

A NEW COLLOCATION METHOD FOR SOLVING CERTAIN HADAMARD FINITE-PART INTEGRAL EQUATION

HUI FENG, YAN GAO, LILI JU, AND XIAOPING ZHANG

Abstract. In this paper, we study a new nodal-type trapezoidal rule for approximating Hadamard finite-part integrals, and its application to numerical solution of certain finite-part integral equation. We start with a nodal-type trapezoidal rule discussed in [21], and then establish its error expansion analysis, from which a new nodal-type trapezoidal rule with higher order accuracy is proposed and corresponding error analysis is also obtained. Based on the proposed rule, a new collocation scheme is then constructed to solve certain finite-part integral equation, with the optimal error estimate being rigorously derived. Some numerical experiments are also performed to verify the theoretical results.

Key words. Hadamard finite-part integral equation, quadrature rule, collocation method, error analysis.

1. Introduction

We consider the following finite-part integral

$$(1) \quad \mathcal{L}u(x) := \int_a^b \frac{u(y)}{|y-x|^{1+2s}} dy, \quad x \in (a, b),$$

where $s \in (0, 1)$ is the singularity index. The integral (1) is divergent in the classic Riemann sense, and should be understood in the Hadamard finite-part sense. There are several equivalent definitions for this finite-part integral in the literatures [15], and we here adopt the following definition:

$$(2) \quad \mathcal{L}u(x) = \lim_{\epsilon \rightarrow 0} \left(\int_{\Omega_\epsilon(x)} \frac{u(y)}{|y-x|^{1+2s}} dy - \frac{\epsilon^{-2s}u(x)}{s} \right), \quad x \in (a, b),$$

where x is the singular point and $\Omega_\epsilon(x) = (a, b) \setminus (x - \epsilon, x + \epsilon)$. A function $u(y)$ is said to be finite-part integrable with respect to the weight $|y-x|^{-1-2s}$ if the limit on the right-hand side of (2) exists. Assuming u is absolutely integrable on (a, b) , then a sufficient condition for $u(x)$ to be finite-part integrable is that $u(x)$ is α -Hölder continuous for some $\alpha \in (2s, 1)$ on (a, b) if $s \in (0, 1/2)$, and $u'(x)$ is α -Hölder continuous for some $\alpha \in (2s - 1, 1)$ on (a, b) if $s \in [1/2, 1)$.

Integrals of this kind appear in many practical problems related to aerodynamics, wave propagation or fluid mechanics, mostly with relation to boundary element methods and finite-part integral equations. Numerous work has been devoted in developing the efficient numerical evaluation method, such as Gaussian (GS) rule [6, 7], Newton-Cotes (NC) rule [9, 12, 14, 17, 20, 21], and some other rules [2, 3, 5]. Amongst them, NC rule is a popular one due to its ease of implementation and flexibility of mesh. NC rule is constructed by replacing u by its Lagrange interpolation in (1), and can be classified into two types: grid-type and nodal-type. The way of distinguishing one type from another is the choice of the singular point's location. Grid-type takes the singular point being located in the interior of a certain grid and nodal-type forces the singular point to be a certain nodal one.

There are some other differences between these two type of rules. Amongst those, a major one is that the two rules are based on different definitions of the finite-part integrals (1) respectively. Since Lagrange interpolation is smooth in the interior of every grid, we can use the definition (2) directly to design grid-type NC rules. However, Lagrange interpolation is only continuous at the nodal points and the definition (2) is invalid to produce nodal-type NC rules, especially for $s \geq 1/2$, due to the rigorous regularity requirement on u for the definition (2). To overcome such problem, one often should adopt the following definitions [15]:

$$(3) \quad \begin{aligned} \mathcal{L}^-u(x) &= \lim_{\epsilon \rightarrow 0} \left(\int_a^{x-\epsilon} \frac{u(y)}{(x-y)^{1+2s}} dy + r^-(x) \right), \\ \mathcal{L}^+u(x) &= \lim_{\epsilon \rightarrow 0} \left(\int_{x+\epsilon}^b \frac{u(y)}{(y-x)^{1+2s}} dy + r^+(x) \right), \end{aligned}$$

where

$$\begin{aligned} r^-(x) &= \begin{cases} \frac{\epsilon^{-2s}}{-2s} u(x^-), & s < 1/2, \\ -\epsilon^{-1} u(x^-) - \ln \epsilon u'(x^-), & s = 1/2, \\ \frac{\epsilon^{-2s}}{-2s} u(x^-) - \frac{\epsilon^{1-2s}}{1-2s} u'(x^-), & s > 1/2, \end{cases} \\ r^+(x) &= \begin{cases} \frac{\epsilon^{-2s}}{-2s} u(x^+), & s < 1/2, \\ -\epsilon^{-1} u(x^+) + \ln \epsilon u'(x^+), & s = 1/2, \\ \frac{\epsilon^{-2s}}{-2s} u(x^+) + \frac{\epsilon^{1-2s}}{1-2s} u'(x^+), & s > 1/2, \end{cases} \end{aligned}$$

and $u(x^-)$ and $u(x^+)$ denote the left and right limits of u at x respectively. Obviously, if u is smooth enough, then $\mathcal{L}u(x) = \mathcal{L}^-u(x) + \mathcal{L}^+u(x)$.

It's well-known that the accuracy of NC rule with k th order piecewise polynomial interpolant for the usual Riemann integrals is $O(h^{k+1})$ for odd k and $O(h^{k+2})$ for even k . However, the rule is less accurate for finite-part integral (1) due to the hyper-singularity of the kernel. For example, general error analysis shows that the accuracy of both types of rules are $O(h^{k+1-2s})$ [4, 8, 9, 12, 14, 18]. A way of obtaining higher order accuracy for grid-type rule is to study its superconvergence property. This property implies that one can get higher order accuracy on the condition that the singular point coincides with some *a priori* known point. A series of outstanding works have been devoted to this field [11, 13, 16, 17, 18, 22, 23].

One goal of this paper is to study a higher order nodal-type rule for evaluation of (1). We start with a nodal-type trapezoidal rule ($k = 1$) investigated in [21]. Instead of estimating the error directly, we turn to analyze its error expansion. Once this expansion is established, a new nodal-type rule can be proposed by making a slight modification on the original one. As discussed in [21], the accuracy of the original rule is always $O(h^{2-2s})$. Excitingly, the new rule behaves more accurate, it reaches $O(h^{4-2s})$ if the singular point is far away from the endpoints, and $O(h^{3-2s})$ if very close to the endpoints, which is at least one order higher than the original rule.

A motivation to study the nodal-type NC rule is to solve the corresponding Hadamard finite-part integral equation defined by

$$(4) \quad \begin{cases} \mathcal{L}u(x) = f(x), & x \in (a, b), \\ u(a) = u_a, u(b) = u_b. \end{cases}$$

Since nodal-type NC rule takes the singular point to be the nodal one, it naturally can be used to construct the collocation scheme for solving (4). One collocation scheme has been investigated in [21] by using a very slight modified version of the original nodal-type trapezoidal rule, where the scheme's stiffness matrix is shown to be a symmetric, strictly diagonally dominant M -matrix with Toeplitz structure. With these properties of the stiffness matrix, the accuracy of the scheme is proved to be $O(h^{2-2s})$. A fast solver is available for implementing the scheme (due to Toeplitz structure). This is very important, especially in some real problems, such as electromagnetic cavity problems [1, 10, 19, 24].

Another goal of this paper is then to use the new nodal-type trapezoidal rule to construct a new collocation scheme for solving (4). Fortunately, the stiffness matrix of the new scheme is also shown to be a strictly diagonally dominant M -matrix, from which the accuracy is proved to be $O(h^{4-2s})$, which is twice order higher than that of the existing scheme in [21]. As we mentioned before, the accuracy of the new rule is affected by the distance between the singular point and the end points, but the accuracy of the scheme does not. In other words, the accuracy of the scheme is better than that of the rule, and the reason why this happens will be explained in our analysis. On the other hand, we note that at the time of acquiring higher order accuracy we regretfully sacrifice the opportunity of using fast solver due to the lack of Toeplitz structure.

The rest of the paper is organized as follows. In Section 2, we start with a nodal-type trapezoidal rule, establish an error expansion analysis, and then propose our ultimate rules for evaluating finite-part integral (1). Based on the new scheme, a collocation scheme is then developed to approximate certain finite-part integral equation (4), and its optimal error estimate is also derived in Sections 3. In Section 4, numerical experiments are reported to demonstrate the efficiency and accuracy of the proposed collocation scheme. Finally concluding remarks are given in Section 5.

2. Nodal-type trapezoidal rules and error analysis

Consider the uniform partition $a = x_0 < x_1 < \dots < x_n < x_{n+1} = b$ of the interval $[a, b]$ with grid sizes $h = (b - a)/(n + 1)$. Let $I_h u(y) = \sum_{j=0}^{n+1} u(x_j) \varphi_j(y)$ be continuous piecewise-linear interpolation of $u(y)$, where $\varphi_j(y)$ is the standard "hat" function with respect to the given partition.

2.1. Original rule. Replacing $u(y)$ by $I_h u(y)$ leads to the trapezoidal rule:

$$(5) \quad \widehat{\mathcal{L}}_h u(x_i) := \sum_{j=0}^{n+1} \widehat{\omega}_{ij} u(x_j),$$

where

$$(6) \quad \widehat{\omega}_{ij} = (1 - \delta_{0j}) h^{-1} \int_{x_{j-1}}^{x_j} \frac{y - x_{j-1}}{|y - x_i|^{1+2s}} dy + (1 - \delta_{n+1,j}) h^{-1} \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - y}{|y - x_i|^{1+2s}} dy.$$

We note that the coefficients (6) include finite-part integrals when $|j - i| \leq 1$, and we should adopt the definition (3) to compute them.

Remark 1. The error analysis of (5) has been given in [21]: assume $u \in C^2(a, b)$, it holds

$$\left| \mathcal{L}u(x_i) - \widehat{\mathcal{L}}_h u(x_i) \right| \leq \begin{cases} Ch |\ln h|, & s = 1/2, \\ Ch^{2-2s}, & \text{otherwise,} \end{cases} \quad i = 1, 2, \dots, n.$$

From the above error estimate we see easily that the rule (5) gradually cease to be in force when s is getting close to 1.

Let us now turn to do error expansion analysis of the rule (5), and the main result is illustrated in the following theorem. It is the stepping stone of proposing our ultimate rule.

Theorem 1. Suppose $u \in C^4(a, b)$, then for $\widehat{\mathcal{L}}_h u(x_i)$ defined by (5), we have

$$(7) \quad \begin{aligned} \mathcal{L}u(x_i) - \widehat{\mathcal{L}}_h u(x_i) = & -h^{-1}\theta[u(x_{i-1}) - 2u(x_i) + u(x_{i+1})] \\ & + \frac{1}{2!} \sum_{j=1}^n \phi_{ij} u''(x_j) + \mathcal{R}_i, \quad i = 1, 2, \dots, n, \end{aligned}$$

where $\theta = \int_0^h z^{-2s} dz$, and

$$(8) \quad \phi_{ij} = \begin{cases} \int_{x_{j-1}}^{x_j} \frac{(y-x_{j-1})(y-x_j)}{(x_i-y)^{1+2s}} dy, & j = 1, \dots, i-1, \\ 2 \int_{x_i}^{x_{i+1}} (y-x_i)^{1-2s} dy, & j = i, \\ \int_{x_j}^{x_{j+1}} \frac{(y-x_j)(y-x_{j+1})}{(y-x_i)^{1+2s}} dy, & j = i+1, \dots, n, \end{cases}$$

$$(9) \quad |\mathcal{R}_i| \leq C (h^{4-2s} + h^3 + h^3 \eta^{-2s}(x_i))$$

with

$$(10) \quad \eta(x_i) = \min\{x_i - a, b - x_i\}.$$

Proof. Define $e_j(y) = u(y) - I_h u(y)$, $y \in (x_j, x_{j+1})$, then the error of (5) can be split into two parts:

$$(11) \quad \mathcal{L}u(x_i) - \widehat{\mathcal{L}}_h u(x_i) = \mathcal{E}_1 + \mathcal{E}_2,$$

where

$$\begin{aligned} \mathcal{E}_1 &= \int_{x_{i-1}}^{x_i} \frac{e_{i-1}(y)}{(x_i-y)^{1+2s}} dy + \int_{x_i}^{x_{i+1}} \frac{e_i(y)}{(y-x_i)^{1+2s}} dy, \\ \mathcal{E}_2 &= \sum_{j=1}^{i-1} \int_{x_{j-1}}^{x_j} \frac{e_{j-1}(y)}{(x_i-y)^{1+2s}} dy + \sum_{j=i+1}^n \int_{x_j}^{x_{j+1}} \frac{e_j(y)}{(y-x_i)^{1+2s}} dy. \end{aligned}$$

Now we estimate \mathcal{E}_i , $i = 1, 2$ term by term. For \mathcal{E}_1 , by noting the facts that

$$\begin{aligned} e_{i-1}(y) &= u(y) - u(x_i) - \frac{u(x_i) - u(x_{i-1})}{h} (y - x_i), \\ e_i(y) &= u(y) - u(x_i) - \frac{u(x_{i+1}) - u(x_i)}{h} (y - x_i) \end{aligned}$$

and $\int_{x_{i-1}}^{x_{i+1}} \frac{(y-x_i)^k}{|y-x_i|^{1+2s}} dy = 0$ for odd k , it can be reformulated as

$$(12) \quad \mathcal{E}_1 = -h^{-1}\theta[u(x_{i-1}) - 2u(x_i) + u(x_{i+1})] + \frac{1}{2!} \phi_{ii} u''(x_i) + \mathcal{R}_i^1,$$

where

$$\mathcal{R}_i^1 = \int_{x_{i-1}}^{x_{i+1}} \frac{u(y) - \sum_{k=0}^3 \frac{u^{(k)}(x_i)(y-x_i)^k}{k!}}{|y-x_i|^{1+2s}} dy.$$

Since $u \in C^4(a, b)$, a fourth order Taylor expansion yields

$$(13) \quad |\mathcal{R}_i^1| \leq C \int_{x_{i-1}}^{x_{i+1}} |y - x_i|^{3-2s} dy \leq Ch^{4-2s}.$$

As for \mathcal{E}_2 , also fourth order Taylor expansions lead us to

$$\begin{aligned} e_{j-1}(y) &= \frac{u''(x_j)}{2!}(y - x_{j-1})(y - x_j) + \frac{u'''(x_j)}{3!}(y - x_{j-1})(y - x_j)(y - x_{j+1}) \\ &\quad + O(h^4), \quad y \in (x_{j-1}, x_j), \quad j = 1, \dots, i-1, \\ e_j(y) &= \frac{u''(x_j)}{2!}(y - x_j)(y - x_{j+1}) + \frac{u'''(x_j)}{3!}(y - x_{j-1})(y - x_j)(y - x_{j+1}) \\ &\quad + O(h^4), \quad y \in (x_j, x_{j+1}), \quad j = i+1, \dots, n. \end{aligned}$$

Therefore, \mathcal{E}_2 can be expanded as

$$(14) \quad \mathcal{E}_2 = \frac{1}{2!} \sum_{j=1, j \neq i}^n \phi_{ij} u''(x_j) + \frac{1}{3!} \sum_{j=1, j \neq i}^n \psi_{ij} u'''(x_j) + \mathcal{R}_i^2,$$

where

$$(15) \quad \psi_{ij} = \begin{cases} \int_{x_{j-1}}^{x_j} \frac{(y - x_{j-1})(y - x_j)(y - x_{j+1})}{(x_i - y)^{1+2s}} dy, & j < i, \\ \int_{x_j}^{x_{j+1}} \frac{(y - x_{j-1})(y - x_j)(y - x_{j+1})}{(y - x_i)^{1+2s}} dy, & j > i, \end{cases}$$

and

$$\begin{aligned} |\mathcal{R}_i^2| &\leq Ch^4 \left(\int_a^{x_{i-1}} \frac{1}{(x_i - y)^{1+2s}} dy + \int_{x_{i-1}}^b \frac{1}{(y - x_i)^{1+2s}} dy \right) \\ &\leq C(h^{4-2s} + h^4 \eta^{-2s}(x_i)). \end{aligned}$$

Without loss of generality, we assume that $i \geq \frac{n+1}{2}$, and then the second sum in the right-hand side of (14) can be rearranged as

$$\sum_{j=1, j \neq i}^n \psi_{ij} u'''(x_j) = \sum_{m=1}^{n-i} \psi_{i, i+m} [u'''(x_{i+m}) - u'''(x_{i-m})] + \sum_{m=n-i+1}^{i-1} \psi_{i, i-m} u'''(x_{i-m}).$$

By appropriate variable's transformation, ψ_{ij} can be reformulated as

$$(16) \quad \psi_{i, i+m} = \operatorname{sgn}(m) \cdot h^{3-2s} \int_0^1 \frac{\tau(\tau^2 - 1)}{(\tau + |m|)^{1+2s}} d\tau, \quad m \neq 0.$$

As a consequence, we have

$$\begin{aligned} &\left| \sum_{m=1}^{n-i} \psi_{i, i+m} [u'''(x_{i+m}) - u'''(x_{i-m})] \right| \\ &\leq Ch^{4-2s} \sum_{m=1}^{n-i} m \int_0^1 \frac{1}{(\tau + m)^{1+2s}} d\tau \\ &\leq Ch^{4-2s} \sum_{m=1}^{n-i} \left[\int_0^1 \frac{1}{(\tau + m)^{2s}} d\tau + \int_0^1 \frac{\tau}{(\tau + m)^{1+2s}} d\tau \right] \\ &\leq C(h^{4-2s} + h^4 \eta^{-2s}(x_i)), \end{aligned}$$

and

$$\left| \sum_{m=n-i+1}^{i-1} \psi_{i,i-m} u'''(x_{i-m}) \right| \leq Ch^{3-2s} \sum_{m=n-i+1}^{i-1} \int_0^1 \frac{1}{(\tau+m)^{1+2s}} d\tau \leq Ch^3(1+\eta^{-2s}(x_i)).$$

Thus

$$(17) \quad \left| \sum_{j=1, j \neq i}^n \psi_{ij} u'''(x_j) \right| \leq C(h^{4-2s} + h^3 + h^3 \eta^{-2s}(x_i)),$$

Putting (11), (12), (13), (14) and (17) together concludes the proof. □

2.2. Modified rule. Based on (7), we next propose a modified trapezoidal rule as follows:

$$(18) \quad \tilde{\mathcal{L}}_h u(x_i) = \widehat{\mathcal{L}}_h u(x_i) - h^{-1} \theta [u(x_{i-1}) - 2u(x_i) + u(x_{i+1})] := \sum_{j=0}^{n+1} \tilde{\omega}_{ij} u(x_j), \quad i = 1, 2, \dots, n,$$

where

$$(19) \quad \tilde{\omega}_{ij} = \widehat{\omega}_{ij} + \begin{cases} 2h^{-1}\theta, & j = i, \\ -h^{-1}\theta, & |j - i| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2. The explicit expression of $\tilde{\omega}_{ij}$ has been given in [21] with the error analysis of (18): assume $u \in C^2(a, b)$, it holds

$$\left| \mathcal{L}u(x_i) - \tilde{\mathcal{L}}_h u(x_i) \right| \leq Ch^{2-2s}, \quad i = 1, 2, \dots, n.$$

Alternatively, we here reformulate $\tilde{\omega}_{ij}$ in a more compact way. Define

$$F(t) = \begin{cases} -\ln t, & s = 1/2, \\ \frac{t^{1-2s}}{(1-2s)(-2s)}, & \text{otherwise,} \end{cases}$$

then $\tilde{\omega}_{ij}$ can be expressed as

$$(20) \quad \begin{aligned} \tilde{\omega}_{i,i+m} &= h^{-2s} \begin{cases} 2F'(1), & m = 0, \\ -F(1) + F(2) - F'(1), & |m| = 1, \\ F(|m| - 1) - 2F(|m|) + F(|m| + 1), & |m| > 1, \end{cases} \\ \tilde{\omega}_{i,0} &= h^{-2s} \begin{cases} 0, & i = 1, \\ F(i - 1) - F(i) + F'(i), & 2 \leq i \leq n, \end{cases} \\ \tilde{\omega}_{i,n+1} &= h^{-2s} \begin{cases} 0, & i = n, \\ F(n - i) - F(n + 1 - i) + F'(n + 1 - i), & 1 \leq i \leq n - 1. \end{cases} \end{aligned}$$

The following corollary is a natural consequence of Theorem 1, which also provide an error expansion of the rule (18).

Corollary 1. Suppose $u \in C^4(a, b)$, then for $\tilde{\mathcal{L}}_h u(x_i)$ defined by (18), we have

$$(21) \quad \mathcal{L}u(x_i) - \tilde{\mathcal{L}}_h u(x_i) = \frac{1}{2} \sum_{j=1}^n \phi_{ij} u''(x_j) + \mathcal{R}_i, \quad i = 1, 2, \dots, n,$$

where ϕ_{ij} is defined by (8) and \mathcal{R}_i is bounded by (9).

2.3. Ultimate rule. Once the error expansion of $\tilde{\mathcal{L}}_h u(x_i)$ is determined, we can approximate $u''(x_j)$ in (21) by its central difference approximation, and then obtain our ultimate trapezoidal rule:

$$(22) \quad \begin{aligned} \mathcal{L}_h u(x_i) &= \tilde{\mathcal{L}}_h u(x_i) + \frac{h^{-2}}{2} \sum_{j=1}^n \phi_{ij} [u(x_{j-1}) - 2u(x_j) + u(x_{j+1})] \\ &:= \sum_{j=0}^{n+1} \omega_{ij} u(x_j), \quad i = 1, 2, \dots, n. \end{aligned}$$

where ϕ_{ij} is defined by (8). Before stating the error estimate of $\mathcal{L}_h u(x_i)$, let us take a closer look at ϕ_{ij} .

Lemma 1. *Let $\Phi_{n \times n}$ be an $n \times n$ matrix with entries $\phi_{ij}, i, j = 1, \dots, n$ defined by (8), then it is a symmetric Toeplitz matrix and $\phi_{ii} > 0$ and $\phi_{ij} < 0$ for $j \neq i$. Moreover, $\{\phi_{i, i+m}\}$ constitutes an increasing sequence for $m > 0$.*

Proof. By using appropriate variable's transformation we get

$$(23) \quad \phi_{i, i+m} = h^{2-2s} \begin{cases} \frac{1}{1-s}, & m = 0, \\ \int_0^1 \frac{\tau(\tau-1)}{(\tau+|m|)^{1+2s}} d\tau, & m \neq 0, \end{cases}$$

from which we can easily determine the sign symbol of ϕ_{ij} . Moreover,

$$\begin{aligned} &\phi_{i, i+m+1} - \phi_{i, i+m} \\ &= h^{2-2s} \int_0^1 \left[\frac{1}{(\tau+m+1)^{1+2s}} - \frac{1}{(\tau+m)^{1+2s}} \right] \tau(\tau-1) d\tau > 0, \quad m > 0, \end{aligned}$$

which implies that $\{\phi_{i, i+m}\}$ constitutes an increasing sequence for $m > 0$, and the proof is completed. \square

Now we give the error estimate of (22).

Theorem 2. *Assume that $u \in C^4(a, b)$, then for $\mathcal{L}_h u(x_i)$ defined in (22), we have*

$$(24) \quad |\mathcal{L}u(x_i) - \mathcal{L}_h u(x_i)| \leq C(h^{4-2s} + h^3 + h^3 \eta^{-2s}(x_i)).$$

Proof. Subtracting (22) from (21) yields

$$\mathcal{L}u(x_i) - \mathcal{L}_h u(x_i) = \frac{1}{2} \sum_{j=1}^n \phi_{ij} \left[u''(x_j) - \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1}))}{h^2} \right] + \mathcal{R}_i.$$

Since $u \in C^4(a, b)$, we have $\left| u''(x_j) - \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1}))}{h^2} \right| \leq Ch^2$, and thus

$$(25) \quad |\mathcal{L}u(x_i) - \mathcal{L}_h u(x_i)| \leq Ch^2 \sum_{j=1}^n |\phi_{ij}| + \mathcal{R}_i.$$

By using the sign symbol of ϕ_{ij} discussed in Lemma 1 and (23), we get

$$\begin{aligned} \sum_{j=1}^n |\phi_{ij}| &= \phi_{ii} - \sum_{j=1, j \neq i}^n \phi_{ij} \leq Ch^{2-2s} + Ch^{2-2s} \left(\sum_{m=1}^{i-1} + \sum_{m=1}^{n-i+1} \right) \int_0^1 \frac{1}{(\tau+m)^{1+2s}} d\tau \\ &\leq C(h^{2-2s} + h^2 \eta^{-2s}(x_i)). \end{aligned}$$

Combining it with (25) concludes the proof. \square

3. New collocation scheme for the integral equation

In this section, we will use the ultimate rule $\mathcal{L}_h u(x_i)$ defined by (22) to construct a collocation scheme for solving the finite-part integral equation (4), and also establish an optimal error estimate for the scheme.

First, we can reformulate (22) at the interior grid points $x_i, i = 1, 2, \dots, n$ as the matrix form

$$(26) \quad \mathbf{W}_{n \times (n+2)} \mathbf{U}_{n+2} \approx \mathbf{F}_n,$$

where $\mathbf{W}_{n \times (n+2)}$ is an $n \times (n + 2)$ matrix with entries ω_{ij} for $i = 1, 2, \dots, n$ and $j = 0, 1, \dots, n + 1$, and

$$\mathbf{F}_n = (f(x_1), f(x_2), \dots, f(x_n))^T$$

denotes the exact value vector of the finite part integrals. From (22) we see that the relationship between $\widetilde{\mathbf{W}}_{n \times (n+2)} = (\tilde{\omega}_{i,j})$ ($\tilde{\omega}_{i,j}$ is defined in (19)) and $\mathbf{W}_{n \times (n+2)}$ can be represented as

$$(27) \quad \mathbf{W}_{n \times (n+2)} = \widetilde{\mathbf{W}}_{n \times (n+2)} + \mathbf{P}_{n \times (n+2)}, \quad \mathbf{P}_{n \times (n+2)} = \frac{h^{-2}}{2} \Phi_{n \times n} \Lambda_{n \times (n+2)},$$

where $\Phi_{n \times n}$ is defined in Lemma 1 and

$$\Lambda_{n \times (n+2)} = \begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 1 & -2 & 1 \end{pmatrix}.$$

By taking the approximately equal symbol as the equal one in (26) and using the boundary conditions, a new collocation scheme for solving (4) is obtained:

$$(28) \quad \mathbf{W}_{n \times n} \mathbf{U}_n^h = \mathbf{G}_n,$$

where $\mathbf{W}_{n \times n}$ is submatrix of $\mathbf{W}_{n \times (n+2)}$ by deleting the first and last columns, and \mathbf{G}_n are obtained from \mathbf{F}_n by using the boundary values.

Lemma 2. $-\widetilde{\mathbf{W}}_{n \times n}$ is a strictly diagonally dominant M-matrix with Toeplitz structure, and moreover,

$$(29) \quad -\sum_{j=1}^n \tilde{\omega}_{ij} \geq C\eta(x_i)^{-2s}.$$

Proof. From (6) and (19) we see that $\tilde{\omega}_{ij} > 0$ for all $|j - i| > 1$, since the integrands in it are regular and $\varphi_j(x)$ is positive. Moreover, through some direct calculations and variable transformations, we can get

$$(30) \quad \tilde{\omega}_{i,i+m} = h^{-2s} \begin{cases} 2 \int_0^1 \tau^{-1-2s} d\tau, & m = 0, \\ \int_0^1 \frac{1 - \tau}{(\tau + 1)^{1+2s}} d\tau, & m = \pm 1, \end{cases}$$

from which we can easily see that $\tilde{\omega}_{ii} < 0$ and $\tilde{\omega}_{i,i \pm 1} > 0$. To prove $-\widetilde{\mathbf{W}}_{n \times n}$ is an M-matrix, it remains us show it is also an strictly diagonally dominant. Obviously, the quadrature rule (18) is exact for any linear function. So, by setting $u \equiv 1$ we can get

$$(31) \quad -\sum_{j=1}^n \tilde{\omega}_{ij} = -\int_a^b \frac{1}{|y - x_i|^{1+2s}} dy + \tilde{\omega}_{i0} + \tilde{\omega}_{i,n+1}.$$

Since $-\int_a^b \frac{1}{|y-x_i|^{1+2s}} dy = \frac{(x_i-a)^{-2s} + (b-x_i)^{-2s}}{2s} > 0$ and $\tilde{\omega}_{i0} > 0, \tilde{\omega}_{i,n+1} > 0$, it means that the sum of each row of $-\tilde{\mathbf{W}}_{n \times n}$ is positive, and thus for $i = 1, \dots, n$ we get $|\tilde{\omega}_{ii}| - \sum_{j=1, j \neq i}^n |\tilde{\omega}_{ij}| = -\sum_{j=1}^n \tilde{\omega}_{ij} > 0$.

Now, it remains for us to prove (31). Let us here only consider the case $s \neq 1/2$ since the other cases can be done similarly. From (20) and (31) we obtain (32)

$$-\sum_{j=1}^n \tilde{\omega}_{ij} = \begin{cases} \frac{(x_1-a)^{-2s}}{2s} + \frac{(b-x_1)^{1-2s} - (b-x_2)^{1-2s}}{2s(1-2s)h}, & i = 1, \\ \frac{(x_i-a)^{1-2s} - (x_{i-1}-a)^{1-2s} + (b-x_i)^{1-2s} - (b-x_{i+1})^{1-2s}}{2s(1-2s)h}, & i = 2, \dots, n-1, \\ \frac{(x_n-a)^{1-2s} - (x_{n-1}-a)^{1-2s}}{2s(1-2s)h} + \frac{(b-x_n)^{-2s}}{2s}, & i = n \end{cases}$$

Furthermore, by using the inequalities

$$\begin{aligned} (1+x)^{1-2s} - 1 &\leq -Cx, & s > 1/2, \\ (1+x)^{1-2s} - 1 &\geq Cx, & s < 1/2, \end{aligned}$$

for $x \in [0, 1]$, we obtain

$$\begin{aligned} \frac{(x_i-a)^{1-2s} - (x_{i-1}-a)^{1-2s}}{(1-2s)h} &= \frac{(x_{i-1}-a)^{1-2s}}{(1-2s)h} \left[\left(1 + \frac{h}{x_{i-1}-a}\right)^{1-2s} - 1 \right] \\ (33) \qquad \qquad \qquad &\geq C(x_{i-1}-a)^{-2s}, \end{aligned}$$

for $i = 2, \dots, n$, and similarly,

$$(34) \qquad \qquad \qquad \frac{(b-x_i)^{1-2s} - (b-x_{i+1})^{1-2s}}{(1-2s)h} \geq C(b-x_{i+1})^{-2s}.$$

for $i = 1, \dots, n-1$. Combining (32), (33) and (34) together leads to (29), which completes the proof. \square

Lemma 3. $-\mathbf{W}_{n \times n}$ is a strictly diagonally dominant M-matrix, and

$$(35) \qquad \qquad \qquad -\sum_{j=1}^n \omega_{ij} \geq C\eta^{-2s}(x_i).$$

As a consequence, the linear system (28) has a unique solution.

Proof. We will first check the sign symbol of each element of $\mathbf{W}_{n \times (n+2)}$. Let p_{ij} be entries of $\mathbf{P}_{n \times (n+2)}$, from the definition of $\mathbf{P}_{n \times (n+2)}$ and Lemma 1, it is easy to check that $p_{ii} < 0$ and $p_{i,i\pm 1} > 0$ for $i = 1, 2, \dots, n$. Combining these with Lemma 2 we see that $\omega_{ii} < 0, \omega_{i,i\pm 1} > 0$ for $i = 1, 2, \dots, n$.

Now it remains for us to prove $\omega_{ij} > 0$ for $|j-i| > 1$, and we only consider the case that $j > i+1$ since the case $j < i-1$ can be done similarly. Through some direct calculations, ω_{ij} can be reformulated as

$$(36) \qquad \omega_{ij} = \tilde{\omega}_{ij} + \frac{h^{-2}}{2} \phi_{i,j-1} + \begin{cases} -\frac{h^{-2}}{2} \phi_{ij} + \frac{h^{-2}}{2} (\phi_{i,j+1} - \phi_{i,j}), & i = 1, \dots, n-3, i+1 < j \leq n-1, \\ -h^{-2} \phi_{ij}, & i = 1, \dots, n-2, j = n, \\ 0, & i = 1, \dots, n-1, j = n+1. \end{cases}$$

For $i + 1 < j \leq n + 1$, we have

$$(37) \quad \tilde{\omega}_{ij} + \frac{h^{-2}}{2} \phi_{i,j-1} = \frac{h^{-2}}{2} \int_{x_{j-1}}^{x_j} \frac{(y - x_{j-1})(y - x_{j-2})}{(y - x_i)^{1+2s}} dy \\ + (1 - \delta_{n+1,j}) h^{-1} \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - y}{(y - x_i)^{1+2s}} dy > 0,$$

Putting (36), (37) and Lemma 1 together leads to $\omega_{ij} > 0$ for $|j - i| > 1$. The strictly diagonal dominance of $-\mathbf{W}_{n \times n}$ can be validated by

$$(38) \quad -\sum_{j=1}^n \omega_{ij} = -\int_a^b \frac{1}{|y - x_i|^{1+2s}} dy + \omega_{i0} + \omega_{i,n+1} > 0,$$

and thus the uniqueness of (28) follows immediately. Combination of all these together leads to the fact that $-\mathbf{W}_{n \times n}$ is an M-matrix.

From (27), (31) and (38) we can easily get

$$-\sum_{j=1}^n \omega_{ij} = -\sum_{j=1}^n \tilde{\omega}_{ij} + \frac{h^{-2}}{2} (\phi_{i1} + \phi_{in}).$$

For $i = 2, \dots, n - 1$, since $\phi_{i1} < 0, \phi_{in} < 0$, we obtain

$$-\sum_{j=1}^n \omega_{ij} \geq -\sum_{j=1}^n \tilde{\omega}_{ij} \geq C\eta(x_i)^{-2s}.$$

In addition, since $\phi_{11} = \frac{h^{2-2s}}{1-s}$ and $\phi_{1n} < 0$, we get

$$-\sum_{j=1}^n \omega_{1j} \geq -\sum_{j=1}^n \tilde{\omega}_{1j} + \frac{h^{-2}}{2} \phi_{11} \geq C\eta(x_1)^{-2s},$$

and similarly we can also obtain

$$-\sum_{j=1}^n \omega_{nj} \geq C\eta(x_n)^{-2s}.$$

Putting all these together yields (35). \square

Theorem 3. *Suppose the solution of (4) belongs to $C^4(a, b)$ and let \mathbf{U}_n be the exact solution vector, then we have*

$$(39) \quad \|\mathbf{U}_n - \mathbf{U}_n^h\|_\infty \leq C(h^{4-2s} + h^3).$$

Proof. It is easy to see

$$\mathbf{W}_{n \times n}(\mathbf{U}_n - \mathbf{U}_n^h) = \mathbf{W}_{n \times n} \mathbf{U}_n - \mathbf{G}_n,$$

which implies

$$\sum_{j=1}^n \omega_{ij} [u(x_j) - u_j] = \mathcal{L}u(x_i) - \mathcal{L}_h u(x_i), \quad i = 1, \dots, n.$$

It is straightforward that there exists some k , such that

$$u(x_k) - u_k = \max_{1 \leq i \leq n} |u(x_i) - u_i| \quad \text{or} \quad u(x_k) - u_k = -\max_{1 \leq i \leq n} |u(x_i) - u_i|,$$

and we only consider the first case. Since $-\mathbf{W}_{n \times n}$ is an M-matrix,

$$\begin{aligned} \mathcal{L}u(x_k) - \mathcal{L}_h u(x_k) &= \sum_{j=1, j \neq k}^n \omega_{kj} [u(x_j) - u_j - (u(x_k) - u_k)] + (u(x_k) - u_k) \sum_{j=1}^n \omega_{kj} \\ &\leq (u(x_k) - u_k) \sum_{j=1}^n \omega_{kj}, \end{aligned}$$

which leads to

$$\|\mathbf{U}_n - \mathbf{U}_n^h\|_\infty = u(x_k) - u_k \leq \frac{1}{|\sum_{j=1}^n \omega_{kj}|} |\mathcal{L}u(x_k) - \mathcal{L}_h u(x_k)|.$$

Combining it with (24) and (35) we have

$$\|\mathbf{U}_n - \mathbf{U}_n^h\|_\infty \leq C\eta(x_k)^{2s}(h^{4-2s} + h^3\eta(x_k)^{-2s}) \leq C(h^{4-2s} + h^3),$$

and the proof is completed. \square

4. Numerical experiments

In this section, we examine numerically the performance of the proposed collocation scheme. In addition, to emphasize higher order accuracy of the new scheme, we compare it with the collocation scheme studied in [21]. Actually, the later one is based on the modified rule $\tilde{\mathcal{L}}_h u(x_i)$ defined in (18) and can be formulated as

$$(40) \quad \tilde{\mathbf{W}}_{n \times n} \tilde{\mathbf{U}}_n^h = \tilde{\mathbf{G}}_n,$$

where $\tilde{\mathbf{W}}_{n \times n}$ is submatrix of $\tilde{\mathbf{W}}_{n \times (n+2)}$ by deleting the first and last columns, and $\tilde{\mathbf{G}}_n$ are obtained from \mathbf{F}_n by using the boundary values.

As an example, we choose $a = 0, b = 1$, and $u(y) = y^2(1 - y^2)$ in (4) and the right-hand side term can be determined accordingly by using the definition (2). We will investigate the truncation errors and the solution errors for both schemes. For example, for our proposed scheme, the truncation error is the error of the quadrature rule $\mathcal{L}_h u(x_i)$, and the solution error is measured by

$$\|\mathbf{U}_n - \mathbf{U}_n^h\|_\infty = \max_{1 \leq i \leq n} |u(x_i) - u_i|.$$

Numerical results for the modified rule $\tilde{\mathcal{L}}_h u(x_i)$ and the collocation scheme (40) are presented in Tables 1 - 3, from which we observe that: (1) no matter where the singular point x_i is located at, the truncation errors behave at the same level and are approximately $O(h^{2-2s})$ for any s , which agrees well with the estimate in Remark 2; (2) the solution error also approximately reach $O(h^{2-2s})$ for any s .

TABLE 1. Numerical results on the modified rule $\tilde{\mathcal{L}}_h u(x_i)$ and the collocation scheme (40) with $s = 1/4$.

n	$ \tilde{\mathcal{L}}u(x_{\frac{n+1}{2}}) - \tilde{\mathcal{L}}_h u(x_{\frac{n+1}{2}}) $		$ \tilde{\mathcal{L}}u(x_n) - \tilde{\mathcal{L}}_h u(x_n) $		$\ \mathbf{U}_n - \tilde{\mathbf{U}}_n^h\ _\infty$	
	Value	Conv. Rate	Value	Conv. Rate	Value	Conv. Rate
127	2.901e-04	-	7.493e-04	-	2.747e-05	-
255	9.663e-05	1.586	2.625e-04	1.513	9.153e-06	1.586
511	3.268e-05	1.564	9.160e-05	1.519	3.099e-06	1.562
1023	1.118e-05	1.547	3.197e-05	1.518	1.062e-06	1.545
2047	3.860e-06	1.534	1.118e-05	1.516	3.671e-07	1.532

TABLE 2. Numerical results on the modified rule $\tilde{\mathcal{L}}_h u(x_i)$ and the collocation scheme (40) with $s = 1/2$.

n	$ \tilde{\mathcal{L}}u(x_{\frac{n+1}{2}}) - \tilde{\mathcal{L}}_h u(x_{\frac{n+1}{2}}) $		$ \tilde{\mathcal{L}}u(x_n) - \tilde{\mathcal{L}}_h u(x_n) $		$\ \mathbf{U}_n - \tilde{\mathbf{U}}_n^h\ _\infty$	
	Value	Conv. Rate	Value	Conv. Rate	Value	Conv. Rate
127	6.626e-03	-	1.393e-02	-	5.303e-04	-
255	3.293e-03	1.009	7.083e-03	0.976	2.626e-04	1.014
511	1.642e-03	1.004	3.568e-03	0.989	1.307e-04	1.006
1023	8.195e-04	1.002	1.790e-03	0.995	6.523e-05	1.003
2047	4.094e-04	1.001	8.964e-04	0.998	3.258e-05	1.001

TABLE 3. Numerical results on the modified rule $\tilde{\mathcal{L}}_h u(x_i)$ and the collocation scheme (40) with $s = 3/4$.

n	$ \tilde{\mathcal{L}}u(x_{\frac{n+1}{2}}) - \tilde{\mathcal{L}}_h u(x_{\frac{n+1}{2}}) $		$ \tilde{\mathcal{L}}u(x_n) - \tilde{\mathcal{L}}_h u(x_n) $		$\ \mathbf{U}_n - \tilde{\mathbf{U}}_n^h\ _\infty$	
	Value	Conv. Rate	Value	Conv. Rate	Value	Conv. Rate
127	1.676e-01	-	3.293e-01	-	7.183e-03	-
255	1.184e-01	0.501	2.381e-01	0.468	4.918e-03	0.547
511	8.373e-02	0.500	1.702e-01	0.484	3.403e-03	0.531
1023	5.920e-02	0.500	1.210e-01	0.492	2.370e-03	0.522
2047	4.186e-02	0.500	8.582e-02	0.496	1.658e-03	0.515

Numerical results for the ultimate rule $\mathcal{L}_h u(x_i)$ and the proposed collocation scheme (28) are given in Tables 4 - 6, from which we can clearly see that:

- (1) when the singular point x_i is located at the middle point of (a, b) , i.e., $i = \frac{n+1}{2}$, the truncation error is approximately $O(h^3)$ for $s = 1/4$ and $s = 1/2$, and $O(h^{5/2})$ for $s = 3/4$. This agrees well with the estimate in Theorem 2, because in this case the factor $\eta(x_i) = \frac{b-a}{2}$ becomes a constant.
- (2) when the singular point x_i is very close to one of the end-points of (a, b) , and here we choose $i = n$, the truncation error is approximately $O(h^{3-2s})$ for all s . This also agrees well with the estimate in Theorem 2 due to the factor $\eta(x_n) = h$ in this case.
- (3) the solution errors can approximately reach $O(h^3)$ when $s \leq 1/2$ and $O(h^{4-2s})$ when $s > 1/2$. Comparing the errors in last three columns in Tables 4 - 6 we find an interesting result: although the truncation errors are effected by the factor $\eta(x_i)$, the minimum distance between the singular point x_i and the endpoints, the solution errors do not. All of these agree well with the estimate in Theorem 3.

5. Concluding remarks

In this paper, we start with a nodal-type trapezoidal rule discussed in [21], and establish its error expansion analysis. Then a new nodal-type quadrature rule with higher order accuracy is proposed with subtle error analysis being obtained. Based on this rule, a new collocation scheme is constructed to solve certain finite-part integral equation, and optimal error estimate for the scheme is also rigorously obtained. An interesting phenomenon for this collocation scheme is that although its truncation error is affected by the factor $\eta(x_i)$, the solution error does not.

TABLE 4. Numerical results on the ultimate rule $\mathcal{L}_h u(x_i)$ and the collocation scheme (28) with $s = 1/4$.

n	$ \mathcal{L}u(x_{\frac{n+1}{2}}) - \mathcal{L}_h u(x_{\frac{n+1}{2}}) $		$ \mathcal{L}u(x_n) - \mathcal{L}_h u(x_n) $		$\ \mathbf{U}_n - \mathbf{U}_n^h\ _\infty$	
	Value	Conv. Rate	Value	Conv. Rate	Value	Conv. Rate
127	1.155e-06		3.832e-06		1.529e-07	
255	1.512e-07	2.933	7.505e-07	2.352	2.011e-08	2.926
511	1.952e-08	2.953	1.419e-07	2.403	2.606e-09	2.948
1023	2.496e-09	2.967	2.623e-08	2.435	3.341e-10	2.964
2047	3.170e-10	2.977	4.781e-09	2.456	4.252e-11	2.974

TABLE 5. Numerical results on the ultimate rule $\mathcal{L}_h u(x_i)$ and the collocation scheme (28) with $s = 1/2$.

n	$ \mathcal{L}u(x_{\frac{n+1}{2}}) - \mathcal{L}_h u(x_{\frac{n+1}{2}}) $		$ \mathcal{L}u(x_n) - \mathcal{L}_h u(x_n) $		$\ \mathbf{U}_n - \mathbf{U}_n^h\ _\infty$	
	Value	Conv. Rate	Value	Conv. Rate	Value	Conv. Rate
127	4.234e-06		2.533e-05		5.822e-07	
255	6.115e-07	2.792	6.721e-06	1.914	8.540e-08	2.769
511	8.674e-08	2.818	1.734e-06	1.954	1.227e-08	2.799
1023	1.213e-08	2.838	4.409e-07	1.976	1.735e-09	2.822
2047	1.678e-09	2.854	1.112e-07	1.987	2.422e-10	2.841

TABLE 6. Numerical results on the ultimate rule $\mathcal{L}_h u(x_i)$ and the collocation scheme (28) with $s = 3/4$.

n	$ \mathcal{L}u(x_{\frac{n+1}{2}}) - \mathcal{L}_h u(x_{\frac{n+1}{2}}) $		$ \mathcal{L}u(x_n) - \mathcal{L}_h u(x_n) $		$\ \mathbf{U}_n - \mathbf{U}_n^h\ _\infty$	
	Value	Conv. Rate	Value	Conv. Rate	Value	Conv. Rate
127	3.330e-05		1.800e-04		2.526e-06	
255	6.025e-06	2.466	6.875e-05	1.389	4.629e-07	2.448
511	1.082e-06	2.477	2.522e-05	1.447	8.395e-08	2.463
1023	1.935e-07	2.484	9.079e-06	1.474	1.512e-08	2.474
2047	3.448e-08	2.489	3.239e-06	1.487	2.707e-09	2.481

Analysis in Theorem 3 show that the nice property (35) of the stiffness matrix can amend the influence of the truncation error on the solution error. Finally, numerical experiments are also performed to verify the theoretical results.

Acknowledgments

This work was partially supported by the National Key Research and Development Program of China under grant number 2017YFC0403301, by the National Natural Science Fund of China under grant number 11671313, and by the Doctoral Fund of Ministry of Education of China under grant number 20130141110026.

References

- [1] G. Bao, W. Sun, A fast algorithm for the electromagnetic scattering from a cavity, SIAM J. Sci. Comput., 27 (2005) 553-574.

- [2] U. Choi, S. Kim, B. Yun, Improvement of the asymptotic behavior of the Euler-Maclaurin formula for Cauchy principal value and Hadamard finite-part integrals, *Int. J. Numer. Methods Engrg.*, 61 (2004) 496-513.
- [3] K. Diethelm, Modified compound quadrature rules for strongly singular integrals, *Computing*, 52 (1994) 337-354.
- [4] K. Diethelm, Asymptotically sharp error bounds for a quadrature rule for Cauchy principal value integrals based on piecewise linear interpolation, *Approx. Theory Appl.*, 11 (1995) 78-89.
- [5] T. Hasegawa, Uniform approximations to finite Hilbert transform and its derivative, *J. Comput. Appl. Math.*, 163 (2004) 127-138.
- [6] C. Hui, D. Shia, Evaluations of hypersingular integrals using Gaussian quadrature, *Int. J. Numer. Methods Engrg.*, 44 (1999) 205-214.
- [7] N. Ioakimidis, On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals and their derivatives, *Math. Comp.*, 44 (1985) 191-198.
- [8] P. Köhler, On the error of quadrature formulae for Cauchy principal value integrals based on piecewise interpolation, *Approx. Theory Appl.*, 13 (1997) 58-67.
- [9] B. Li, W. Sun, Newton-Cotes rules for Hadamard finite-part integrals on an interval, *IMA J. Numer. Anal.*, 30 (2010) 1235-1255.
- [10] C. Li, Z. Qiao, A fast preconditioned iterative algorithm for the electromagnetic scattering from a large cavity, *J. Sci. Comput.*, 53 (2012) 435-450.
- [11] J. Li, X. Zhang, D. Yu, Superconvergence and ultraconvergence of Newton-Cotes rules for supersingular integrals, *J. Comput. Appl. Math.*, 233 (2010), 2841-2854.
- [12] P. Linz, On the approximate computation of certain strongly singular integrals, *Computing*, 35 (1985) 345-353.
- [13] D. Liu, J. Wu, D. Yu, The superconvergence of the Newton-Cotes rule for Cauchy principal value integrals, *J. Comput. Appl. Math.*, 235 (2010) 696-707.
- [14] W. Sun, J. Wu, Newton-Cotes formulae for the numerical evaluation of certain hypersingular integral, *Computing*, 75 (2005) 297-309.
- [15] G. Vainikko, I. Lifanov, On the notion of the finite part of divergent integrals in integral equations, *Diff. Eq.*, 38 (2002) 1313-1326.
- [16] J. Wu, Z. Dai, X. Zhang, The superconvergence of the composite midpoint rule for the finite-part integral, *J. Comput. Appl. Math.*, 233 (2010) 1954-1968.
- [17] J. Wu, W. Sun, The superconvergence of the composite trapezoidal rule for Hadamard finite part integrals, *Numer. Math.*, 102 (2005) 343-363.
- [18] J. Wu, W. Sun, The superconvergence of the Newton-Cotes rules for the Hadamard finite part integral on an interval, *Numer. Math.*, 109 (2008) 143-165.
- [19] J. Wu, Y. Wang, W. Li, W. Sun, Toeplitz-type approximations to the Hadamard integral operator and their applications to electromagnetic cavity problems, *Appl. Numer. Math.*, 58 (2008) 101-121.
- [20] D. Yu, *Natural Boundary Integral Method and its Applications*, Kluwer Academic Publishers, 2002.
- [21] X. Zhang, M. Gunzhuriger, L. Ju, Nodal-type collocation methods for hypersingular integral equations and nonlocal diffusion problems, *Comput. Methods Appl. Mech. Engrg.*, 299 (2016) 401-420.
- [22] X. Zhang, J. Wu, D. Yu, The superconvergence of composite Newton-Cotes rules for Hadamard finite-part integral on a circle, *Computing*, 85 (2009) 219-244.
- [23] X. Zhang, J. Wu, D. Yu, The superconvergence of composite trapezoidal rule for Hadamard finite-part integral on a circle and its application, *Int. J. Comput. Math.*, 87 (2010) 855-876.
- [24] M. Zhao, Z. Qiao, T. Tang, A fast high order method for electromagnetic scattering by large open cavities, *J. Comput. Math.*, 29 (2011) 287-304.

School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, P.R. China;
Computational Science Hubei Key Laboratory, Wuhan University, Wuhan, Hubei 430072, P.R.
China

E-mail: `hfeng.math@whu.edu.cn`

School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, P.R. China

E-mail: `sabrina8128@163.com`

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

E-mail: `ju@math.sc.edu`

School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, P.R. China;
Computational Science Hubei Key Laboratory, Wuhan University, Wuhan, Hubei 430072, P.R.
China

E-mail: `xpzhang.math@whu.edu.cn`