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SECOND-ORDER TWO-SCALE ANALYSIS METHOD FOR DYNAMIC THERMO-MECHANICAL PROBLEMS OF COMPOSITE STRUCTURES WITH CYLINDRICAL PERIODICITY

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Abstract. In this paper, a novel second-order two-scale (SOTS) analysis method and corresponding numerical algorithm is developed for dynamic thermo-mechanical problems of composite structures with cylindrical periodicity. The formal SOTS solutions are successfully constructed by the multiscale asymptotic analysis. Then we theoretically explain the necessity of developing the SOTS solutions by the error analysis in the pointwise sense. Futhermore, the convergence result with an explicit rate for the SOTS solutions is obtained. In addition, a SOTS numerical algorithm is presented to effectively solve these multiscale problems. Finally, some numerical examples verify the feasibility and validity of the SOTS numerical algorithm we proposed. This study offers a unified multiscale framework that enables the simulation and analysis of thermo-mechanical coupled behavior of composite structures with cylindrical periodicity.

Key words. Dynamic thermo-mechanical problem, multiscale asymptotic analysis, composite structure, cylindrical periodicity, SOTS numerical algorithm.

1. Introduction

In the past decades, composite materials have been widely used in engineering applications owing to their attractive physical and mechanical properties. With the appearance of various complex and extreme environments, composite materials usually served under multi-physics coupled circumstances, such as electro-mechanical, thermo-electrical, thermo-mechanical and magneto-electro-thermo-elastic, etc. Due to a great application prospect, the thermo-mechanical performances of composite materials have been a research hotspot of scientists and engineers. To the best of our knowledge, some studies have performed on dynamic thermo-mechanical problems of composites. However, most of these studies focused on one-way thermo-mechanical coupled problems [1–5], namely only the thermal effects affect the mechanical field. Besides, some researchers devoted to the two-way thermo-mechanical coupled problems which are fully coupled hyperbolic and parabolic systems, but their researches were based on the cartesian coordinate system [6–10]. To the best of our knowledge, the structures made of the composites with cylindrical periodic configurations have a great application value in practical engineering, such as composite shells, composite cylinder, composite tube, etc. In recent years, some research results for composite structures with cylindrical periodicity have appeared [5, 11–15]. However, up to now there is a lack of adequate research on dynamic thermo-mechanical problems of composite structures with cylindrical periodicity.

The subject of this paper is to develop a SOTS analysis method and associated numerical algorithm for dynamic thermo-mechanical problems of composite

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structures with cylindrical periodicity. In such cases, the direct numerical computation of these multiscale problems needs a tremendous amount of computational resources to capture the micro-scale behaviors due to large heterogeneities (caused by inclusions or holes) in composite structures. Furthermore, the stability of numerical scheme for these coupled systems with cylindrical periodic configurations is also a difficult problem to handle. From the point of view of theoretical analysis, the error estimate of SOTS solutions with an explicit convergence rate is hard to gain due to lack of a prior estimate for wave equations with nonhomogeneous boundary condition. In order to deal with these difficulties, we develop a SOTS method to overcome numerical difficulties based on asymptotic homogenization method (AHM), finite element method (FEM) and finite difference method (FDM). On the other hand, we impose the homogeneous Dirichlet condition on auxiliary cell problems. At this case, the explicit convergence rate of SOTS solutions is easily obtained because the SOTS solutions will satisfy automatically the boundary condition of governing equations under some assumptions.

This paper is organized as follows. In Sections 2, the detailed construction of the SOTS solutions for dynamic thermo-mechanical problems of composite structures with cylindrical periodicity is given by multiscale asymptotic analysis. Moreover, the error analysis in the pointwise sense of first-order two-scale (FOTS) solutions and SOTS solutions is obtained, respectively. Through the above analysis, we theoretically explain the importance of developing the SOTS solutions in capturing micro-scale information. In Section 3, an explicit convergence rate for the SOTS solutions are derived under some hypotheses. In Section 4, a SOTS numerical algorithm based on FEM and FDM is presented to solve these multiscale problems effectively. In Section 5, some numerical results are given to verify the feasibility and validity of our SOTS algorithm. Finally, some conclusions are given in Section 6.

For convenience, we use the Einstein summation convention on repeated indices in this paper. Besides, the notation δ_{ij} is the Kronecker symbol, and if $i = j$, $\delta_{ij} = 1$, or $\delta_{ij} = 0$.

2. The multiscale asymptotic analysis of governing equations

Consider governing equations for dynamic thermo-mechanical problems of composite structures with cylindrical periodicity as follows

$$
(1)\quad \begin{cases}
\rho^\varepsilon \dfrac{\partial^2 u_r^\varepsilon}{\partial t^2} - (\dfrac{\partial \sigma_{rr}^\varepsilon}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\theta}^\varepsilon}{\partial \theta} + \dfrac{\partial \sigma_{rz}^\varepsilon}{\partial z} + \dfrac{\sigma_{rr}^\varepsilon - \sigma_{\theta\theta}^\varepsilon}{r}) = f_r \ \text{ in } \ \Omega\times(0,T],\\
\rho^\varepsilon \dfrac{\partial^2 u_\theta^\varepsilon}{\partial t^2} - (\dfrac{\partial \sigma_{r\theta}^\varepsilon}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{\theta\theta}^\varepsilon}{\partial \theta} + \dfrac{\partial \sigma_{z\theta}^\varepsilon}{\partial z} + 2\dfrac{\sigma_{r\theta}^\varepsilon}{r}) = f_\theta \ \text{ in } \ \Omega\times(0,T],\\
\rho^\varepsilon \dfrac{\partial^2 u_z^\varepsilon}{\partial t^2} - (\dfrac{\partial \sigma_{zr}^\varepsilon}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{z\theta}^\varepsilon}{\partial \theta} + \dfrac{\partial \sigma_{zz}^\varepsilon}{\partial z} + \dfrac{\sigma_{zr}^\varepsilon}{r}) = f_z \ \text{ in } \ \Omega\times(0,T],\\
\rho^\varepsilon c^\varepsilon \dfrac{\partial T^\varepsilon}{\partial t} + (\dfrac{\partial q_r^\varepsilon}{\partial r} + \dfrac{1}{r}\dfrac{\partial q_\theta^\varepsilon}{\partial \theta} + \dfrac{\partial q_z^\varepsilon}{\partial z} + \dfrac{q_r^\varepsilon}{r}) + \widetilde{T}\beta_{ij}^\varepsilon \dfrac{\partial \varepsilon_{ij}^\varepsilon}{\partial t} = h \ \text{ in } \ \Omega\times(0,T],\\
\mathbf{u}^\varepsilon(\mathbf{x},t) = \widehat{\mathbf{u}}(\mathbf{x},t), \ \ T^\varepsilon(\mathbf{x},t) = \widehat{T}(\mathbf{x},t) \ \text{ on } \ \partial\Omega\times(0,T],\\
\mathbf{u}^\varepsilon(\mathbf{x},0) = \mathbf{u}^0, \ \ \dfrac{\partial \mathbf{u}^\varepsilon(\mathbf{x},t)}{\partial t}\Big|_{t=0} = \mathbf{u}^1(\mathbf{x}), \ \ T^\varepsilon(\mathbf{x},0) = \widetilde{T} \ \text{ in } \ \Omega.
\end{cases}
$$

where Ω is a bounded convex domain ($0 < r < \infty$) in $\mathbb{R}^3$ with a boundary $\partial\Omega$; The u_r^ε, u_θ^ε, u_z^ε and T^ε in (1) are undetermined displacement and temperature fields; $\widehat{\mathbf{u}}(\mathbf{x},t)$, $\widehat{T}(\mathbf{x},t)$ and $\mathbf{u}^1(\mathbf{x})$ are known functions with macro-coordinates $\mathbf{x} = (r,\theta,z)$; ε represents the characteristic periodic unit cell size; ρ^ε and c^ε are the mass density and specific heat; f_r, f_θ, f_z and h are the the body forces in three directions and internal heat source; $\mathbf{u}^0$ is the initial displacement field; $\widetilde{T}$ is the initial temperature when the composites are stress-free. In this paper, each constituent of the composites is assumed to exhibit linear thermo-mechanical coupled response. For governing equations, the strains $\boldsymbol{\varepsilon}^\varepsilon$ are given in terms of the displacement field $\mathbf{u}^\varepsilon$ as follows

$$\begin{aligned} &\varepsilon_{rr}^\varepsilon = \frac{\partial u_r^\varepsilon}{\partial r}, \quad \varepsilon_{\theta\theta}^\varepsilon = \frac{1}{r}\big(\frac{\partial u_\theta^\varepsilon}{\partial \theta} + u_r^\varepsilon\big), \qquad \varepsilon_{r\theta}^\varepsilon = \frac{1}{2}\big(\frac{1}{r}\frac{\partial u_r^\varepsilon}{\partial \theta} + \frac{\partial u_\theta^\varepsilon}{\partial r} - \frac{u_\theta^\varepsilon}{r}\big),\\ &\varepsilon_{zz}^\varepsilon = \frac{\partial u_z^\varepsilon}{\partial z}, \quad \varepsilon_{z\theta}^\varepsilon = \frac{1}{2}\big(\frac{\partial u_\theta^\varepsilon}{\partial z} + \frac{1}{r}\frac{\partial u_z^\varepsilon}{\partial \theta}\big), \quad \varepsilon_{rz}^\varepsilon = \frac{1}{2}\big(\frac{\partial u_z^\varepsilon}{\partial r} + \frac{\partial u_r^\varepsilon}{\partial z}\big). \end{aligned} \tag{2}$$

If we apply the substitutions: $1 \to r, 2 \to \theta, 3 \to z$ and $\psi_1 = \dfrac{\partial}{\partial r}, \psi_2 = \dfrac{1}{r}\dfrac{\partial}{\partial \theta}, \psi_3 = \dfrac{\partial}{\partial z}$ to simplify the notations in our paper, the constitutive laws of problem (1) are given by

$$\sigma_{ij}^\varepsilon = C_{ijkl}^\varepsilon \varepsilon_{kl}^\varepsilon - \beta_{ij}^\varepsilon (T^\varepsilon - \widetilde{T}) \tag{3}$$

and

$$q_i^\varepsilon = -k_{ij}^\varepsilon \psi_j (T^\varepsilon) \tag{4}$$

where $\{C_{ijkl}^\varepsilon\}$ is the fourth order elastic tensor, $\{\beta_{ij}^\varepsilon\}$ is the second order thermal modulus tensor and $\{k_{ij}^\varepsilon\}$ is the second order thermal conductivity tensor (i, j, k, l=1,2,3).

Now, let us set $\mathbf{y} = \dfrac{\mathbf{x}}{\varepsilon} = (\dfrac{r}{\varepsilon}, \dfrac{\theta}{\varepsilon}, \dfrac{z}{\varepsilon}) = (\tilde{r}, \tilde{\theta}, \tilde{z})$ as micro-coordinates of periodic unit cell $Y = (0,1)^3$. Then material parameters $\rho^\varepsilon(\mathbf{x})$, $c^\varepsilon(\mathbf{x})$, $C_{ijkl}^\varepsilon(\mathbf{x})$, $k_{ij}^\varepsilon(\mathbf{x})$ and $\beta_{ij}^\varepsilon(\mathbf{x})$ can be rewritten as $\rho(\mathbf{y})$, $c(\mathbf{y})$, $C_{ijkl}(\mathbf{y})$, $k_{ij}(\mathbf{y})$ and $\beta_{ij}(\mathbf{y})$. Additionally, the operators ψ_i for the macro-scale and $\widetilde{\psi}_i$ for the micro-scale are defined as follows

$$\begin{cases} \psi_1 = \dfrac{\partial}{\partial r}, \quad \psi_2 = \dfrac{1}{r}\dfrac{\partial}{\partial \theta}, \quad \psi_3 = \dfrac{\partial}{\partial z} \\ \widetilde{\psi}_1 = \dfrac{\partial}{\partial \widetilde{r}}, \quad \widetilde{\psi}_2 = \dfrac{1}{r}\dfrac{\partial}{\partial \widetilde{\theta}}, \quad \widetilde{\psi}_3 = \dfrac{\partial}{\partial \widetilde{z}} \end{cases} \tag{5}$$

Hence, the chain rule of AHM for original problem (1) can be expressed as

$$\psi_i = \psi_i + \varepsilon^{-1}\widetilde{\psi}_i \tag{6}$$

For obtaining the convergence result in this paper, we make the following hypotheses [7,18,19]

$$\text{(A)}\quad \begin{cases} C_{ijkl}^\varepsilon,\ \beta_{ij}^\varepsilon,\ k_{ij}^\varepsilon \in L^\infty(\Omega), \\ C_{ijkl}^\varepsilon = C_{ijlk}^\varepsilon = C_{klij}^\varepsilon,\ \beta_{ij}^\varepsilon = \beta_{ji}^\varepsilon,\ k_{ij}^\varepsilon = k_{ji}^\varepsilon, \\ \exists \gamma_0, \gamma_1 > 0, \gamma_0 \eta_{ij}\eta_{ij} \le C_{ijkl}^\varepsilon \eta_{ij}\eta_{kl} \le \gamma_1 \eta_{ij}\eta_{ij},\ \forall \{\eta_{ij}\} \in \mathbb{R}^{3\times 3}, \\ \gamma_0|\boldsymbol{\xi}|^2 \le \beta_{ij}^\varepsilon \xi_i\xi_j \le \gamma_1|\boldsymbol{\xi}|^2, \gamma_0|\boldsymbol{\xi}|^2 \le k_{ij}^\varepsilon \xi_i\xi_j \le \gamma_1|\boldsymbol{\xi}|^2,\ \forall \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \\ C_{ijkl}^\varepsilon,\ \beta_{ij}^\varepsilon \text{ and } k_{ij}^\varepsilon \text{ are } 1-\text{periodic functions in } \mathbf{y}. \end{cases}$$

$$\text{(B)}\quad \begin{cases} \rho^\varepsilon,\ c^\varepsilon \in L^\infty(\Omega), \\ 0 < \rho^0 \le \rho^\varepsilon,\ 0 < c^0 \le c^\varepsilon, \text{ where } \rho^0 \text{ and } c^0 \text{ are constants}, \\ \rho^\varepsilon \text{ and } c^\varepsilon \text{ are } 1-\text{periodic functions in } \mathbf{y}. \end{cases}$$

(C) $f_i \in L^2(\Omega \times (0,T))$, $h \in L^2(\Omega \times (0,T))$, $\widehat{\mathbf{u}}(\mathbf{x}, t) \in (L^2(\Omega \times (0,T)))^3$, $\widehat{T}(\mathbf{x}, t) \in L^2(\Omega \times (0,T))$, $\mathbf{u}^1 \in (L^2(\Omega))^3$.

In this section, we firstly give the specific construction process of first-order two-scale (FOTS) solutions and second-order two-scale (SOTS) solutions to problem

(1). After that, the error analysis in the pointwise sense of FOTS solutions and SOTS solutions is given which neatly illustrates the extreme necessity of developing SOTS solutions. On the other hand, it will give the residual equations which are of vital significance for us to prove the main convergence theorem in Section 3.

2.1. Second-order two-scale analysis for governing equations. To the original problem (1), we suppose that $u_i^\varepsilon(\mathbf{x},t)$ and $T^\varepsilon(\mathbf{x},t)$ can be expressed as the following asymptotic expansion forms

$$(7)\qquad \begin{cases} u_i^\varepsilon(\mathbf{x},t)=u_i^{(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon u_i^{(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2u_i^{(2)}(\mathbf{x},\mathbf{y},t)+\mathrm{O}(\varepsilon^3),\\ T^\varepsilon(\mathbf{x},t)=T^{(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon T^{(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2T^{(2)}(\mathbf{x},\mathbf{y},t)+\mathrm{O}(\varepsilon^3).\end{cases}$$

It is worth stressing that SOTS analysis in cylindrical coordinates is quite different from that of cartesian coordinates [7,8]. Due to lack of a consistent form of $\varepsilon_{ij}^\varepsilon(\mathbf{x},t)$ in cylindrical coordinates, the SOTS analysis in cylindrical coordinates should start from the basic physical quantities $\varepsilon_{ij}^\varepsilon(\mathbf{x},t)$, $\sigma_{ij}^\varepsilon(\mathbf{x},t)$ and $q_i^\varepsilon(\mathbf{x},t)$. Using (2) and (7), the basic quantities $\varepsilon_{ij}^\varepsilon(\mathbf{x},t)$ can be expanded as the following forms

$$(8)\qquad \varepsilon_{ij}^\varepsilon(\mathbf{x},t)=\varepsilon^{-1}\varepsilon_{ij}^{(-1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^0\varepsilon_{ij}^{(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon^1\varepsilon_{ij}^{(1)}(\mathbf{x},\mathbf{y},t)+\mathrm{O}(\varepsilon^2).$$

where

$$(9)\qquad \begin{aligned} &\varepsilon_{ij}^{(-1)}=\frac{1}{2}\big(\widetilde{\psi}_i(u_j^{(0)})+\widetilde{\psi}_j(u_i^{(0)})\big),\ \varepsilon_{ij}^{(s)}=\varepsilon_{ij}^{(s)*}+\frac{1}{2}\big(\widetilde{\psi}_i(u_j^{(s+1)})+\widetilde{\psi}_j(u_i^{(s+1)})\big),\\ &\varepsilon_{11}^{(s)*}=\psi_1(u_1^{(s)}),\ \varepsilon_{22}^{(s)*}=\psi_2(u_2^{(s)})+\frac{u_1^{(s)}}{r},\\ &\varepsilon_{12}^{(s)*}=\frac{1}{2}\big(\psi_2(u_1^{(s)})+\psi_1(u_2^{(s)})-\frac{u_2^{(s)}}{r}\big),\varepsilon_{33}^{(s)*}=\psi_3(u_3^{(s)}),\\ &\varepsilon_{23}^{(s)*}=\frac{1}{2}\big(\psi_3(u_2^{(s)})+\psi_2(u_3^{(s)})\big),\ \varepsilon_{13}^{(s)*}=\frac{1}{2}\big(\psi_1(u_3^{(s)})+\psi_3(u_1^{(s)})\big).\end{aligned}$$

And then, we assume that $\sigma_{ij}^\varepsilon(\mathbf{x},t)$ and $q_i^\varepsilon(\mathbf{x},t)$ have the detailed asymptotic expansion forms as below

$$(10)\qquad \begin{cases} \sigma_{ij}^\varepsilon(\mathbf{x},t)=\varepsilon^{-1}\sigma_{ij}^{(-1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^0\sigma_{ij}^{(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon^1\varepsilon_{ij}^{(1)}(\mathbf{x},\mathbf{y},t)+\mathrm{O}(\varepsilon^2),\\ q_i^\varepsilon(\mathbf{x},t)=\varepsilon^{-1}q_i^{(-1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^0q_i^{(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon^1q_i^{(1)}(\mathbf{x},\mathbf{y},t)+\mathrm{O}(\varepsilon^2).\end{cases}$$

By virtue of (9), and constitutive laws (3) and (4), the specific expressions of each asymptotic expansion term in (10) can be derived as follows

$$(11)\qquad \begin{aligned} &\sigma_{ij}^{(-1)}=C_{ijkl}\widetilde{\psi}_k(u_l^{(0)}),\ \sigma_{ij}^{(0)}=C_{ijkl}\varepsilon_{kl}^{(0)*}+C_{ijkl}\widetilde{\psi}_k(u_l^{(1)})-\beta_{ij}(T^{(0)}-\widetilde{T}),\\ &\sigma_{ij}^{(1)}=C_{ijkl}\varepsilon_{kl}^{(1)*}+C_{ijkl}\widetilde{\psi}_k(u_l^{(2)})-\beta_{ij}T^{(1)}.\end{aligned}$$

and

$$(12)\qquad q_i^{(-1)}=-k_{ij}\widetilde{\psi}_j(T^{(0)}),\ q_i^{(s)}=-k_{ij}\psi_j(T^{(s)})-k_{ij}\widetilde{\psi}_j(T^{(s+1)}),\ s=0,1.$$

Then substituting (7), (8) and (10) into original problem (1), expanding the derivatives and matching terms with the same order of small periodic parameter ε, we can immediately obtain

$$
(13)\quad \begin{cases}
\rho\dfrac{\partial^2 u_1^{(0)}}{\partial t^2} = \varepsilon^{-2}\widetilde{\psi}_j\sigma_{1j}^{(-1)} + \varepsilon^{-1}\big(\psi_j\sigma_{1j}^{(-1)} + \widetilde{\psi}_j\sigma_{1j}^{(0)} + \dfrac{\sigma_{11}^{(-1)} - \sigma_{22}^{(-1)}}{r}\big) \\
\qquad + \varepsilon^{0}\big(\psi_j\sigma_{1j}^{(0)} + \widetilde{\psi}_j\sigma_{1j}^{(1)} + \dfrac{\sigma_{11}^{(0)} - \sigma_{22}^{(0)}}{r} + f_1\big) + \mathrm{O}(\varepsilon), \\
\rho\dfrac{\partial^2 u_2^{(0)}}{\partial t^2} = \varepsilon^{-2}\widetilde{\psi}_j\sigma_{2j}^{(-1)} + \varepsilon^{-1}\big(\psi_j\sigma_{2j}^{(-1)} + \widetilde{\psi}_j\sigma_{2j}^{(0)} + 2\dfrac{\sigma_{12}^{(-1)}}{r}\big) \\
\qquad + \varepsilon^{0}\big(\psi_j\sigma_{2j}^{(0)} + \widetilde{\psi}_j\sigma_{2j}^{(1)} + 2\dfrac{\sigma_{12}^{(0)}}{r} + f_2\big) + \mathrm{O}(\varepsilon), \\
\rho\dfrac{\partial^2 u_3^{(0)}}{\partial t^2} = \varepsilon^{-2}\widetilde{\psi}_j\sigma_{3j}^{(-1)} + \varepsilon^{-1}\big(\psi_j\sigma_{3j}^{(-1)} + \widetilde{\psi}_j\sigma_{3j}^{(0)} + \dfrac{\sigma_{31}^{(-1)}}{r}\big) \\
\qquad + \varepsilon^{0}\big(\psi_j\sigma_{3j}^{(0)} + \widetilde{\psi}_j\sigma_{3j}^{(1)} + \dfrac{\sigma_{31}^{(0)}}{r} + f_3\big) + \mathrm{O}(\varepsilon), \\
\rho c\dfrac{\partial T^{(0)}}{\partial t} = -\varepsilon^{-2}\widetilde{\psi}_i q_i^{(-1)} - \varepsilon^{-1}\big[\psi_i q_i^{(-1)} + \widetilde{\psi}_i q_i^{(0)} + \dfrac{q_1^{(-1)}}{r} + \widetilde{T}\beta_{ij}\dfrac{\partial\varepsilon_{ij}^{(-1)}}{\partial t}\big] \\
\qquad - \varepsilon^{0}\big[\psi_i q_i^{(0)} + \widetilde{\psi}_i q_i^{(1)} + \dfrac{q_1^{(0)}}{r} + \widetilde{T}\beta_{ij}\dfrac{\partial\varepsilon_{ij}^{(0)}}{\partial t} - h\big] + \mathrm{O}(\varepsilon).
\end{cases}
$$

From (13), a series of equations with the same order of small parameter ε are derived as follows according to the classical procedure of AHM [20, 21]

$$
(14)\qquad \mathrm{O}(\varepsilon^{-2}) : \begin{cases} \widetilde{\psi}_j\sigma_{ij}^{(-1)} = 0, \\ \widetilde{\psi}_i q_i^{(-1)} = 0. \end{cases}
$$

$$
(15)\qquad \mathrm{O}(\varepsilon^{-1}) : \begin{cases}
\psi_j\sigma_{1j}^{(-1)} + \widetilde{\psi}_j\sigma_{1j}^{(0)} + \dfrac{\sigma_{11}^{(-1)} - \sigma_{22}^{(-1)}}{r} = 0, \\
\psi_j\sigma_{2j}^{(-1)} + \widetilde{\psi}_j\sigma_{2j}^{(0)} + 2\dfrac{\sigma_{12}^{(-1)}}{r} = 0, \\
\psi_j\sigma_{3j}^{(-1)} + \widetilde{\psi}_j\sigma_{3j}^{(0)} + \dfrac{\sigma_{31}^{(-1)}}{r} = 0, \\
\psi_i q_i^{(-1)} + \widetilde{\psi}_i q_i^{(0)} + \dfrac{q_1^{(-1)}}{r} + \widetilde{T}\beta_{ij}\dfrac{\partial\varepsilon_{ij}^{(-1)}}{\partial t} = 0.
\end{cases}
$$

$$
(16)\qquad \mathrm{O}(\varepsilon^{0}) : \begin{cases}
\rho\dfrac{\partial^2 u_1^{(0)}}{\partial t^2} = \psi_j\sigma_{1j}^{(0)} + \widetilde{\psi}_j\sigma_{1j}^{(1)} + \dfrac{\sigma_{11}^{(0)} - \sigma_{22}^{(0)}}{r} + f_1, \\
\rho\dfrac{\partial^2 u_2^{(0)}}{\partial t^2} = \psi_j\sigma_{2j}^{(0)} + \widetilde{\psi}_j\sigma_{2j}^{(1)} + 2\dfrac{\sigma_{12}^{(0)}}{r} + f_2, \\
\rho\dfrac{\partial^2 u_3^{(0)}}{\partial t^2} = \psi_j\sigma_{3j}^{(0)} + \widetilde{\psi}_j\sigma_{3j}^{(1)} + \dfrac{\sigma_{31}^{(0)}}{r} + f_3, \\
\rho c\dfrac{\partial T^{(0)}}{\partial t} = -\psi_i q_i^{(0)} - \widetilde{\psi}_i q_i^{(1)} - \dfrac{q_1^{(0)}}{r} - \widetilde{T}\beta_{ij}\dfrac{\partial\varepsilon_{ij}^{(0)}}{\partial t} + h.
\end{cases}
$$

Now, we start to recursively solve the asymptotic expansion terms of u_i^{ε} and T^{ε}. From the equations with order $\mathrm{O}(\varepsilon^{-2})$, the following equations can be obtained by substituting (11) and (12) into (14)

$$\left\{\begin{array}{l}\widetilde{\psi}_j\big[C_{ijkl}\widetilde{\psi}_k(u_l^{(0)})\big]=0,\\ \widetilde{\psi}_i\big[k_{ij}\widetilde{\psi}_j(T^{(0)})\big]=0.\end{array}\right. \tag{17}$$

Subsequently, it follows by virtue of the periodicity of the $u_i^{(0)}$ and $T^{(0)}$, and the linearity in the micro-scale of (17) that

$$u_i^{(0)}(\mathbf{x},\mathbf{y},t)=u_i^{(0)}(\mathbf{x},t),T^{(0)}(\mathbf{x},\mathbf{y},t)=T^{(0)}(\mathbf{x},t). \tag{18}$$

That is, $u_i^{(0)}$ and $T^{(0)}$ are independent of micro-scale variable $\mathbf{y}$. Hence, it can be concluded that $\sigma_{ij}^{(-1)}=0$. Due to this property, (15) can be further simplified as the following equations by using (11) and (12)

$$\left\{\begin{array}{l}\widetilde{\psi}_j\big[C_{ijkl}\widetilde{\psi}_k(u_l^{(1)})\big]=-\widetilde{\psi}_j\big[C_{ijkl}\varepsilon_{kl}^{(0)*}-\beta_{ij}(T^{(0)}-\widetilde{T})\big],\\ \widetilde{\psi}_i\big[k_{ij}\widetilde{\psi}_j(T^{(1)})\big]=-\widetilde{\psi}_i\big[k_{ij}\psi_j(T^{(0)})\big].\end{array}\right. \tag{19}$$

After that, we construct

$$\left\{\begin{array}{l}u_i^{(1)}=N_i^{mn}(r,\mathbf{y})\varepsilon_{mn}^{(0)*}-P_i(r,\mathbf{y})(T^{(0)}-\widetilde{T}),\\ T^{(1)}=M_m(r,\mathbf{y})\psi_m\big(T^{(0)}\big),\quad m,n=1,2,3.\end{array}\right. \tag{20}$$

where N_i^{mn}, P_i and M_m are the first-order auxiliary cell functions defined in unit cell Y.

Remark 1 It is important to mention that the first-order auxiliary cell functions are quasi-periodic functions which all depend on the macro-coordinate r. This is a significant difference compared to classical composites with micro-scale periodicity in cartesian coordinates.

Now substituting (20) into (19), the following equations with homogeneous Dirichlet boundary condition are obtained after simplification and calculation

$$\left\{\begin{array}{ll}\widetilde{\psi}_j\big[C_{ijkl}\widetilde{\psi}_k(N_l^{mn})\big]=-\widetilde{\psi}_j(C_{ijmn}), & \mathbf{y}\in Y\\ N_l^{mn}(r,\mathbf{y})=0, & \mathbf{y}\in\partial Y\end{array}\right. \tag{21}$$

$$\left\{\begin{array}{ll}\widetilde{\psi}_j\big[C_{ijkl}\widetilde{\psi}_k(P_l)\big]=-\widetilde{\psi}_j(\beta_{ij}), & \mathbf{y}\in Y\\ P_l(r,\mathbf{y})=0, & \mathbf{y}\in\partial Y\end{array}\right. \tag{22}$$

$$\left\{\begin{array}{ll}\widetilde{\psi}_i\big[k_{ij}\widetilde{\psi}_j(M_m)\big]=-\widetilde{\psi}_i(k_{im}), & \mathbf{y}\in Y\\ M_m(r,\mathbf{y})=0, & \mathbf{y}\in\partial Y\end{array}\right. \tag{23}$$

Then, one can obtain the following equations by making the volume integral and using the Green's formula on (16) inspired by Refs. [7, 8, 10, 12]

$$\left\{\begin{array}{l}\langle\rho\rangle\dfrac{\partial^2u_1^{(0)}}{\partial t^2}=\psi_j\langle\sigma_{1j}^{(0)}\rangle+\dfrac{\langle\sigma_{11}^{(0)}\rangle-\langle\sigma_{22}^{(0)}\rangle}{r}+f_1,\\ \langle\rho\rangle\dfrac{\partial^2u_2^{(0)}}{\partial t^2}=\psi_j\langle\sigma_{2j}^{(0)}\rangle+2\dfrac{\langle\sigma_{12}^{(0)}\rangle}{r}+f_2,\\ \langle\rho\rangle\dfrac{\partial^2u_3^{(0)}}{\partial t^2}=\psi_j\langle\sigma_{3j}^{(0)}\rangle+\dfrac{\langle\sigma_{31}^{(0)}\rangle}{r}+f_3,\\ \langle\rho c\rangle\dfrac{\partial T^{(0)}}{\partial t}=-\psi_i\langle q_i^{(0)}\rangle-\dfrac{\langle q_1^{(0)}\rangle}{r}-\widetilde{T}\dfrac{\partial\langle\beta_{ij}\varepsilon_{ij}^{(0)}\rangle}{\partial t}+h.\end{array}\right. \tag{24}$$

where the different operators in (24) are defined as follows

$$
\begin{aligned}
&\big\langle \phi(\mathbf{y})\big\rangle = \frac{1}{|Y|}\int_Y \phi(\mathbf{y})dY,\ \langle q_i^{(0)}\rangle = -\big\langle k_{im} + k_{ij}\widetilde{\psi}_j(M_m)\big\rangle \psi_m(T^{(0)}),\\
&\langle \sigma_{ij}^{(0)}\rangle = \big\langle C_{ijmn} + C_{ijkl}\widetilde{\psi}_k(N_l^{mn})\big\rangle \varepsilon_{mn}^{(0)*} - \big\langle \beta_{ij} + C_{ijkl}\widetilde{\psi}_k(P_l)\big\rangle (T^{(0)} - \widetilde{T}),\\
&\langle \beta_{ij}\varepsilon_{ij}^{(0)}\rangle = \big\langle \beta_{ij} + \beta_{mn}\widetilde{\psi}_n(N_m^{ij})\big\rangle \varepsilon_{ij}^{(0)*} - \big\langle \beta_{ij}\widetilde{\psi}_j(P_i)\big\rangle (T^{(0)} - \widetilde{T}).
\end{aligned}
\tag{25}
$$

Thus, the homogenized mass density $\hat{\rho}$, elasticity coefficients $\hat{C}_{ijmn}$, thermal modulus $\hat{\beta}_{ij}$ and $\hat{\beta}_{ij}^{*}$, specific heat capacity $\hat{S}$ and thermal conductivity $\hat{k}_{ij}$ can be defined as follows

$$
\begin{aligned}
&\hat{\rho} = \langle \rho\rangle,\ \hat{C}_{ijmn}(r) = \langle C_{ijmn} + C_{ijkl}\widetilde{\psi}_k(N_l^{mn})\rangle,\\
&\hat{\beta}_{ij}(r) = \big\langle \beta_{ij} + C_{ijkl}\widetilde{\psi}_k(P_l)\big\rangle,\ \hat{\beta}_{ij}^{*}(r) = \langle \beta_{ij} + \beta_{mn}\widetilde{\psi}_n(N_m^{ij})\rangle,\\
&\hat{S}(r) = \big\langle \rho c - \widetilde{T}\beta_{ij}\widetilde{\psi}_j(P_i)\big\rangle,\ \hat{k}_{ij}(r) = \big\langle k_{ij} + k_{im}\widetilde{\psi}_m(M_j)\big\rangle.
\end{aligned}
\tag{26}
$$

Remark 2 It is easy to prove that $\hat{\beta}_{ij} = \hat{\beta}_{ij}^{*}$ for any fixed macro-coordinate r according to Refs. [7, 8, 10].

Further, one can define the homogenized problems attached with the same initial-boundary value condition as the original problem (1)

$$
\begin{cases}
\hat{\rho}\dfrac{\partial^2 u_1^{(0)}}{\partial t^2} = \psi_j\langle \sigma_{1j}^{(0)}\rangle + \dfrac{\langle \sigma_{11}^{(0)}\rangle - \langle \sigma_{22}^{(0)}\rangle}{r} + f_1 \ \text{ in } \ \Omega\times(0,T],\\
\hat{\rho}\dfrac{\partial^2 u_2^{(0)}}{\partial t^2} = \psi_j\langle \sigma_{2j}^{(0)}\rangle + 2\dfrac{\langle \sigma_{12}^{(0)}\rangle}{r} + f_2 \ \text{ in } \ \Omega\times(0,T],\\
\hat{\rho}\dfrac{\partial^2 u_3^{(0)}}{\partial t^2} = \psi_j\langle \sigma_{3j}^{(0)}\rangle + \dfrac{\langle \sigma_{31}^{(0)}\rangle}{r} + f_3 \ \text{ in } \ \Omega\times(0,T],\\
\hat{S}\dfrac{\partial T^{(0)}}{\partial t} = -\psi_i\langle q_i^{(0)}\rangle - \dfrac{\langle q_1^{(0)}\rangle}{r} - \widetilde{T}\hat{\beta}_{ij}\dfrac{\partial \varepsilon_{ij}^{(0)*}}{\partial t} + h \ \text{ in } \ \Omega\times(0,T],\\
\mathbf{u}^{(0)}(\mathbf{x},t) = \widehat{\mathbf{u}}(\mathbf{x},t),\ \ T^{(0)}(\mathbf{x},t) = \widehat{T}(\mathbf{x},t) \ \text{ on } \ \partial\Omega\times(0,T],\\
\mathbf{u}^{(0)}(\mathbf{x},0) = \mathbf{u}^0,\ \ \dfrac{\partial \mathbf{u}^{(0)}(\mathbf{x},t)}{\partial t}\Big|_{t=0} = \mathbf{u}^1(\mathbf{x}),\ \ T^{(0)}(\mathbf{x},0) = \widetilde{T} \ \text{ in } \ \Omega.
\end{cases}
\tag{27}
$$

Next, we start to solve the vital second-order auxiliary cell functions. Firstly, the following equations are obtained by subtracting (16) from (27)

$$
\begin{cases}
\widetilde{\psi}_j\sigma_{1j}^{(1)} = (\rho - \hat{\rho})\,\dfrac{\partial^2 u_1^{(0)}}{\partial t^2} + \psi_j\langle \sigma_{1j}^{(0)}\rangle - \psi_j\sigma_{1j}^{(0)} + \dfrac{\langle \sigma_{11}^{(0)}\rangle - \langle \sigma_{22}^{(0)}\rangle}{r}\\
\qquad\qquad - \dfrac{\sigma_{11}^{(0)} - \sigma_{22}^{(0)}}{r},\\
\widetilde{\psi}_j\sigma_{2j}^{(1)} = (\rho - \hat{\rho})\,\dfrac{\partial^2 u_2^{(0)}}{\partial t^2} + \psi_j\langle \sigma_{2j}^{(0)}\rangle - \psi_j\sigma_{2j}^{(0)} + 2\dfrac{\langle \sigma_{12}^{(0)}\rangle}{r} - 2\dfrac{\sigma_{12}^{(0)}}{r},\\
\widetilde{\psi}_j\sigma_{3j}^{(1)} = (\rho - \hat{\rho})\,\dfrac{\partial^2 u_3^{(0)}}{\partial t^2} + \psi_j\langle \sigma_{3j}^{(0)}\rangle - \psi_j\sigma_{3j}^{(0)} + \dfrac{\langle \sigma_{31}^{(0)}\rangle}{r} - \dfrac{\sigma_{31}^{(0)}}{r},\\
\widetilde{\psi}_i q_i^{(1)} = (\hat{S} - \rho c)\dfrac{\partial T^{(0)}}{\partial t} + \psi_i\big(\langle q_i^{(0)}\rangle - q_i^{(0)}\big) +\\
\qquad \dfrac{\langle q_1^{(0)}\rangle - q_1^{(0)}}{r} + \widetilde{T}(\hat{\beta}_{ij}\dfrac{\partial \varepsilon_{ij}^{(0)*}}{\partial t} - \beta_{ij}\dfrac{\partial \varepsilon_{ij}^{(0)}}{\partial t}).
\end{cases}
\tag{28}
$$

Secondly, the following equations can be easily gotten by combining (9) and (20) together

$$
\begin{aligned}
&\beta_{ij}\varepsilon_{ij}^{(0)} = \big[\beta_{ij} + \beta_{mn}\widetilde{\psi}_n(N_m^{ij})\big]\varepsilon_{ij}^{(0)*} - \beta_{ij}\widetilde{\psi}_j(P_i)(T^{(0)} - \widetilde{T}),\\
&C_{ijkl}\widetilde{\psi}_k(u_l^{(1)}) = C_{ijkl}\widetilde{\psi}_k(N_l^{mn})\varepsilon_{mn}^{(0)*} - C_{ijkl}\widetilde{\psi}_k(P_l)(T^{(0)} - \widetilde{T}),\\
&\varepsilon_{ij}^{(1)*} = D_{ijmn}\varepsilon_{mn}^{(0)*} + \frac{1}{2}\big[N_i^{mn}\psi_j(\varepsilon_{mn}^{(0)*}) + N_j^{mn}\psi_i(\varepsilon_{mn}^{(0)*})\big]\\
&\qquad - E_{ij}(T^{(0)} - \widetilde{T}) - \frac{1}{2}\big[P_i\psi_j(T^{(0)} - \widetilde{T}) + P_j\psi_i(T^{(0)} - \widetilde{T})\big],\\
&k_{ij}\widetilde{\psi}_j(T^{(1)}) = k_{ij}\widetilde{\psi}_j(M_m)\psi_m(T^{(0)}).
\end{aligned}
\tag{29}
$$

where D_{ijmn} and E_{ij} are defined as follows

$$
\begin{aligned}
&D_{11mn} = \psi_1(N_1^{mn}),\ D_{22mn} = \psi_2(N_2^{mn}) + \frac{N_1^{mn}}{r},\\
&D_{12mn} = \frac{1}{2}\big[\psi_2(N_1^{mn}) + \psi_1(N_2^{mn}) - \frac{N_2^{mn}}{r}\big],\ D_{33mn} = \psi_3(N_3^{mn}),\\
&D_{23mn} = \frac{1}{2}\big[\psi_3(N_2^{mn}) + \psi_2(N_3^{mn})\big],\ D_{13mn} = \frac{1}{2}\big[\psi_1(N_3^{mn}) + \psi_3(N_1^{mn})\big],\\
&E_{11} = \psi_1(P_1),\ E_{22} = \psi_2(P_2) + \frac{P_1}{r},\ E_{12} = \frac{1}{2}\big[\psi_2(P_1) + \psi_1(P_2) - \frac{P_2}{r}\big],\\
&E_{33} = \psi_3(P_3),\ E_{23} = \frac{1}{2}\big[\psi_3(P_2) + \psi_2(P_3)\big],\ E_{13} = \frac{1}{2}\big[\psi_1(P_3) + \psi_3(P_1)\big].
\end{aligned}
\tag{30}
$$

Then, we replace the terms $\sigma_{ij}^{(1)}$, $q_i^{(1)}$, $\langle\sigma_{ij}^{(0)}\rangle$, $\langle q_i^{(0)}\rangle$ and $\varepsilon_{ij}^{(0)}$ in (28) with (11), (12), (25), (26) and (29). After computation, (28) can be rewritten as the following two equations

$$
\begin{aligned}
&\widetilde{\psi}_j\big[C_{ijkl}\widetilde{\psi}_k(u_l^{(2)})\big] = \widetilde{\psi}_j(\beta_{ij}M_m)\psi_m(T^{(0)}) + (\rho - \hat{\rho})\frac{\partial^2 u_i^{(0)}}{\partial t^2}\\
&+ \Big[\psi_j(\hat{C}_{ijmn}) - \psi_j\big(C_{ijkl}\widetilde{\psi}_k(N_l^{mn})\big) - \widetilde{\psi}_j(C_{ijkl}D_{klmn})\Big]\varepsilon_{mn}^{(0)*}\\
&+ \big[\hat{C}_{ijmn} - C_{ijmn} - C_{ijkl}\widetilde{\psi}_k(N_l^{mn}) - \widetilde{\psi}_k(C_{ikjl}N_l^{mn})\big]\psi_j(\varepsilon_{mn}^{(0)*})\\
&+ \Big[\psi_j\big(C_{ijkl}\widetilde{\psi}_k(P_l)\big) - \psi_j(\hat{\beta}_{ij}) + \widetilde{\psi}_j(C_{ijkl}E_{kl})\Big](T^{(0)} - \widetilde{T})\\
&+ \big[\beta_{ij} + C_{ijkl}\widetilde{\psi}_k(P_l) - \hat{\beta}_{ij} + \widetilde{\psi}_k(C_{ikjl}P_l)\big]\psi_j(T^{(0)} - \widetilde{T})\\
&+ \frac{\delta_{i1}}{r}\Big\{\big[\hat{C}_{11mn} - C_{11mn} - C_{11kl}\widetilde{\psi}_k(N_l^{mn})\big]\varepsilon_{mn}^{(0)*}\\
&+ \big[\beta_{11} - \hat{\beta}_{11} + C_{11kl}\widetilde{\psi}_k(P_l)\big](T^{(0)} - \widetilde{T})\Big\}\\
&- \frac{\delta_{i1}}{r}\Big\{\big[\hat{C}_{22mn} - C_{22mn} - C_{22kl}\widetilde{\psi}_k(N_l^{mn})\big]\varepsilon_{mn}^{(0)*}\\
&+ \big[\beta_{22} - \hat{\beta}_{22} + C_{22kl}\widetilde{\psi}_k(P_l)\big](T^{(0)} - \widetilde{T})\Big\}\\
&+ \frac{2\delta_{i2}}{r}\Big\{\big[\hat{C}_{12mn} - C_{12mn} - C_{12kl}\widetilde{\psi}_k(N_l^{mn})\big]\varepsilon_{mn}^{(0)*}\\
&+ \big[\beta_{12} - \hat{\beta}_{12} + C_{12kl}\widetilde{\psi}_k(P_l)\big](T^{(0)} - \widetilde{T})\Big\}\\
&+ \frac{\delta_{i3}}{r}\Big\{\big[\hat{C}_{31mn} - C_{31mn} - C_{31kl}\widetilde{\psi}_k(N_l^{mn})\big]\varepsilon_{mn}^{(0)*}\\
&+ \big[\beta_{31} - \hat{\beta}_{31} + C_{31kl}\widetilde{\psi}_k(P_l)\big](T^{(0)} - \widetilde{T})\Big\}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\widetilde{\psi}_i\big[k_{ij}\widetilde{\psi}_j(T^{(2)})\big] = -\big[\hat{S} - \rho c + \widetilde{T}\beta_{ij}\widetilde{\psi}_j(P_i)\big]\frac{\partial T^{(0)}}{\partial t}\\
&- \Big[\psi_i\big(k_{ij}\widetilde{\psi}_j(M_m)\big) - \psi_i(\hat{k}_{im}) + \widetilde{\psi}_i\big(k_{ij}\psi_j(M_m)\big)\Big]\psi_m(T^{(0)})\\
&- \big[k_{mn} - \hat{k}_{mn} + k_{mj}\widetilde{\psi}_j(M_n) + \widetilde{\psi}_j(k_{mj}M_n)\big]\psi_m\psi_n(T^{(0)})\\
&- \widetilde{T}\big[\hat{\beta}_{mn} - \beta_{mn} - \beta_{ij}\widetilde{\psi}_j(N_i^{mn})\big]\frac{\partial \varepsilon_{mn}^{(0)*}}{\partial t}\\
&- \frac{\big[k_{1m} - \hat{k}_{1m} + k_{1j}\widetilde{\psi}_j(M_m)\big]\psi_m(T^{(0)})}{r}
\end{aligned} \tag{32}
$$

According to (31) and (32), we construct

$$
\begin{cases}
u_i^{(2)} = N_i^{jmn}(r,\mathbf{y})\psi_j(\varepsilon_{mn}^{(0)*}) + H_i^j(r,\mathbf{y})\psi_j(T^{(0)}) + F_i^j(r,\mathbf{y})\dfrac{\partial^2 u_j^{(0)}}{\partial t^2}\\
\qquad + M_i^{mn}(r,\mathbf{y})\varepsilon_{mn}^{(0)*} + Q_i(r,\mathbf{y})(T^{(0)} - \widetilde{T}),\\
T^{(2)} = S(r,\mathbf{y})\dfrac{\partial T^{(0)}}{\partial t} + R_m(r,\mathbf{y})\psi_m(T^{(0)})\\
\qquad + M_{mn}(r,\mathbf{y})\psi_m\psi_n(T^{(0)}) + G_{mn}(r,\mathbf{y})\dfrac{\partial \varepsilon_{mn}^{(0)*}}{\partial t}.
\end{cases} \tag{33}
$$

where N_i^{jmn}, H_i^j, F_i^j, M_i^{mn}, Q_i, S, R_m, M_{mn} and G_{mn} are the second-order auxiliary cell functions defined in unit cell Y. Substituting (33) into (31) and (32), a series of equations, which are attached with the homogeneous Dirichlet boundary condition, are derived as follows

$$
\begin{cases}
\widetilde{\psi}_p\big[C_{ipkl}\widetilde{\psi}_k(N_l^{jmn})\big] = \hat{C}_{ijmn} - C_{ijmn}\\
- C_{ijkl}\widetilde{\psi}_k\left(N_l^{mn}\right) - \widetilde{\psi}_k\left(C_{ikjl}N_l^{mn}\right), & \mathbf{y}\in Y\\
N_l^{jmn}(r,\mathbf{y}) = 0, & \mathbf{y}\in\partial Y
\end{cases} \tag{34}
$$

$$
\begin{cases}
\widetilde{\psi}_p\big[C_{ipkl}\widetilde{\psi}_k(H_l^j)\big] = \beta_{ij} + C_{ijkl}\widetilde{\psi}_k\left(P_l\right)\\
- \hat{\beta}_{ij} + \widetilde{\psi}_k\left(C_{ikjl}P_l\right) + \widetilde{\psi}_k(\beta_{ik}M_j), & \mathbf{y}\in Y\\
H_l^j(r,\mathbf{y}) = 0, & \mathbf{y}\in\partial Y
\end{cases} \tag{35}
$$

$$
\begin{cases}
\widetilde{\psi}_p\big[C_{ipkl}\widetilde{\psi}_k(F_l^j)\big] = \delta_{ij}\left(\rho - \hat{\rho}\right), & \mathbf{y}\in Y\\
F_l^j(r,\mathbf{y}) = 0, & \mathbf{y}\in\partial Y
\end{cases} \tag{36}
$$

$$
\begin{cases}
\widetilde{\psi}_j\big[C_{ijkl}\widetilde{\psi}_k(M_l^{mn})\big] = \psi_j(\hat{C}_{ijmn}) - \psi_j\big[C_{ijkl}\widetilde{\psi}_k(N_l^{mn})\big]\\
- \widetilde{\psi}_j\big(C_{ijkl}D_{klmn}\big) + \dfrac{\delta_{i1}}{r}\Big\{\big[\hat{C}_{11mn} - C_{11mn} - C_{11kl}\widetilde{\psi}_k\left(N_l^{mn}\right)\big]\\
- \big[\hat{C}_{22mn} - C_{22mn} - C_{22kl}\widetilde{\psi}_k\left(N_l^{mn}\right)\big]\Big\}\\
+ \dfrac{2\delta_{i2}}{r}\big[\hat{C}_{12mn} - C_{12mn} - C_{12kl}\widetilde{\psi}_k(N_l^{mn})\big]\\
+ \dfrac{\delta_{i3}}{r}\big[\hat{C}_{31mn} - C_{31mn} - C_{31kl}\widetilde{\psi}_k(N_l^{mn})\big], & \mathbf{y}\in Y\\
M_l^{mn}(r,\mathbf{y}) = 0, & \mathbf{y}\in\partial Y
\end{cases} \tag{37}
$$

$$
\begin{cases}
\widetilde{\psi}_j\big[C_{ijkl}\widetilde{\psi}_k(Q_l)\big] = \psi_j\big(C_{ijkl}\widetilde{\psi}_k(P_l)\big) - \psi_j(\hat{\beta}_{ij}) + \widetilde{\psi}_j(C_{ijkl}E_{kl}) \\
\quad + \dfrac{\delta_{i1}}{r}\Big\{\big[\beta_{11} - \hat{\beta}_{11} + C_{11kl}\widetilde{\psi}_k(P_l)\big] - \big[\beta_{22} - \hat{\beta}_{22} + C_{22kl}\widetilde{\psi}_k(P_l)\big]\Big\} \\
\quad + \dfrac{2\delta_{i2}}{r}\big[\beta_{12} - \hat{\beta}_{12} + C_{12kl}\widetilde{\psi}_k(P_l)\big] + \dfrac{\delta_{i3}}{r}\big[\beta_{31} - \hat{\beta}_{31} + C_{31kl}\widetilde{\psi}_k(P_l)\big], & \mathbf{y} \in Y \\
Q_l(r, \mathbf{y}) = 0, & \mathbf{y} \in \partial Y
\end{cases} \tag{38}
$$

$$
\begin{cases}
\widetilde{\psi}_i\big[k_{ij}\widetilde{\psi}_j(S)\big] = -\hat{S} + \rho c - \widetilde{T}\beta_{ij}\widetilde{\psi}_j(P_i), & \mathbf{y} \in Y \\
S(r, \mathbf{y}) = 0, & \mathbf{y} \in \partial Y
\end{cases} \tag{39}
$$

$$
\begin{cases}
\widetilde{\psi}_i\big[k_{ij}\widetilde{\psi}_j(R_m)\big] = -\psi_i\big(k_{ij}\widetilde{\psi}_j(M_m)\big) + \psi_i(\hat{k}_{im}) \\
\quad - \widetilde{\psi}_i\big(k_{ij}\psi_j(M_m)\big) - \dfrac{\big[k_{1m} - \hat{k}_{1m} + k_{1j}\widetilde{\psi}_j(M_m)\big]}{r}, & \mathbf{y} \in Y \\
R_m(r, \mathbf{y}) = 0, & \mathbf{y} \in \partial Y
\end{cases} \tag{40}
$$

$$
\begin{cases}
\widetilde{\psi}_i\big[k_{ij}\widetilde{\psi}_j(M_{mn})\big] = -k_{mn} + \hat{k}_{mn} - k_{mj}\widetilde{\psi}_j(M_n) - \widetilde{\psi}_j(k_{mj}M_n), & \mathbf{y} \in Y \\
M_{mn}(r, \mathbf{y}) = 0, & \mathbf{y} \in \partial Y
\end{cases} \tag{41}
$$

$$
\begin{cases}
\widetilde{\psi}_i\big[k_{ij}\widetilde{\psi}_j(G_{mn})\big] = -\widetilde{T}\big[\hat{\beta}_{mn} - \beta_{mn} - \beta_{ij}\widetilde{\psi}_j(N_i^{mn})\big], & \mathbf{y} \in Y \\
G_{mn}(r, \mathbf{y}) = 0, & \mathbf{y} \in \partial Y
\end{cases} \tag{42}
$$

where $p = 1, 2, 3$.

Remark 3 According to Lax-Milgram theorem and the hypotheses (A)-(C), it is easy to prove that problems (21)-(23) and (34)-(42) have a unique solution for any fixed macro-coordinate r.

In conclusion, the following theorem is obtained based on SOTS analysis for multiscale problem (1).

Theorem 1. *The dynamic thermo-mechanical problems of composite structures with cylindrical periodicity have SOTS asymptotic expansion solutions as follows*

$$
\begin{aligned}
u_i^{\varepsilon}(\mathbf{x}, t) \cong\ & u_i^{(0)} + \varepsilon\big[N_i^{mn}(r, \mathbf{y})\varepsilon_{mn}^{(0)*} - P_i(r, \mathbf{y})(T^{(0)} - \widetilde{T})\big] \\
& + \varepsilon^2\Big[N_i^{jmn}(r, \mathbf{y})\psi_j(\varepsilon_{mn}^{(0)*}) + H_i^j(r, \mathbf{y})\psi_j(T^{(0)}) + F_i^j(r, \mathbf{y})\frac{\partial^2 u_j^{(0)}}{\partial t^2} \\
& + M_i^{mn}(r, \mathbf{y})\varepsilon_{mn}^{(0)*} + Q_i(r, \mathbf{y})(T^{(0)} - \widetilde{T})\Big]
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
T^{\varepsilon}(\mathbf{x}, t) \cong\ & T^{(0)} + \varepsilon M_m(r, \mathbf{y})\psi_m(T^{(0)}) + \varepsilon^2\Big[M_{mn}(r, \mathbf{y})\psi_m\psi_n(T^{(0)}) \\
& + S(r, \mathbf{y})\frac{\partial T^{(0)}}{\partial t} + R_m(r, \mathbf{y})\psi_m(T^{(0)}) + G_{mn}(r, \mathbf{y})\frac{\partial \varepsilon_{mn}^{(0)*}}{\partial t}\Big]
\end{aligned} \tag{44}
$$

where $u_i^{(0)}$ and $T^{(0)}$ are the solutions of the homogenized problem (27), and N_i^{mn}, P_i and M_m are the first-order auxiliary cell functions defined by (21)-(23), N_i^{jmn}, H_i^j, F_i^j, M_i^{mn}, Q_i, S, R_m, M_{mn} and G_{mn} are the second-order auxiliary cell functions defined by (34)-(42).

2.2. Error analysis in the pointwise sense. In this subsection, the specific error analysis of FOTS solutions and SOTS solutions in the pointwise sense is given. Firstly, the FOTS solutions $u_i^{(1\varepsilon)}$ and $T^{(1\varepsilon)}$, and SOTS solutions $u_i^{(2\varepsilon)}$ and $T^{(2\varepsilon)}$ for original problem (1) are defined as below

$$\begin{aligned} u_i^{(1\varepsilon)} &= u_i^{(0)} + \varepsilon u_i^{(1)},\ \ T^{(1\varepsilon)} = T^{(0)} + \varepsilon T^{(1)} \\ u_i^{(2\varepsilon)} &= u_i^{(0)} + \varepsilon u_i^{(1)} + \varepsilon^2 u_i^{(2)},\ \ T^{(2\varepsilon)} = T^{(0)} + \varepsilon T^{(1)} + \varepsilon^2 T^{(2)} \end{aligned} \tag{45}$$

Furthermore, the error functions of the FOTS solutions and SOTS solutions are defined as follows

$$\begin{aligned} u_{\Delta i}^{(1\varepsilon)} &= u_i^{\varepsilon} - u_i^{(1\varepsilon)},\ \ T_{\Delta}^{(1\varepsilon)} = T^{\varepsilon} - T^{(1\varepsilon)} \\ u_{\Delta i}^{(2\varepsilon)} &= u_i^{\varepsilon} - u_i^{(2\varepsilon)},\ \ T_{\Delta}^{(2\varepsilon)} = T^{\varepsilon} - T^{(2\varepsilon)} \end{aligned} \tag{46}$$

Before giving the detailed results of error analysis, some assumptions about original problem (1) need to be presented. Suppose that Ω is composed of the entire periodic cells, i.e. $\bar{\Omega} = \cup_{\mathbf{z}\in T_\varepsilon} \varepsilon(\mathbf{z} + \bar{Y})$, where the index set $T_\varepsilon = \{\mathbf{z} = (z_1, z_2, z_3) \in Z^3, \varepsilon(\mathbf{z} + \bar{Y}) \in \bar{\Omega}\}$. Besides, let $E_{\mathbf{z}} = \varepsilon(\mathbf{z} + Y)$ and $\partial E_{\mathbf{z}}$ be the boundary of $E_{\mathbf{z}}$.

To compare $u_i^{(1\varepsilon)}$ and $T^{(1\varepsilon)}$ with the exact solutions u_i^{ε} and T^{ε}, we substitute $u_{\Delta i}^{(1\varepsilon)}$ and $T_{\Delta}^{(1\varepsilon)}$ into original problem (1), and obtain the following residual equations of the FOTS solutions

$$\left\{\begin{aligned} &\rho^{\varepsilon}\frac{\partial^2 u_{\Delta i}^{(1\varepsilon)}}{\partial t^2} - \Big\{\psi_j \sigma_{ij}^{\varepsilon}(\mathbf{u}_{\Delta}^{(1\varepsilon)}, T_{\Delta}^{(1\varepsilon)}) + \delta_{i1}\big[\frac{\sigma_{11}^{\varepsilon}(\mathbf{u}_{\Delta}^{(1\varepsilon)}, T_{\Delta}^{1\varepsilon}) - \sigma_{22}^{\varepsilon}(\mathbf{u}_{\Delta}^{(1\varepsilon)}, T_{\Delta}^{(1\varepsilon)})}{r}\big] \\ &+ \delta_{i2}\frac{2\sigma_{12}^{\varepsilon}(\mathbf{u}_{\Delta}^{(1\varepsilon)}, T_{\Delta}^{(1\varepsilon)})}{r} + \delta_{i3}\frac{\sigma_{31}^{\varepsilon}(\mathbf{u}_{\Delta}^{(1\varepsilon)}, T_{\Delta}^{(1\varepsilon)})}{r}\Big\} \\ &= S_{0i}(\mathbf{x},\mathbf{y},t) + \varepsilon S_{1i}(\mathbf{x},\mathbf{y},t) \quad \text{in } \ \Omega\times(0,T], \\ &\rho^{\varepsilon}c^{\varepsilon}\frac{\partial T_{\Delta}^{(1\varepsilon)}}{\partial t} + \big[\psi_i q_i^{\varepsilon}(T_{\Delta}^{(1\varepsilon)}) + \frac{q_1^{\varepsilon}(T_{\Delta}^{(1\varepsilon)})}{r} + \widetilde{T}\beta_{ij}^{\varepsilon}\frac{\partial \varepsilon_{ij}^{\varepsilon}(\mathbf{u}_{\Delta}^{(1\varepsilon)})}{\partial t}\big] = F_0(\mathbf{x},\mathbf{y},t) \\ &+ \varepsilon F_1(\mathbf{x},\mathbf{y},t) \quad \text{in } \ \Omega\times(0,T], \\ &\mathbf{u}_{\Delta}^{(1\varepsilon)}(\mathbf{x},t) = \mathbf{0},\ \ T_{\Delta}^{(1\varepsilon)}(\mathbf{x},t) = 0 \quad \text{on } \ \partial\Omega\times(0,T], \\ &u_{\Delta i}^{(1\varepsilon)}(\mathbf{x},0) = -\varepsilon N_i^{mn}\varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}(\mathbf{x},0)\big) = \varepsilon\widehat{\psi}_{1i}(\mathbf{x}), \\ &\frac{\partial u_{\Delta i}^{(1\varepsilon)}}{\partial t}\Big|_{t=0} = -\varepsilon N_i^{mn}\frac{\partial \varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}\big)}{\partial t}\Big|_{t=0} + \varepsilon P_i\frac{\partial T^{(0)}}{\partial t}\Big|_{t=0} = \varepsilon\widehat{\psi}_{2i}(\mathbf{x}), \\ &T_{\Delta}^{(1\varepsilon)}(\mathbf{x},0) = -\varepsilon M_m\psi_m\big(T^{(0)}(\mathbf{x},0)\big) = \varepsilon\widehat{\psi}_3(\mathbf{x}) \quad \text{in } \ \Omega. \end{aligned}\right. \tag{47}$$

where the operators $\sigma_{ij}^{\varepsilon}(\mathbf{u}_{\Delta}^{(1\varepsilon)}, T_{\Delta}^{(1\varepsilon)}) = C_{ijkl}^{\varepsilon}\varepsilon_{kl}^{\varepsilon}(\mathbf{u}_{\Delta}^{(1\varepsilon)}) - \beta_{ij}^{\varepsilon}T_{\Delta}^{(1\varepsilon)}$ and $q_i^{\varepsilon}(T_{\Delta}^{(1\varepsilon)}) = -k_{ij}^{\varepsilon}\psi_j(T_{\Delta}^{(1\varepsilon)})$.

Analogously, substituting $u_{\Delta i}^{(2\varepsilon)}$ and $T_{\Delta}^{(2\varepsilon)}$ into original problem (1), we derive the following residual equations of the SOTS solutions

(48)
$$\left\{\begin{aligned}
&\rho^{\varepsilon}\frac{\partial^2 u_{\Delta i}^{(2\varepsilon)}}{\partial t^2}-\Big\{\psi_j\sigma_{ij}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)},T_{\Delta}^{(2\varepsilon)})+\delta_{i1}\big[\frac{\sigma_{11}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)},T_{\Delta}^{(2\varepsilon)})-\sigma_{22}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)},T_{\Delta}^{(2\varepsilon)})}{r}\big]\\
&+\delta_{i2}\frac{2\sigma_{12}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)},T_{\Delta}^{(2\varepsilon)})}{r}+\delta_{i3}\frac{\sigma_{31}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)},T_{\Delta}^{(2\varepsilon)})}{r}\Big\}=\varepsilon H_i(\mathbf{x},\mathbf{y},t)\text{ in }\Omega\times(0,T],\\
&\rho^{\varepsilon}c^{\varepsilon}\frac{\partial T_{\Delta}^{(2\varepsilon)}}{\partial t}+\big[\psi_i q_i^{\varepsilon}(T_{\Delta}^{(2\varepsilon)})+\frac{q_1^{\varepsilon}(T_{\Delta}^{(2\varepsilon)})}{r}+\widetilde{T}\beta_{ij}^{\varepsilon}\frac{\partial\varepsilon_{ij}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)})}{\partial t}\big]\\
&\qquad\qquad=\varepsilon G(\mathbf{x},\mathbf{y},t)\quad\text{in }\ \Omega\times(0,T],\\
&\mathbf{u}_{\Delta}^{(2\varepsilon)}(\mathbf{x},t)=\mathbf{0},\ \ T_{\Delta}^{(2\varepsilon)}(\mathbf{x},t)=0\quad\text{on }\ \partial\Omega\times(0,T],\\
&u_{\Delta i}^{(2\varepsilon)}(\mathbf{x},0)=-\varepsilon N_i^{mn}\frac{\partial\varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}\big)}{\partial t}\Big|_{t=0}-\varepsilon^2\Big\{N_i^{jmn}\psi_j\Big[\varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}(\mathbf{x},0)\big)\Big]\\
&+H_i^j\psi_j\big(T^{(0)}(\mathbf{x},0)\big)+F_i^j\frac{\partial^2 u_j^{(0)}}{\partial t^2}+M_i^{mn}\varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}(\mathbf{x},0)\big)\Big\}=\varepsilon\tilde{\psi}_{1i}(\mathbf{x}),\\
&\frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t}\Big|_{t=0}=-\varepsilon\Big[N_i^{mn}\frac{\partial\varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}\big)}{\partial t}\Big|_{t=0}-P_i\frac{\partial T^{(0)}}{\partial t}\Big|_{t=0}\Big]\\
&-\varepsilon^2\Big\{N_i^{jmn}\frac{\psi_j\Big[\varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}(\mathbf{x})\big)\Big]}{\partial t}+H_i^j\frac{\partial\psi_j(T^{(0)})}{\partial t}\Big|_{t=0}+F_i^j\frac{\partial^3 u_j^{(0)}}{\partial t^3}\Big|_{t=0}\\
&+M_i^{mn}\frac{\partial\varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}\big)}{\partial t}\Big|_{t=0}+Q_i\frac{\partial T^{(0)}}{\partial t}\Big|_{t=0}\Big\}=\varepsilon\tilde{\psi}_{2i}(\mathbf{x}),\\
&T_{\Delta}^{(2\varepsilon)}(\mathbf{x},0)=-\varepsilon M_m\psi_m\big(T^{(0)}(\mathbf{x},0)\big)-\varepsilon^2\Big[S\frac{\partial T^{(0)}}{\partial t}\Big|_{t=0}+R_m\psi_m\big(T^{(0)}(\mathbf{x},0)\big)\\
&+M_{mn}\psi_m\psi_n\big(T^{(0)}(\mathbf{x},0)\big)+G_{mn}\frac{\partial\varepsilon_{mn}^{(0)*}\big(\mathbf{u}^{(0)}\big)}{\partial t}\Big|_{t=0}\Big]=\varepsilon\tilde{\psi}_3(\mathbf{x})\quad\text{in }\ \Omega.
\end{aligned}\right.$$

where the detailed expressions of S_{0i}, S_{1i}, F_0, F_1 and H_i, G are not given in this paper because they are tediously long. However, it is easy to get their specific forms and worth noting that the highest order terms of $H_i(\mathbf{x},\mathbf{y},t)$ and $G(\mathbf{x},\mathbf{y},t)$ are $\psi_i\psi_j\psi_k\psi_l(\mathbf{u}^{(0)})$, $\dfrac{\partial^4\mathbf{u}^{(0)}}{\partial t^4}$, $\psi_i\psi_j\psi_k\psi_l(T^{(0)})$ and $\dfrac{\partial^2 T^{(0)}}{\partial t^2}$.

Now we can give a conclusion about the error analysis in the pointwise sense. From the residual equation (47), one can easily find that the residual of FOTS solutions is order $O(1)$ in the pointwise sense due to the terms $S_{0i}(\mathbf{x},\mathbf{y},t)$ and $F_0(\mathbf{x},\mathbf{y},t)$. In addition, it is clear to see that the residual of SOTS solutions is order $O(\varepsilon)$ in the pointwise sense from the residual equation (48). This means that the SOTS solutions can satisfy the original equation (1) in the pointwise sense. Thus even ε is a small constant, the SOTS solutions can still provide the required accuracy of engineering application and capture the micro-scale oscillating behavior of composite materials. This is the main reason and motivation to develop the SOTS solutions.

3. Main convergence theorem and its proof

In this section, the specific proof of the explicit convergence rate of the SOTS solutions in the integral sense is presented. It is known to all that the classical auxiliary cell functions are defined with periodic boundary condition [7, 12, 20, 21].

At this case, the auxiliary cell functions have enough regularity on the boundary of unit cell Y. However, the auxiliary cell functions in this paper are all imposed with homogeneous Dirichlet boundary condition. For this kind of auxiliary cell functions, the normal derivatives only are continuous on the boundary of unit cell Y under the geometric symmetry and regularity assumptions of material property parameters. So we firstly give some hypotheses similar to literatures [3, 6, 22–24] as follows

(i) $\rho(\mathbf{y}),\ C_{ijkl}(\mathbf{y}),\ \beta_{ij}(\mathbf{y}),\ c(\mathbf{y})$ and $k_{ij}(\mathbf{y})$ are the functions with piecewise constants in Y.
(ii) Let $\Delta_1 \dots \Delta_3$ be the middle hyperplanes of the reference cell $Y = (0,1)^3$. Assume that $\rho(\mathbf{y}),\ C_{ijij}(\mathbf{y}),\ \beta_{ii}(\mathbf{y}),\ c(\mathbf{y})\ ,k_{ii}(\mathbf{y})$ are symmetric with respect to $\Delta_1 \dots \Delta_3$ and $C_{ijkl}(\mathbf{y}),\ \beta_{ij}(\mathbf{y}),\ k_{ij}(\mathbf{y})$ are anti-symmetric with respect to $\Delta_1 \dots \Delta_3$ in Y.

Lemma 1. *Denote the operators $\sigma_{iY}(\boldsymbol{\chi}) = n_j C_{ijkl}(\mathbf{y})\varepsilon_{kl}^{(-1)}(\boldsymbol{\chi})$ and $\sigma_{TY}(\phi) = n_i k_{ij}(\mathbf{y})\tilde{\psi}_j(\phi)$, where $n_1 = r\cdot n_r$, $n_2 = n_\theta$ and $n_3 = r\cdot n_z$. Then under assumptions (A)-(C) and (i)-(ii), the normal derivatives $\sigma_{iY}(\mathbf{N}^{mn})$, $\sigma_{iY}(\mathbf{P})$, $\sigma_{iY}(\mathbf{N}^{jmn})$, $\sigma_{iY}(\mathbf{H}^j)$, $\sigma_{iY}(\mathbf{F})$, $\sigma_{iY}(\mathbf{M}^{mn})$, $\sigma_{iY}(\mathbf{Q})$ and $\sigma_{TY}(M_m)$, $\sigma_{TY}(S)$, $\sigma_{TY}(R_m)$, $\sigma_{TY}(M_{mn})$, $\sigma_{TY}(G_{mn})$ can be proved to be continuous on the boundary of unit cell Y by using the same method in Refs. [6, 23, 24].*

Lemma 2. *In [16–18], the following Korn's inequality holds for curvilinear coordinates*

$$\|\mathbf{u}\|_{H_0^1(\Omega)} \le C\Big\{\sum_{i,j} \big|e_{i||j}(\mathbf{u})\big|_{L^2(\Omega)}^2\Big\}^{\frac{1}{2}}$$

where $e_{i||j}(\mathbf{u})$ represents the strain in curvilinear coordinate system. Then it is easy to know that this inequality still holds for cylindrical coordinates because cylindrical coordinate is a special curvilinear coordinate.

Theorem 2. *Suppose that $\Omega \subset \mathbb{R}^3$ is the union of entire periodic cells, i.e. $\bar{\Omega} = \cup_{\mathbf{z}\in T_\varepsilon}\varepsilon(\mathbf{z}+\bar{Y})$, where the index set $T_\varepsilon = \{\mathbf{z}\in Z^3, \varepsilon(\mathbf{z}+\bar{Y})\in\bar{\Omega}\}$. Let $\mathbf{u}^\varepsilon(\mathbf{x},t)$, $T^\varepsilon(\mathbf{x},t)$ and $\mathbf{u}^{(0)}(\mathbf{x},t)$, $T^{(0)}(\mathbf{x},t)$ be the weak solutions of model problem (1) and associated homogenized problem (27), respectively. The specific expressions of SOTS solutions are defined in Theorem 1. Under the assumptions (A)-(C), (i)-(ii), Lemma 1 and 2, if $\mathbf{u}^{(0)} \in L^\infty(0,T;(H^4(\Omega))^3)$, $\dfrac{\partial \mathbf{u}^{(0)}}{\partial t} \in L^\infty(0,T;(H^3(\Omega))^3)$, $\dfrac{\partial^2 \mathbf{u}^{(0)}}{\partial t^2} \in L^\infty(0,T;(H^2(\Omega))^3)$, $\dfrac{\partial^3 \mathbf{u}^{(0)}}{\partial t^3} \in L^\infty(0,T;(H^1(\Omega))^3)$, $\dfrac{\partial^4 \mathbf{u}^{(0)}}{\partial t^4} \in L^\infty(0,T;(L^2(\Omega))^3)$, $T^{(0)} \in L^\infty(0,T;H^4(\Omega))$, $\dfrac{\partial T^{(0)}}{\partial t} \in L^\infty(0,T;H^2(\Omega))$, $\dfrac{\partial^2 T^{(0)}}{\partial t^2} \in L^\infty(0,T;H^1(\Omega))$; then we have the following error estimate of SOTS solutions*

$$\begin{aligned}\Big\|\frac{\partial \mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t)}{\partial t}\Big\|_{L^\infty(0,T;(L^2(\Omega))^3)} &+ \big\|\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t)\big\|_{L^\infty(0,T;(H_0^1(\Omega))^3)} \\ &+ \big\|T_\Delta^{(2\varepsilon)}(\mathbf{x},t)\big\|_{L^\infty(0,T;L^2(\Omega))} + \big\|T_\Delta^{(2\varepsilon)}(\mathbf{x},t)\big\|_{L^2(0,T;H_0^1(\Omega))} \le C(T)\varepsilon \end{aligned} \tag{49}$$

where $C(T)$ is a constant independent of ε, but dependent of T.
Proof : Firstly, the following two equalities can be obtained from (43) and (44)

$$
\begin{aligned}
&\sigma_i(\mathbf{u}^{(2\varepsilon)}) = n_j C^{\varepsilon}_{ijkl}(\mathbf{x})\varepsilon^{\varepsilon}_{kl}(\mathbf{u}^{(2\varepsilon)})\\
&= n_j C_{ijkl}(\mathbf{y})\Big\{\varepsilon^{-1}\widetilde{\psi}_k(u_l^{(0)}) + \varepsilon^0\Big[\varepsilon^{(0)*}_{kl}(\mathbf{u}^{(0)}) + \widetilde{\psi}_k(u_l^{(1)})\Big]\\
&+ \varepsilon^1\Big[\varepsilon^{(1)*}_{kl}(\mathbf{u}^{(1)}) + \widetilde{\psi}_k(u_l^{(2)})\Big]\Big\}\\
&= n_j C_{ijkl}(\mathbf{y})\Big\{\varepsilon^0\Big[\varepsilon^{(0)*}_{kl}(\mathbf{u}^{(0)}) + \widetilde{\psi}_k\big(N_l^{mn}\varepsilon^{(0)*}_{mn} - P_l(T^{(0)} - \widetilde{T})\big)\Big]\\
&+ \varepsilon^1\Big[\varepsilon^{(1)*}_{kl}\big(\mathbf{N}^{mn}\varepsilon^{(0)*}_{mn} - \mathbf{P}(T^{(0)} - \widetilde{T})\big) + \widetilde{\psi}_k\big(N_l^{jmn}\psi_j(\varepsilon^{(0)*}_{mn}) + H_l^j\psi_j(T^{(0)})\\
&+ F_l^j\frac{\partial^2 u_j^{(0)}}{\partial t^2} + M_l^{mn}\varepsilon^{(0)*}_{mn} + Q_l(T^{(0)} - \widetilde{T})\big)\Big]\Big\}\\
&= n_j C_{ijkl}(\mathbf{y})\Big\{\Big[\varepsilon^{(0)*}_{kl}(\mathbf{u}^{(0)}) + \widetilde{\psi}_k\big(N_l^{mn}\big)\varepsilon^{(0)*}_{mn} - \widetilde{\psi}_k\big(P_l\big)(T^{(0)} - \widetilde{T})\Big]\\
&+ \varepsilon^1\Big[\varepsilon^{(1)*}_{kl}\big(\mathbf{N}^{mn}\varepsilon^{(0)*}_{mn} - \mathbf{P}(T^{(0)} - \widetilde{T})\big) + \widetilde{\psi}_k\big(N_l^{jmn}\big)\psi_j(\varepsilon^{(0)*}_{mn})\\
&+ \widetilde{\psi}_k\big(H_l^j\big)\psi_j(T^{(0)}) + \widetilde{\psi}_k\big(F_l^j\big)\frac{\partial^2 u_j^{(0)}}{\partial t^2} + \widetilde{\psi}_k\big(M_l^{mn}\big)\varepsilon^{(0)*}_{mn}\\
&+ \widetilde{\psi}_k\big(Q_l\big)(T^{(0)} - \widetilde{T})\Big]\Big\}\\
&= n_j C_{ijkl}(\mathbf{y})\Big[\varepsilon^{(0)*}_{kl}(\mathbf{u}^{(0)}) + \varepsilon^1\varepsilon^{(1)*}_{kl}\big(\mathbf{N}^{mn}\varepsilon^{(0)*}_{mn} - \mathbf{P}(T^{(0)} - \widetilde{T})\big)\Big]\\
&+ \sigma_{iY}(\mathbf{N}^{mn})\varepsilon^{(0)*}_{mn} - \sigma_{iY}(\mathbf{P})(T^{(0)} - \widetilde{T}) + \varepsilon\sigma_{iY}(\mathbf{N}^{jmn})\psi_j(\varepsilon^{(0)*}_{mn})\\
&+ \varepsilon\sigma_{iY}(\mathbf{H}^j)\psi_j(T^{(0)}) + \varepsilon\sigma_{iY}(\mathbf{F})\frac{\partial^2\mathbf{u}^{(0)}}{\partial t^2} + \varepsilon\sigma_{iY}(\mathbf{M}^{mn})\varepsilon^{(0)*}_{mn}\\
&+ \varepsilon\sigma_{iY}(\mathbf{Q})(T^{(0)} - \widetilde{T})
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
&\sigma_T(T^{(2\varepsilon)}) = n_j k^{\varepsilon}_{ij}(\mathbf{x})\psi_j(T^{(2\varepsilon)})\\
&= n_j k_{ij}(\mathbf{y})(\psi_j + \varepsilon^{-1}\tilde{\psi}_j)\big[T^{(0)} + \varepsilon M^m(r,\mathbf{y})\psi_m(T^{(0)}) + \varepsilon^2 S(r,\mathbf{y})\frac{\partial T^{(0)}}{\partial t}\\
&+ \varepsilon^2 R^m(r,\mathbf{y})\psi_m(T^{(0)}) + \varepsilon^2 M^{mn}(r,\mathbf{y})\psi_m\psi_n(T^{(0)}) + \varepsilon^2 G^{mn}(r,\mathbf{y})\frac{\partial\varepsilon^{(0)*}_{mn}}{\partial t}\big]\\
&= n_j k_{ij}(\mathbf{y})\psi_j\big[T^{(0)} + \varepsilon M^m\psi_m(T^{(0)}) + \varepsilon^2 S\frac{\partial T^{(0)}}{\partial t} + \varepsilon^2 R^m\psi_m(T^{(0)})\\
&+ \varepsilon^2 M^{mn}\psi_m\psi_n(T^{(0)}) + \varepsilon^2 G^{mn}\frac{\partial\varepsilon^{(0)*}_{mn}}{\partial t}\big] + \big[\sigma_{TY}(M^m)\psi_m(T^{(0)})\\
&+ \varepsilon\sigma_{TY}(S)\frac{\partial T^{(0)}}{\partial t} + \varepsilon\sigma_{TY}(R^m)\psi_m(T^{(0)})\\
&+ \varepsilon\sigma_{TY}(M^{mn})\psi_m\psi_n(T^{(0)}) + \varepsilon\sigma_{TY}(G^{mn})\frac{\partial\varepsilon^{(0)*}_{mn}}{\partial t}\big]
\end{aligned}
\tag{51}
$$

These two formulas will be used in the following proof process.

Secondly, we use the residual equation (48) to complete the error estimate. Since $\dfrac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t}$ is in $L^{\infty}(0,T;L^2(\Omega))$, $\dfrac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t}$ cannot be directly used as test function for residual equation (48). To overcome this difficulty, we use the density argument [21,22]. In order to simplify the process of proof, this process is omitted. We assume that $r \in [r_1, r_2]$, where r_1 and r_2 denote the inner and outer radius of composite structures, respectively. Then, multiplying by $\dfrac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r$ and $T_{\Delta}^{(2\varepsilon)} r$ on both sides of (48) and integrating on Ω, the following equations are derived

(52)

$$\begin{cases} \displaystyle\int_{\Omega} \rho^{\varepsilon} \frac{\partial^2 u_{\Delta i}^{(2\varepsilon)}}{\partial t^2} \frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Omega - \int_{\Omega} \Big\{ \psi_j \sigma_{ij}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)}, T_{\Delta}^{(2\varepsilon)}) \\ +\delta_{i1} [\dfrac{\sigma_{11}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)}, T_{\Delta}^{(2\varepsilon)}) - \sigma_{22}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)}, T_{\Delta}^{(2\varepsilon)})}{r}] \\ + \delta_{i2} \dfrac{2\sigma_{12}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)}, T_{\Delta}^{(2\varepsilon)})}{r} + \delta_{i3} \dfrac{\sigma_{31}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)}, T_{\Delta}^{(2\varepsilon)})}{r} \Big\} \dfrac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Omega \\ = \displaystyle\int_{\Omega} \varepsilon H_i(\mathbf{x}, \mathbf{y}, t) \frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Omega \\ \displaystyle\int_{\Omega} \rho^{\varepsilon} c^{\varepsilon} \frac{\partial T_{\Delta}^{(2\varepsilon)}}{\partial t} T_{\Delta}^{(2\varepsilon)} r d\Omega + \int_{\Omega} \Big[\psi_i q_i^{\varepsilon}(T_{\Delta}^{(2\varepsilon)}) + \frac{q_1^{\varepsilon}(T_{\Delta}^{(2\varepsilon)})}{r} + \widetilde{T} \beta_{ij}^{\varepsilon} \frac{\partial \varepsilon_{ij}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)})}{\partial t} \Big] T_{\Delta}^{(2\varepsilon)} r d\Omega \\ = \displaystyle\int_{\Omega} \varepsilon G(\mathbf{x}, \mathbf{y}, t) T_{\Delta}^{(2\varepsilon)} r d\Omega \end{cases}$$

Using the Green's formula and integrating by parts on (52), the above identity can be simplified as follows

(53)

$$\begin{cases} \displaystyle\int_{\Omega} \rho^{\varepsilon} \frac{\partial^2 u_{\Delta i}^{(2\varepsilon)}}{\partial t^2} \frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Omega + \int_{\Omega} \Big[C_{ijkl}^{\varepsilon} \varepsilon_{kl}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)}) - \beta_{ij}^{\varepsilon} T_{\Delta}^{(2\varepsilon)} \Big] \varepsilon_{ij}^{\varepsilon}(\frac{\partial \mathbf{u}_{\Delta}^{(2\varepsilon)}}{\partial t}) r d\Omega \\ = \displaystyle\int_{\Omega} \varepsilon H_i \frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Omega + \int_{\cup_{\mathbf{z} \in T_{\varepsilon}} \partial E_{\mathbf{z}}} \Phi_i \frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Gamma_{\mathbf{y}} \\ \displaystyle\int_{\Omega} \rho^{\varepsilon} c^{\varepsilon} \frac{\partial T_{\Delta}^{(2\varepsilon)}}{\partial t} T_{\Delta}^{(2\varepsilon)} r d\Omega - \int_{\Omega} q_i^{\varepsilon}(T_{\Delta}^{(2\varepsilon)}) \psi_i(T_{\Delta}^{(2\varepsilon)}) r d\Omega + \int_{\Omega} \widetilde{T} \beta_{ij}^{\varepsilon} \frac{\partial \varepsilon_{ij}^{\varepsilon}(\mathbf{u}_{\Delta}^{(2\varepsilon)})}{\partial t} T_{\Delta}^{(2\varepsilon)} r d\Omega \\ = \displaystyle\int_{\Omega} \varepsilon G(\mathbf{x}, \mathbf{y}, t) T_{\Delta}^{(2\varepsilon)} r d\Omega + \int_{\cup_{\mathbf{z} \in T_{\varepsilon}} \partial E_{\mathbf{z}}} \varphi T_{\Delta}^{(2\varepsilon)} r d\Gamma_{\mathbf{y}} \end{cases}$$

where Φ_i and φ result from using the Green's formula on $\partial E_{\mathbf{z}}$.

Combining (50), (51) and Lemma 1 together, we can obtain

(54)
$$\begin{cases} \langle \Phi_i, \dfrac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} \rangle = \displaystyle\int_{\cup_{\mathbf{z}\in T_\varepsilon}\partial E_{\mathbf{z}}} \Phi_i \dfrac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Gamma_{\mathbf{y}} = \sum_{\mathbf{z}\in T_\varepsilon}\int_{\partial E_{\mathbf{z}}} \sigma_i(\mathbf{u}^\varepsilon - \mathbf{u}^{(2\varepsilon)}) \dfrac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} d\Gamma_{\mathbf{y}} \\ \qquad = -\displaystyle\sum_{\mathbf{z}\in T_\varepsilon}\int_{\partial E_{\mathbf{z}}} \sigma_i(\mathbf{u}^{(2\varepsilon)}) \dfrac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} d\Gamma_{\mathbf{y}} = 0 \\ \langle \varphi, T_{\Delta}^{(2\varepsilon)} \rangle = \displaystyle\int_{\cup_{\mathbf{z}\in T_\varepsilon}\partial E_{\mathbf{z}}} \varphi T_{\Delta}^{(2\varepsilon)} r d\Gamma_{\mathbf{y}} = \sum_{\mathbf{z}\in T_\varepsilon}\int_{\partial E_{\mathbf{z}}} \sigma_T(T^\varepsilon - T^{(2\varepsilon)}) T_{\Delta}^{(2\varepsilon)} d\Gamma_{\mathbf{y}} \\ \qquad = -\displaystyle\sum_{\mathbf{z}\in T_\varepsilon}\int_{\partial E_{\mathbf{z}}} \sigma_T(T^{(2\varepsilon)}) T_{\Delta}^{(2\varepsilon)} d\Gamma_{\mathbf{y}} = 0 \end{cases}$$

Afterwards, it is easy to derive the following two identities by combining (53) and (54)

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\Big[\int_\Omega \rho^\varepsilon (\frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t})^2 r d\Omega + \int_\Omega C_{ijkl}^\varepsilon \varepsilon_{kl}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}) \varepsilon_{ij}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}) r d\Omega\Big] \\ &= \int_\Omega \varepsilon H_i \frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Omega + \int_\Omega \beta_{ij}^\varepsilon T_\Delta^{(2\varepsilon)} \varepsilon_{ij}^\varepsilon (\frac{\partial \mathbf{u}_\Delta^{(2\varepsilon)}}{\partial t}) r d\Omega \end{aligned} \tag{55}$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\Big[\int_\Omega \rho^\varepsilon c^\varepsilon (T_\Delta^{(2\varepsilon)})^2 r d\Omega\Big] - \int_\Omega q_i^\varepsilon (T_\Delta^{(2\varepsilon)}) \psi_i (T_\Delta^{(2\varepsilon)}) r d\Omega \\ &= \int_\Omega \varepsilon G(\mathbf{x},\mathbf{y},t) T_\Delta^{(2\varepsilon)} r d\Omega - \int_\Omega \widetilde{T} \beta_{ij}^\varepsilon \frac{\partial \varepsilon_{ij}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)})}{\partial t} T_\Delta^{(2\varepsilon)} r d\Omega \end{aligned} \tag{56}$$

Then, the following equation is obtained by taking the sum of (56) and the product of (55) and initial temperature $\widetilde{T}$

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\Big[\int_\Omega \rho^\varepsilon (\frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t})^2 \widetilde{T} r d\Omega + \int_\Omega C_{ijkl}^\varepsilon \varepsilon_{kl}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}) \varepsilon_{ij}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}) \widetilde{T} r d\Omega \\ &+ \int_\Omega \rho^\varepsilon c^\varepsilon (T_\Delta^{(2\varepsilon)})^2 r d\Omega\Big] - \int_\Omega q_i^\varepsilon (T_\Delta^{(2\varepsilon)}) \psi_i (T_\Delta^{(2\varepsilon)}) r d\Omega = \int_\Omega \varepsilon \widetilde{T} H_i \frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial t} r d\Omega \\ &+ \int_\Omega \varepsilon G T_\Delta^{(2\varepsilon)} r d\Omega \end{aligned} \tag{57}$$

Subsequently, we integrate both sides of (57) from 0 to t ($0 < t \leq T$) and it follows that

(58)
$$\begin{aligned} &\Big[\int_\Omega \rho^\varepsilon (\frac{\partial u_{\Delta i}^{(2\varepsilon)}(\mathbf{x},t)}{\partial t})^2 \widetilde{T} r d\Omega + \int_\Omega C_{ijkl}^\varepsilon \varepsilon_{kl}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t)) \varepsilon_{ij}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t)) \widetilde{T} r d\Omega \\ &+ \int_\Omega \rho^\varepsilon c^\varepsilon (T_\Delta^{(2\varepsilon)}(\mathbf{x},t))^2 r d\Omega\Big] - 2\int_0^t \int_\Omega q_i^\varepsilon (T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau)) \psi_i (T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau)) r d\Omega d\tau \\ &= 2\int_0^t \int_\Omega \varepsilon \widetilde{T} H_i \frac{\partial u_{\Delta i}^{(2\varepsilon)}(\mathbf{x},\tau)}{\partial \tau} r d\Omega d\tau + 2\int_0^t \int_\Omega \varepsilon G T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau) r d\Omega d\tau \\ &+ \Big[\int_\Omega \rho^\varepsilon (\frac{\partial u_{\Delta i}^{(2\varepsilon)}(\mathbf{x},t)}{\partial t}\Big|_{t=0})^2 \widetilde{T} r d\Omega + \int_\Omega C_{ijkl}^\varepsilon \varepsilon_{kl}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},0)) \varepsilon_{ij}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},0)) \widetilde{T} r d\Omega \\ &+ \int_\Omega \rho^\varepsilon c^\varepsilon (T_\Delta^{(2\varepsilon)}(\mathbf{x},0))^2 r d\Omega\Big] \end{aligned}$$

Substituting the initial conditions and boundary conditions of residual equation (48) into the above identity (58), the following equality is derived

$$\begin{aligned}
&\Big[\int_\Omega \rho^\varepsilon\big(\frac{\partial u_{\Delta i}^{(2\varepsilon)}(\mathbf{x},t)}{\partial t}\big)^2\widetilde{T}rd\Omega+\int_\Omega C_{ijkl}^\varepsilon\varepsilon_{kl}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t))\varepsilon_{ij}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t))\widetilde{T}rd\Omega\\
&+\int_\Omega \rho^\varepsilon c^\varepsilon(T_\Delta^{(2\varepsilon)}(\mathbf{x},t))^2rd\Omega\Big]-2\int_0^t\int_\Omega q_i^\varepsilon(T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau))\psi_i(T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau))rd\Omega d\tau\\
&=2\int_0^t\int_\Omega \varepsilon\widetilde{T}H_i\frac{\partial u_{\Delta i}^{(2\varepsilon)}(\mathbf{x},\tau)}{\partial\tau}rd\Omega d\tau+2\int_0^t\int_\Omega \varepsilon G T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau)rd\Omega d\tau\\
&+\Big[\int_\Omega \rho^\varepsilon(\varepsilon\tilde{\psi}_{2i}(\mathbf{x}))^2\widetilde{T}rd\Omega+\int_\Omega C_{ijkl}^\varepsilon\varepsilon_{kl}^\varepsilon(\varepsilon\tilde{\boldsymbol{\psi}}_1(\mathbf{x}))\varepsilon_{ij}^\varepsilon(\varepsilon\tilde{\boldsymbol{\psi}}_1(\mathbf{x}))\widetilde{T}rd\Omega\\
&+\int_\Omega \rho^\varepsilon c^\varepsilon(\varepsilon\tilde{\psi}_3(\mathbf{x}))^2rd\Omega\Big]
\end{aligned}\tag{59}$$

Until now, the vital identity (59) for Theorem 2 are gained. Starting from here, we will get the final proof.

Owing to (A) and (B), and using Poincar$\acute{e}$-Friedrichs inequality in curvilinear coordinates [16–18], Lemma 2 and $r\in[r_1,r_2]$, it is easy to acquire the following inequality by transforming left side of equation (59)

(60)

$$\begin{aligned}
&\int_\Omega \rho^\varepsilon\big(\frac{\partial u_{\Delta i}^{(2\varepsilon)}(\mathbf{x},t)}{\partial t}\big)^2\widetilde{T}rd\Omega+\int_\Omega C_{ijkl}^\varepsilon\varepsilon_{kl}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t))\varepsilon_{ij}^\varepsilon(\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t))\widetilde{T}rd\Omega\\
&+\int_\Omega \rho^\varepsilon c^\varepsilon(T_\Delta^{(2\varepsilon)}(\mathbf{x},t))^2rd\Omega-2\int_0^t\int_\Omega q_i^\varepsilon(T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau))\psi_i(T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau))rd\Omega d\tau\\
&\geq\rho^0\widetilde{T}r_1\Big\|\frac{\partial\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t)}{\partial t}\Big\|_{(L^2(\Omega))^3}^2+\widetilde{T}r_1C_1\Big\|\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t)\Big\|_{(H_0^1(\Omega))^3}^2\\
&+\rho^0c^0r_1\Big\|T_\Delta^{(2\varepsilon)}(\mathbf{x},t)\Big\|_{L^2(\Omega)}^2+r_1C_2\int_0^t\Big\|T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau)\Big\|_{H_0^1(\Omega)}^2d\tau\\
&\geq\lambda_1\Big(\Big\|\frac{\partial\mathbf{u}_\Delta^{(2\varepsilon)}}{\partial t}\Big\|_{(L^2(\Omega))^3}^2+\Big\|\mathbf{u}_\Delta^{(2\varepsilon)}\Big\|_{(H_0^1(\Omega))^3}^2+\Big\|T_\Delta^{(2\varepsilon)}\Big\|_{L^2(\Omega)}^2+\int_0^t\Big\|T_\Delta^{(2\varepsilon)}\Big\|_{H_0^1(\Omega)}^2d\tau\Big)
\end{aligned}$$

where C_1 and C_2 result from Korn's inequality and Poincar$\acute{e}$-Friedrichs inequality in curvilinear coordinates respectively, $\lambda_1=min(\rho^0\widetilde{T}r_1,\widetilde{T}r_1C_1,\rho^0c^0r_1,r_1C_2)$.

After that, using Schwarz's inequality and Young's inequality, we obtain the following inequality by transforming the right side of equation (59)

(61)

$$
\begin{aligned}
&2\int_0^t\int_\Omega \varepsilon\widetilde{T}H_i\frac{\partial u_{\Delta i}^{(2\varepsilon)}(\mathbf{x},\tau)}{\partial t}rd\Omega d\tau \\
&+2\int_0^t\int_\Omega \varepsilon G T_\Delta^{(2\varepsilon)}(\mathbf{x},\tau)rd\Omega d\tau+\int_\Omega \rho^\varepsilon(\varepsilon\tilde{\psi}_{2i}(\mathbf{x}))^2\widetilde{T}rd\Omega \\
&+\int_\Omega C_{ijkl}^\varepsilon\varepsilon_{kl}^\varepsilon(\varepsilon\tilde{\boldsymbol{\psi}}_1(\mathbf{x}))\varepsilon_{ij}^\varepsilon(\varepsilon\tilde{\boldsymbol{\psi}}_1(\mathbf{x}))\widetilde{T}rd\Omega+\int_\Omega \rho^\varepsilon c^\varepsilon(\varepsilon\tilde{\psi}_3(\mathbf{x}))^2 rd\Omega \\
&\le\sum_{i=1}^3 2\int_0^t\int_\Omega\frac{(\varepsilon\widetilde{T}H_i r)^2+(\frac{\partial u_{\Delta i}^{(2\varepsilon)}}{\partial\tau})^2}{2}d\Omega d\tau+2\int_0^t\int_\Omega\frac{(\varepsilon G r)^2+(T_\Delta^{(2\varepsilon)})^2}{2}d\Omega d\tau+C\varepsilon^2 \\
&\le C\Big[\frac{1}{2}\varepsilon^2+\frac{1}{2}\int_0^t\Big\|\frac{\partial\mathbf{u}_\Delta^{(2\varepsilon)}}{\partial\tau}\Big\|_{(L^2(\Omega))^3}^2 d\tau\Big]+C\Big[\frac{1}{2}\varepsilon^2+\frac{1}{2}\int_0^t\Big\|T_\Delta^{(2\varepsilon)}\Big\|_{L^2(\Omega)}^2 d\tau\Big]+C\varepsilon^2 \\
&\le C\varepsilon^2+C\Big[\int_0^t\Big\|\frac{\partial\mathbf{u}_\Delta^{(2\varepsilon)}}{\partial\tau}\Big\|_{(L^2(\Omega))^3}^2 d\tau+\int_0^t\Big\|\mathbf{u}_\Delta^{(2\varepsilon)}\Big\|_{(H_0^1(\Omega))^3}^2 d\tau+\int_0^t\Big\|T_\Delta^{(2\varepsilon)}\Big\|_{L^2(\Omega)}^2 d\tau \\
&+\int_0^t\int_0^\tau\Big\|T_\Delta^{(2\varepsilon)}\Big\|_{H_0^1(\Omega)}^2 ds d\tau\Big]
\end{aligned}
$$

where C will denote a positive generic constant and have different values in different places in this paper.

Denote $C=C/\lambda_1$, without loss of generality and setting $\Theta(t)=\|\frac{\partial\mathbf{u}_\Delta^{(2\varepsilon)}}{\partial t}\|_{(L^2(\Omega))^3}^2+\|\mathbf{u}_\Delta^{(2\varepsilon)}\|_{(H_0^1(\Omega))^3}^2+\|T_\Delta^{(2\varepsilon)}\|_{L^2(\Omega)}^2+\int_0^t\|T_\Delta^{(2\varepsilon)}\|_{H_0^1(\Omega)}^2 d\tau$, then we have $\Theta(t)\le C\varepsilon^2+C\int_0^t\Theta(\tau)d\tau$ by combining (60) and (61) together. It follows from Gronwall's inequality [21,22] that $\Theta(t)\le C(T)\varepsilon^2$. Subsequently, there holds the following result

(62)

$$
\Big\|\frac{\partial\mathbf{u}_\Delta^{(2\varepsilon)}}{\partial t}\Big\|_{(L^2(\Omega))^3}^2+\|\mathbf{u}_\Delta^{(2\varepsilon)}\|_{(H_0^1(\Omega))^3}^2+\|T_\Delta^{(2\varepsilon)}\|_{L^2(\Omega)}^2+\int_0^t\|T_\Delta^{(2\varepsilon)}\|_{H_0^1(\Omega)}^2 d\tau\le C(T)\varepsilon^2
$$

Then using the AM-GM inequality $\frac{a+b+c+d}{4}\le\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$ to the left side of (62) and squaring root on both sides of the inequality (62), the following inequality is obtained

(63)

$$
\begin{aligned}
&\Big\|\frac{\partial\mathbf{u}_\Delta^{(2\varepsilon)}}{\partial t}\Big\|_{(L^2(\Omega))^3}+\|\mathbf{u}_\Delta^{(2\varepsilon)}\|_{(H_0^1(\Omega))^3}+\|T_\Delta^{(2\varepsilon)}\|_{L^2(\Omega)}+\|T_\Delta^{(2\varepsilon)}(\mathbf{x},t)\|_{L^2(0,t;H_0^1(\Omega))} \\
&\le C(T)\varepsilon
\end{aligned}
$$

With the arbitrariness of time t, we get the final convergence result as follows

$$
\begin{aligned}
&\Big\|\frac{\partial\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t)}{\partial t}\Big\|_{L^\infty(0,T;(L^2(\Omega))^3)}+\|\mathbf{u}_\Delta^{(2\varepsilon)}(\mathbf{x},t)\|_{L^\infty(0,T;(H_0^1(\Omega))^3)} \\
&+\|T_\Delta^{(2\varepsilon)}(\mathbf{x},t)\|_{L^\infty(0,T;L^2(\Omega))}+\|T_\Delta^{(2\varepsilon)}(\mathbf{x},t)\|_{L^2(0,T;H_0^1(\Omega))}\le C(T)\varepsilon
\end{aligned}
\tag{64}
$$

where $C(T)$ is a constant independent of ε, but dependent of T.

4. Second-order two-scale numerical algorithm

In this section, we present the SOTS numerical algorithm for the multiscale problem (1). The detailed procedures of the SOTS numerical algorithm for two-dimensional and three-dimensional multiscale problem (1) are listed as follows

(1) Define the geometric structure of the unit cell $Y=(0,1)^{\mathcal{N}}(\mathcal{N}=2,3)$ and homogenized macroscopic region Ω in $\mathbb{R}^{\mathcal{N}}$, and verify the material parameters of composite materials. Then, generate the triangular finite element mesh in $\mathbb{R}^2$ and tetrahedral mesh in $\mathbb{R}^3$, and define the linear conforming finite element spaces $V_{h_1}(Y)$ and $V_{h_0}(\Omega)$ for the above two regions respectively, where h_1 and h_0 represent the finite element mesh sizes of unit cell Y and homogenized macroscopic region Ω.

(2) Solve the first-order auxiliary cell functions defined by (21)-(23) on $V_{h_1}(Y)$ corresponding to different representative macro-coordinate parameters $r^{s_1}\in[r_1,r_2]$, where $s_1=1,2,\cdots,\bar{L}$. The specific FEM scheme for solving auxiliary cell function defined by (23) is given as follows

$$-\int_Y k_{ij}\widetilde{\psi}_j(M_m)\widetilde{\psi}_i(v^{h_1})dY=\int_Y k_{im}\widetilde{\psi}_i(v^{h_1})dY,\ \forall v^{h_1}\in V_{h_1}(Y). \tag{65}$$

Other first-order auxiliary cell functions can be solved similarly. And then, the homogenized material parameters $\hat{\rho}$, $\hat{C}_{ijmn}$, $\hat{\beta}_{ij}$, $\hat{S}$ and $\hat{k}_{ij}$ are evaluated by making integral of (26) corresponding to different macro-coordinate parameters r^{s_1}. After that, the homogenized material coefficients can be computed by interpolation method on each nodes of $V_{h_0}(\Omega)$.

(3) Using the uniform time step $\Delta t=\dfrac{T}{M}$ to discretize time-domain $(0,T)$ as $0=t_0<t_1<\cdots<t_M=T$ and $t_N=N\Delta t(N=0,\cdots,M)$, we denote $f_i^N=f_i(\mathbf{x},t_N)$. Then, the homogenized problem (27) is solved by coupled FDM-FEM method in a coarse mesh and with a large time step on the whole domain $\Omega\times(0,T)$. The concrete hybrid FDM-FEM scheme is given as follows

$$\left\{\begin{aligned}
&\int_\Omega \hat{\rho}\frac{u_i^{(0),N+1}-2u_i^{(0),N}+u_i^{(0),N-1}}{(\Delta t)^2}\nu_i^{h_0}rd\Omega\\
&+\int_\Omega \hat{C}_{ijkl}\varepsilon_{kl}^{(0)*}(\mathbf{u}^{(0),N+1})\varepsilon_{ij}^{(0)*}(\boldsymbol{\nu}^{h_0})rd\Omega\\
&-\int_\Omega \hat{\beta}_{ij}(T^{(0),N+1}-\widetilde{T})\varepsilon_{ij}^{(0)*}(\boldsymbol{\nu}^{h_0})rd\Omega\\
&=\int_\Omega f_i^{N+1}\nu_i^{h_0}rd\Omega,\ \forall\boldsymbol{\nu}^{h_0}\in\left(V_{h_0}(\Omega)\right)^{\mathcal{N}},\\
&\int_\Omega \hat{S}\frac{T^{(0),N+1}-T^{(0),N}}{\Delta t}\tilde{\varphi}^{h_0}rd\Omega\\
&+\int_\Omega \hat{k}_{ij}\psi_j(T^{(0),N+1})\psi_i(\tilde{\varphi}^{h_0})rd\Omega\\
&+\int_\Omega \widetilde{T}\hat{\beta}_{ij}\frac{\varepsilon_{ij}^{(0)*}(\mathbf{u}^{(0),N+1})-\varepsilon_{ij}^{(0)*}(\mathbf{u}^{(0),N})}{\Delta t}\tilde{\varphi}^{h_0}rd\Omega\\
&=\int_\Omega h^{N+1}\tilde{\varphi}^{h_0}rd\Omega,\ \forall\tilde{\varphi}^{h_0}\in V_{h_0}(\Omega).
\end{aligned}\right. \tag{66}$$

It is easy to know that the dynamic system (27) is a strongly coupled hyperbolic and parabolic system. In order to maintain the unconditional stability of our SOTS numerical algorithm, the implicit FDM scheme is employed in time domain of hybrid FDM-FEM scheme (66).

(4) Using the same mesh as first-order auxiliary cell functions, the second-order auxiliary cell functions defined by (34)-(42), which correspond to different macro-coordinate parameters r^{s_1}, are solved by the similar FEM scheme to (65) on $V_{h_1}(Y)$, respectively.

(5) For arbitrary point $(\mathbf{x},t)\in\Omega\times[0,T]$, we use the interpolation method to get the corresponding values of first-order auxiliary cell functions, second-order auxiliary cell functions and homogenized solutions. The spatial derivatives $\varepsilon_{mn}^{(0)*}$, $\psi_j(\varepsilon_{mn}^{(0)*})$, $\psi_m(T^{(0)})$ and $\psi_m\psi_n(T^{(0)})$ in Theorem 1 are evaluated by the average technique on relative elements [25,26], and the temporal derivatives $\dfrac{\partial T^{(0)}}{\partial t}$, $\dfrac{\partial^2 u_i^{(0)}}{\partial t^2}$ and $\dfrac{\partial \varepsilon_{mn}^{(0)*}}{\partial t}$ in Theorem 1 are evaluated by using the hybrid FDM-FEM schemes (66) at every time steps. Then, the displacement field $\mathbf{u}^{(2\varepsilon)}(\mathbf{x},t)$ and temperature field $T^{(2\varepsilon)}(\mathbf{x},t)$ can be solved by the formulas (43) and (44). Moreover, we can further use the higher-order interpolation method to gain the high-precision SOTS solutions [22].

5. Numerical examples and discussion

In this section, three numerical examples are given to verify the validity and feasibility of the SOTS numerical algorithm we developed. Since it is difficult to find the analytic solutions for the two-way coupled system (1), we replace $\mathbf{u}^\varepsilon(\mathbf{x},t)$ and $T^\varepsilon(\mathbf{x},t)$ with $\mathbf{u}_e(\mathbf{x},t)$ and $T_e(\mathbf{x},t)$ which are precise FEM solutions for original problem (1) on a very fine mesh. Without confusion, some notations are introduced as follows

(67)
$$\mathbf{T}error0=\frac{||T_e-T^{(0)}||_{L^2}}{||T_e||_{L^2}},\mathbf{T}error1=\frac{||T_e-T^{(1\varepsilon)}||_{L^2}}{||T_e||_{L^2}},\mathbf{T}error2=\frac{||T_e-T^{(2\varepsilon)}||_{L^2}}{||T_e||_{L^2}}.$$

(68)
$$\mathbf{T}Error0=\frac{|T_e-T^{(0)}|_{H^1}}{|T_e|_{H^1}},\mathbf{T}Error1=\frac{|T_e-T^{(1\varepsilon)}|_{H^1}}{|T_e|_{H^1}},\mathbf{T}Error2=\frac{|T_e-T^{(2\varepsilon)}|_{H^1}}{|T_e|_{H^1}}.$$

(69)
$$\mathbf{u}error0=\frac{||\mathbf{u}_e-\mathbf{u}^{(0)}||_{L^2}}{||\mathbf{u}_e||_{L^2}},\mathbf{u}error1=\frac{||\mathbf{u}_e-\mathbf{u}^{(1\varepsilon)}||_{L^2}}{||\mathbf{u}_e||_{L^2}},\mathbf{u}error2=\frac{||\mathbf{u}_e-\mathbf{u}^{(2\varepsilon)}||_{L^2}}{||\mathbf{u}_e||_{L^2}}.$$

(70)
$$\mathbf{u}Error0=\frac{|\mathbf{u}_e-\mathbf{u}^{(0)}|_{H^1}}{|\mathbf{u}_e|_{H^1}},\mathbf{u}Error1=\frac{|\mathbf{u}_e-\mathbf{u}^{(1\varepsilon)}|_{H^1}}{|\mathbf{u}_e|_{H^1}},\mathbf{u}Error2=\frac{|\mathbf{u}_e-\mathbf{u}^{(2\varepsilon)}|_{H^1}}{|\mathbf{u}_e|_{H^1}}.$$

where $|\mathbf{u}_e-\mathbf{u}^{(0)}|_{H^1}=\Big(\sum_{i,j=1}^{\mathcal{N}}\big|\big|\varepsilon_{ij}(\mathbf{u}_e-\mathbf{u}^{(0)})\big|\big|_{L^2}\Big)^{\frac{1}{2}}$.

5.1. Example 1: The planar dynamic thermo-mechanical problem. In this example, a fiber reinforced cylinder shell with periodicity in radial and hoop directions is considered. The macrostructure Ω and unit cell Y are shown in Fig. 1, where $\Omega=(r,\theta)=[\pi,3\pi/2]\times[0,\pi]$ and $\varepsilon=\pi/10$.

The non-dimensional material property parameters of this example are listed in Table 1.

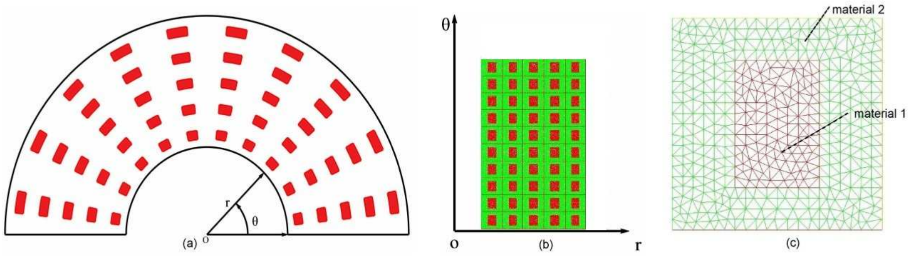

Figure 1. Example 1: (a) Actual physical area; (b) Computational domain Ω; (c) Unit cell Y.

Table 1. Material property parameters.

Property	Matrix	Inclusion
Young's modulus	E=3.0×10^7	E=1.5×10^4
Poisson's ratio	ν=0.3	ν=0.25
Mass density	ρ=10.0	ρ=1.0
Specific heat	c=1.0	c=0.1
Thermal modulus	β_{ij}=50.0(i=j) or 0	β_{ij}=1.0(i=j) or 0
Thermal conductivity	k_{ij}=3.3(i=j) or 0	k_{ij}=0.01(i=j) or 0

The data in original problem (1) of this example are given as follows

$$\begin{aligned} & f_1(\mathbf{x},t) = -2000,\ f_2(\mathbf{x},t) = -2000,\ h(\mathbf{x},t) = 200, \\ & \hat{\mathbf{u}}(\mathbf{x},t) = 0,\ \hat{T}(\mathbf{x},t) = 20.0 \ \text{ in } \ \partial\Omega, \\ & \mathbf{u}^0 = 0,\ \mathbf{u}^1(\mathbf{x}) = 0,\ \widetilde{T} = 20.0 \ \text{ in } \ \Omega. \end{aligned} \tag{71}$$

Now, we implement the triangular mesh generation to original problem (1), auxiliary cell problems and corresponding homogenized problem (27). Then, the computational cost of FEM elements and nodes is listed in Table 2.

Table 2. Comparison of computational cost (Δt=0.002, $t \in$ [0,1]).

	Original equation	Cell problem	Homogenized equation
number of elements	33500	854	3600
number of nodes	17021	468	1891

The SOTS algorithm is implemented for this example. After numerical computation, Fig. 2 and Fig. 3 show the numerical results for different types of solutions $T^{(0)}$, $T^{(1\varepsilon)}$, $T^{(2\varepsilon)}$, T^{ε} and $u_2^{(0)}$, $u_2^{(1\varepsilon)}$, $u_2^{(2\varepsilon)}$, u_2^{ε} at $t = 1.0$, respectively.

The evolutive relative errors of temperature and displacement fields are shown in Fig. 4.

From Table 2, one can easily see that the computational cost of SOTS algorithm is much less than that of precise FEM. It means that SOTS solutions can greatly save computer memory, which is meaningful and important to practical engineering computation. From Fig. 2 and Fig. 3, one can see that only SOTS solutions, which are almost the same as the precise FEM solutions, can accurately capture the microscale fluctuation information due to the heterogeneities of composite materials. The homogenized and FOTS solutions are far from enough to provide a high accuracy solution. Besides, it is clear to see that the accuracy of SOTS solutions is much

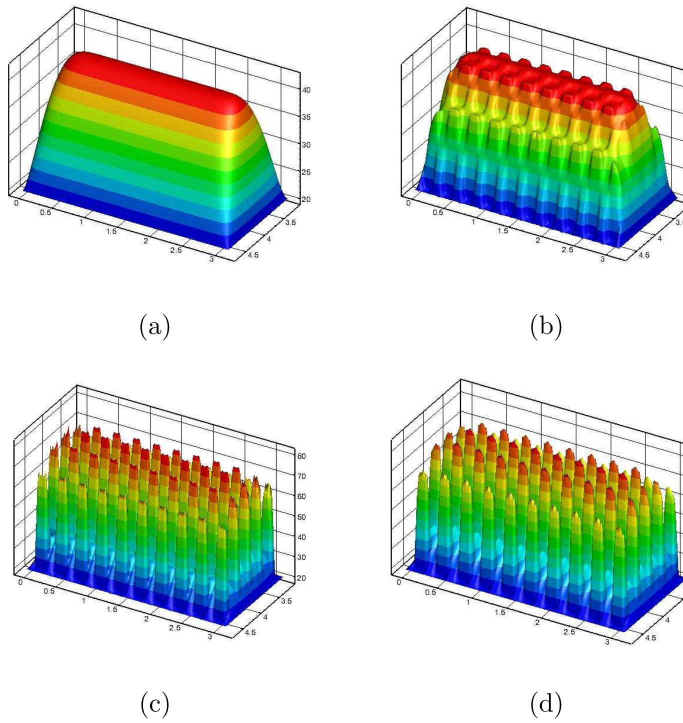

FIGURE 2. Example 1: (a) $T^{(0)}$; (b) $T^{(1\varepsilon)}$; (c) $T^{(2\varepsilon)}$; (d) T^{ε} at $t = 1.0$.

better than homogenized and FOTS solutions from Fig. 4 whether it is the relative errors of temperature or displacement fields. From Fig. 4, we can also find that our SOTS numerical algorithm is stable and effective after long-time numerical computation.

5.2. Example 2: The axisymmetric dynamic thermo-mechanical problem. This example studies the axisymmetric dynamic thermo-mechanical problem with periodicity in axial and radial directions. The macrostructure Ω and unit cell Y are shown in Fig. 5, where $\Omega = (r, z) = [1, 1.5] \times [0, 1]$ and $\varepsilon = 1/12$. It is worth noting that the computational domain Ω of this axisymmetric problem is a cross-section of actual physical area Fig. 5(a) because there is no need to carry out the geometric transformation for axisymmetric problem.

The non-dimensional material property parameters of this example are listed in Table 3.

TABLE 3. Material property parameters.

Property	Matrix	Inclusion
Young's modulus	E=3.0×10^6	E=1.5×10^4
Poisson's ratio	ν=0.3	ν=0.25
Mass density	ρ=10.0	ρ=1.0
Specific heat	c=1.0	c=0.1
Thermal modulus	β_{ij}=50.0(i=j) or 0	β_{ij}=1.0(i=j) or 0
Thermal conductivity	k_{ij}=3.3(i=j) or 0	k_{ij}=0.01(i=j) or 0

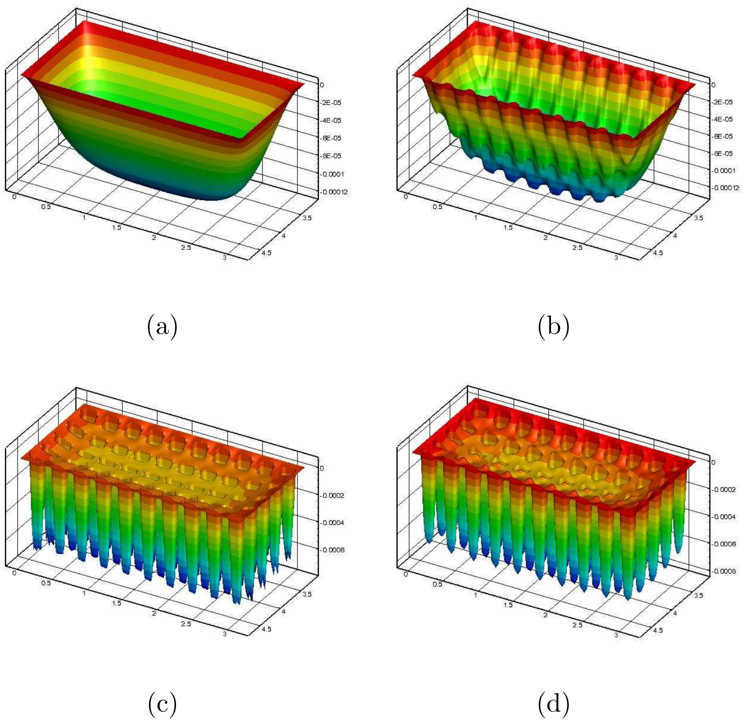

FIGURE 3. Example 1: (a) $u_2^{(0)}$; (b) $u_2^{(1\varepsilon)}$; (c) $u_2^{(2\varepsilon)}$; (d) u_2^{ε} at $t = 1.0$.

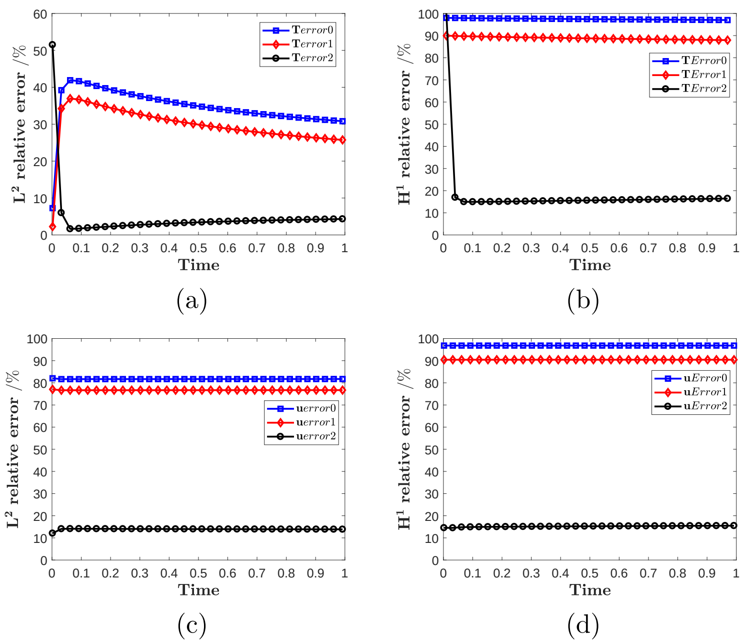

FIGURE 4. Example 1: (a) $Terror$; (b) $TError$; (c) $\mathbf{u}error$; (d) $\mathbf{u}Error$.

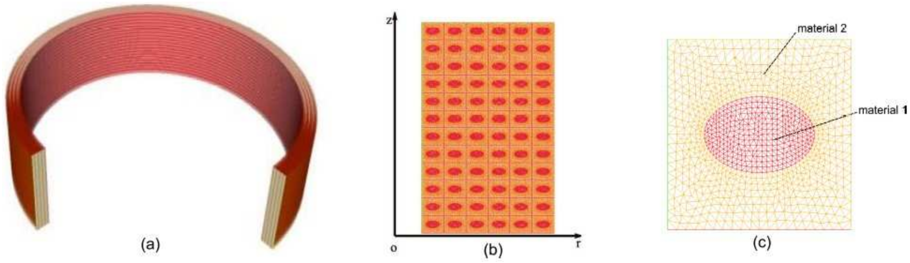

FIGURE 5. Example 2: (a) Actual physical area; (b) Computational domain Ω; (c) Unit cell Y.

The data in problem (1) of this example are given as follows

$$\begin{aligned}&f_1(\mathbf{x},t)=-2000,\ f_3(\mathbf{x},t)=-2000,\ h(\mathbf{x},t)=500,\\&\hat{\mathbf{u}}(\mathbf{x},t)=0,\ \hat{T}(\mathbf{x},t)=10.0\ \text{ in }\ \partial\Omega,\\&\mathbf{u}^0=0,\ \mathbf{u}^1(\mathbf{x})=0,\ \widetilde{T}=10.0\ \text{ in }\ \Omega.\end{aligned}\tag{72}$$

Moreover, the computational cost of FEM elements and nodes is listed in Table 4 after mesh generation.

TABLE 4. Comparison of computational cost (Δt=0.002, $t\in$ [0,1]).

	Original equation	Cell problem	Homogenized equation
number of elements	65088	904	3600
number of nodes	32905	493	1891

Next, the SOTS algorithm is adopted for computing and simulating this example. After numerical computation, Fig. 6 and Fig. 7 depict the numerical results for different types of solutions $T^{(0)}$, $T^{(1\varepsilon)}$, $T^{(2\varepsilon)}$, T^{ε} and $u_3^{(0)}$, $u_3^{(1\varepsilon)}$, $u_3^{(2\varepsilon)}$, u_3^{ε} at $t=1.0$, respectively.

Then, we show the evolutive relative errors of temperature and displacement fields in Fig. 8.

From Table 4, we can conclude that the SOTS numerical algorithm consumes less computational resources compared to precise FEM method, which is of great significance for engineering computation. Fig. 6 and Fig. 7 demonstrate that only SOTS solutions are in good agreement with the precise FEM solutions both in temperature and displacement fields. The homogenized solutions and FOST solutions can not accurately capture the thermo-mechanical responses in composite structures. Furthermore, it is easy to find that only the SOTS solutions can provide enough accuracy for engineering applications from Fig. 8. The accuracy of homogenized solutions and FOTS solutions is far from enough especially for the H^1 semi-norm. Hence, it is of great practical value to develop the SOTS solutions for multiscale problem (1).

5.3. Example 3: Example 2 with a large number of inclusions. In order to further validate the stability and effectiveness of our SOTS algorithms, this example continues to discuss Example 2 with a large number of inclusions. In this example, we define $\Omega=(r,z)=[1,1.5]\times[0,1]$ and $\varepsilon=1/24$. The total number of unit cells in this example is four times as Example 2.

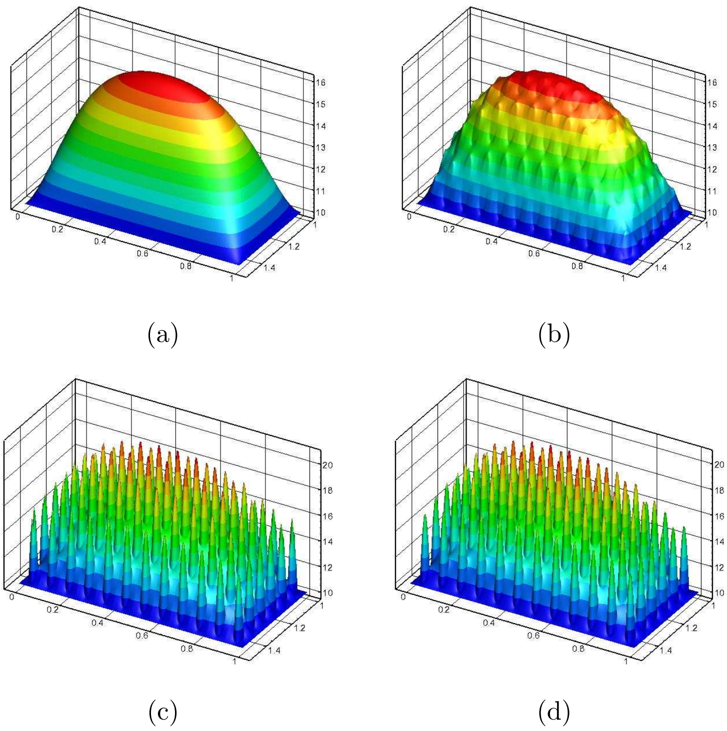

FIGURE 6. Example 2: (a) $T^{(0)}$; (b) $T^{(1\varepsilon)}$; (c) $T^{(2\varepsilon)}$; (d) T^{ε} at $t = 1.0$.

Then, the triangular mesh generation is implemented to original problem (1), auxiliary cell problems and corresponding homogenized problem (27). The detailed information of FEM elements and nodes is listed in Table 5.

TABLE 5. Comparison of computational cost (Δt=0.002, $t \in$ [0,1]).

	Original equation	Cell problem	Homogenized equation
number of elements	260352	904	3600
number of nodes	130897	493	1891

The SOTS algorithm is implemented to this example with a large number of inclusions. After numerical computation, Fig. 9 exhibits the numerical results for solutions $T^{(2\varepsilon)}$, T^{ε}, $u_1^{(2\varepsilon)}$, u_1^{ε}, $u_3^{(2\varepsilon)}$ and u_3^{ε} at $t = 1.0$, respectively.

In addition, the evolutive relative errors of temperature and displacement fields are shown in Fig. 10.

By analyzing the mesh data in Table 5, we can find that our SOTS method is very cheap to simulate original problem (1) compared to precise FEM, which can greatly save computer memory without reducing numerical precision. According to the numerical result in Fig. 9, it concludes that the SOTS solutions of temperature and displacement fields agree well with precise FEM solutions. From Fig. 10, it is easy to see that only SOTS solutions can provide enough numerical accuracy not only in L^2 norm but also in H^1 semi-norm. It means that our SOTS algorithm is stable and effective for composite structures with a large number of inclusions. Furthermore, it can be easily seen that the SOTS solutions own the

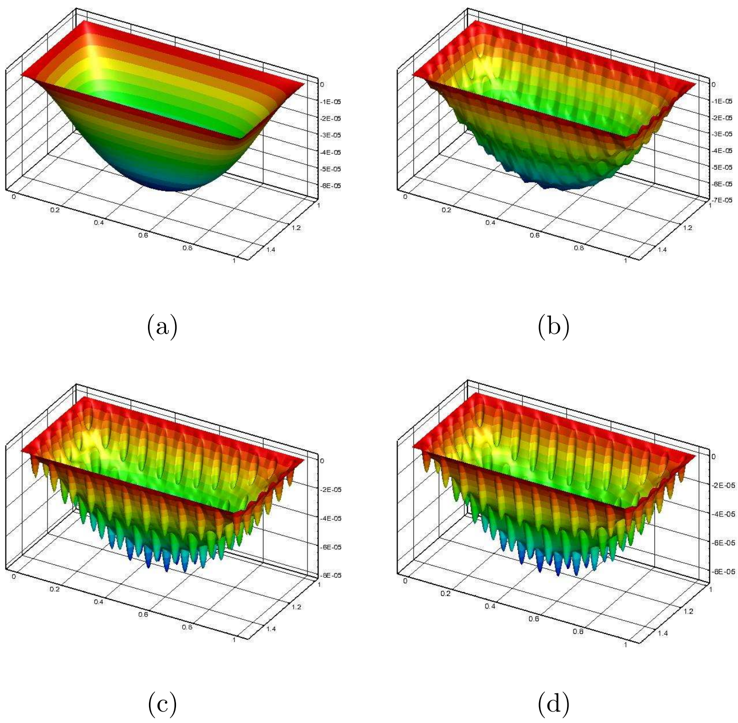

FIGURE 7. Example 2: (a) $u_3^{(0)}$; (b) $u_3^{(1\varepsilon)}$; (c) $u_3^{(2\varepsilon)}$; (d) u_3^{ε} at $t = 1.0$.

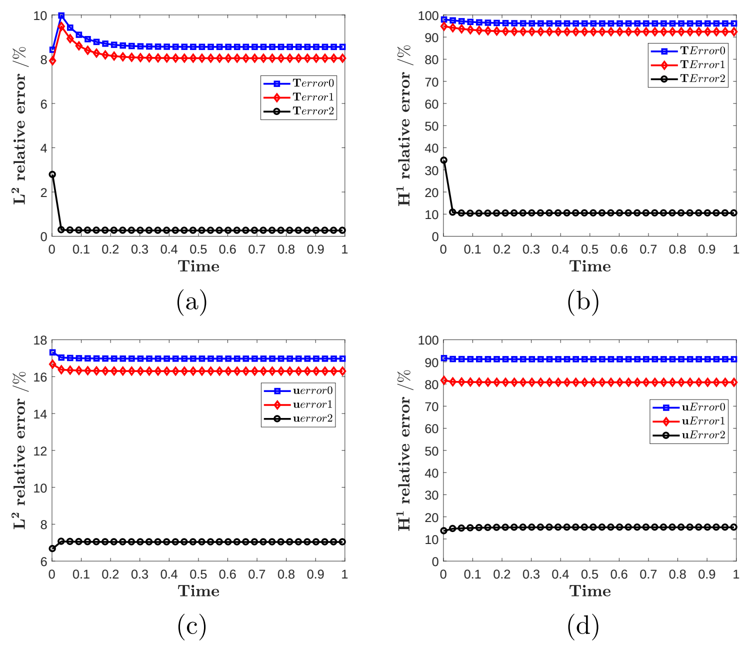

FIGURE 8. Example 2: (a) $Terror$; (b) $TError$; (c) $\mathbf{u}error$; (d) $\mathbf{u}Error$.

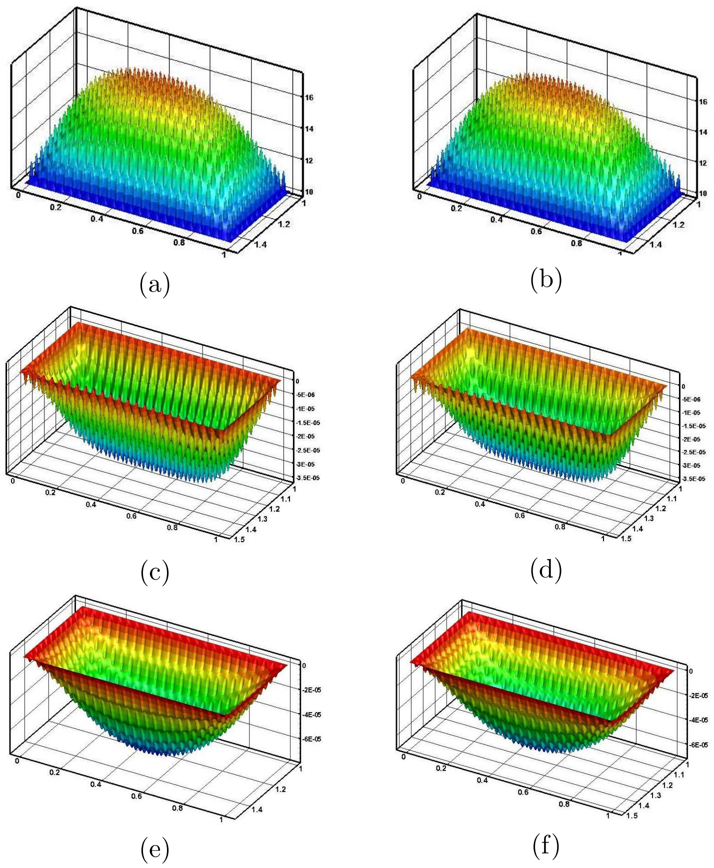

FIGURE 9. Example 3: (a) $T^{(2\varepsilon)}$; (b) T^{ε}; (c) $u_1^{(2\varepsilon)}$; (d) u_1^{ε}; (e) $u_3^{(2\varepsilon)}$; (f) u_3^{ε} at $t = 1.0$.

highest numerical accuracy compare with the homogenized solutions and FOST solutions especially in the H^1 semi-norm. In practical applications, engineers are more concerned about the gradients of temperature and displacement fields, which represent the heat flux and strain fields. Therefore, developing the SOTS solutions for simulating the thermo-mechanical behaviors of composite structures is of great importance in engineering application.

6. Conclusions

In this paper, we develop a novel SOTS analysis method and corresponding numerical algorithm for dynamic thermo-mechanical problems of composite structures with cylindrical periodicity. The new contributions of this paper are the SOTS analysis, the convergence result with an explicit rate for the SOTS solutions, and corresponding SOTS numerical algorithm. Numerical experiments show that the SOTS numerical method we proposed is stable and effective for multiscale problem (1). Furthermore, numerical results show that only SOTS solutions can accurately capture the micro-scale oscillating information and provide enough numerical accuracy for engineering applications, which support the theoretical results of this paper. It also should be underlined that, in order to verify the validity of our SOTS method, we use precise FEM solutions of the original problem (1) in a very fine

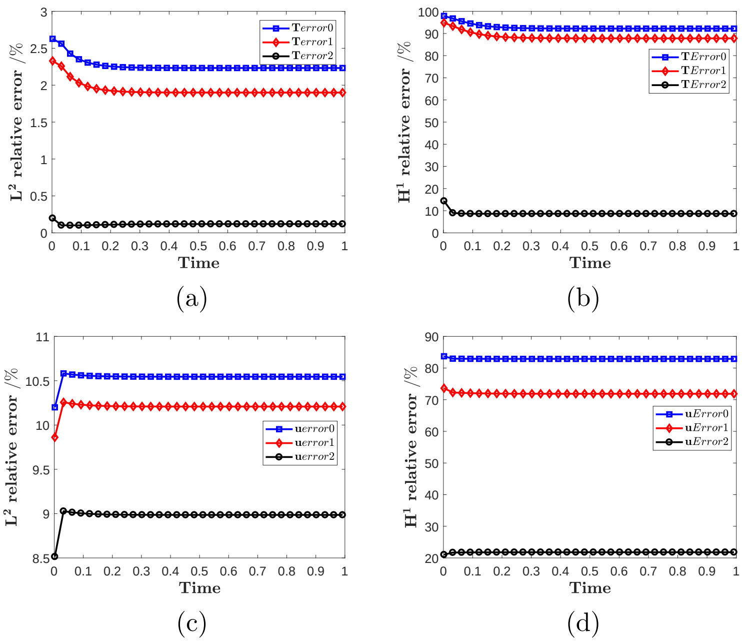

FIGURE 10. Example 3: (a) $Terror$; (b) $TError$; (c) $\mathbf{u}error$; (d) $\mathbf{u}Error$.

mesh as reference solutions. In the actual engineering applications, we do not need to numerically solve the original problem (1) in a very fine mesh. Sometimes it is scarcely possible to obtain FEM reference solutions for large-scale engineering problems. However, we can analyze and compute these complex large-scale problems by using the SOTS method we proposed. Moreover, the high-accuracy solutions provided by our SOTS method encourage the application of proposed SOTS method to deal with the thermal deformation and failure analysis of complicated composite structures. The unified multiscale framework developed in this paper can be extended to other multi-physics coupled problems.

Acknowledgments

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