D\eta^2 + c\lambda_* + d - D\lambda_*^2 - b'(0)e^{-\lambda_* c_* r} = D\eta^2, \quad \text{for } c = c_*.$$

Then, from (56) we have

$$\begin{aligned}
|e^{-A(\eta)(t+r)} e_r^{\bar{B}(\eta)t}| &\le e^{-k_1(c,\eta)(t+r)} e_r^{\bar{k}_2(c,\eta)t} \\
&\le C e^{-\varepsilon_1[k_1(c,\eta) - k_2(c)]t} \\
&= \begin{cases} C e^{-\varepsilon_1(D\eta^2 + \mu_0)t} & \text{for } c > c_*, \\ C e^{-\varepsilon_1 D\eta^2 t} & \text{for } c = c_*. \end{cases} \qquad (70)
\end{aligned}$$

Applying (70), we derive the optimal estimate for $I_1(t)$:

$$\begin{aligned}
I_1(t) &= \int_{-\infty}^{\infty} \Big| e^{-A(\eta)(t+r)} e_r^{\bar{B}(\eta)t} \hat{u}_0^+(-r,\eta) \Big| d\eta \\
&\le \begin{cases} C\int_{-\infty}^{\infty} e^{-\varepsilon_1 D\eta^2 t} e^{-\varepsilon_1 \mu_0 t} |\hat{u}_0^+(-r,\eta)| d\eta & \text{for } c > c_* \\ C\int_{-\infty}^{\infty} e^{-\varepsilon_1 D\eta^2 t} |\hat{u}_0^+(-r,\eta)| d\eta & \text{for } c = c_* \end{cases}
\end{aligned}$$

$$\leq \begin{cases} Ce^{-\varepsilon_1\mu_0 t}\|\hat{u}_0^+(-r)\|_{L^\infty}\int_{-\infty}^{\infty}e^{-\varepsilon_1 D\eta^2 t}d\eta & \text{for } c>c_* \\ C\|\hat{u}_0^+(-r)\|_{L^\infty}\int_{-\infty}^{\infty}e^{-\varepsilon_1 D\eta^2 t}d\eta & \text{for } c=c_* \end{cases}$$

$$\leq \begin{cases} Ce^{-\mu_1 t}t^{-\frac{1}{2}}\|u_0^+(-r)\|_{L^1} & \text{for } c>c_*, \\ Ct^{-\frac{1}{2}}\|u_0^+(-r)\|_{L^1} & \text{for } c=c_*, \end{cases} \tag{71}$$

with $\mu_1 := \varepsilon_1\mu_0 < c\lambda + d - D\lambda^2 - b'(0)e^{-c\lambda r}$.

Similarly, we may estimate $I_2(t)$. We note that

$$\begin{aligned} \sup_{\eta\in R}|A(\eta)\hat{u}_0^+(s,\eta)| &= \sup_{\eta\in R}\Big|[D\eta^2 + a_1(c) + \mathrm{i}a_0(c)\eta]\hat{u}_0^+(s,\eta)\Big| \\ &\leq C\|u_0^+(s)\|_{W^{2,1}(\mathbb{R})}. \end{aligned} \tag{72}$$

Thus, we can derive the decay rate for $I_2(t)$ as follows

$$\begin{aligned} I_2(t) &= \int_{-r}^{0}\int_{-\infty}^{\infty}\Big|e^{-A(\eta)(t-s)}e_r^{\bar{B}(\eta)(t-r-s)}\Big[\frac{d}{ds}\hat{u}_0^+(s,\eta) + A(\eta)\hat{u}_0^+(s,\eta)\Big]\Big|d\eta ds \\
&\leq \begin{cases} C\int_{-r}^{0}\int_{-\infty}^{\infty}e^{-\varepsilon_1 D\eta^2(t-s)}e^{-\varepsilon_1\mu_0(t-s)}\Big[\frac{d}{ds}\hat{u}_0^+(s,\eta) + A(\eta)\hat{u}_0^+(s,\eta)\Big]\Big|d\eta ds \\ \qquad\qquad \text{for } c>c_* \\ C\int_{-r}^{0}\int_{-\infty}^{\infty}e^{-\varepsilon_1 D\eta^2(t-s)}\Big[\frac{d}{ds}\hat{u}_0^+(s,\eta) + A(\eta)\hat{u}_0^+(s,\eta)\Big]\Big|d\eta ds \\ \qquad\qquad \text{for } c=c_* \end{cases} \\
&\leq \begin{cases} Ce^{-\varepsilon_1\mu_1 t}\int_{-r}^{0}e^{\varepsilon_1\mu_0 s}\sup_{\eta\in R}\Big|\frac{d}{ds}\hat{u}_0^+(s,\eta) + A(\eta)\hat{u}_0^+(s,\eta)\Big| \\ \qquad \times\int_{-\infty}^{\infty}e^{-\varepsilon_1 D\eta^2(t-s)}d\eta ds \qquad \text{for } c>c_* \\ C\int_{-r}^{0}\sup_{\eta\in R}\Big|\frac{d}{ds}\hat{u}_0^+(s,\eta) + A(\eta)\hat{u}_0^+(s,\eta)\Big|\int_{-\infty}^{\infty}e^{-\varepsilon_1 D\eta^2(t-s)}d\eta ds \\ \qquad\qquad \text{for } c=c_* \end{cases} \\
&\leq \begin{cases} Ce^{-\varepsilon_1\mu_0 t}\int_{-r}^{0}(t-s)^{-\frac{1}{2}}[\|{u_0^+}'(s)\|_{L^1(\mathbb{R})} + \|u_0^+(s)\|_{W^{2,1}(\mathbb{R})}]ds \\ \qquad\qquad \text{for } c>c_*, \\ C\int_{-r}^{0}(t-s)^{-\frac{1}{2}}[\|{u_0^+}'(s)\|_{L^1(\mathbb{R})} + \|u_0^+(s)\|_{W^{2,1}(\mathbb{R})}]ds \\ \qquad\qquad \text{for } c=c_*, \end{cases} \\
&\leq \begin{cases} Ce^{-\varepsilon_1\mu_1 t}(1+t)^{-\frac{1}{2}}\int_{-r}^{0}\frac{(1+t)^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}}[\|{u_0^+}'(s)\|_{L^1(\mathbb{R})} + \|u_0^+(s)\|_{W^{2,1}(\mathbb{R})}]ds \\ \qquad\qquad \text{for } c>c_*, \\ C(1+t)^{-\frac{1}{2}}\int_{-r}^{0}\frac{(1+t)^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}}[\|{u_0^+}'(s)\|_{L^1(\mathbb{R})} + \|u_0^+(s)\|_{W^{2,1}(\mathbb{R})}]ds \\ \qquad\qquad \text{for } c=c_*, \end{cases} \\
&\leq \begin{cases} Ce^{-\mu_1 t}(1+t)^{-\frac{1}{2}}[\|{u_0^+}'(s)\|_{L^1(-r,0;L^1(\mathbb{R}))} + \|u_0^+(s)\|_{L^1(-r,0;W^{2,1}(\mathbb{R}))}] \\ \qquad\qquad \text{for } c>c_*, \\ C(1+t)^{-\frac{1}{2}}[\|{u_0^+}'(s)\|_{L^1(-r,0;L^1(\mathbb{R}))} + \|u_0^+(s)\|_{L^1(-r,0;W^{2,1}(\mathbb{R}))}] \\ \qquad\qquad \text{for } c=c_*. \end{cases} \end{aligned} \tag{73}$$

Substituting (71) and (73) to (64), we obtain the decay rates:

$$\|u^+(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} C(1+t)^{-\frac{1}{2}}e^{-\mu_1 t}, & \text{for } c>c_*, \\ C(1+t)^{-\frac{1}{2}}, & \text{for } c=c_*. \end{cases}$$

The proof is complete. □

Let us choose $u_0^+(s,\xi)$ such that $u_0^+ \in L^1([-r,0];W^{2,1}(\mathbb{R}))$ and $\partial_s u_0^+ \in L^1([-r,0];L^1(\mathbb{R}))$ and

$$u_0^+(s,\xi) \geq |\tilde{u}_0(s,\xi)|.$$

Combining Lemmas 2.4 and 2.7, we immediately get the convergence rates for $\tilde{u}(t,\xi)$.

Lemma 2.8. *When $\tilde{u}_0 \in L^1([-r,0];W^{2,1}(\mathbb{R}))$ and $\partial_s\tilde{u}_0 \in L^1([-r,0];L^1(\mathbb{R}))$, then*

$$\|\tilde{u}(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} C(1+t)^{-\frac{1}{2}}e^{-\mu_1 t}, & \text{for } c > c_*, \\ C(1+t)^{-\frac{1}{2}}, & \text{for } c = c_*. \end{cases} \tag{74}$$

Since $\tilde{u}(t,\xi) = \sqrt{w(\xi)}u(t,\xi) = e^{-\lambda\xi}u(t,\xi)$ for $c \geq c_*$, and $e^{-\lambda\xi} \to 0$ as $\xi \to \infty$, from (74) we cannot guarantee any decay estimate for $u(t,\xi)$ at far field $\xi \sim \infty$. In order to get the optimal decay rates for $u(t,\xi)$ in cases of $c > c_*$ and $c = c_*$, let us first investigate the convergence of $u(t,\xi)$ at far field of $\xi = \infty$.

Lemma 2.9. *When $d \geq |b'(v_+)|$ with arbitrary time-delay $r > 0$, or $d < |b'(v_+)|$ but with a small time-delay $0 < r < \overline{r}$, where $\overline{r}$ is defined in* (11)*, then there exists a large number $x_0 \gg 1$ such that the solution $u(t,\xi)$ of* (20) *satisfies*

$$\sup_{[x_0,\infty)} |u(t,\xi)| \leq Ce^{-\mu_2 t}, \quad t > 0, \quad c \geq c_*, \tag{75}$$

for some $0 < \mu_2 = \mu_2(p,d,r,b'(v_+)) < d$.

Proof. Since $u \in C_{unif}(0,\infty)$, namely $\lim_{\xi\to+\infty} u(t,\xi) = u(t,\infty) =: z^+(t)$ exists uniformly for $t \in [-r,\infty)$ and $\lim_{\xi\to+\infty} u_\xi(t,\xi) = \lim_{\xi\to+\infty} u_{\xi\xi}(t,\xi) = 0$ are uniformly for $t \in [-r,\infty)$, let us take the limits to (20) as $\xi \to \infty$, then we have

$$\begin{cases} \dfrac{d}{dt}z^+(t) + dz^+(t) - b'(v_+)z^+(t-r) = Q(z^+(t-r)), \\ z^+(s) = z_0^+(s), \ \ s \in [-r,0], \end{cases} \tag{76}$$

where

$$Q(z^+) = b(v_+ + z^+) - b(v_+) - b'(v_+)z^+.$$

As shown in [25], it is well-known that, when $d \geq |b'(v_+)|$ with arbitrary time-delay $r > 0$, or $d < |b'(v_+)|$ but with a small time-delay $0 < r < \overline{r}$, where $\overline{r}$ is defined in (11), then the above equation (76) satisfies

$$|u(t,\infty)| = |z^+(t)| \leq Ce^{-\mu_2 t}, \ t > 0, c \geq c_*, \tag{77}$$

for some $0 < \mu_2 = \mu_2(p,d,r,b'(v_+)) < d$, provided with $|z_0^+| \ll 1$.

From the continuity and the uniform convergence of $u(t,\xi)$ as $\xi \to +\infty$, there exists a large $x_0 \gg 1$ such that (77) implies the following convergence immediately

$$\sup_{\xi\in[x_0,+\infty)} |u(t,\xi)| \leq Ce^{-\mu_2 t}, \ t > 0, c \geq c_*, \tag{78}$$

provided $\sup_{\xi\in[x_0,+\infty)} |u_0(s,\xi)| \ll 1$ for $s \in [-r,0]$. Such a smallness for the initial perturbation u_0 near $\xi = +\infty$ can be automatically verified, because $\lim_{x\to+\infty} v_0(s,x) = v_+$, which implies $\lim_{\xi\to+\infty} u_0(s,\xi) = \lim_{\xi\to+\infty}[v_0(s,\xi) - \phi(\xi)] = v_+ - v_+ = 0$ uniformly for $s \in [-r,0]$. Thus, the proof is complete. □

Lemma 2.10. *It holds*

$$\sup_{\xi\in(-\infty,x_0]} |u(t,\xi)| \leq \begin{cases} C(1+t)^{-\frac{1}{2}}e^{-\mu_1 t}, & \text{for } c > c_*, \\ C(1+t)^{-\frac{1}{2}}, & \text{for } c = c_*. \end{cases} \tag{79}$$

Proof. Since

$$w(\xi)=\begin{cases}e^{2\lambda|\xi|}, & \text{for } c>c_*\\ e^{2\lambda_*|\xi|}, & \text{for } c=c_*\end{cases}\geq\begin{cases}e^{2\lambda x_0}, & \text{for } c>c_*,\ \ \xi\le x_0,\\ e^{2\lambda_* x_0}, & \text{for } c=c_*,\ \ \xi\le x_0.\end{cases}$$

and $\tilde{u}(t,\xi)=\sqrt{w(\xi)}u(t,\xi)$, then from (74) we get

$$\sup_{\xi\in(-\infty,x_0]}|u(t,\xi)|\le\begin{cases}C(1+t)^{-\frac{1}{2}}e^{-\mu_1 t}, & \text{for } c>c_*,\\ C(1+t)^{-\frac{1}{2}}, & \text{for } c=c_*.\end{cases}$$

The proof is complete. □

Proof of Proposition 2.2. Based on Lemmas 2.9 and 2.10, we immediately prove (26) and (27) for the convergence rates of $u(t,\xi)$ for $\xi\in\mathbb{R}$, where $0<\mu<\min\{\mu_1,\mu_2\}$. □

3. Numerical computations

This section is devoted to carrying out numerical simulations on the stability of oscillatory traveling waves, which will numerically confirm our theoretical results, that is, by suitably setting, the solution of (1) will behave like a certain critical/non-critical oscillatory traveling wave after long time. Here, we will present two cases, one is that the solution behaves like a non-critical oscillatory traveling wave with $c>c_*$, and the other is for the case of critical oscillatory wave with $c=c_*$.

The targeted equation is chosen as Nicholson's blowflies equation

$$\begin{cases}\dfrac{\partial v(t,x)}{\partial t}-D\dfrac{\partial^2 v(t,x)}{\partial x^2}+dv(t,x)=pv(t-r,x)e^{-av(t-r,x)},\\ v|_{t=s}=v_0(s,x),\quad (s,x)\in[-r,0]\times\mathbb{R}.\end{cases}\tag{80}$$

It possesses two constant equilibria $v_-=0$ and $v_+=\frac{1}{a}\ln\frac{p}{d}$. When $\frac{p}{d}>e$, the birth rate function $b(v)=pve^{-av}$ is unimodal, and satisfies $b'(0)>|b'(v)|$ for $v\in(0,\infty)$. The condition $d\ge|b'(v_+)|$ is equivalent to $e<\frac{p}{d}\le e^2$, and $d<|b'(v_+)|$ is equivalent to $\frac{p}{d}>e^2$.

Throughout this section we fix $D=d=a=1$, $p=10$, $r=1$, and leave the initial data $v_0(s,x)$ free. Thus, we have $v_+=2.302585\cdots$ and it satisfies $d<|b'(v_+)|$, namely $\frac{p}{d}>e^2$; and from (9) and (18), we have $\underline{r}=0.225423518774018\cdots$ and $\overline{r}=3.034694045748307\cdots$, so $\underline{r}<r<\overline{r}$. From Corollary 1.3, the solution of (1) is expected to behave like a certain critical/non-critical oscillatory traveling wave based on the different choice of the initial data.

For critical traveling wave $\phi(x+c_*t)$, from (6), the critical wave speed $c_*=c_*(\lambda_*)$ is uniquely determined by

$$c_*\lambda_*-D\lambda_*^2+d=b'(0)e^{-\lambda_*c_*r},\quad c_*-2D\lambda_*=-c_*rb'(0)e^{-\lambda_*c_*r},\tag{81}$$

where $\lambda_*>0$ is called the eigenvalue such that the critical wave $\phi(x+c_*t)=\phi(\xi)$ spatial-exponentially decays with such an eigenvalue in the form

$$\phi(\xi)=O(1)|\xi|e^{-\lambda_*|\xi|}\quad \text{as}\ \ \xi\to-\infty.\tag{82}$$

Here, from (81) we can exactly calculate

$$\lambda_*=1.517427128889966\cdots\quad\text{and}\quad c_*=1.517427123733230\cdots.\tag{83}$$

It happens that, with the above parameters setting, both the critical eigenvalue λ_* and the critical wave speed c_* are very close. But, in most cases these two numbers are totally different.

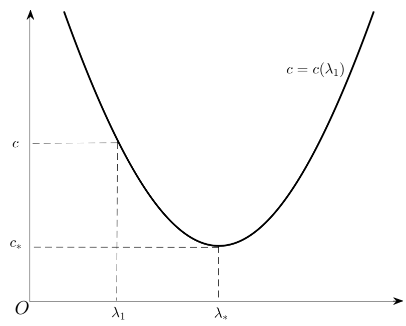


FIGURE 1. The sketch graph of function $c = c(\lambda_1)$.

For non-critical traveling wave $\phi(x + ct)$ with $c > c_*$, the wave speed $c = c(\lambda_1)$ satisfies

$$c\lambda_i - D\lambda_i^2 + d = b'(0)e^{-\lambda_i cr}, \qquad \text{for } i = 1, 2, \text{ and } 0 < \lambda_1 < \lambda_2, \tag{84}$$

and

$$c\lambda - D\lambda^2 + d > b'(0)e^{-\lambda cr}, \qquad \text{for } \lambda \in (\lambda_1, \lambda_2), \tag{85}$$

where $\lambda_1 > 0$ is the eigenvalue such that the non-critical wave $\phi(x + ct) = \phi(\xi)$ behaves at the far field as

$$\phi(\xi) = O(1)e^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty. \tag{86}$$

Implicitly determined by (84), namely

$$c = D\lambda_1 + \frac{b'(0)e^{-\lambda_1 cr} - d}{\lambda_1}, \tag{87}$$

the function $c = c(\lambda_1) \sim O(\frac{1}{\lambda_1})$ for $\lambda_1 \in (0, \lambda_*]$ can be sketched in Figure 1. For given $0 < \lambda_1 < \lambda_* = 1.517427128889966\cdots$, we can uniquely determine the wave speed $c = c(\lambda_1) > c_*$.

From our stability theorem, the initial data of equation (80) is expected to be

$$\lim_{x\to-\infty} v_0(s, x) = 0, \ \lim_{x\to\infty} v_0(s, x) = v_+ \ \text{ uniformly in } s \in [-r, 0],$$

and particularly,

$$\begin{cases} e^{-\lambda_1 x}|v_0(s, x) - \phi(x + cs)| \to 0, \text{ for } c > c_*, \\ e^{-\lambda_* x}|v_0(s, x) - \phi(x + c_* s)| \to 0, \text{ for } c = c_*, \end{cases} \quad \text{as } x \to -\infty,$$

uniformly in $s \in [-r, 0]$.

Case 1. Convergence to the non-critical oscillatory traveling wave. Let us select the initial data in the form of

$$v_0(s, x) = \frac{v_+}{1 + e^{-\lambda_1 x}} + 2e^{-0.001(x-200)^2} \sin^2 x, \tag{88}$$

and take the eigenvalue λ_1 as

$$\lambda_1 = 0.5 < \lambda_* = 1.517427128889966\cdots. \tag{89}$$

With the selected initial data in (88) and the selected λ_1 in (89), calculated from (87), the wave speed for the expected traveling wave $\phi(x + ct)$ can be specified as

$$c = c(\lambda_1) = 2.988449782668396\cdots > c_* = 1.517427123733230\cdots, \tag{90}$$

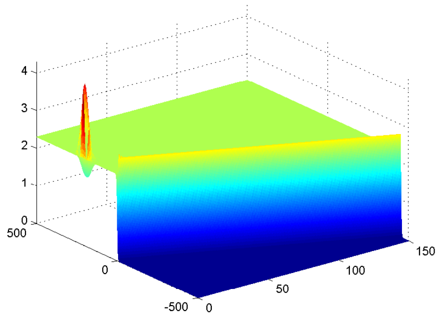


FIGURE 2. 3D-graph of $v(t,x)$ for Case 1 with $\frac{p}{d} = 10 > e^2$ and time-delay $r = 1 \in (\underline{r}, \overline{r})$ and $\lambda_1 < \lambda_*$. The solution behaves like a stable non-critical oscillatory wavefront traveling from right to left with $c > c_*$.

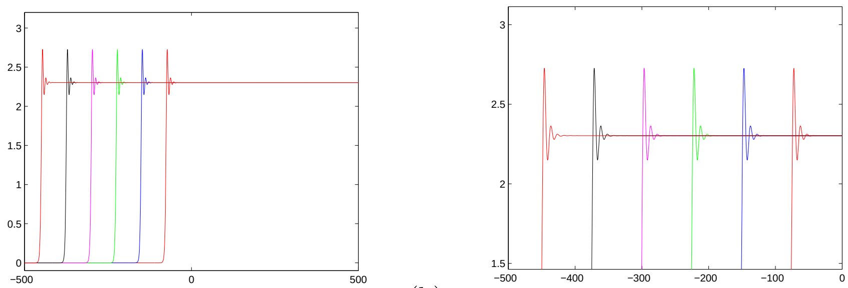
(a)  (b)

FIGURE 3. (Case 1) The solution behaves like a stable non-critical oscillatory wavefront traveling from right to left with $c > c_*$. (a) 2D-graphs of $v(t,x)$ at $t = 25,\ 50,\ 75,\ 100,\ 125,\ 150.$ (b) Enlarge graphs on oscillation parts.

and this targeted wave $\phi(x + ct)$ is such that

$$\lim_{x\to-\infty} \sup_{s\in[-r,0]} \frac{v_0(s,x)}{\phi(x+cs)} = 1 \tag{91}$$

and

$$\lim_{x\to-\infty} \sup_{s\in[-r,0]} e^{\lambda_1|x|}|v_0(s,x) - \phi(x+cs)| = 0.$$

Thus, from Corollary 1.3, we expect that the solution of (1) converges exponentially to the non-critical oscillatory traveling wave $\phi(x + ct)$ with a specified wave speed $c = 2.9884497826683 96\cdots$. Now we are going to present the numerical results in Figures 2 and 3 which exactly confirm such a convergence to the oscillatory traveling wave $\phi(x + ct)$ with $c = 2.988449782668396\cdots$.

In Figure 2 and Figure 3, we see that the solution $v(t,x)$ behaves like an oscillatory wave and travels from right to left with some oscillations. The oscillations of $v(t,x)$ around v_+ are never disappeared. The shapes of $v(t,x)$ at different times $t = 25,\ 50,\ 75,\ 100,\ 125,\ 150$ are the same. This indicates that the oscillatory traveling wave is stable. Furthermore, from the contour graph (Figure 4), we know that the slope of the contour line is just the wave speed. Since the contour line

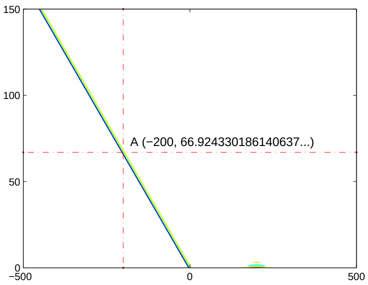


FIGURE 4. (Case 1) $\frac{p}{d} = 10 > e^2$ with time-delay $r = 1 \in (\underline{r}, \overline{r})$ and $\lambda_1 < \lambda_*$. The contour line showed in above indicates that the solution $v(t,x)$ travels with a speed of $c = 2.988449782668396\cdots > c_* = 1.517427123733230\cdots$, which implies that $v(t,x)$ behaves like the non-critical oscillatory traveling wave $\phi(x+ct)$ with $c = 2.988449782668396\cdots$.

passes through the points $(0,0)$ and $(-200, 66.924330186140637\cdots)$, so the speed can be estimated as

$$\begin{aligned} c &= \frac{|x_2 - x_1|}{|t_2 - t_1|} = \frac{|-200-0|}{|66.924330186140637\cdots - 0|} \\ &= 2.988449782668396\cdots \\ &> 1.517427123733230\cdots = c_*, \end{aligned}$$

which is exactly the same as we predicated in (90). Thus, we verify that the solution $v(t,x)$ behaves like the non-critical oscillatory traveling wave $\phi(x+ct)$ with $c = 2.988449782668396\cdots$.

Case 2. Convergence to the critical oscillatory traveling wave. Now we choose the initial data $v_0(s,x)$ as

$$v_0(s,x) = \begin{cases} v_+(1+|x|)e^{-\lambda_*|x|}, & \text{as } x \le 0,\ s \in [-r,0], \\ \frac{v_+}{1+e^{-x}}, & \text{as } x \ge 0,\ s \in [-r,0]. \end{cases} \tag{92}$$

Clearly, such a function $v_0(s,x)$ is continuous in $[-r,0]\times\mathbb{R}$. The expected critical wave speed is $c_* = c_*(\lambda_*)$ determined by (6), and the targeted wave is $\phi(x+c_*t)$ such that

$$\lim_{x\to-\infty} \sup_{s\in[-r,0]} \frac{v_0(s,x)}{\phi(x+c_*s)} = 1$$

$$\lim_{x\to-\infty} \sup_{s\in[-r,0]} e^{\lambda_*|x|}|v_0(s,x) - \phi(x+c_*s)| = 0.$$

Again, from Corollary 1.3, we expect that the solution of (1) converges exponentially to the critical oscillatory traveling wave $\phi(x+c_*t)$ with a specified wave speed $c_* = 1.517427123733230\cdots$. This can be verified from the numerical reports presented in Figure 5, Figure6 and Figure7.

In fact, Figure 5 and Figure 6 show that the solution $v(t,x)$ travels from right to left with some oscillations, those oscillations around v_+ are never disappeared, and the shapes of $v(t,x)$ at different times are the same. This implies that the solution $v(t,x)$ behaves like a stable oscillatory traveling wave. Now we further confirm that it is just the critical traveling wave. In fact, from Figure 7, the contour line passes

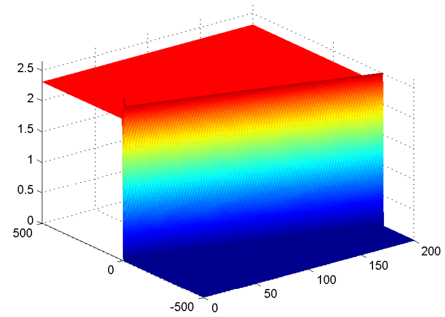
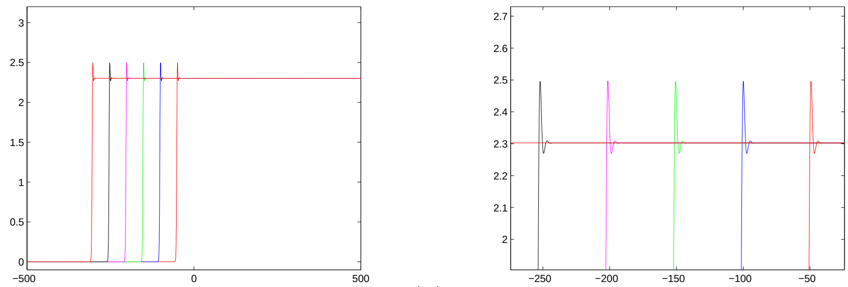


FIGURE 5. 3D-graph of $v(t, x)$ for Case 2 with $\frac{p}{d} = 10 > e^2$ and time-delay $r = 1 \in (\underline{r}, \overline{r})$. The solution $v(t, x)$ behaves like a stable critical oscillatory wavefront traveling from right to left with $c = c_*$.



(a) (b)

FIGURE 6. (Case 2) The solution $v(t, x)$ behaves like a stable critical oscillatory wavefront traveling from right to left with $c = c_*$. (a) 2D-graphs of $v(t, x)$ at $t = 25, \ 50, \ 75, \ 100, \ 125, \ 150.$ (b) Enlarge graphs on oscillation parts.

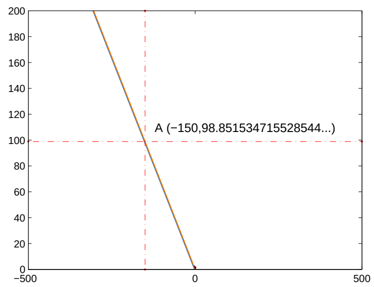


FIGURE 7. (Case 2) $\frac{p}{d} = 10 > e^2$ with time-delay $r = 1 \in (\underline{r}, \overline{r})$ and $\lambda_1 = \lambda_*$. The contour line showed in above indicates that the solution $v(t, x)$ travels with a speed of $c = c_* = 1.517427123733230 \cdots$, which implies that $v(t, x)$ behaves like the critical oscillatory traveling wave $\phi(x + c_* t)$.

through the points $(0,0)$ and $(-150, 98.851534715528544\cdots)$, so the traveling speed of the solution $v(t,x)$ can be estimated as

$$c = \frac{|x_2 - x_1|}{|t_2 - t_1|} = \frac{|-150-0|}{|98.851534715528544\cdots - 0|} = 1.517427123733230\cdots = c^*,$$

which is exactly the same to what we calculated before in (83). Therefore, we can conclude that the solution $v(t,x)$ behaves like the critical oscillatory traveling wave $\phi(x + c_* t)$ with $c_* = 1.517427123733230\cdots$.

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Department of Mathematics, Champlain College Saint-Lambert, Quebec, J4P 3P2, Canada and Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada

E-mail: `mmei@champlaincollege.qc.ca and ming.mei@mcgill.ca`

URL: `http://www.math.mcgill.ca/mei`

School of Mathematics and Statistics, Northeast Normal University Changchun, Jilin, 130024, China

E-mail: `zhangkj201@nenu.edu.cn`

Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China

E-mail: `zhangqifeng0504@163.com`