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A TIME SECOND-ORDER CHARACTERISTIC FINITE ELEMENT METHOD FOR NONLINEAR ADVECTION-DIFFUSION EQUATIONS

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Abstract. In this paper, a new time second-order characteristic finite element method is proposed and analyzed for solving the advection-diffusion equations involving a nonlinear right side term. In order to obtain a time second-order characteristic scheme, the global derivative term transferred from the time derivative and advection terms is discretized by using difference operator along the characteristics, the diffusion term is discretized by the average operator along the characteristics, while specially a second-order extrapolation along the characteristics is applied to the right side nonlinear function. We analyze and prove that the proposed scheme for the nonlinear advection-diffusion equations has second order accuracy in time step size, which improves the first order accuracy in time of the classical characteristic methods. The proposed characteristic FEM scheme allows to use large time step sizes in computation. Numerical tests are taken to show the accuracy of our proposed scheme, and the case of single-species population dynamics is further simulated and analyzed by using our method and numerical results show its advantage and effectiveness.

Key words. Nonlinear advection-diffusion equations, characteristic method, characteristic average, characteristic extrapolation, time second-order, error estimate, population dynamics.

1. Introduction

Nonlinear advection-diffusion equations have been widely applied in science and engineering computing such as in heat and mass transfer, oil reservoir simulation, groundwater modelling, atmospheric pollution, aerodynamics, biological population, and so on (see, for examples, [3, 4, 6, 9, 11, 20, 21, 22, 23, 24], etc). The nonlinear right side terms are of significance, which describe the different important physical and chemical processes such as the nonlinear reaction in physics, the biological nonlinear predator-prey and competition interaction in biological population, and the nonlinear coagulation process in atmospheric aerosol dynamics and so on. The nonlinear function of population dynamics results from the interaction of birth rate, death rate and the environment factor, which can lead to the density change of population in the global dynamics. Studying such problems has been playing more and more important role in discovering the relation between species and their environment, and in understanding the dynamic processes involved in such areas as the spread and control of diseases and viruses, predator-prey and competition interaction, evolution of pesticide-resistant strains, biological pest control, and so on. The study can also be used to describe, predict and adjust the developing trend of the population. So this research subject has an actual meaning very much.

Classical finite element methods and finite difference methods can resolve the diffusion problems well when diffusion dominates the physical process. However, solving advection-dominated diffusion equations presents non-physical oscillations or excessive numerical dispersions at steep fronts. In the framework of numerical

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solutions of advection-dominated diffusion problems, the discretization along characteristics is a good technique, which takes advantage of the physics characteristics of the convection-diffusion equations, it can not only reduce the non-physical oscillation and excessive numerical dispersion but also has no stability constraints required on the time step. Douglas and Russell [10] first proposed and analyzed the modified method of characteristics (MMOC) methods for solving one dimensional convection diffusion problems, where they combined a time first order characteristics scheme with both finite differences and classical Lagrange finite elements for time dependent convection-diffusion problems. Lately, combined with the mixed finite element methods, [2, 12, 16, 25] developed the characteristic mixed element methods for convection-diffusion problems, miscible displacement flows and immiscible displacement flows. Further developments of characteristics methods were carried out in [1, 5, 7, 17, 19], etc. The characteristic methods avoid the grid distortion, significantly reduce the truncation errors in time and eliminate the excessive numerical dispersions. However, these above characteristic methods are only of time first order accuracy. In order to improve the accuracy, recently, for nonlinear atmospheric aerosol dynamics, [18] developed a time second-order characteristic finite element scheme, where the mathematical model is the advection equation with a nonlinear coagulation integration right side term, but the equation has no diffusion term. Hence, there is of importance to develop and analyze the time second-order characteristic finite element methods to solve nonlinear advection diffusion problems with nonlinear right side terms.

In this paper, we develop and analyze a time second-order characteristic finite element method for solving nonlinear advection-diffusion problems, the time derivative term and the advection term are first transferred into the global derivative term and it is then discretized by the central difference operator along the characteristic curve. The diffusion term is approximated by the second-order average operator along the characteristic curve. For treating the nonlinear right side term, the second-order extrapolation along the characteristics is applied, where two previous level values are used along the characteristics. Both the average operator and the extrapolation are different from the normal Crank-Nicoson average and extrapolation along the time direction, they are proposed to be along the characteristics in our scheme. The developed scheme has second-order accuracy in time and can provide efficiently high accuracy solutions when using large time step sizes. Using the theory of variation and prior estimates, we analyze theoretically the developed characteristic finite element method for the nonlinear advection diffusion equations. We prove our developed characteristic scheme to have the error estimate of $O((\Delta t)^2 + h^l)$. Numerical tests for nonlinear advection diffusion equations first show that our methods have the second-order accuracy in time, which confirm the theoretical analysis results. Meanwhile, we compute the moving of a sharp front gradient and our scheme performs excellently. Finally, in simulation of the population dynamics, considering the one-dimensional single-species spatio-temporal population dynamic models which demonstrate the intricate interplay between mixing, advection, boundary conditions and other factors which are critical for the persistence of a population. We assume the population model is the nonlinear logistic growth. The change of density occurs as a result of the nonlinear logistic death-birth processes and of the spatial movement of organisms due to diffusion and advection. With different values of environmental and biological parameters, the simulated results show that changes of parameters can make the different shapes and propagations of population density distribution, thus we can control the population size at a particular place through adjusting the above parameters.

The paper is structured as follows. In Section 2, the time second-order characteristic finite element method for nonlinear convection-diffusion equations will be presented. In Section 3, we analyze theoretically the error estimate of the scheme. In Section 4, numerical experiments for nonlinear advection-diffusion equations are given. Finally, some conclusions are addressed in Section 5.

2. The mathematical model and the time second-order characteristic FE method

Consider the following nonlinear advection-diffusion equations

$$(1) \qquad \begin{cases} \frac{\partial c}{\partial t} + u(x,t)\frac{\partial c}{\partial x} - \frac{\partial}{\partial x}(k\frac{\partial c}{\partial x}) = f(x,t,c(x,t)), & x \in \Omega, t \in (0,T], \\ c(0,t) = c(b,t) = 0, & t \in (0,T], \\ c(x,0) = c^0(x), & x \in \Omega, \end{cases}$$

where $\Omega = [0,b]$ is the finite interval and $T > 0$ be the time period. The unknown $c(x,t) : \Omega \times (0,T] \to R$, $u(x,t)$ is the velocity of the field, $u(x,t) : \Omega \times (0,T] \to R$ and $u \in c^0(W^{1,\infty}(\Omega))$. $k > 0$ is diffusion coefficient. The initial function $c_0(x)$ is a smooth function. The right side function $f(x,y,c)$ is a smooth function, satisfying local Lipschitz condition with respect to variable c. For example, in biological population dynamics, $c(x,t)$ denote the population density of a population of interest at time t and position x. The nonlinear right side function normally is nonlinear logical growth function as

$$f(x,t,c(x,t)) = \beta(x,t)c(x,t)\Big(1 - \frac{c(x,t)}{K_P}\Big),$$

where $K_P > 0$ describes the carrying capacity of the environment, and $\beta(x,t)$ describes the growth rate of the biological species, which may be a spatially nutrient or an homogeneous illumination pattern and so on.

The problem (1) is well posed ([5, 13, 15]). In many applications, the advection-diffusion problems, especially for advection-dominated equations, have strongly hyperbolic feature, which lead to the difficulty of numerical solutions due to numerical oscillation or dispersion.

Let Ω_h be the quasi-uniform mesh of the interval $\Omega = [0,b]$, and $h = \max\{h_i\}, i = 1,2\cdots I$. Let $\triangle t = T/N$ be the time step, and the time level $t^n = n\triangle t, n = 1,2\cdots N$. Let $L^2(\Omega)$ be the usual space of square integrated functions on Ω, where the inner product is $(\cdot,\cdot)$, and norm is $||\cdot||$. Let $H^s(\Omega)$ be the normal Sobolev space with norm $||\cdot||_s$, and define the space $H_0^1(\Omega) = \{\psi \in H^1(\Omega) : \psi(0) = \psi(b) = 0\}$, $W_h \subset H_0^1(\Omega)$ is the standard finite element space with index $l \geq 1$ and associated with Ω_h.

We now develop the time second-order characteristic method for problem (1). Let the characteristic direction be denoted by τ, then the characteristic curve $X(\tau;x,t)$ from the point (x,t) satisfies

$$(2) \qquad \begin{cases} \frac{dX(\tau;x,t)}{d\tau} = u(X(\tau;x,t),\tau), \\ X(t;x,t) = x. \end{cases}$$

The global derivative along the characteristics is defined as

$$(3) \qquad \frac{dc}{d\tau}(X(\tau;x,t),\tau) = c'(X(\tau;x,t),\tau) + u(X(\tau;x,t),\tau)\frac{\partial c}{\partial x}(X(\tau;x,t),\tau).$$

By re-writing the first equation of (1) at point $X(\tau;x,t)$ and time τ, and using (3), we have that

$$\begin{aligned}\frac{dc}{d\tau}\left(X(\tau;x,t),\tau\right)-\frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x}\right)\left(X(\tau;x,t),\tau\right)\\ =f\left(X(\tau;x,t),\tau,c(X(\tau;x,t),\tau)\right).\end{aligned}\tag{4}$$

For obtaining the time second-order accuracy, along the characteristics, we apply the time second-order central difference to approximate the global derivative and propose to use the average of Crank-Nicolson along the characteristics to treat the diffusion term. Meanwhile, the second order extrapolation along the characteristics is proposed to approximate the nonlinear right side term as

$$\begin{aligned}&f\left(X(\tau;x,t^{n+1}),\tau,c(X(\tau;x,t^{n+1}),\tau)\right)\\ &:=\frac{3}{2}f(\bar{x}^n,t^n,c(\bar{x}^n,t^n))-\frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c(\bar{\bar{x}}^{n-1},t^{n-1})),\end{aligned}\tag{5}$$

where $\bar{x}^n=X(t^n;x,t^{n+1})$ is the intersection point of the characteristics $X(\tau;x,t^{n+1})$ with $t=t^n$ and $\bar{\bar{x}}^{n-1}=X(t^{n-1};x,t^{n+1})$ is the intersection point of the characteristic curve with time level $t=t^{n-1}$. Thus, we get the time discrete scheme for problem (1) as

$$\begin{aligned}\frac{c(x,t^{n+1})-c(\bar{x}^n,t^n)}{\Delta t}&-\frac{1}{2}\frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x}\right)(x,t^{n+1})-\frac{1}{2}\frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x}\right)(\bar{x}^n,t^n)\\ &=\frac{3}{2}f(\bar{x}^n,t^n,c(\bar{x}^n,t^n))-\frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c(\bar{\bar{x}}^{n-1},t^{n-1})).\end{aligned}\tag{6}$$

In order to get the fully discrete scheme, we will then give a weak formulation. We multiply (6) by a test function $\psi(x)\in H_0^1(\Omega)$ and integrate it in Ω.

We need to further treat the third term of of (6). According to the integration by parts, the third term of (6) becomes

$$\begin{aligned}&\frac{1}{2}\int_\Omega\frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x}\right)(\bar{x}^n,t^n)\psi(x)dx\\ &=\frac{1}{2}\int_\Omega(\frac{\partial X}{\partial x})^{-1}(x,t^n)\psi(x)d\left(k\frac{\partial c}{\partial x}(\bar{x}^n,t^n)\right)\\ &=-\frac{1}{2}\int_\Omega k\frac{\partial c}{\partial x}(\bar{x}^n,t^n)\frac{\partial}{\partial x}\left((\frac{\partial X}{\partial x})^{-1}(x,t^n)\psi(x)\right)dx\\ &=-\frac{1}{2}\int_\Omega k(\frac{\partial X}{\partial x})^{-1}(x,t^n)\frac{\partial c}{\partial x}(\bar{x}^n,t^n)\frac{\partial\psi(x)}{\partial x}dx\\ &\quad-\frac{1}{2}\int_\Omega k\frac{\partial}{\partial x}(\frac{\partial X}{\partial x})^{-1}(x,t^n)\frac{\partial c}{\partial x}(\bar{x}^n,t^n)\psi(x)dx.\end{aligned}\tag{7}$$

Meanwhile, from (2), and using Taylor's expansion of $X(\tau;x,t)$, the $(\frac{\partial X}{\partial x})^{-1}(x,t^n)$ and $\frac{\partial}{\partial x}(\frac{\partial X}{\partial x})^{-1}(x,t^n)$ can be approximated by

$$\begin{aligned}(\frac{\partial X}{\partial x})^{-1}(x,t^n)&=1+\Delta tL(\bar{x}^n,t^n)+O(\Delta t^2),\\ \frac{\partial}{\partial x}(\frac{\partial X}{\partial x})^{-1}(x,t^n)&=\Delta tL_x(\bar{x}^n,t^n)+O(\Delta t^2),\end{aligned}\tag{8}$$

where $L(x,t)=\frac{\partial u}{\partial x}(x,t)$ and $L_x(x,t)=\frac{\partial^2 u}{\partial x^2}(x,t)$.

Let $c_h(x,t)\in W_h$ be the approximate solution of $c(x,t)$. Using the above notations, the time second-order characteristic finite element scheme is proposed as:

find $c_h \in W_h$, such that

$$
\begin{aligned}
&\left(\frac{c_h(x,t^{n+1})-c_h(\bar{x}^n,t^n)}{\Delta t},\psi_h\right)+\frac{1}{2}\left(k\frac{\partial c_h}{\partial x}(x,t^{n+1})+k\frac{\partial c_h}{\partial x}(\bar{x}^n,t^n),\frac{\partial \psi_h}{\partial x}\right)\\
&+\frac{\Delta t}{2}\left(k(L\frac{\partial c_h}{\partial x})(\bar{x}^n,t^n),\frac{\partial \psi_h}{\partial x}\right)+\frac{\Delta t}{2}\left(k(L_x\frac{\partial c_h}{\partial x})(\bar{x}^n,t^n),\psi_h\right)\\
&=\left(\frac{3}{2}f(\bar{x}^n,t^n,c_h(\bar{x}^n,t^n))-\frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c_h(\bar{\bar{x}}^{n-1},t^{n-1})),\psi_h\right),\psi_h\in W_h
\end{aligned}
\tag{9}
$$

with the initial value $c_h(0)=Q_hc^0(x)$, where the operator Q_h is the approximation project operator to W_h, also, the operator Q_h ensures the high accuracy for $c_h(0)$. In scheme (9), we use the zero extension of $c_h(\bar{x}^n)=0$, when $\bar{x}^n$ goes out of Ω.

As we have used the second-order extrapolation along characteristics to treat the nonlinear right side term, the proposed scheme (9) is a linearized scheme. Thus, it is clear that the system (9) has the existence and uniqueness of the solution.

Throughout this paper, the symbols M and ε will denote, respectively, a generic constant and a generic small positive constant, which could have different values at different appearances.

3. Error analysis

Let m be a non-negative integer, $H^m(\Omega)$ be the *Sobolev* space with norm $||\cdot||_m$, when $m=0$, it is the L^2 space, the product is $(\cdot)$, the norm is $||\cdot||_0$. We use the functions space $C^m([0,T];X)$ and $H^m([0,T];X)$ for positive number and Banach space X, which will be abbreviated as $C^m(X)$ and $H^m(X)$, respectively, and we introduce their norms defined by

$$
\begin{aligned}
||g||_{C^m(X)} &:= \max_{t\in[0,T]}\left\{\max_{j=0,\cdots,m}||g^{(j)}(t)||_X\right\},\\
||g||_{H^m(X)} &:= \left(\int_0^T\sum_{j=0}^m||g^{(j)}(t)||_X^2dt\right)^{\frac{1}{2}},
\end{aligned}
\tag{10}
$$

where $g^{(j)}$ denotes the jth derivative of g with respect to time. For a non-negative integer m, we introduce a function space Z^m defined by

$$
Z^m=\left\{g\in C^j(H^{m-j}(\Omega)),j=0,1,\cdots,m,|||g|||_m<+\infty\right\},
\tag{11}
$$

where

$$
|||g|||_m=\max\{||g||_{C^j(H^{m-j}(\Omega))},0\le j\le m\}.
\tag{12}
$$

Lemma 1. *For the finite space W_h with index $l\ge 1$, there exists an interpolation operator $\pi_h: L_\infty(\Omega)\to W_h$ satisfying*

$$
\begin{aligned}
&||c-\pi_hc||_r\le Mh^{l+1-r}||c||_{l+1}, \quad \forall c\in H_0^1(\Omega)\cap H^{l+1}(\Omega), \qquad r=0,1,\\
&||c-\pi_hc||_\infty\le Mh^{l+1}||c||_{W^{l+1,\infty}}, \ \forall c\in H_0^1(\Omega)\cap W^{l+1,\infty}(\Omega),
\end{aligned}
\tag{13}
$$

with a positive constant M independence of h.

Theorem 2. *Let c_h be the solution of (9) over W_h with index $l\ge 1$ subject to the initial value $c_h^0=\pi_hc^0$. Let c be the solution of (1) which satisfies $c\in C^0(H^{l+1})\cap H^2(H^l)\cap Z^3,\Delta c\in Z^2$, with the assumption that $f(x,y,c)$ is smooth function, satisfying local Lipschitz condition with respect to variable c. Then there exists a positive constant M independent of $h,\Delta t$ such that*

$$
\max_{0\le n\le N}||c^n-c_h^n||_0\le M\left((\Delta t)^2+h^l\right).
\tag{14}
$$

Proof: Let π_h be the interpolation operator, let $e_h^n = c_h^n - \pi_h c^n, \eta^n = c^n - \pi_h c^n$. Using these notations, we have the following error equation

(15)
$$
\begin{aligned}
&\left(\frac{e_h(x,t^{n+1}) - e_h(\bar{x}^n,t^n)}{\Delta t}, \psi\right) + \frac{1}{2}\left(k\frac{\partial e_h}{\partial x}(x,t^{n+1}) + k\frac{\partial e_h}{\partial x}(\bar{x}^n,t^n), \frac{\partial \psi}{\partial x}\right)\\
&= \left(\frac{\eta(x,t^{n+1}) - \eta(\bar{x}^n,t^n)}{\Delta t}, \psi\right) + \frac{1}{2}\left(k\frac{\partial \eta}{\partial x}(x,t^{n+1}) + k\frac{\partial \eta}{\partial x}(\bar{x}^n,t^n), \frac{\partial \psi}{\partial x}\right)\\
&\quad - \frac{\Delta t}{2}\left(k(L\frac{\partial e_h}{\partial x})(\bar{x}^n,t^n), \frac{\partial \psi}{\partial x}\right) + \frac{\Delta t}{2}\left(k(L\frac{\partial \eta}{\partial x})(\bar{x}^n,t^n), \frac{\partial \psi}{\partial x}\right)\\
&\quad - \frac{\Delta t}{2}\left(k(L_x\frac{\partial e_h}{\partial x})(\bar{x}^n,t^n), \psi\right) + \frac{\Delta t}{2}\left(k(L_x\frac{\partial \eta}{\partial x})(\bar{x}^n,t^n), \psi\right)\\
&\quad + \left((\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x})(\frac{x+\bar{x}^n}{2}, t^{n+\frac{1}{2}}), \psi\right) - \left(\frac{c(x,t^{n+1}) - c(\bar{x}^n,t^n)}{\Delta t}, \psi\right)\\
&\quad - \left(\frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(\frac{x+\bar{x}^n}{2}, t^{n+\frac{1}{2}}), \psi\right) - \frac{1}{2}\left(k\frac{\partial c}{\partial x}(x,t^{n+1}) + k\frac{\partial c}{\partial x}(\bar{x}^n,t^n), \frac{\partial \psi}{\partial x}\right)\\
&\quad - \frac{\Delta t}{2}\left(k(L\frac{\partial c}{\partial x})(\bar{x}^n,t^n), \frac{\partial e_h^{n+1}}{\partial x}\right) - \frac{\Delta t}{2}\left(k(L_x\frac{\partial c}{\partial x})(\bar{x}^n,t^n), e_h^{n+1}\right)\\
&\quad - \left(f(\frac{x+\bar{x}^n}{2}, t^{n+\frac{1}{2}}, c(\frac{x+\bar{x}^n}{2}, t^{n+\frac{1}{2}})) - \frac{3}{2}f(\bar{x}^n,t^n,c_h(\bar{x}^n,t^n))\right.\\
&\quad\quad \left. + \frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c_h(\bar{\bar{x}}^{n-1},t^{n-1})), \psi\right)\\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9.
\end{aligned}
$$

In the following analysis, we use the zero extension of $c^n(\bar{x}) = 0$ *and* $\pi_h c^n(\bar{x}) = 0$, when $\bar{x} < 0$.

Choosing the test function $\psi = e_h^{n+1}$, we estimate the terms on the right-hand of (15), firstly, noting that

$$
\begin{aligned}
||e_h(\bar{x}^n,t^n)||_0^2 &= \int_\Omega e_h^2(\bar{x}^n,t^n)dx = \int_\Omega e_h^2(y,t^n)(\frac{\partial X}{\partial x})^{-1}dy\\
&\leq (1+M\Delta t)||e_h^n||_0^2.
\end{aligned} \tag{16}
$$

Similarly, we can get the following estimate

$$
||\frac{\partial e_h}{\partial x}(\bar{x}^n,t^n)||_0^2 \leq (1+M\Delta t)||\frac{\partial e_h}{\partial x}||_0^2. \tag{17}
$$

For the term of I_1, we can get

$$
\begin{aligned}
I_1 &= \left(\frac{\eta(x,t^{n+1}) - \eta(\bar{x}^n,t^n)}{\Delta t}, e_h^{n+1}\right)\\
&= \left(\frac{\eta(x,t^{n+1}) - \eta(x,t^n)}{\Delta t}, e_h^{n+1}\right) + \left(\frac{\eta(x,t^n) - \eta(\bar{x}^n,t^n)}{\Delta t}, e_h^{n+1}\right)\\
&= I_{11} + I_{12}.
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
I_{11} &= \left(\frac{\eta(x,t^{n+1}) - \eta(x,t^n)}{\Delta t}, e_h^{n+1}\right) = \frac{1}{\Delta t}\int_\Omega \left[\int_{t^n}^{t^{n+1}} \frac{\partial \eta}{\partial t}dt\right] e_h^{n+1}dx\\
&\leq M||e_h^{n+1}||_0^2 + \frac{M}{\Delta t}||\frac{\partial \eta}{\partial t}||_{L^2(t^n,t^{n+1};L^2)}^2.
\end{aligned} \tag{19}
$$

$$I_{12} = \left(\frac{\eta(x,t^n) - \eta(\bar{x}^n,t^n)}{\Delta t}, e_h^{n+1} \right) \leq M(h^{2l} + ||e_h^{n+1}||_0^2). \tag{20}$$

Thus

$$I_1 \leq Mh^{2l} + M||e_h^{n+1}||_0^2 + \frac{M}{\Delta t}||\frac{\partial \eta}{\partial t}||_0^2. \tag{21}$$

Considering the term of I_2

$$\begin{aligned}
I_2 &= \frac{1}{2}\left(k\frac{\partial \eta}{\partial x}(x,t^{n+1}) + k\frac{\partial \eta}{\partial x}(\bar{x}^n,t^n), \frac{\partial e_h^{n+1}}{\partial x} \right) \\
&= \frac{k}{2}\left((\frac{\partial \eta}{\partial x}(x,t^{n+1}), \frac{\partial e_h^{n+1}}{\partial x}) - (\frac{\partial \eta}{\partial x}(x,t^n), \frac{\partial e_h^n}{\partial x}) \right) \\
&+ \frac{k}{2}\left((\frac{\partial \eta}{\partial x}(\bar{x}^n,t^n), \frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n,t^n) \right) \\
&+ \frac{k}{2}\left((\frac{\partial \eta}{\partial x}(x,t^n), \frac{\partial e_h^n}{\partial x}) - (\frac{\partial \eta}{\partial x}(\bar{x}^n,t^n), \frac{\partial e_h^n}{\partial x}(\bar{x}^n,t^n)) \right) \\
&= I_{21} + I_{22} + I_{23}.
\end{aligned} \tag{22}$$

For the term of I_{22}, we have

$$\begin{aligned}
I_{22} &= \frac{k}{2}\left((\frac{\partial \eta}{\partial x}(\bar{x}^n,t^n), \frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n,t^n) \right) \\
&\leq \frac{k}{4}\left(||\frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n,t^n)||_0^2 + ||\frac{\partial \eta}{\partial x}(\bar{x}^n,t^n)||_0^2 \right) \\
&\leq \frac{k}{4}||\frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n,t^n)||_0^2 + Mh^{2l}.
\end{aligned} \tag{23}$$

The term I_{23} is estimated as

$$\begin{aligned}
I_{23} &= \frac{k}{2}\left((\frac{\partial \eta}{\partial x}(x,t^n), \frac{\partial e_h^n}{\partial x}) - (\frac{\partial \eta}{\partial x}(\bar{x}^n,t^n), \frac{\partial e_h^n}{\partial x}(\bar{x}^n,t^n)) \right) \\
&= \frac{k}{2}\left(\int_{\Omega_i} \frac{\partial \eta}{\partial x}(x,t^n)\frac{\partial e_h^n}{\partial x}dx - \int_{X^n(\Omega_i)} \frac{\partial \eta}{\partial x}(x,t^n)\frac{\partial e_h^n}{\partial x}(\frac{\partial X^n}{\partial x})^{-1}dy \right) \\
&= \frac{k}{2}\int_{\Omega_i} \frac{\partial \eta}{\partial x}(x,t^n)\frac{\partial e_h^n}{\partial x}(1 - (\frac{\partial X^n}{\partial x})^{-1})dx \\
&= M\Delta t(||\frac{\partial \eta}{\partial x}||_0^2 + ||\frac{\partial e_h^n}{\partial x}||_0^2) \\
&= M\Delta t h^{2l} + M\Delta t||\frac{\partial e_h^n}{\partial x}||_0^2.
\end{aligned} \tag{24}$$

It is clear that

$$\begin{aligned}
I_2 &\leq \frac{k}{2}\left((\frac{\partial \eta^{n+1}}{\partial x}, \frac{\partial e_h^{n+1}}{\partial x}) - (\frac{\partial \eta^n}{\partial x}, \frac{\partial e_h^n}{\partial x}) \right) \\
&+ \frac{k}{4}||\frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n,t^n)||_0^2 + M\Delta t||\frac{\partial e_h^n}{\partial x}||_0^2 + M(1+\Delta t)h^{2l}.
\end{aligned} \tag{25}$$

Using Minkowski inequality and (18), we can get

$$\begin{aligned}
I_3 &= -\frac{\Delta t}{2}\left(k(L\frac{\partial e_h}{\partial x})(\bar{x}^n,t^n), \frac{\partial e_h^{n+1}}{\partial x} \right) \\
&\leq M\Delta t\left((1+\Delta t)||\frac{\partial e_h^n}{\partial x}||_0^2 + ||\frac{\partial e_h^{n+1}}{\partial x}||_0^2 \right).
\end{aligned} \tag{26}$$

Similarly, we can estimate I_4, I_5, I_6 using same method, yields

$$
\begin{aligned}
I_4 &= \frac{\Delta t}{2}\left(k(L\frac{\partial \eta}{\partial x})(\bar{x}^n, t^n), \frac{\partial e_h^{n+1}}{\partial x}\right) \\
&\le M\Delta t\left((1+\Delta t)||\frac{\partial \eta}{\partial x}||_0^2 + ||\frac{\partial e_h^{n+1}}{\partial x}||_0^2\right) \\
&\le M\Delta t\left(h^{2l} + ||\frac{\partial e_h^{n+1}}{\partial x}||_0^2\right).
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
I_5 &= -\frac{\Delta t}{2}\left(k(\frac{\partial^2 u}{\partial x^2}\frac{\partial e_h}{\partial x})(\bar{x}^n, t^n), e_h^{n+1}\right) \\
&\le M\Delta t\left((1+\Delta t)||\frac{\partial e_h^n}{\partial x}||_0^2 + ||e_h^{n+1}||_0^2\right).
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
I_6 &= \frac{\Delta t}{2}\left(k(\frac{\partial^2 u}{\partial x^2}\frac{\partial \eta}{\partial x})(\bar{x}^n, t^n), e_h^{n+1}\right) \\
&\le M\Delta t\left((1+\Delta t)||\frac{\partial \eta}{\partial x}||_0^2 + ||e_h^{n+1}||_0^2\right) \\
&\le M\Delta t\left(h^{2l} + ||e_h^{n+1}||_0^2\right).
\end{aligned} \tag{29}
$$

Now, we estimate the term of

$$
I_7 = \left((\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x})(\frac{x+\bar{x}^n}{2}, t^{n+\frac{1}{2}}), e_h^{n+1}\right) - \left(\frac{c(x,t^{n+1}) - c(\bar{x}^n, t^n)}{\Delta t}, e_h^{n+1}\right). \tag{30}
$$

By the following Taylor's expansion of $c(x, t^{n+1})$ and $c(\bar{x}, t^n)$ at the point $(\frac{x+\bar{x}^n}{2}, t^{n+\frac{1}{2}})$, let $x^* = \frac{x+\bar{x}^n}{2}$.

$$
\begin{aligned}
c(x, t^{n+1}) &= c(x^*, t^{n+\frac{1}{2}}) + \frac{\Delta l}{2}\left(\frac{\partial c}{\partial \tau}\right)(x^*, t^{n+\frac{1}{2}}) \\
&\quad + \frac{(\frac{\Delta l}{2})^2}{2!}\left(\frac{\partial^2 c}{\partial \tau^2}\right)^{n+\frac{1}{2}}(x^*) + \frac{(\frac{\Delta l}{2})^3}{3!}\left(\frac{\partial^3 c}{\partial \tau^3}\right)(x', t'),
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
c(\bar{x}^n, t^n) &= c(x^*, t^{n+\frac{1}{2}}) - \frac{\Delta l}{2}\left(\frac{\partial c}{\partial \tau}\right)(x^*, t^{n+\frac{1}{2}}) \\
&\quad + \frac{(\frac{\Delta l}{2})^2}{2!}\left(\frac{\partial^2 c}{\partial \tau^2}\right)^{n+\frac{1}{2}}(x^*) - \frac{(\frac{\Delta l}{2})^3}{3!}\left(\frac{\partial^3 c}{\partial \tau^3}\right)(x'', t'').
\end{aligned} \tag{32}
$$

where $\Delta l = ((x-\bar{x}^n)^2 + (\Delta t)^2)^{\frac{1}{2}}$. Thus, we have that

$$
c(x, t^{n+1}) - c(\bar{x}^n, t^n) = \Delta l\left(\frac{\partial c}{\partial \tau}\right)(x^*, t^{n+\frac{1}{2}}) + M(\Delta l)^3\frac{\partial^3 c}{\partial \tau^3}(x''', t'''), \tag{33}
$$

and

$$
\frac{c(x, t^{n+1}) - c(\bar{x}^n, t^n)}{\Delta t} = \left(\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x}\right)(x^*, t^{n+\frac{1}{2}}) + \frac{M(\Delta l)^3}{\Delta t}\frac{\partial^3 c}{\partial \tau^3}(x''', t'''). \tag{34}
$$

In addition

$$
\Delta l = \sqrt{(x-\bar{x}^n)^2 + (\Delta t)^2} = M\Delta t. \tag{35}
$$

It leads to the following estimate

$$
I_7 \le M((\Delta t)^4 |||c|||_3^2 + ||e_h^{n+1}||_0^2). \tag{36}
$$

Considering the term of I_8

$$
\begin{aligned}
I_8 = &-\left(\frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(\frac{x+\bar{x}^n}{2}, t^{n+\frac{1}{2}}), e_h^{n+1}\right)\\
&-\frac{1}{2}\left(k\frac{\partial c}{\partial x}(x,t^{n+1}) + k\frac{\partial c}{\partial x}(\bar{x}^n,t^n), \frac{\partial e_h^{n+1}}{\partial x}\right)\\
&-\frac{\Delta t}{2}\left(k(L\frac{\partial c}{\partial x})(\bar{x}^n,t^n), \frac{\partial e_h^{n+1}}{\partial x}\right) - \frac{\Delta t}{2}\left(k(L_x\frac{\partial c}{\partial x})(\bar{x}^n,t^n), e_h^{n+1}\right)\\
= &\frac{1}{2}\left(\frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(x,t^{n+1}) + \frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(\bar{x}^n,t^n), e_h^{n+1}\right)\\
&-\left(\frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(x^*, t^{n+\frac{1}{2}}), e_h^{n+1}\right) + O(\Delta t^4 h).
\end{aligned}
\tag{37}
$$

By the following Taylor's expansion of $\frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(x,t^{n+1})$ and $\frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(\bar{x}^n,t^n)$ at the point $(x^*, t^{n+\frac{1}{2}})$.

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x})(x,t^{n+1}\right) &= \frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x}\right)(x^*, t^{n+\frac{1}{2}}) + \frac{k\Delta l}{2}\frac{\partial^3 c}{\partial x^2\partial\tau}(x^*, t^{n+\frac{1}{2}})\\
&\quad + \frac{k(\frac{\Delta l}{2})^2}{2!}\frac{\partial^4 c}{\partial x^2\partial\tau^2}(x', t'),\\
\frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x}\right)(\bar{x}^n,t^n) &= \frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x}\right)(x^*, t^{n+\frac{1}{2}}) - \frac{k\Delta l}{2}\frac{\partial^3 c}{\partial x^2\partial\tau}(x^*, t^{n+\frac{1}{2}})\\
&\quad + \frac{k(\frac{\Delta l}{2})^2}{2!}\frac{\partial^4 c}{\partial x^2\partial\tau^2}(x'', t'').
\end{aligned}
\tag{38}
$$

we have that

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(x,t^{n+1}) + \frac{\partial}{\partial x}(k\frac{\partial c}{\partial x})(\bar{x}^n,t^n)\right)\\
&\quad = \frac{\partial}{\partial x}\left(k\frac{\partial c}{\partial x}\right)(x^*, t^{n+\frac{1}{2}}) + \frac{k(\frac{\Delta l}{2})^2}{2!}\frac{\partial^4 c}{\partial x^2\partial\tau^2}(x''', t''').
\end{aligned}
\tag{39}
$$

It leads to the following estimate of I_8

$$
I_8 \le M\left((\Delta t)^4 |||c|||_4^2 + ||e_h^{n+1}||_0^2\right).
\tag{40}
$$

Finally, we estimate the term of I_9 by using the function f satisfying the local Lipschitz condition with respect to variable c. We use the zero extension of $f(x,t,c(x,t)) = 0$ and $f(x,t,c_h(x,t)) = 0$, when $x < 0$.

For doing this, we make an induction hypothesis that there exists a positive constant $M^* > 0$ such that

$$
\sup_{0\le n\le N} ||c_h^n||_\infty \le M^*.
\tag{41}
$$

which will be proved later.

we have

$$
\begin{aligned}
I_9 &= \left(\frac{3}{2}f(\bar{x}^n,t^n,c_h(\bar{x}^n,t^n)) - \frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c_h(\bar{\bar{x}}^{n-1},t^{n-1})),e_h^{n+1}\right)\\
&\quad - \left(f(x^*,t^{n+\frac{1}{2}},c(x^*,t^{n+\frac{1}{2}})),e_h^{n+1}\right)\\
&= \left(\frac{3}{2}f(\bar{x}^n,t^n,c_h(\bar{x}^n,t^n)) - \frac{3}{2}f(\bar{x}^n,t^n,c(\bar{x}^n,t^n)),e_h^{n+1}\right)\\
&\quad - \Big[(\frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c_h(\bar{\bar{x}}^{n-1},t^{n-1}))\\
&\quad -\frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c(\bar{\bar{x}}^{n-1},t^{n-1})),e_h^{n+1})\Big]\\
&\quad + \Big(\frac{3}{2}f(\bar{x}^n,t^n,c(\bar{x}^n,t^n)) - \frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c(\bar{\bar{x}}^{n-1},t^{n-1}))\\
&\quad\quad -f(x^*,t^{n+\frac{1}{2}},c(x^*,t^{n+\frac{1}{2}})),e_h^{n+1}\Big)\\
&:= I_{91}+I_{92}+I_{93}.
\end{aligned}
\tag{42}
$$

For the term of I_{91}, we have

$$
\begin{aligned}
I_{91} &= \left(\frac{3}{2}f(\bar{x}^n,t^n,c_h(\bar{x}^n,t^n)) - \frac{3}{2}f(\bar{x}^n,t^n,c(\bar{x}^n,t^n)),e_h^{n+1}\right)\\
&\le M\left(|c_h(\bar{x}^n,t^n)-c(\bar{x}^n,t^n)|,e_h^{n+1}\right)\\
&\le M\left(||e_h^n(\bar{x}^n)||_0^2+||\eta^n(\bar{x}^n)||_0^2+||e_h^{n+1}||_0^2\right)\\
&\le M\left((1+\Delta t)(||e_n^n||_0^2+||\eta^n||_0^2)+||e_h^{n+1}||_0^2\right)\\
&\le M\left((1+\Delta t)h^{2l+2}+(1+\Delta t)||e_h^n||_0^2+||e_h^{n+1}||_0^2\right).
\end{aligned}
\tag{43}
$$

For the term of I_{92}, we can similarly derive that

$$
\begin{aligned}
I_{92} &= -\Big[(\frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c_h(\bar{\bar{x}}^{n-1},t^{n-1})) - \frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c(\bar{\bar{x}}^{n-1},t^{n-1})),e_h^{n+1})\Big]\\
&\le M\left((1+\Delta t)h^{2l+2}+(1+\Delta t)||e_h^{n-1}||_0^2+||e_h^{n+1}||_0^2\right).
\end{aligned}
\tag{44}
$$

Then, for the term of I_{93}, we apply Taylor's expansion to $f(c,t,c(x,t))$

$$
\begin{aligned}
I_{93} &= \Big(\frac{3}{2}f(\bar{x}^n,t^n,c(\bar{x}^n,t^n)) - \frac{1}{2}f(\bar{\bar{x}}^{n-1},t^{n-1},c(\bar{\bar{x}}^{n-1},t^{n-1}))\\
&\quad -f(x^*,t^{n+\frac{1}{2}},c(x^*,t^{n+\frac{1}{2}})),e_h^{n+1}\Big),
\end{aligned}
\tag{45}
$$

where

$$
\begin{aligned}
f\left(\bar{x}^n,t^n,c(\bar{x}^n,t^n)\right) &= f\left(x^*,t^n,c(x^*,t^n)\right) + (-\frac{\Delta l}{2})\left(\frac{\partial f}{\partial \tau}\right)(x^*,t^{n+\frac{1}{2}})\\
&\quad + \frac{(-\frac{\Delta l}{2})^2}{2!}\left(\frac{\partial^2 f}{\partial \tau^2}\right)^{n+\frac{1}{2}}(x^*) + \frac{(-\frac{\Delta l}{2})^3}{3!}\left(\frac{\partial^3 f}{\partial \tau^3}\right)(x^{'},t^{'}),
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&f\left(\bar{\bar{x}}^{n-1}, t^{n-1}, c(\bar{\bar{x}}^{n-1}, t^{n-1})\right) \\
&\quad = f\left(x^*, t^n, c(x^*, t^n)\right) + (-\frac{3\Delta l}{2})\left(\frac{\partial f}{\partial \tau}\right)(x^*, t^{n+\frac{1}{2}}) \\
&\qquad + \frac{(-\frac{3\Delta l}{2})^2}{2!}\left(\frac{\partial^2 f}{\partial \tau^2}\right)^{n+\frac{1}{2}}(x^*) + \frac{(-\frac{3\Delta l}{2})^3}{3!}\left(\frac{\partial^3 f}{\partial \tau^3}\right)(x^{''}, t^{''}).
\end{aligned} \tag{47}
$$

Thus, we have that

$$
\begin{aligned}
&\frac{3}{2} f\left(\bar{x}^n, t^n, c(\bar{x}^n, t^n)\right) - \frac{1}{2} f\left(\bar{\bar{x}}^{n-1}, t^{n-1}, c(\bar{\bar{x}}^{n-1}, t^{n-1})\right) \\
&\qquad\qquad = f\left(x^*, t^n, c(x^*, t^n)\right) - \frac{3\Delta l^2}{8}\frac{\partial^2 f}{\partial \tau^2}(x^{'''}, t^{'''}).
\end{aligned} \tag{48}
$$

then, it holds that

$$
I_{93} \le M\left((\Delta t)^4 |||f|||_2^2 + ||e_h^{n+1}||_0^2\right). \tag{49}
$$

Combining the estimates of I_{91}, I_{92} and I_{93} together, we have finally the estimate to I_9

$$
\begin{aligned}
I_9 \le M &\left((1+\Delta t)h^{2l+2} + (1+\Delta t)(||e_h^n||_0^2 + ||e_h^{n-1}||_0^2)\right. \\
&\left. + (\Delta t)^4 |||f|||_2^2 + ||e_h^{n+1}||_0^2\right).
\end{aligned} \tag{50}
$$

Next, we have the estimate for the left-hand side term in (15)

$$
\begin{aligned}
&\left(\frac{e_h(x, t^{n+1}) - e_h(\bar{x}^n, t^n)}{\Delta t}, e_h^{n+1}\right) \\
&= \left(\frac{e_h(x, t^{n+1}) - e_h(\bar{x}^n, t^n)}{\Delta t}, \frac{e_h(x, t^{n+1}) - e_h(\bar{x}^n, t^n) + e_h(x, t^{n+1}) + e_h(\bar{x}^n, t^n)}{2}\right) \\
&= \frac{1}{2\Delta t}\left(||e_h(x, t^{n+1}) - e_h(\bar{x}^n, t^n)||_0^2 + ||e_h(x, t^{n+1})||_0^2 - ||e_h(\bar{x}^n, t^n)||_0^2\right) \\
&\ge \frac{1}{2\Delta t}\left(||e_h(x, t^{n+1}) - e_h(\bar{x}^n, t^n)||_0^2\right) + \frac{1}{2\Delta t}\left(||e_h^{n+1}||_0^2 - ||e_h^n||_0^2\right) - \frac{M}{2}||e_h^n||_0^2.
\end{aligned} \tag{51}
$$

The last inequality can be calculated by formula (18).

Using the similar method, we can get

$$
\begin{aligned}
&\left(\frac{k\frac{\partial e_h}{\partial x}(x, t^{n+1}) + k\frac{\partial e_h}{\partial x}(\bar{x}^n, t^n)}{2}, \frac{\partial e_h^{n+1}}{\partial x}\right) \\
&= \frac{k}{2}\left(\frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n), \frac{\frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n) + \frac{\partial e_h^{n+1}}{\partial x} - \frac{\partial e_h^n}{\partial x}(\bar{x}^n)}{2}\right) \\
&= \frac{k}{4}\left(||\frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n)||_0^2 + ||\frac{\partial e_h^{n+1}}{\partial x}||_0^2 - ||\frac{\partial e_h^n}{\partial x}(\bar{x}^n)||_0^2\right) \\
&\ge \frac{k}{4}||\frac{\partial e_h^{n+1}}{\partial x} + \frac{\partial e_h^n}{\partial x}(\bar{x}^n)||_0^2 + \frac{k}{4}\left(||\frac{\partial e_h^{n+1}}{\partial x}||_0^2 - ||\frac{\partial e_h^n}{\partial x}(x^n)||_0^2\right) - \frac{kM\Delta t}{4}||\frac{\partial e_h^n}{\partial x}||_0^2.
\end{aligned} \tag{52}
$$

From the estimates of $I_1 \sim I_9$, and (51),(52), we can get

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(||e_h^{n+1}||_0^2 - ||e_h^n||_0^2\right) + \frac{k}{4}\left(||\frac{\partial e_h^{n+1}}{\partial x}||_0^2 - ||\frac{\partial e_h^n}{\partial x}(x^n)||_0^2\right) \\
&\le \frac{k}{2}\left((\frac{\partial \eta^{n+1}}{\partial x}, \frac{\partial e_h^{n+1}}{\partial x}) - (\frac{\partial \eta^n}{\partial x}, \frac{\partial e_h^n}{\partial x})\right) \\
&+ M\left(\frac{1}{\Delta t}||\frac{\partial \eta}{\partial t}||_0^2 + \Delta t(||\frac{\partial e_h^n}{\partial x}||_0^2 + ||\frac{\partial e_h^{n+1}}{\partial x}||_0^2)\right) \\
&+ M(\Delta t)^4\left(|||c|||_3^2 + |||c|||_4^2\right) + M(1+\Delta t)(h^{2l} + h^{2l+2}) \\
&+ M\left((1+\Delta t)(||e_h^n||_0^2 + ||e_h^{n-1}||_0^2 + ||e_h^{n+1}||_0^2) + (\Delta t)^4|||f|||_2^2\right).
\end{aligned} \tag{53}$$

Multiplying the above equations by $2\Delta t$ and summing them from $i = 1$ to $m-1$, we can get

$$\begin{aligned}
&||e_h^m||_0^2 + \frac{k\Delta t}{2}||\frac{\partial e_h^m}{\partial x}||_0^2 \le k\Delta t(\frac{\partial \eta^m}{\partial x}, \frac{\partial e_h^m}{\partial x}) \\
&\quad + M\Delta t\left(\sum_{i=1}^{m}||e_h^i||_0^2 + \Delta t\sum_{i=1}^{m}||\frac{\partial e_h^i}{\partial x}||_0^2\right) + M\left(h^{2l} + (\Delta t)^4\right).
\end{aligned} \tag{54}$$

Using the estimates

$$(\frac{\partial \eta^m}{\partial x}, \frac{\partial e_h^m}{\partial x}) \le Mh^{2l} + \frac{1}{4}||\frac{\partial e_h^m}{\partial x}||_0^2. \tag{55}$$

when $m = 1$, using the implicit scheme or small time step to calculate time error, we can get second order accuracy, and an application of $Gronwall's$ lemma shows that

$$||e_h^m||_0^2 + \frac{k\Delta t}{4}||\frac{\partial e_h^m}{\partial x}||_0^2 \le M\left(h^{2l} + (\Delta t)^4\right). \tag{56}$$

So

$$||e_h^m||_0 \le M\left((h^l + (\Delta t)^2\right), \tag{57}$$

for $m = 1, 2, \cdots, N$.

At the end, we need to show that the induction hypothesis (41) is true. First, it is clear that

$$\begin{aligned}
||c_h^0||_\infty &= ||\pi_h c^0||_\infty \le ||c^0 - \pi_h c^0||_\infty + ||c^0||_\infty \\
&\le Mh^{l+1}||c^0||_{W^{l+1,\infty}} + ||c^0||_\infty \le M^*.
\end{aligned} \tag{58}$$

If (41) is false, there exists an integer q^* such that

$$||c_h^q||_\infty \le M^*\ , for\ 0 \le q \le q^* - 1\ ; ||c_h^{q^*}||_\infty > M^*. \tag{59}$$

Similarly to (57), we can get that when h and Δt are small enough, and providing with $\Delta t^2 = O(h^{\frac{1}{2}})$,

$$\begin{aligned}
||c_h^{q^*}||_\infty &\le ||c^{q^*} - \pi_h c^{q^*}||_\infty + ||\pi_h c^{q^*} - c_h^{q^*}||_\infty + ||c^{q^*}||_\infty \\
&\le Mh^{l+1} + Mh^{-\frac{1}{2}}\left((\Delta t)^2 + h^l\right) + M \\
&\le M^*.
\end{aligned} \tag{60}$$

which is a contradiction to (59). This thus proves the induction hypothesis (41). Therefore, we have the error estimate

$$\max_{0 \le n \le N}||c^n - c_h^n||_0 \le M\left((\Delta t)^2 + h^l\right). \tag{61}$$

Remark 1. *In the theorem, we have proved the error estimate of $O\left((\Delta t)^2 + h^l\right)$, where the time order is optimal second-order while the space error is $O(h^l)$. However, numerical tests in Section 4 have shown the space error to be optimal $O(h^2)$ when the piecewise linear polynomial space ($l = 1$) is used. There is a big difficulty of analyzing the terms of diffusions at the n-level characteristic points for our time second-order characteristic scheme, for obtaining the error estimate of $O(h^{l+1})$, which is much different to the classical time first-order characteristic scheme. This leaves us a future research work in the near future.*

4. Numerical experiments

In this section, we will carry out numerical tests to observe the performance of our time second-order characteristic finite element method for solving the nonlinear advection-diffusion equations. We first consider four different nonlinear advection-diffusion equations with exact solutions, where errors and ratios for different time steps and space sizes are compared, we can see clearly from the results that our method has second-order accuracy in time. At last, a numerical simulation of nonlinear population dynamics is carried out by using our method, and the numerical results show the excellent performance of our method.

Let $c(x(i), T)$ and $c_h(x(i), T)$ be the analytical solution and the numerical solution value at node $x(i)$ and time T. The errors of numerical solutions are measured in L_2 norms and L_∞ norms defined as follows:

$$
\begin{aligned}
E_{2,\Delta t} &= \sqrt{h\sum_{i=1}^{M}\left(c(x(i),T) - c_h(x(i),T)\right)^2},\\
E_{\infty,\Delta t} &= \max_i \Big| c(x(i),T) - c_h(x(i),T)\Big|.
\end{aligned}
\tag{62}
$$

The ratio of convergence in time is calculated by

$$
\log\left(\frac{E_{l,\Delta t_1}}{E_{l,\Delta t_2}}\right)\left[\log(\frac{\Delta t_1}{\Delta t_2})\right]^{-1}, \quad l = 2, \infty. \tag{63}
$$

where Δt_1 and Δt_2 are two time steps. Similarly, the ratios of convergence in space can be calculated.

Example 1: Consider the following convection-diffusion equation with linear right side term

$$
\begin{aligned}
&\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} - k\frac{\partial^2 c}{\partial x^2} = f(x,t), && 0 \le x \le 1, 0 \le t \le 1,\\
&c(0,t) = c(1,t) = 0, && 0 < t \le 1,\\
&c(x,0) = x(1-x), && 0 \le x \le 1.
\end{aligned}
\tag{64}
$$

The problem admits the exact solution

$$
c(x,t) = x(1-x)e^{-t}.
$$

Take parameters $u = 0.01$, and $k = 10^{-3}, 10^{-5}, 10^{-7}$. $f(x,t)$ is decided from the equation by using analytical solution. The finite element space is the piecewise linear polynomial space, and take $\Delta t = h$, the errors and ratios at time $T = 1$ are shown in Table 1. We can see clearly that our scheme is of second order in time step for the advection diffusion problems with different small diffusion coefficients, which is same as the theoretical result in Theorem 3. The different time steps are used as $\Delta t = \frac{1}{20}, \frac{1}{30}, \frac{1}{40}$, and $\frac{1}{50}$, when $\Delta t = \frac{1}{20}$, the L^2 error is $1.3219e{-}04$, which also shows that our method can use a large time step to get the high-order approximation.

TABLE 1. Errors and ratios of our method in time and space with $u = 0.01, k = 10^{-l}, l = 3, 5, 7$.

	$k = 10^{-3}$		$k = 10^{-5}$		$k = 10^{-7}$	
$\Delta t = h$	L^2 error	Ratio	L^2 error	Ratio	L^2 error	Ratio
1/20	1.3219e-04	-	1.4602e-04	-	1.4630e-04	-
1/30	5.9623e-05	1.9638	6.7127e-05	1.9167	6.7347e-05	1.9133
1/40	3.3767e-05	1.9764	3.8352e-05	1.9458	3.8538e-05	1.9404
1/50	2.1753e-05	1.9706	2.4755e-05	1.9620	2.4917e-05	1.9543

Moreover, for the linear finite element space (piecewise linear polynomial space), our method is also of optimal second-order accuracy in spatial step h, as shown in Table 1.

Example 2: Consider the following convection-diffusion equation with nonlinear source term

$$\begin{aligned} &\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} - k\frac{\partial^2 c}{\partial x^2} = f(x,t,c(x,t)), \quad 0 \le x \le 1, 0 \le t \le 1,\\ &c(0,t) = c(1,t) = 0, \qquad\qquad\qquad\qquad 0 < t \le 1,\\ &c(x,0) = x(1-x), \qquad\qquad\qquad\qquad\qquad 0 \le x \le 1. \end{aligned} \tag{65}$$

The problem admits the exact solution

$$c(x,t) = x(1-x)e^{-t},$$

and the nonlinear right side function is

$$f(x,t,c(x,t)) = 0.1(c(x,t))^2 + e^{-t}(-x + x^2 + u - 2ux + 2k) - 0.1e^{-2t}x^2(1-x)^2.$$

Take parameters $u = 0.01$, and $k = 10^{-l}, l = 3, 5, 7$, the finite element space is the piecewise linear polynomial space, and take $\Delta t = h$. Table 2 shows the errors

TABLE 2. Errors and ratios of our method in time and space with $u = 0.01, k = 10^{-l}, l = 3, 5, 7$.

	$k = 10^{-3}$		$k = 10^{-5}$		$k = 10^{-7}$	
$\Delta t = h$	L^2 error	Ratio	L^2 error	Ratio	L^2 error	Ratio
1/20	1.3803e-04	-	1.5139e-04	-	1.5166e-04	-
1/30	6.2374e-05	1.9591	6.9604e-05	1.9164	6.9816e-05	1.9132
1/40	3.5363e-05	1.9726	3.9769e-05	1.9456	3.9949e-05	1.9405
1/50	2.2796e-05	1.9677	2.5672e-05	1.9616	2.5828e-05	1.9545

and ratios of our method in time and space. It is shown in Table 2 that our method has second-order error accuracy in time, and it also is the optimal-order error accuracy in spatial step when the piecewise linear polynomials are used. In this example, the right source term is nonlinear, the accuracies of time and space show the extrapolation along the characteristics is second order. The different time steps are used as $\Delta t = \frac{1}{20}, \frac{1}{30}, \frac{1}{40}$, and $\frac{1}{50}$, and the table shows L^2 error in the order of magnitude of 10^{-5}, which can provide efficiently high accuracy solutions when using large time step sizes.

Example 3: Consider the following convection-diffusion equation with nonlinear right side function and variable velocity

$$
(66)\qquad
\begin{aligned}
&\frac{\partial c}{\partial t}+u(x)\frac{\partial c}{\partial x}-k\frac{\partial^2 c}{\partial x^2}=f(x,t,c(x,t)), && 0\le x\le 1, 0\le t\le 1,\\
&c(0,t)=c(1,t)=0, && 0<t\le 1,\\
&c(x,0)=x(1-x), && 0\le x\le 1.
\end{aligned}
$$

The exact solution is

$$c(x,t)=x(1-x)e^{-t},$$

where the nonlinear source term follows

$$f(x,t,c(x,t))=0.1\sin(c)+e^{-t}(x^2-x+u-2ux+2k)-0.1\sin(x(1-x)e^{-t}).$$

Take parameters $u=1+0.3x$, and $k=0.1$. In this example, we use nonlinear right side function and variable velocity and test numerically the ratios in space and time of the proposed scheme. The different time steps and space steps are used as $\Delta t=h=\frac{1}{20},\frac{1}{30},\frac{1}{40}$, and $\frac{1}{50}$. The finite element space is the piecewise linear polynomial space. Table 3 shows the numerical results of errors and ratios in time steps at time $T=1$. From the numerical results shown in Table 3, we can see that

TABLE 3. Errors and ratios of our method in time and space with $u=1+0.3x, k=0.1$.

$\Delta t=h$	1/20	1/30	1/40	1/50
L^2 error	2.6350e-03	1.1145e-03	6.1735e-04	3.7497e-04
Ratio	-	2.0550	2.1482	2.2344
L^∞ error	5.6224e-03	2.5511e-03	1.4484e-03	9.3180e-04
Ratio	-	1.9490	1.9678	1.9766

our method has second-order accuracy in time and space, and the table shows L^2 error and L^∞ error in the order of magnitude of 10^{-4}, which indicate that we can get high accuracy solutions with large time step sizes.

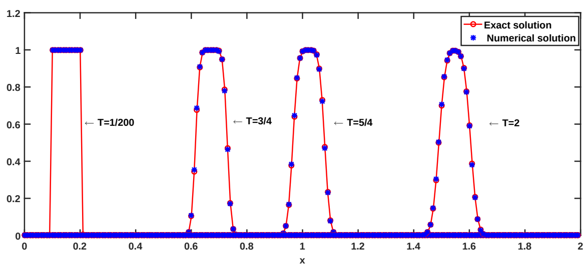

FIGURE 1. The exact solution and numerical solution with $u=0.3, k=9\times 10^{-5}$.

Example 4: Consider the following convection-diffusion equation with a sharp front gradient

$$
(67)\qquad
\begin{aligned}
&\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}-k\frac{\partial^2 c}{\partial x^2}=0, && 0\le x\le 1, 0\le t\le 1,\\
&c(0,t)=c(1,t)=0, && 0<t\le 1,
\end{aligned}
$$

with

$$c(x,0) = \begin{cases} 1, & x_0 \le x \le x_1 \\ 0, & 0 \le x < x_0, x_1 < x \le 1. \end{cases}$$

The analytical solution to this equation is given by

$$c(x,t) = \frac{1}{2}\{erf(\frac{x_1 - x + ut}{2\sqrt{kt}}) + erf(\frac{-x_0 + x - ut}{2\sqrt{kt}})\},$$

where $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2} d\eta$ is the error function.

Let $x_0 = 0.0905, x_1 = 0.205$ and take parameters $u = 0.1, k = 5 \times 10^{-4}$. The finite element space is the piecewise linear polynomial space, and take $\Delta t = h$. By choosing different time step sizes and space step sizes of $\Delta t = h = \frac{1}{30}, \frac{1}{40}, \frac{1}{50}$, and $\frac{1}{60}$, the results of errors and ratios are listed in Table 4. It is clearly shown that the discrete L^2-norm error is second order in time step t, which is same as the theoretical result in Theorem 3. Meanwhile, we can see that our method has second order accuracy in space size step. This example was solved in [14], where the error orders of space and time were not given. The exact solution and numerical results with $u = 0.7, k = 9 \times 10^{-5}$ are shown in Fig.1. In order to describe more trend of the equation, we choose $T = L = 2$. From the figure, we can see that our method can accurately simulate this problem without nonphysical oscillation phenomenon, and as time goes on, the curve shifts to the right and becomes smooth due to the influence of the advection and diffusion.

Table 4. Errors and ratios of our method with Example 4.

$\Delta t = h$	1/30	1/40	1/50	1/60
L^2 error	4.2069e-02	2.480e-02	1.5311e-02	1.0535e-02
Ratio	-	1.8368	2.1617	2.0505

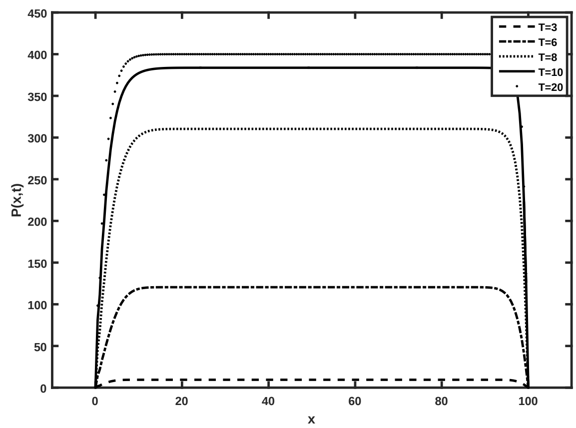

Figure 2. The numerical solutions at different time.

Example 5: In this example, we consider the nonlinear population dynamic problems ([22, 23, 9, 6], etc). Consider an extension of the Fisher-Kolmogorov equation of a biological population subject to an advective flow with constant velocity of field as

$$(68)\qquad \begin{aligned} &\frac{\partial P(x,t)}{\partial t} + v\frac{\partial P}{\partial x} - D\frac{\partial^2 P}{\partial x^2} = u_0 P(1-\frac{P}{K}),\\ &P(0,t)=\alpha, \quad P(L,t)=0, \quad 0<t\le T,\\ &P(x,0)=0, \qquad\qquad 0\le x\le L. \end{aligned}$$

$P(x,t)$ denotes the population density of a population of interest at time t and position x, v is the advection velocity and D is the diffusivity. The advective term $v\frac{\partial P}{\partial x}$ describes the drift of the population with the constant velocity v. The nonlinear right side term $u_0P(1-\frac{P}{K})$ is a nonlinear logical model, which takes into account for density dependent regulatory affection associated with resource depletion or conditions of over-crowding, where K is the carrying capacity of the environment, and it describes the maximal population density that can be sustained by the system, and the growth rate u_0 is a constant which describes a spatially nutrient or an homogeneous illumination pattern and so on. The first type (or Dirichlet) boundary condition merely specifies the value of the solution at the boundary. It assumes that the population density vanishes at the habitat edges, which corresponds to an absolutely hostile environment outside the segment $[0, L]$.

In Fig.2, we take $v = 0.6, D = 1, u_0 = 1, K = 400, L = 100, \alpha = 0$, it shows the distributions of the population dynamics. The resulting population density has a characteristic shape, with large densities in the central part of the patch and a decay of population numbers, the closer one comes to the hostile border. the conditions are sufficiently far from the critical region, so the carrying capacity is almost reached in the middle of the patch, we achieve the same results as the article of [22].

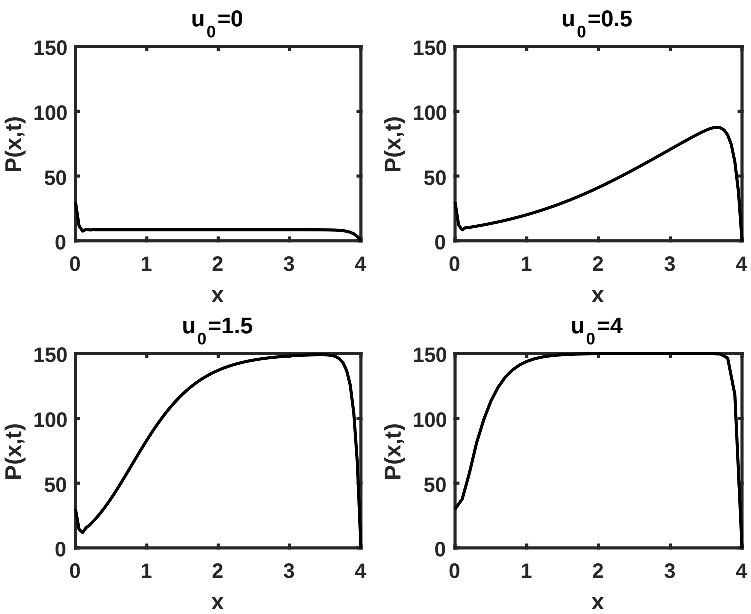

FIGURE 3. The numerical solutions with different growth rate u_0.

The following figures show the tendency change of the population dynamics systems under the influence of the varying parameters. In Fig.3, we choose difference growth rate u_0 to test the tendency of the distribution curve, and other parameters are taken as $v = 0.6, D = 6 \times 10^{-2}, \alpha = 30, K = 150, L = 4$. If u_0 is a positive constant, then a small initial population grows to occupy the entire domain so that ultimately the population density is uniform ($i.e. \lim_{t\to\infty} P = K$). Simulation results show that the population density is accelerated with the increasing of the growth rate parameter. From the first picture of Fig 3, the distribution curve first declines, then moves in a steady position and finally decreases to zero when the growth rate $u_0 = 0$. It has not reproduction with the growth rate $u_0 = 0$. When the growth rate is $u_0 = 0.5$, the growing will need long time to achieve high position as shown in the second picture of Fig. 3. However, with big growth rate u_0, the population density will reach carrying capacity in a short time at the patch near the left boundary, where we can see from the pictures with the growth rates $u_0 = 1.5, u_0 = 4$ in Fig. 3, and the growing has short time and patch to get maximum density.

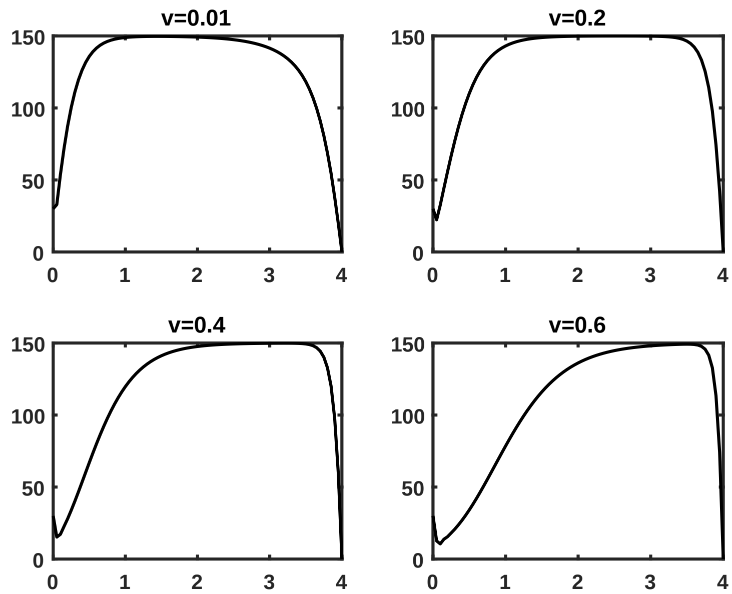

FIGURE 4. The numerical solutions with different advection velocity v.

In Fig 4, We choose different advection velocity v. The diffusion coefficient is chosen as $D = 5 \times 10^{-2}$, the growth rate is chosen as $u_0 = 1.5$ and the other parameters are same as used for Fig. 3. From the pictures of Fig. 4, we can see that the patch where population density gets to maximum is closer to the left boundary as the decrease of velocity v, and they all can reach the maximal population density in a certain time. From the simulated results in Fig. 3 and Fig. 4, we can see that small changes of above parameters can make the big difference of population density, then we can controlling the population size at different place and different time through adjusting different parameters.

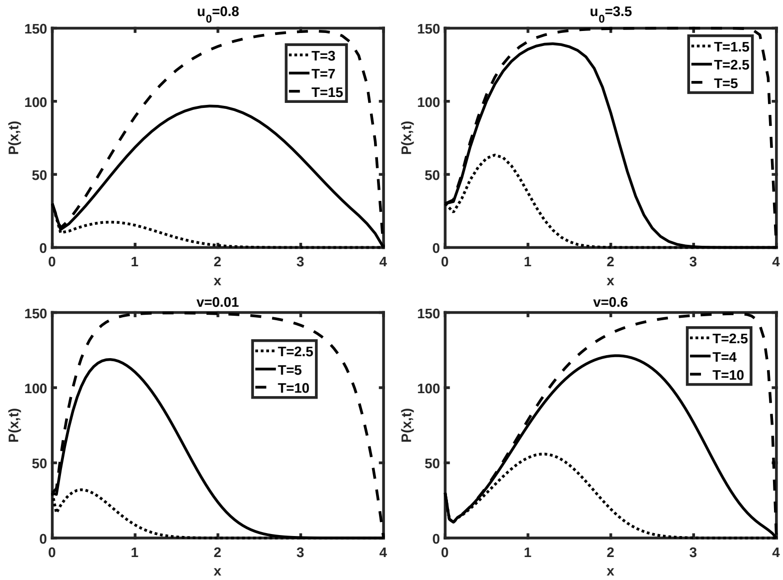

FIGURE 5. The numerical solutions at different time.

The Fig. 5 thoroughly shows the trends of population densities at different time. The first and second pictures of Fig. 5 have different growth rates, and the other parameters are chosen as same. The first picture presents that it first declines then goes up, next, up to a certain height, smoothly shifts and decreases to zero at the right boundary, which explains that the population density first declines due to the environment suffers overload at low growth rate, it will increase and reach the maximum population density following the nonlinear growth term affection but needs a long time. The right boundary condition is an absolutely hostile environment, which leads to the population density vanishes at the habitat edges. For the second picture, we use the growth rate $u_0 = 3.5$, it shows that the population density increases firstly then remains in a high level, and then decreases to zero. It is clear that due to the high growth rate, the distribution curve increases at the very beginning. and the population density will reach carrying capacity in a short time with the big growth rate u_0. The third and fourth pictures of Fig. 5 have different advection coefficients, and the other parameters are chosen as same. From these figures we can see that they all can reach the maximal population density in a certain time, and with the big velocity v, the patch where population density gets to maximum is closer to the right boundary. Through analyzing in Fig. 5, we could conclude that the population can also be regulated and controlled by changing the environmental factors where the biological species live.

5. Conclusion

In this work, an efficient time second-order characteristic finite element method was developed for solving nonlinear convection diffusion problems. For efficiently and accurately simulating the time derivative term and the advection term, we proposed to use the characteristic method to approximate the global derivative term. For having second-order accuracy to the diffusion term, the average along the characteristics is applied to treat it. Moreover, the second-order extrapolation along the characteristics is proposed to approximate the nonlinear right side function.

The developed method was further proved that it has second-order accuracy in time. The use of this method in the practical applications of nonlinear population dynamics can help us to understand nature, develop natural resources rationally and protect environment.

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References

[1] T. Arbogast and C.S. Huang, A fully conservative Eulerian-Lagrangian method for a convection-diffusion problem in a solenoidal field. J. Comput. Phys., 229 (2010), 3415-3427.

[2] T. Arbogast and M.F. Wheeler, A characteristics-mixed finite element method for advection-dominated transport problems. SIAM J. Numer. Anal., 32 (1995), 404-424.

[3] K. Aziz and A. Settari, Petroleum reservoir simulation. Applied Science Publisher, Ltd., London, 1979.

[4] J. Bear, Hydraulics of Groundwater. McGraw-Hill, New York, 1978.

[5] R. Bermejo, A Galerkin-characteristic algorithm for transport-diffusion equations. SIAM J. Numer. Anal., 32 (1995), 425-454.

[6] D.A. Birch, Y.K. Tsang and W.R. Young, Bounding biomass in the Fisher equation. Phys. Rev. E., 75(6)(2007), 066304.

[7] M. A. Celia, T. F. Russell, I. Herrera and R.E. Ewing, An Eulerian-Lagrangian localized adjoint method for the advection-diffusion equation. Adv. Water Res., 13 (1990), 187-206.

[8] P.G. Ciarlet, The finite element method for elliptic problems, 1978.

[9] K.A. Dahmen, D.R. Nelson and N.M. Shnerb, Life and death near a windy oasis. J. Math. Biol., 41(1) (2000), 1-23.

[10] J. Douglas Jr and T. F. Russell, Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. SIAM J. Numer. Anal., 19 (1982), 871-885.

[11] R.W. Ewing, The mathematics of reservoir simulation. Frontiers in Applied Mathematics, 1 (1988), 294-318.

[12] R.E. Ewing, T.F. Russell, and M.F. Wheeler, Simulation of miscible displacement using mixed methods and a modified method of characteristics. Soc. Pet. Eng. Aime, Pap., 1983, SPE 11241.

[13] L.C. Evans, Partial differential equations. Grad. Stu. Math. 19, AMS, Providence, RI, 1998.

[14] F. Gao and D. Liang, A new weighted upwind finite volume element method based on non-standard co-volume for time-dependent convection-diffusion problems. Int. J. Numer. Methods Fluids, 73(11) (2013), 953-975.

[15] P. Houston, C. Schwab and E. Suli, Discontinuous hp-finite element methods for advection-diffusion problems. SIAM J. Numer. Anal., 39 (2002), 2133-2163.

[16] D. Liang, A characteristics mixed finite element method of numerical simulation for 2-phase immiscible flows. Sci. in China, ser. A, 34 (1991), 1281-1289.

[17] D. Liang, C. Du and H. Wang, A fractional step ELLAM approach to high-dimensional convection-diffusion problems with forward particle tracking. J. Comput. Phys., 221 (2007), 198-225.

[18] D. Liang, W. Wang and Y. Cheng, An efficient second-order characteristic finite element method for non-linear aerosol dynamic equations. Int. J. Numer. Meth. Eng., 80 (2009), 338-354.

[19] D. Pokrajac and R. Lazic, An efficient algorithm for high accuracy particle tracking in finite elements. Adv. Water Res., 25 (2002), 353-369.

[20] M. Reeves and R.M. Cranwell, User's manual for the sandia waste-isolation flow and transport model(swift) release 4.81, Sandia Report Nureg/CR-2324, SAND 81-2516, GR, Albuquerque, NM, 1981.

[21] W. Rivera, J. Zhu and D. Huddleston, An efficient parallel algorithm with application to computational fluid dynamics. Comput. Math. Appl., 45 (2003), 165-188.

[22] A. B. Ryabov, Population growth and persistence in a heterogeneous environment: the role of diffusion and advection. Math Model. Nat. Pheno., 3 (2008), 42-86.

[23] D.C. Speirs and W. S. Gurney, Population persistence in rivers and estuaries. Ecology, 82 (2001), 1219-1237.

[24] J. Seinfeld and S. Pandis, Atmospheric chemical and physics: from air pollution to climate change. Second Edition, Wiley-Interscience, New York, 2006.

[25] J. Wang, Z. Si and W. Sun, A new error analysis of characteristics mixed FEMs for miscible displacement in porous media in porous media. SIAM J. Numer. Anal., 52 (2014), 3000-3020.

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