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CONSTRUCTION AND ANALYSIS OF WEIGHTED SEQUENTIAL SPLITTING FDTD METHODS FOR THE 3D MAXWELL'S EQUATIONS

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Abstract. In this paper, we present a one parameter family of fully discrete Weighted Sequential Splitting (WSS)-finite difference time-domain (FDTD) methods for Maxwell's equations in three dimensions. In one time step, the Maxwell WSS-FDTD schemes consist of two substages each involving the solution of several 1D discrete Maxwell systems. At the end of a time step we take a weighted average of solutions of the substages with a weight parameter θ, $0 \leq \theta \leq 1$. Similar to the Yee-FDTD method, the Maxwell WSS-FDTD schemes stagger the electric and magnetic fields in space in the discrete mesh. However, the Crank-Nicolson method is used for the time discretization of all 1D Maxwell systems in our splitting schemes. We prove that for all values of θ, the Maxwell WSS-FDTD schemes are unconditionally stable, and the order of accuracy is of first order in time when $\theta \neq 0.5$, and of second order when $\theta = 0.5$. The Maxwell WSS-FDTD schemes are of second order accuracy in space for all values of θ. We prove the convergence of the Maxwell WSS-FDTD methods for all values of the weight parameter θ and provide error estimates. We also analyze the discrete divergence of solutions to the Maxwell WSS-FDTD schemes for all values of θ and prove that for $\theta \neq 0.5$ the discrete divergence of electric and magnetic field solutions is approximated to first order, while for $\theta = 0.5$ we obtain a third order approximation to the exact divergence. Numerical experiments and examples are given that illustrate our theoretical results.

Key words. Maxwell's equations, Yee scheme, Crank-Nicolson method, operator splitting, weighted sequential splitting.

1. Introduction

The electric and magnetic fields inside a material are governed by the macroscopic Maxwell's equations along with constitutive laws that account for the response of the material to the incident electromagnetic (EM) field. The computational simulation of electromagnetic interrogation problems, for the determination of the dielectric properties of materials (such as permittivities and permeabilities), requires the use of highly efficient forward simulations of the propagation of transient electromagnetic waves in these media. Thus, a lot of research has concentrated on the development of fully discrete forward solvers of Maxwell's equations that are accurate, consistent, stable, and computationally efficient.

The Yee scheme is a simple and efficient finite difference time domain (FDTD) method [28], and one of the most important numerical techniques for solving Maxwell's equations in the time domain. The Yee-FDTD method, first proposed by Yee in 1966 [28], is an explicit scheme that employs staggered (uniform) grids in both space and time for the electric and magnetic field components. On the staggered grids, central difference approximations in space and time are constructed for each component of the electric and magnetic fields in Maxwell's equations which gives second order accuracy in both space and time. The scheme is non-dissipative for EM wave propagation in vacuum. The Yee scheme along with other discrete methods have been extended to numerically solve Maxwell's equations for EM wave propagation in a variety of linear and nonlinear materials [1, 2, 3, 27], and applied

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to a wide variety of applications in nondestructive evaluation, optical simulations, bioelectromagnetic simulations among others [11, 21, 25]. The Yee scheme has also been extended to nonuniform meshes for EM propagation in a variety of materials [15, 19, 20].

The most limiting aspect of the Yee scheme is the fact that the time step $\triangle t$ and the spatial step sizes $\triangle x, \triangle y$ and $\triangle z$ must satisfy a Courant-Friedrichs-Lewy (CFL) stability condition [26, 27]. The conditionally stable Yee scheme has a stability condition that is determined by the smallest cell size in the domain. For geometries which include features that are smaller than the wavelengths of typical interrogating pulses, fine scale spatial resolution is required to resolve small features. For example, to study the effect of microwaves on brain cells, the size of geometrical features can be five orders of magnitude smaller than a typical wave length [14]. In this case, the conditionally stable Yee scheme requires a very small time step in the *entire* domain to resolve the smallest spatial scales. Thus, the FDTD analysis of very fine geometric structures via the Yee scheme can require a large number of time iterations and long computation times.

The Crank-Nicolson (CN) FDTD method for the numerical simulation of the time domain Maxwell's equations is an implicit FDTD technique and is unconditionally stable [22, 23, 24]. Unconditionally stable schemes are well suited for problems involving geometries needing different details of discretization such as narrow slots [13]. For geometries requiring fine scale spatial resolution, non-uniform meshing techniques can be created by using locally small spatial increments which do not require extremely small time steps in the entire domain in an unconditionally stable scheme [9]. The implicit nature of the CN-FDTD method allows the time step to be chosen based on just accuracy requirements and not stability, and is thus a well suited scheme for the simulation of EM wave propagation in geometries with fine details. However, the CN method is computationally more intensive than the Yee scheme as it requires the solution of a large linear (3D spatial) system at every time step rather than a matrix vector multiplication as in the explicit Yee scheme [22, 23, 24].

The operator splitting method [10] is a powerful tool to solve multi-dimensional and multi-physics problems. In this approach, we replace the original problem involving a complicated operator into a sequence of sub-problems each involving a single operator that models a single physical process. The sub-problems communicate via their initial conditions and an approximation to the solution of the original problem is obtained by combining the solutions of sub-problems. Operator splitting methods are classified based on how the sequence of sub-problems are solved and how these sub or intermediate solutions are combined to approximate the solution of the original problem, which also determines the accuracy of the splitting technique. The classical operator splittings, which include sequential splitting, the Strang-Marchuk splitting, the alternating direction implicit (ADI-FDTD) scheme [13], among others, are popular splitting methods for solving complex time-dependent problems. These splitting techniques can offer additional reductions in computational time over fully implicit methods like the CN scheme while preserving the property of unconditional stability. By using the sequential and symmetrized splitting methods, Chen, Li, and Liang presented the energy-conserved splitting FDTD methods for the free space Maxwell's equations in two- [6] and three-dimensions [7]. The sequential splitting method gives low accuracy in time although its algorithm has a simple structure. As shown in [6, 7], the order of accuracy of a sequential splitting FDTD method for Maxwell's equations is of the first order in time and the second order in space; The Strang-Marchuk splitting FDTD method is more accurate than the sequential splitting FDTD method but is more computationally intensive [10]. Operator splitting methods for Maxwell's equations in complex materials have also been considered; see for e.g., [4, 16, 17].

In this paper we use the *Weighted Sequential Splitting* (WSS) technique [10], that is a generalization of the sequential splitting method, to construct a *family of fully discrete* WSS-FDTD schemes (`Maxwell WSS-FDTD`) for the numerical approximation of the three dimensional time domain Maxwell's equations in a linear non-dispersive, non-dissipative medium such as a vacuum. The `Maxwell WSS-FDTD` methods are constructed using a weighted average of two sequential splitting methods with a weight parameter $\theta \in [0, 1]$. For the spatial discretization we stagger the electric and magnetic field components in space, as is done in the Yee-FDTD method. As opposed to the Yee-FDTD method, we use the Crank-Nicolson discretization in time. For the trivial case of the weight parameter $\theta = 0$ or 1 we recover the sequential splitting schemes constructed in [6, 7], which are first order accurate in time. The interesting case of $\theta = 0.5$, is called the *Symmetrically Weighted Sequential Splitting* (`Maxwell SWSS-FDTD`), and results in a scheme that is second order accurate in time. For values of θ other than 0.5 we obtain discrete schemes that have first order accuracy in time. For all values of $\theta \in [0, 1]$, the `Maxwell WSS-FDTD` schemes are second order accurate in space due to the staggered nature of the spatial discretization.

We show that our fully discrete `Maxwell WSS-FDTD` methods are unconditionally stable via an energy analysis and prove the convergence of the methods for all values of the weight parameter θ. The `Maxwell WSS-FDTD` schemes are dissipative for $\theta \in (0, 1)$. However, we show that the energy decay in time is of higher order accuracy than the temporal and spatial accuracy of the schemes. We also analyze the discrete divergence of the electric and the magnetic field solutions produced by our discrete schemes. The `Maxwell WSS-FDTD` schemes approximate the divergence of the electromagnetic fields to first order for $\theta \neq 0.5$ and to third order for $\theta = 0.5$, a distinct advantage over the Strang symmetrized scheme in [7], that approximates the exact divergence to second order accuracy. All our theoretical results for the `Maxwell WSS-FDTD` schemes are illustrated by numerical experiments.

The outline of this paper is as follows. In Section 2, we present the Weighted Sequential Splitting (WSS) technique for a general initial value problem. In Section 3, we consider the three dimensional time domain Maxwell's equations modeling electromagnetic wave propagation in a linear non-dispersive, non-dissipative medium. We then present discrete in time and continuous in space WSS schemes (`Maxwell WSS`) for Maxwell's equations in Section 4. The fully discrete `Maxwell WSS-FDTD` schemes are presented in Section 5. The stability, and convergence analysis of the fully discrete schemes are presented in Sections 6 and 7, respectively. In Section 6, we also analyze in detail the discrete energy decay in our methods. In Section 8, we analyze the discrete divergence of the solution produced by the `Maxwell WSS-FDTD` methods. In Section 9 we present numerical simulations that demonstrate our theoretical results, and conclude in Section 10.

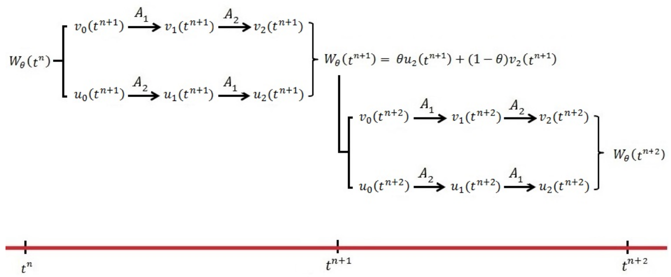


FIGURE 1. Two time iterations of the Weighted Sequential Splitting Algorithm for $d = 2$.

2. Weighted Sequential Splitting (WSS) Schemes

Let $d \in \mathbb{N}, d \geq 2$. Consider the following system of differential equations in the Banach space X:

$$(1) \qquad \begin{cases} \dfrac{d}{dt}W(t) = AW(t) = \left(\displaystyle\sum_{i=1}^{d} A_i\right) W(t); \quad t \in (0,T], \\ W(0) = W_0, \end{cases}$$

where $W : [0,T] \to X$, for $T > 0$, is the unknown solution, $W_0 \in X$, and the operators $A, A_i : X \to X$ $(i = 1, 2, ..., d)$ are such that $A = \displaystyle\sum_{i=1}^{d} A_i$.

The classical operator splitting methods that are commonly used for the time discretization of partial differential equations (PDEs) are the first order sequential splitting scheme, and the second order Strang-Marchuk symmetrized scheme [10]. These splitting methods require the solution of several sub-problems corresponding to the sub-operators A_i. However, the low accuracy of the sequential splitting method and the computational complexity of the Strang-Marchuk symmetrized splitting motivates the investigation of alternative splitting methods.

In this paper, we consider the weighted sequential splitting (WSS) method [10] which generalizes the sequential splitting technique by using a weighted average of two sequential splitting solutions. In other words, for each time step we apply two sequential splittings in the following manner: $A_1 \to A_2 \to ... \to A_d$ and $A_d \to A_{d-1} \to ... \to A_1$. At the end of the time step we take a weighted average of solutions of the two different sequential splittings with a weight parameter θ, $0 \leq \theta \leq 1$.

We choose a time step $\Delta t > 0$ and define $t^n = n\Delta t$ for $n = 0, 1, \ldots, N$, with $N = T/\Delta t$. Using Δt we partition the time interval $[0,T]$ as

$$0 = t^0 < t^1 < t^2 < \ldots < t^n < \ldots < t^N = N\Delta t = T.$$

For $0 \leq \theta \leq 1$, let $W_\theta(t^n)$ denote the solution of the WSS method applied to problem (1) at time $t^n \in [0,T]$ at the nth step. The algorithm for the WSS method is given in Algorithm 1.

In Figure 1, we illustrate the WSS scheme over two time intervals for $d = 2$. The weighted parameters $\theta = 0$ and $\theta = 1$ recover the basic sequential splitting. The WSS scheme for the case $\theta = 0.5$ is called the *Symmetrically Weighted Sequential Splitting (SWSS)*.

Data: Given $W_\theta(t^0 = 0) = W_0$
Result: Obtain $W_\theta(t^N)$
begin

```
For n = 0,1,2,...,N-1;
   v_0(t^{n+1}) = u_0(t^{n+1}) = W_θ(t^n);
   For i = 1,2,...,d ;
       Step A_i: Solve for v_i(t), t ∈ [t^n, t^{n+1}] in
```

$$\begin{cases} \dfrac{d}{dt} v_i(t) = A_i v_i(t), \\ \\ v_i(t^n) = v_{i-1}(t^{n+1}). \end{cases}$$

```
       Step B_i: Solve for u_i(t), t ∈ [t^n, t^{n+1}] in
```

$$\begin{cases} \dfrac{d}{dt} u_i(t) = A_{d+1-i} u_i(t), \\ \\ u_i(t^n) = u_{i-1}(t^{n+1}), \end{cases}$$

```
   end
W_θ(t^{n+1}) = θ u_d(t^{n+1}) + (1-θ) v_d(t^{n+1}).
end
```

end

Algorithm 1: Weighted Sequential Splitting (WSS).

Operator Splitting methods are special time discretization techniques. Thus, in analogy with local truncation error (LTE), we define the notion of *local splitting error* [10].

Definition 1. *For an operator splitting method, the error that arises at the end of the first step, denoted by τ_θ, defined as*

$$\tau_\theta(W_0, \Delta t) = W(t^1) - W_\theta(t^1),$$

is defined as the local splitting error of the operator splitting method.

An operator splitting method is of order p (in time) for some $p > 0$ if its local splitting error $\tau_\theta(W_0, \Delta t) = \mathcal{O}((\Delta t)^{p+1})$ where Δt is the time step. As shown in ([8],[10]), the local splitting error of the weighted sequential splitting can be expressed as

$$\tau_\theta(W_0, \Delta t) = (\Delta t)^2(\theta - 0.5) \sum_{\substack{i,j=1 \\ i<j}}^{d} \big[A_i, A_j\big] W_0 + \mathcal{O}(\Delta t^3), \tag{2}$$

where $\big[A_i, A_j\big] = A_i A_j - A_j A_i$, is the commutator of the operators A_i and A_j. Thus, we have the following result.

Theorem 1. *(Theorem 3.3 in* [10]*) For arbitrary operators, the weighted sequential splitting method has first order of accuracy (in time) for any $\theta \neq 0.5$. The choice $\theta = 0.5$ (SWSS) provides a second order accurate splitting method in time.*

In the next section we present Maxwell's equations in a non-dispersive, non-conductive (non-dissipative) linear dielectric and then construct a weighted sequential splitting scheme for this system.

3. Maxwell's Equations

The classical three dimensional Maxwell's equations are derived from physical laws that govern the evolution of the electromagnetic field, connecting the electric field intensity $\mathbf{E} = (E_x, E_y, E_z)^T$, the magnetic field intensity $\mathbf{H} = (H_x, H_y, H_z)^T$, the magnetic flux density $\mathbf{B} = (B_x, B_y, B_z)^T$ and the electric flux density $\mathbf{D} = (D_x, D_y, D_z)^T$. Let $\Omega \subset \mathbb{R}^3$ be a bounded region. On the space-time region $\Omega \times (0, T]$ with $T > 0$, Maxwell's equations, assuming a source free region Ω, are given as

$$\frac{\partial}{\partial t}\mathbf{D} = \mathbf{curl}\,\mathbf{H}, \tag{3a}$$

$$\frac{\partial}{\partial t}\mathbf{B} = -\mathbf{curl}\,\mathbf{E}, \tag{3b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3c}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{3d}$$

along with initial conditions on all fields. All vector field variables (and their scalar components) are functions of space, $\mathbf{x} = (x, y, z)^T$ and time, t. The magnetic ($\mathbf{B}$) and electric ($\mathbf{D}$) flux densities defined on $\Omega \times [0, T]$ satisfy constitutive relations for linear, isotropic and non-dispersive materials given as

$$\mathbf{B} = \mu_0 \mathbf{H} \quad \text{and} \quad \mathbf{D} = \epsilon \mathbf{E}, \tag{4}$$

where ϵ is the frequency-independent (non-dispersive) electric permittivity of the dielectric and μ_0 is the magnetic permeability of free space. In this paper, we will assume that ϵ is a constant depending on the linear, isotropic, and non-dispersive material through which the EM waves propagate. We assume the *Perfect Electrical Conductor* (PEC) boundary condition

$$\mathbf{E} \times \vec{\mathbf{n}} = \mathbf{0}, \text{ on, } \partial\Omega \times [0, T], \tag{5}$$

where $\vec{\mathbf{n}}$ is the unit outward normal vector to the boundary $\partial\Omega$. We can rewrite Maxwell's equations (3) on $\Omega \times (0, T]$ in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} \epsilon \mathbf{E} \\ \mu_0 \mathbf{H} \end{pmatrix} = \mathbf{A}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{6}$$

where the operator $\mathbf{A}$ is defined as

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{curl} \\ -\mathbf{curl} & \mathbf{0} \end{pmatrix}, \ \mathbf{curl} = \begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

Poynting's theorem states that the energy of electromagnetic waves in lossless media with the PEC boundary condition is constant for all time (energy conservation). We state the following lemmas without proof, and refer the reader to the papers [5, 7, 12, 18] for details.

Lemma 1. *(Energy conservation I). Let* $\mathbf{E}$ *and* $\mathbf{H}$ *be the solution to the Maxwell's equations (6) in a lossless medium and satisfy the PEC boundary condition* (5). *Then, for all* $t \geq 0$ *we have*

$$\int_\Omega \left(\epsilon|\mathbf{E}(\mathbf{x}, t)|^2 + \mu_0|\mathbf{H}(\mathbf{x}, t)|^2\right)\, d\mathbf{x} \equiv \textit{constant}.$$

Lemma 2. *(Energy conservation II). Let* $\mathbf{E}$ *and* $\mathbf{H}$ *be the solution to the Maxwell's equations (6) in a lossless medium and satisfy the PEC boundary condition. Then, for all* $t \geq 0$ *we have*

$$\int_\Omega \left(\epsilon \Big| \frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t} \Big|^2 + \mu_0 \Big| \frac{\partial \mathbf{H}(\mathbf{x},t)}{\partial t} \Big|^2 \right) d\mathbf{x} \equiv \textit{constant.}$$

In the rest of the paper, without loss of generality we will assume that $\epsilon = 1$ and $\mu_0 = 1$. Since the operator $\mathbf{A}$ generates a C_0 semigroup [10], the exact solution of the problem (6) can be expressed as

$$\begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = e^{\mathbf{A}t} \begin{pmatrix} \mathbf{E}(0) \\ \mathbf{H}(0) \end{pmatrix}, \ \forall t \in [0, T].$$

As done in [7, 18], we decompose the curl operator **curl** into two sub-operators $\mathbf{curl}^+$ and $\mathbf{curl}^-$ as

$$\mathbf{curl} = \mathbf{curl}^+ + \mathbf{curl}^-, \tag{7}$$

where $\mathbf{curl}^+$ and $\mathbf{curl}^-$ are, respectively, defined as

$$\mathbf{curl}^+ := \begin{pmatrix} 0 & 0 & \partial_y \\ \partial_z & 0 & 0 \\ 0 & \partial_x & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{curl}^- := \begin{pmatrix} 0 & -\partial_z & 0 \\ 0 & 0 & -\partial_x \\ -\partial_y & 0 & 0 \end{pmatrix}.$$

The operator **curl** is antisymmetric, and operators $\mathbf{curl}^+$ and $\mathbf{curl}^-$ are related as

$$\mathbf{curl}^+ = -(\mathbf{curl}^-)^T.$$

Another possible decomposition of the operator **curl** is [7, 18]

$$\mathbf{curl} = \mathbf{curl}_x + \mathbf{curl}_y + \mathbf{curl}_z, \tag{8}$$

where $\mathbf{curl}_\alpha$ is the operator related to ∂_α with $\alpha \in \{x, y, z\}$ (see [7]):

$$\mathbf{curl}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_x \\ 0 & \partial_x & 0 \end{pmatrix}, \mathbf{curl}_y = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & 0 \\ -\partial_y & 0 & 0 \end{pmatrix}, \text{ and } \mathbf{curl}_z = \begin{pmatrix} 0 & -\partial_z & 0 \\ \partial_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The WSS algorithm that is constructed in this paper is for decomposition (7). This algorithm can be extended to the decomposition (8). However, since the number of sub-operators are different in the two decompositions additional details will be involved in the analysis of the second decomposition (8). We do not consider decomposition (8) or its analysis in this paper. Henceforth, we restrict our attention to decomposition (7) of the curl operator.

4. Weighted Sequential Splittings for Maxwell's Equations

In this section, we apply the weighted sequential splitting (WSS) method, given in Algorithm 1, to the three dimensional Maxwell's equations (6) by extending the ideas from [6] and [7] in which sequential splitting for two and three dimensional Maxwell's equations, respectively, were constructed based on the decomposition of the operator **curl** given by equation (7). We decompose operator $\mathbf{A}$ in (6) as $\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^-$ where

$$\mathbf{A}^+ = \begin{pmatrix} \mathbf{0} & \mathbf{curl}^+ \\ (\mathbf{curl}^+)^T & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{A}^- = \begin{pmatrix} \mathbf{0} & \mathbf{curl}^- \\ (\mathbf{curl}^-)^T & \mathbf{0} \end{pmatrix}. \tag{9}$$

For $0 \le \theta \le 1$, we denote the solution of the corresponding weighted sequential splitting for Maxwell's curl equations (6) with the decomposition (9) as $\mathbf{W}_\theta(t)$, at time $t \in (0, t]$, where

$$\mathbf{W}_\theta(t) = \begin{pmatrix} \mathbf{E}_\theta(t) \\ \mathbf{H}_\theta(t) \end{pmatrix}.$$

For $i = 0, 1, 2$, we also denote intermediate variables $\mathbf{V}_i(t)$, and $\mathbf{U}_i(t)$ at time $t \in [0, t]$ as

$$\mathbf{V}_i(t) = \begin{pmatrix} \mathbf{E}^{(i)}(t) \\ \mathbf{H}^{(i)}(t) \end{pmatrix}, \mathbf{U}_i(t) = \begin{pmatrix} \widehat{\mathbf{E}}^{(i)}(t) \\ \widehat{\mathbf{H}}^{(i)}(t) \end{pmatrix}.$$

The weighted sequential splitting algorithm for Maxwell's equations (6), which we call `Maxwell WSS`, corresponding to the splitting (7) and (9) of the operator $\mathbf{A}$, can be written as the following two stage method which we describe below in Algorithm 2.

Thus, in every time interval $[t^n, t^{n+1}]$, the `Maxwell WSS` algorithm solves the following sub-systems in the two stages with proper initialization.

Stage 1 :

$$\overbrace{\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{E}^{(1)} \\ \mathbf{H}^{(1)} \end{pmatrix} = \mathbf{A}^+ \begin{pmatrix} \mathbf{E}^{(1)} \\ \mathbf{H}^{(1)} \end{pmatrix}}^{\mathbf{V}_1} \quad \text{and} \quad \overbrace{\frac{\partial}{\partial t}\begin{pmatrix} \widehat{\mathbf{E}}^{(1)} \\ \widehat{\mathbf{H}}^{(1)} \end{pmatrix} = \mathbf{A}^- \begin{pmatrix} \widehat{\mathbf{E}}^{(1)} \\ \widehat{\mathbf{H}}^{(1)} \end{pmatrix}}^{\mathbf{U}_1}. \tag{10}$$

Stage 2 :

$$\overbrace{\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{E}^{(2)} \\ \mathbf{H}^{(2)} \end{pmatrix} = \mathbf{A}^- \begin{pmatrix} \mathbf{E}^{(2)} \\ \mathbf{H}^{(2)} \end{pmatrix}}^{\mathbf{V}_2} \quad \text{and} \quad \overbrace{\frac{\partial}{\partial t}\begin{pmatrix} \widehat{\mathbf{E}}^{(2)} \\ \widehat{\mathbf{H}}^{(2)} \end{pmatrix} = \mathbf{A}^+ \begin{pmatrix} \widehat{\mathbf{E}}^{(2)} \\ \widehat{\mathbf{H}}^{(2)} \end{pmatrix}}^{\mathbf{U}_2}. \tag{11}$$

Written in scalar form, and redistributed, the above systems result in six decoupled one-dimensional Maxwell systems, three in each stage. These are:

Stage 1:

$$\begin{array}{l} \mathbf{A}^+ \\ \\ \mathbf{A}^- \end{array}\left\{\begin{array}{l} \frac{\partial}{\partial t}\begin{pmatrix} E_x^{(1)} \\ H_z^{(1)} \end{pmatrix} = \frac{\partial}{\partial y}\begin{pmatrix} H_z^{(1)} \\ E_x^{(1)} \end{pmatrix}, \\ \\ \frac{\partial}{\partial t}\begin{pmatrix} \widehat{E}_z^{(1)} \\ \widehat{H}_x^{(1)} \end{pmatrix} = -\frac{\partial}{\partial y}\begin{pmatrix} \widehat{H}_x^{(1)} \\ \widehat{E}_z^{(1)} \end{pmatrix}, \end{array}\right. \left\{\begin{array}{l} \frac{\partial}{\partial t}\begin{pmatrix} E_y^{(1)} \\ H_x^{(1)} \end{pmatrix} = \frac{\partial}{\partial z}\begin{pmatrix} H_x^{(1)} \\ E_y^{(1)} \end{pmatrix}, \\ \\ \frac{\partial}{\partial t}\begin{pmatrix} \widehat{E}_x^{(1)} \\ \widehat{H}_y^{(1)} \end{pmatrix} = -\frac{\partial}{\partial z}\begin{pmatrix} \widehat{H}_y^{(1)} \\ \widehat{E}_x^{(1)} \end{pmatrix}, \end{array}\right. \left\{\begin{array}{l} \frac{\partial}{\partial t}\begin{pmatrix} E_z^{(1)} \\ H_y^{(1)} \end{pmatrix} = \frac{\partial}{\partial x}\begin{pmatrix} H_y^{(1)} \\ E_z^{(1)} \end{pmatrix}, \\ \\ \frac{\partial}{\partial t}\begin{pmatrix} \widehat{E}_y^{(1)} \\ \widehat{H}_z^{(1)} \end{pmatrix} = -\frac{\partial}{\partial x}\begin{pmatrix} \widehat{H}_z^{(1)} \\ \widehat{E}_y^{(1)} \end{pmatrix}, \end{array}\right.$$

Stage 2:

$$\begin{array}{l} \mathbf{A}^- \\ \\ \mathbf{A}^+ \end{array}\left\{\begin{array}{l} \frac{\partial}{\partial t}\begin{pmatrix} E_x^{(2)} \\ H_y^{(2)} \end{pmatrix} = -\frac{\partial}{\partial z}\begin{pmatrix} H_y^{(2)} \\ E_x^{(2)} \end{pmatrix}, \\ \\ \frac{\partial}{\partial t}\begin{pmatrix} \widehat{E}_y^{(2)} \\ \widehat{H}_x^{(2)} \end{pmatrix} = \frac{\partial}{\partial z}\begin{pmatrix} \widehat{H}_x^{(2)} \\ \widehat{E}_y^{(2)} \end{pmatrix}, \end{array}\right. \left\{\begin{array}{l} \frac{\partial}{\partial t}\begin{pmatrix} E_y^{(2)} \\ H_z^{(2)} \end{pmatrix} = -\frac{\partial}{\partial x}\begin{pmatrix} H_z^{(2)} \\ E_y^{(2)} \end{pmatrix}, \\ \\ \frac{\partial}{\partial t}\begin{pmatrix} \widehat{E}_z^{(2)} \\ \widehat{H}_y^{(2)} \end{pmatrix} = \frac{\partial}{\partial x}\begin{pmatrix} \widehat{H}_y^{(2)} \\ \widehat{E}_z^{(2)} \end{pmatrix}, \end{array}\right. \left\{\begin{array}{l} \frac{\partial}{\partial t}\begin{pmatrix} E_z^{(2)} \\ H_x^{(2)} \end{pmatrix} = -\frac{\partial}{\partial y}\begin{pmatrix} H_x^{(2)} \\ E_z^{(2)} \end{pmatrix}, \\ \\ \frac{\partial}{\partial t}\begin{pmatrix} \widehat{E}_x^{(2)} \\ \widehat{H}_z^{(2)} \end{pmatrix} = \frac{\partial}{\partial y}\begin{pmatrix} \widehat{H}_z^{(2)} \\ \widehat{E}_x^{(2)} \end{pmatrix}. \end{array}\right.$$

5. Fully Discrete WSS Methods for Maxwell's Equations

5.1. Spatial and Temporal Discretization. We consider a cubic spatial domain $\Omega = [0, a] \times [0, b] \times [0, c] \subset \mathbb{R}^3$ for $a, b, c > 0$ and time interval $[0, T]$ with $T > 0$. We discretize Ω by using spatial step sizes $\triangle x > 0, \triangle y > 0$ and $\triangle z > 0$ along the x, y, and z axis, respectively. The time interval $[0, T]$ is discretized with time step $\triangle t > 0$. Let I, J, K and N be positive integers such that $I = a/\Delta x$, $J = b/\Delta y$, $K = c/\Delta z$, and $N = T/\Delta t$.

Data: Given $\theta \in [0,1]$, $\mathbf{W}_\theta(t^0 = 0) = \mathbf{W}_0 = (\mathbf{E}_0, \mathbf{H}_0)^T$, with $\mathbf{E}_\theta(t^0 = 0) = \mathbf{E}_0$ satisfying (5).

Result: Obtain $\mathbf{W}_\theta(t^N) = (\mathbf{E}_\theta(t^N), \mathbf{H}_\theta(t^N))^T$, with $\mathbf{E}_\theta(t^N)$ satisfying (5).

begin

`For` $n = 0, 1, 2, ..., N-1$;

$\mathbf{V}_0(t^{n+1}) = \mathbf{U}_0(t^{n+1}) = \mathbf{W}_\theta(t^n)$;

Stage 1:

`Step` A_1: Solve for $\mathbf{V}_1(t), t \in [t^n, t^{n+1}]$ in

$$\begin{cases} \dfrac{d}{dt}\mathbf{V}_1(t) = \mathbf{A}^+\mathbf{V}_1(t), \\ \\ \mathbf{V}_1(t^n) = \mathbf{V}_0(t^{n+1}). \end{cases}$$

`Step` B_1: Solve for $\mathbf{U}_1(t), t \in [t^n, t^{n+1}]$ in

$$\begin{cases} \dfrac{d}{dt}\mathbf{U}_1(t) = \mathbf{A}^-\mathbf{U}_1(t), \\ \\ \mathbf{U}_1(t^n) = \mathbf{U}_0(t^{n+1}). \end{cases}$$

Stage 2:

`Step` A_2: Solve for $\mathbf{V}_2(t), t \in [t^n, t^{n+1}]$ in

$$\begin{cases} \dfrac{d}{dt}\mathbf{V}_2(t) = \mathbf{A}^-\mathbf{V}_2(t), \\ \\ \mathbf{V}_2(t^n) = \mathbf{V}_1(t^{n+1}). \end{cases}$$

`Step` B_2: Solve for $\mathbf{U}_2(t), t \in [t^n, t^{n+1}]$ in

$$\begin{cases} \dfrac{d}{dt}\mathbf{U}_2(t) = \mathbf{A}^+\mathbf{U}_2(t), \\ \\ \mathbf{U}_2(t^n) = \mathbf{U}_1(t^{n+1}). \end{cases}$$

$\mathbf{W}_\theta(t^{n+1}) = \theta\mathbf{U}_2(t^{n+1}) + (1-\theta)\mathbf{V}_2(t^{n+1})$.

`end`

end

Algorithm 2: Maxwell Weighted Sequential Splitting (`Maxwell WSS`).

For $0 \le i \le I$, $0 \le j \le J$, $0 \le k \le K$, and $0 \le n \le N$, we define the following discrete grid points in time and space:

$$t^n = n\triangle t, \qquad t^{n+\frac{1}{2}} = \Big(n + \frac{1}{2}\Big)\triangle t, \qquad n = 0, 1, 2, ..., N-1, \qquad t^N = N\triangle t = T,$$
$$x_i = i\triangle x, \qquad x_{i+\frac{1}{2}} = \Big(i + \frac{1}{2}\Big)\triangle x, \qquad i = 0, 1, 2, ..., I-1, \qquad x_I = I\triangle x = a,$$

$$y_j = j\triangle y, \qquad y_{j+\frac{1}{2}} = \Big(j + \frac{1}{2}\Big)\triangle y, \qquad j = 0, 1, 2, ..., J-1, \qquad y_J = J\triangle y = b,$$
$$z_k = k\triangle z, \qquad z_{k+\frac{1}{2}} = \Big(k + \frac{1}{2}\Big)\triangle z, \qquad k = 0, 1, 2, ..., K-1, \qquad z_K = K\triangle z = c,$$

Let $i, j, k, n \in \mathbb{N}$. We discretize the x, y, z, and t axes, respectively, as

$$0 = x_0 \le x_1 \le \cdots \le x_i \le \cdots \le x_I = a, \tag{12}$$

$$0 = y_0 \le y_1 \le \cdots \le y_j \le \cdots \le y_J = b, \tag{13}$$

$$0 = z_0 \le z_1 \le \cdots \le z_k \le \cdots \le z_K = c, \tag{14}$$

$$0 = t^0 \le t^1 \le \cdots \le t^n \le \cdots \le t^N = T. \tag{15}$$

Let F be one of the field variables H_κ, or E_κ, for $\kappa \in \{x, y, z\}$. In our discrete splitting scheme we stagger the electric and magnetic fields in space. We define the discrete meshes

$$\Omega_h^{E_x} := \left\{\left(x_{i+\frac{1}{2}}, y_j, z_k\right) \mid 0 \le i \le I-1, 0 \le j \le J, 0 \le k \le K\right\}, \tag{16}$$

$$\Omega_h^{E_y} := \left\{\left(x_i, y_{j+\frac{1}{2}}, z_k\right) \mid 0 \le i \le I, 0 \le j \le J-1, 0 \le k \le K\right\}, \tag{17}$$

$$\Omega_h^{E_z} := \left\{\left(x_i, y_j, z_{k+\frac{1}{2}}\right) \mid 0 \le i \le I, 0 \le j \le J, 0 \le k \le K-1\right\}, \tag{18}$$

$$\Omega_h^{H_x} := \left\{\left(x_i, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}}\right) \mid 0 \le i \le I, 0 \le j \le J-1, 0 \le k \le K-1\right\}, \tag{19}$$

$$\Omega_h^{H_y} := \left\{\left(x_{i+\frac{1}{2}}, y_j, z_{j+\frac{1}{2}}\right) \mid 0 \le i \le I-1, 0 \le j \le J, 0 \le k \le K-1\right\}, \tag{20}$$

$$\Omega_h^{H_z} := \left\{\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k\right) \mid 0 \le i \le I-1, 0 \le j \le J-1, 0 \le k \le K\right\}, \tag{21}$$

where Ω_h^F is the set of spatial grid points on which the F field is discretized. These discretized values are the Degrees of Freedom (DoF) of F. Similar to the Yee scheme, for the discretization of the three dimensional Maxwell's equations, the DoF for the electric field component E_κ are at the midpoint of edges oriented in the $\kappa-$direction while the DoF for the magnetic field H_κ are at centers of faces of cubes which are normal to the $\kappa-$direction. In other words, E_κ is discretized at a half-integer grid point in the κ direction and at integer grid points in the other two directions; H_κ is discretized at an integer grid point in the κ direction and at half-integer grid points in the other two directions.

Let m be either n or $n+\frac{1}{2}$, $n \in \mathbb{N}$. We use the notation $F(t^m)$ to denote the continuous function F on the domain Ω at time t^m, and the notation F^m to denote the corresponding *grid function* on its discrete spatial mesh Ω_h^F at time t^m. Let $(x_\alpha, y_\beta, z_\gamma) \in \Omega_h^F$, where depending on F, α is either i or $i+\frac{1}{2}$, β is either j or $j+\frac{1}{2}$, γ is either k or $k+\frac{1}{2}$, and $i, j, k \in \mathbb{N}$. For $m \in \mathbb{N}$, we define the grid function F^m to be an approximation to $F(t^m)$ on the grid Ω_h^F. Thus, F^m is the set of values

$$\{F^m_{\alpha,\beta,\gamma} \approx F(x_\alpha, y_\beta, z_\gamma, t^m), (x_\alpha, y_\beta, z_\gamma) \in \Omega_h^F\}. \tag{22}$$

On the discretized mesh the PEC boundary condition (5) on the electric field can be expressed in scalar form as the following sets of conditions on the DoF of components of the electric field:

$$\begin{aligned} E^n_{x_{i+\frac{1}{2},0,k}} &= E^n_{x_{i+\frac{1}{2},J,k}} = E^n_{x_{i+\frac{1}{2},j,0}} = E^n_{x_{i+\frac{1}{2},j,K}} = 0, \\ E^n_{y_{0,j+\frac{1}{2},k}} &= E^n_{y_{I,j+\frac{1}{2},k}} = E^n_{y_{i,j+\frac{1}{2},0}} = E^n_{y_{i,j+\frac{1}{2},K}} = 0, \\ E^n_{z_{0,j,k+\frac{1}{2}}} &= E^n_{z_{I,j,k+\frac{1}{2}}} = E^n_{z_{i,0,k+\frac{1}{2}}} = E^n_{z_{i,J,k+\frac{1}{2}}} = 0. \end{aligned} \tag{23}$$

for $i = 0, 1, 2, ..., I-1$, $j = 0, 1, 2, ..., J-1$, and $k = 0, 1, 2, ..., K-1$.

5.2. Discrete Differential Operators and Discrete Norms. The centered difference operators in time and space are defined in a standard way (see [7]) as

$$(\delta_t F)^m_{\alpha,\beta,\gamma} = \frac{F^{m+\frac{1}{2}}_{\alpha,\beta,\gamma} - F^{m-\frac{1}{2}}_{\alpha,\beta,\gamma}}{\Delta t}, \qquad (\delta_x F)^m_{\alpha,\beta,\gamma} = \frac{F^m_{\alpha+\frac{1}{2},\beta,\gamma} - F^m_{\alpha-\frac{1}{2},\beta,\gamma}}{\Delta x},$$

$$(\delta_y F)^m_{\alpha,\beta,\gamma} = \frac{F^m_{\alpha,\beta+\frac{1}{2},\gamma} - F^m_{\alpha,\beta-\frac{1}{2},\gamma}}{\Delta y}, \qquad (\delta_z F)^m_{\alpha,\beta,\gamma} = \frac{F^m_{\alpha,\beta,\gamma+\frac{1}{2}} - F^m_{\alpha,\beta,\gamma-\frac{1}{2}}}{\Delta z}.$$

We define the discrete curl operators corresponding to the continuous curl operators as:

$$\mathbf{curl}_h^+ = \begin{pmatrix} 0 & 0 & \delta_y \\ \delta_z & 0 & 0 \\ 0 & \delta_x & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{curl}_h^- = \begin{pmatrix} 0 & -\delta_z & 0 \\ 0 & 0 & -\delta_x \\ -\delta_y & 0 & 0 \end{pmatrix},$$

then $\mathbf{curl}_h$ and $\mathbf{A}_h$ are defined respectively as

$$\mathbf{curl}_h = \mathbf{curl}_h^+ + \mathbf{curl}_h^-, \quad \text{and} \quad \mathbf{A}_h = \mathbf{A}_h^+ + \mathbf{A}_h^-,$$

where

$$\mathbf{A}_h^+ = \begin{pmatrix} \mathbf{0} & \mathbf{curl}_h^+ \\ (\mathbf{curl}_h^+)^T & \mathbf{0} \end{pmatrix}, \quad \text{and} \quad \mathbf{A}_h^- = \begin{pmatrix} \mathbf{0} & \mathbf{curl}_h^- \\ (\mathbf{curl}_h^-)^T & \mathbf{0} \end{pmatrix}.$$

Let $F = (F_{\alpha,\beta,\gamma})$, and $G = (G_{\alpha,\beta,\gamma})$ be scalar fields, both being components of electric fields or both components of magnetic fields, with α, β, γ as described above. We define a discrete inner product of the two fields as

$$\langle F, G \rangle = \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} F_{\alpha,\beta,\gamma} G_{\alpha,\beta,\gamma} \Delta x \Delta y \Delta z. \tag{24}$$

For vector fields $\mathbf{F} = (F_x, F_y, F_z)^T$, and $\mathbf{G} = (G_x, G_y, G_z)^T$, (either both electric fields or both magnetic fields), their inner product is defined using (24) as

$$\langle \mathbf{F}, \mathbf{G} \rangle = \langle F_x, G_x \rangle + \langle F_y, G_y \rangle + \langle F_z, G_z \rangle. \tag{25}$$

Each inner product induces a normed vector space on the spaces of discrete electric or magnetic fields, with a discrete L^2 norm for a scalar field, F_ℓ, defined as

$$\|F_\ell\|^2 = \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \left|F_{\ell_{\alpha,\beta,\gamma}}\right|^2 \Delta x \Delta y \Delta z, \; \ell \in \{x, y, z\}, \tag{26}$$

and, using (26), a discrete L^2 norm for the vector field $\mathbf{F}$ defined as

$$\|\mathbf{F}\|^2 = \|F_x\|^2 + \|F_y\|^2 + \|F_z\|^2. \tag{27}$$

5.3. The Maxwell WSS-FDTD Schemes. We will denote the solution of the fully discretized `Maxwell WSS-FDTD` scheme for Maxwell's equations as $\mathbf{W}_\theta^n = (\mathbf{E}_\theta^n, \mathbf{H}_\theta^n)^T \approx \mathbf{W}_\theta(t^n)$ at time t^n, for $\theta \in [0, 1]$, initialized by

$$\mathbf{E}_\theta^0 = \mathbf{E}_0 \approx \mathbf{E}(0) \quad \text{and} \quad \mathbf{H}_\theta^0 = \mathbf{H}_0 \approx \mathbf{H}(0),$$

where $\mathbf{E}_\theta^n = (E_{\theta,x}^n, E_{\theta,y}^n, E_{\theta,z}^n)^T$, and $\mathbf{H}_\theta^n = (H_{\theta,x}^n, H_{\theta,y}^n, H_{\theta,z}^n)^T$. To keep the notation simple, we continue to denote the components of the intermediate variables in the fully discretized `Maxwell WSS-FDTD` scheme as $\mathbf{E}^{(i)}$ and $\mathbf{H}^{(i)}$, for $i = 1, 2$. On every time sub-interval $[t^n, t^{n+1}]$, using the Crank Nicolson scheme to discretize in time and staggering the electric and magnetic fields in space, the fully discretized `Maxwell-WSS-FDTD` schemes are described in Algorithm 3

In Algorithm 3, we compute the solution of the `Maxwell WSS-FDTD` method for given $\theta \in [0, 1]$ at time t^{n+1} by taking a weighted average of solutions obtained at the end of the two stages with weight parameter θ, $0 \le \theta \le 1$. Scalar forms of the systems to be solved in the two stages are:

Data: Given $\theta \in [0,1]$, $\mathbf{W}_\theta^0 = (\mathbf{E}_\theta^0, \mathbf{H}_\theta^0)^T$, with $\mathbf{E}_\theta^0 = \mathbf{E}_0$ satisfying (23), and $\mathbf{H}_\theta^0 = \mathbf{H}_0$.
Result: Obtain $\mathbf{W}_\theta^N = (\mathbf{E}_\theta^N, \mathbf{H}_\theta^N)^T$ with $\mathbf{E}_\theta^N$ satisfying (23)
begin

`For` $n = 0, 1, 2, ..., N-1$;
Solve for $\mathbf{E}_\theta^{n+1}$, and $\mathbf{H}_\theta^{n+1}$, from the two stages
Stage 1:

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\mathbf{E}^{(1)}\\ \mathbf{H}^{(1)}\end{pmatrix} - \begin{pmatrix}\mathbf{E}_\theta^n\\ \mathbf{H}_\theta^n\end{pmatrix}\right] = \frac{1}{2}\mathbf{A}_h^+\left[\begin{pmatrix}\mathbf{E}^{(1)}\\ \mathbf{H}^{(1)}\end{pmatrix} + \begin{pmatrix}\mathbf{E}_\theta^n\\ \mathbf{H}_\theta^n\end{pmatrix}\right], \tag{28a}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\widehat{\mathbf{E}}^{(1)}\\ \widehat{\mathbf{H}}^{(1)}\end{pmatrix} - \begin{pmatrix}\mathbf{E}_\theta^n\\ \mathbf{H}_\theta^n\end{pmatrix}\right] = \frac{1}{2}\mathbf{A}_h^-\left[\begin{pmatrix}\widehat{\mathbf{E}}^{(1)}\\ \widehat{\mathbf{H}}^{(1)}\end{pmatrix} + \begin{pmatrix}\mathbf{E}_\theta^n\\ \mathbf{H}_\theta^n\end{pmatrix}\right]. \tag{28b}$$

Stage 2:

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\mathbf{E}^{(2)}\\ \mathbf{H}^{(2)}\end{pmatrix} - \begin{pmatrix}\mathbf{E}^{(1)}\\ \mathbf{H}^{(1)}\end{pmatrix}\right] = \frac{1}{2}\mathbf{A}_h^-\left[\begin{pmatrix}\mathbf{E}^{(2)}\\ \mathbf{H}^{(2)}\end{pmatrix} + \begin{pmatrix}\mathbf{E}^{(1)}\\ \mathbf{H}^{(1)}\end{pmatrix}\right], \tag{29a}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\widehat{\mathbf{E}}^{(2)}\\ \widehat{\mathbf{H}}^{(2)}\end{pmatrix} - \begin{pmatrix}\widehat{\mathbf{E}}^{(1)}\\ \widehat{\mathbf{H}}^{(1)}\end{pmatrix}\right] = \frac{1}{2}\mathbf{A}_h^+\left[\begin{pmatrix}\widehat{\mathbf{E}}^{(2)}\\ \widehat{\mathbf{H}}^{(2)}\end{pmatrix} + \begin{pmatrix}\widehat{\mathbf{E}}^{(1)}\\ \widehat{\mathbf{H}}^{(1)}\end{pmatrix}\right]. \tag{29b}$$

$$\mathbf{E}_\theta^{n+1} = \theta\mathbf{E}^{(2)} + (1-\theta)\widehat{\mathbf{E}}^{(2)} \quad \text{and} \quad \mathbf{H}_\theta^{n+1} = \theta\mathbf{H}^{(2)} + (1-\theta)\widehat{\mathbf{H}}^{(2)}.$$

`end`

end

Algorithm 3: Maxwell WSS FDTD Scheme (`Maxwell WSS-FDTD`) for $\theta \in [0,1]$.

Stage 1:

$$\frac{1}{\Delta t}\begin{pmatrix} E_{x_{i+\frac12,j,k}}^{(1)} - E_{\theta,x_{i+\frac12,j,k}}^n \\ H_{z_{i+\frac12,j+\frac12,k}}^{(1)} - H_{\theta,z_{i+\frac12,j+\frac12,k}}^n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\delta_y H_z^{(1)})_{i+\frac12,j,k} + (\delta_y H_{\theta,z}^n)_{i+\frac12,j,k} \\ (\delta_y E_x^{(1)})_{i+\frac12,j+\frac12,k} + (\delta_y E_{\theta,x}^n)_{i+\frac12,j+\frac12,k} \end{pmatrix}, \tag{30}$$

$$\frac{1}{\Delta t}\begin{pmatrix} \widehat{E}_{z_{i,j,k+\frac12}}^{(1)} - E_{\theta,z_{i,j,k+\frac12}}^n \\ \widehat{H}_{x_{i,j+\frac12,k+\frac12}}^{(1)} - H_{\theta,x_{i,j+\frac12,k+\frac12}}^n \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} (\delta_y \widehat{H}_x^{(1)})_{i,j,k+\frac12} + (\delta_y H_{\theta,x}^n)_{i,j,k+\frac12} \\ (\delta_y \widehat{E}_z^{(1)})_{i,j+\frac12,k+\frac12} + (\delta_y E_{\theta,z}^n)_{i,j+\frac12,k+\frac12} \end{pmatrix}, \tag{31}$$

$$\frac{1}{\Delta t}\begin{pmatrix} E_{y_{i,j+\frac12,k}}^{(1)} - E_{\theta,y_{i,j+\frac12,k}}^n \\ H_{x_{i,j+\frac12,k+\frac12}}^{(1)} - H_{\theta,x_{i,j+\frac12,k+\frac12}}^n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\delta_z H_x^{(1)})_{i,j+\frac12,k} + (\delta_z H_{\theta,x}^n)_{i,j+\frac12,k} \\ (\delta_z E_y^{(1)})_{i,j+\frac12,k+\frac12} + (\delta_z E_{\theta,y}^n)_{i,j+\frac12,k+\frac12} \end{pmatrix}, \tag{32}$$

$$\frac{1}{\Delta t}\begin{pmatrix} \widehat{E}_{x_{i+\frac12,j,k}}^{(1)} - E_{\theta,x_{i+\frac12,j,k}}^n \\ \widehat{H}_{y_{i+\frac12,j,k+\frac12}}^{(1)} - H_{\theta,y_{i+\frac12,j,k+\frac12}}^n \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} (\delta_z \widehat{H}_y^{(1)})_{i+\frac12,j,k} + (\delta_z H_{\theta,y}^n)_{i+\frac12,j,k} \\ (\delta_z \widehat{E}_x^{(1)})_{i+\frac12,j,k+\frac12} + (\delta_z E_{\theta,x}^n)_{i+\frac12,j,k+\frac12} \end{pmatrix}, \tag{33}$$

$$\frac{1}{\Delta t}\begin{pmatrix} E_{z_{i,j,k+\frac12}}^{(1)} - E_{\theta,z_{i,j,k+\frac12}}^n \\ H_{y_{i+\frac12,j,k+\frac12}}^{(1)} - H_{\theta,y_{i+\frac12,j,k+\frac12}}^n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\delta_x H_y^{(1)})_{i,j,k+\frac12} + (\delta_x H_{\theta,y}^n)_{i,j,k+\frac12} \\ (\delta_x E_z^{(1)})_{i+\frac12,j,k+\frac12} + (\delta_x E_{\theta,z}^n)_{i+\frac12,j,k+\frac12} \end{pmatrix}, \tag{34}$$

$$\frac{1}{\Delta t}\begin{pmatrix} \widehat{E}_{y_{i,j+\frac12,k}}^{(1)} - E_{\theta,y_{i,j+\frac12,k}}^n \\ \widehat{H}_{z_{i+\frac12,j+\frac12,k}}^{(1)} - H_{\theta,z_{i+\frac12,j+\frac12,k}}^n \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} (\delta_x \widehat{H}_z^{(1)})_{i,j+\frac12,k} + (\delta_x H_{\theta,z}^n)_{i,j+\frac12,k} \\ (\delta_x \widehat{E}_y^{(1)})_{i+\frac12,j+\frac12,k} + (\delta_x E_{\theta,y}^n)_{i+\frac12,j+\frac12,k} \end{pmatrix}. \tag{35}$$

Stage 2:

$$\text{(36)}\quad \frac{1}{\Delta t}\begin{pmatrix} E_{x_{i+\frac12,j,k}}^{(2)} - E_{x_{i+\frac12,j,k}}^{(1)} \\ H_{y_{i+\frac12,j,k+\frac12}}^{(2)} - H_{y_{i+\frac12,j,k+\frac12}}^{(1)} \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} (\delta_z H_y^{(2)})_{i+\frac12,j,k} + (\delta_z H_y^{(1)})_{i+\frac12,j,k} \\ (\delta_z E_x^{(2)})_{i+\frac12,j,k+\frac12} + (\delta_z E_x^{(1)})_{i+\frac12,j,k+\frac12} \end{pmatrix},$$

$$\text{(37)}\quad \frac{1}{\Delta t}\begin{pmatrix} \widehat{E}_{y_{i,j+\frac12,k}}^{(2)} - \widehat{E}_{y_{i,j+\frac12,k}}^{(1)} \\ \widehat{H}_{x_{i,j+\frac12,k+\frac12}}^{(2)} - \widehat{H}_{x_{i,j+\frac12,k+\frac12}}^{(1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\delta_z \widehat{H}_x^{(2)})_{i,j+\frac12,k} + (\delta_z \widehat{H}_x^{(1)})_{i,j+\frac12,k} \\ (\delta_z \widehat{E}_y^{(2)})_{i,j+\frac12,k+\frac12} + (\delta_z \widehat{E}_y^{(1)})_{i,j+\frac12,k+\frac12} \end{pmatrix},$$

$$\text{(38)}\quad \frac{1}{\Delta t}\begin{pmatrix} E_{y_{i,j+\frac12,k}}^{(2)} - E_{y_{i,j+\frac12,k}}^{(1)} \\ H_{z_{i+\frac12,j+\frac12,k}}^{(2)} - H_{z_{i+\frac12,j+\frac12,k}}^{(1)} \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} (\delta_x H_z^{(2)})_{i,j+\frac12,k} + (\delta_x H_z^{(1)})_{i,j+\frac12,k} \\ (\delta_x E_y^{(2)})_{i+\frac12,j+\frac12,k} + (\delta_x E_y^{(1)})_{i+\frac12,j+\frac12,k} \end{pmatrix},$$

$$\text{(39)}\quad \frac{1}{\Delta t}\begin{pmatrix} \widehat{E}_{z_{i,j,k+\frac12}}^{(2)} - \widehat{E}_{z_{i,j,k+\frac12}}^{(1)} \\ \widehat{H}_{y_{i+\frac12,j,k+\frac12}}^{(2)} - \widehat{H}_{y_{i+\frac12,j,k+\frac12}}^{(1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\delta_x \widehat{H}_y^{(2)})_{i,j,k+\frac12} + (\delta_x \widehat{H}_y^{(1)})_{i,j,k+\frac12} \\ (\delta_x \widehat{E}_z^{(2)})_{i+\frac12,j,k+\frac12} + (\delta_x \widehat{E}_z^{(1)})_{i+\frac12,j,k+\frac12} \end{pmatrix},$$

(40)

$$\frac{1}{\Delta t}\begin{pmatrix} E_{z_{i,j,k+\frac12}}^{(2)} - E_{z_{i,j,k+\frac12}}^{(1)} \\ H_{x_{i,j+\frac12,k+\frac12}}^{(2)} - H_{x_{i,j+\frac12,k+\frac12}}^{(1)} \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} (\delta_y H_x^{(2)})_{i,j,k+\frac12} + (\delta_y H_x^{(1)})_{i,j,k+\frac12} \\ (\delta_y E_z^{(2)})_{i,j+\frac12,k+\frac12} + (\delta_y E_z^{(1)})_{i,j+\frac12,k+\frac12} \end{pmatrix},$$

$$\text{(41)}\quad \frac{1}{\Delta t}\begin{pmatrix} \widehat{E}_{x_{i+\frac12,j,k}}^{(2)} - \widehat{E}_{x_{i+\frac12,j,k}}^{(1)} \\ \widehat{H}_{z_{i+\frac12,j+\frac12,k}}^{(2)} - \widehat{H}_{z_{i+\frac12,j+\frac12,k}}^{(1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\delta_y \widehat{H}_z^{(2)})_{i+\frac12,j,k} + (\delta_y \widehat{H}_z^{(1)})_{i+\frac12,j,k} \\ (\delta_y \widehat{E}_x^{(2)})_{i+\frac12,j+\frac12,k} + (\delta_y \widehat{E}_x^{(1)})_{i+\frac12,j+\frac12,k} \end{pmatrix}.$$

6. Stability Analysis of the Maxwell WSS-FDTD Schemes

In this section, we will prove the stability of the fully discrete `Maxwell WSS-FDTD` schemes by using the energy technique. An important property required in proving stability of our discrete schemes is the *summation by parts* (SBP) property, which is a discrete analogue of integration by parts (IBP). Here, we present a modified and expanded form of Lemma 9 of ([6]), which describes the SBP property.

Lemma 3. *Let the scalar grid functions U, V, W (representing x, y, z components, respectively, of electric fields) and Q, R, S (representing x, y, z components, respectively, of magnetic fields) be defined on their respective staggered grids, and suppose that U, V and W satisfy the boundary conditions*

$$\begin{aligned} U_{i+\frac12,0,k} = U_{i+\frac12,J,k} = U_{i+\frac12,j,0} = U_{i+\frac12,j,K} = 0, \\ V_{0,j+\frac12,k} = V_{I,j+\frac12,k} = V_{i,j+\frac12,0} = V_{i,j+\frac12,K} = 0, \\ W_{0,j,k+\frac12} = W_{I,j,k+\frac12} = W_{i,0,k+\frac12} = W_{i,J,k+\frac12} = 0, \end{aligned}$$

for any integers $0 \le i \le I-1$, $0 \le j \le J-1$, and $0 \le k \le K-1$. Then, we have the SBP properties

$$\text{(42a)}\quad \sum_{j=0}^{J-1} S_{i+\frac12,j+\frac12,k}(\delta_y U)_{i+\frac12,j+\frac12,k} = -\sum_{j=1}^{J-1} U_{i+\frac12,j,k}(\delta_y S)_{i+\frac12,j,k},$$

$$\text{(42b)}\quad \sum_{k=0}^{K-1} R_{i+\frac12,j,k+\frac12}(\delta_z U)_{i+\frac12,j,k+\frac12} = -\sum_{k=1}^{K-1} U_{i+\frac12,j,k}(\delta_z R)_{i+\frac12,j,k},$$

$$\text{(43a)}\quad \sum_{i=0}^{I-1} S_{i+\frac12,j+\frac12,k}(\delta_x V)_{i+\frac12,j+\frac12,k} = -\sum_{i=1}^{I-1} V_{i,j+\frac12,k}(\delta_x S)_{i,j+\frac12,k},$$

$$\text{(43b)}\quad \sum_{k=0}^{K-1} Q_{i,j+\frac12,k+\frac12}(\delta_z V)_{i,j+\frac12,k+\frac12} = -\sum_{k=1}^{K-1} V_{i,j+\frac12,k}(\delta_z Q)_{i,j+\frac12,k},$$

$$\sum_{i=0}^{I-1} R_{i+\frac{1}{2},j,k+\frac{1}{2}} (\delta_x W)_{i+\frac{1}{2},j,k+\frac{1}{2}} = -\sum_{i=1}^{I-1} W_{i,j,k+\frac{1}{2}} (\delta_x R)_{i,j,k+\frac{1}{2}}, \tag{44a}$$

$$\sum_{j=0}^{J-1} Q_{i,j+\frac{1}{2},k+\frac{1}{2}} (\delta_y W)_{i,j+\frac{1}{2},k+\frac{1}{2}} = -\sum_{j=1}^{J-1} W_{i,j,k+\frac{1}{2}} (\delta_y Q)_{i,j,k+\frac{1}{2}}, \tag{44b}$$

Proof. Using the boundary conditions on U, we have

$$\begin{aligned}
\sum_{j=0}^{J-1} S_{i+\frac{1}{2},j+\frac{1}{2},k} (\delta_y U)_{i+\frac{1}{2},j+\frac{1}{2},k} &= \frac{1}{\triangle y} \sum_{j=0}^{J-1} S_{i+\frac{1}{2},j+\frac{1}{2},k} \Big(U_{i+\frac{1}{2},j+1,k} - U_{i+\frac{1}{2},j,k} \Big) \\
&= \frac{1}{\triangle y} \Big(\sum_{j=0}^{J-2} S_{i+\frac{1}{2},j+\frac{1}{2},k} U_{i+\frac{1}{2},j+1,k} - \sum_{j=1}^{J-1} S_{i+\frac{1}{2},j+\frac{1}{2},k} U_{i+\frac{1}{2},j,k} \Big) \\
&= -\frac{1}{\triangle y} \sum_{j=1}^{J-1} U_{i+\frac{1}{2},j,k} \Big(S_{i+\frac{1}{2},j+\frac{1}{2},k} - S_{i+\frac{1}{2},j-\frac{1}{2},k} \Big) \\
&= -\sum_{j=1}^{J-1} U_{i+\frac{1}{2},j,k} (\delta_y S)_{i+\frac{1}{2},j,k}.
\end{aligned}$$

This proves the SBP property (42a). Similar analysis can be applied to prove properties (42b), (43), and (44) . ☐

Theorem 2. (Discrete Energy Conservation and decay) *For the integers $n \geq 0$, let $\mathbf{E}_\theta^n, \mathbf{H}_\theta^n$ be the solution of the* `Maxwell WSS-FDTD` *schemes given in Algorithm 3 or equivalently the solution of the system of equations (30)-(41). Then for the case $\theta \in \{0,1\}$, the solution of the* `Maxwell WSS-FDTD` *schemes satisfy the discrete energy conservation property*

$$\mathcal{E}_\theta^{n+1} = \mathcal{E}_\theta^n, \tag{45}$$

while for $\theta \in (0,1)$, the solution of the `Maxwell WSS-FDTD` *scheme satisfies the energy decay*

$$\mathcal{E}_\theta^{n+1} < \mathcal{E}_\theta^n, \tag{46}$$

where, for any $\theta \in [0,1]$ we define the discrete energy $\mathcal{E}_\theta^n$ of the corresponding `Maxwell WSS-FDTD` *scheme as*

$$\mathcal{E}_\theta^n := \left(\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2 \right)^{\frac{1}{2}}.$$

Proof. For the case $\theta = 0$ and 1, the `Maxwell WSS`, and `Maxwell WSS-FDTD` schemes reduce to the sequential splitting method, and its fully discrete version, respectively. So the proof of discrete energy conservation follows from Theorem 2 in [7]. We will analyze the stability of the `Maxwell WSS-FDTD` schemes for each $\theta \in (0,1)$.

We define $\Delta = \Delta t \Delta x \Delta y \Delta z$. Multiplying both sides of the two equations in system (30) by

$$\Delta \left(E^{(1)}_{x_{i+\frac{1}{2},j,k}} + E^n_{\theta,x_{i+\frac{1}{2},j,k}} \right), \text{ and } \Delta \left(H^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}} + H^n_{\theta,z_{i+\frac{1}{2},j+\frac{1}{2},k}} \right),$$

respectively, and then summing over all E_x and H_z degrees of freedom on their respective grid in the two equations in (30), we will get

$$\begin{aligned}
&\|E_x^{(1)}\|^2 + \|H_z^{(1)}\|^2 - \|E_{\theta,x}^n\|^2 - \|H_{\theta,z}^n\|^2 \\
&\quad = \frac{\Delta}{2}\sum_{i,j,k} \delta_y H_{z_{i+\frac{1}{2},j,k}}^{(1)} E_{x_{i+\frac{1}{2},j,k}}^{(1)} + \frac{\Delta}{2}\sum_{i,j,k} \delta_y H_{\theta,z_{i+\frac{1}{2},j,k}}^{n} E_{x_{i+\frac{1}{2},j,k}}^{(1)} \\
&\quad + \frac{\Delta}{2}\sum_{i,j,k} \delta_y H_{z_{i+\frac{1}{2},j,k}}^{(1)} E_{\theta,x_{i+\frac{1}{2},j,k}}^{n} + \frac{\Delta}{2}\sum_{i,j,k} \delta_y H_{\theta,z_{i+\frac{1}{2},j,k}}^{n} E_{\theta,x_{i+\frac{1}{2},j,k}}^{n} \\
&\quad + \frac{\Delta}{2}\sum_{i,j,k} \delta_y E_{x_{i+\frac{1}{2},j+\frac{1}{2},k}}^{(1)} H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{(1)} + \frac{\Delta}{2}\sum_{i,j,k} \delta_y E_{\theta,x_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n} H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{(1)} \\
&\quad + \frac{\Delta}{2}\sum_{i,j,k} \delta_y E_{x_{i+\frac{1}{2},j+\frac{1}{2},k}}^{(1)} H_{\theta,z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n} + \frac{\Delta}{2}\sum_{i,j,k} \delta_y E_{\theta,x_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n} H_{\theta,z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n}.
\end{aligned}$$

Rearranging terms by using Lemma 3 and the fact that $\mathbf{E}_\theta^n$ and $\mathbf{E}^{(1)}$ satisfy the PEC boundary condition (23), the right hand side in the above equation will be zero and we get

$$\|E_x^{(1)}\|^2 + \|H_z^{(1)}\|^2 = \|E_{\theta,x}^n\|^2 + \|H_{\theta,z}^n\|^2. \tag{47}$$

By a similar process system applied to systems (31)-(40), we get the following identities

$$\|\widehat{H}_x^{(1)}\|^2 + \|\widehat{E}_z^{(1)}\|^2 = \|H_{\theta,x}^n\|^2 + \|E_{\theta,z}^n\|^2, \tag{48}$$

$$\|E_y^{(1)}\|^2 + \|H_x^{(1)}\|^2 = \|E_{\theta,y}^n\|^2 + \|H_{\theta,x}^n\|^2, \tag{49}$$

$$\|\widehat{H}_y^{(1)}\|^2 + \|\widehat{E}_x^{(1)}\|^2 = \|H_{\theta,y}^n\|^2 + \|E_{\theta,x}^n\|^2, \tag{50}$$

$$\|E_z^{(1)}\|^2 + \|H_y^{(1)}\|^2 = \|E_{\theta,z}^n\|^2 + \|H_{\theta,y}^n\|^2, \tag{51}$$

$$\|\widehat{H}_z^{(1)}\|^2 + \|\widehat{E}_y^{(1)}\|^2 = \|H_{\theta,z}^n\|^2 + \|E_{\theta,y}^n\|^2, \tag{52}$$

$$\|E_x^{(2)}\|^2 + \|H_y^{(2)}\|^2 = \|E_x^{(1)}\|^2 + \|H_y^{(1)}\|^2, \tag{53}$$

$$\|\widehat{H}_x^{(2)}\|^2 + \|\widehat{E}_y^{(2)}\|^2 = \|\widehat{H}_x^{(1)}\|^2 + \|\widehat{E}_y^{(1)}\|^2, \tag{54}$$

$$\|E_y^{(2)}\|^2 + \|H_z^{(2)}\|^2 = \|E_y^{(1)}\|^2 + \|H_z^{(1)}\|^2, \tag{55}$$

$$\|\widehat{H}_y^{(2)}\|^2 + \|\widehat{E}_z^{(2)}\|^2 = \|\widehat{H}_y^{(1)}\|^2 + \|\widehat{E}_z^{(1)}\|^2, \tag{56}$$

$$\|E_z^{(2)}\|^2 + \|H_x^{(2)}\|^2 = \|E_z^{(1)}\|^2 + \|H_x^{(1)}\|^2, \tag{57}$$

$$\|\widehat{H}_z^{(2)}\|^2 + \|\widehat{E}_x^{(2)}\|^2 = \|\widehat{H}_z^{(1)}\|^2 + \|\widehat{E}_x^{(1)}\|^2. \tag{58}$$

Adding equations (47), (49) and (51), we get

$$\|\mathbf{E}^{(1)}\|^2 + \|\mathbf{H}^{(1)}\|^2 = \|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2. \tag{59}$$

Similarly, adding equations (48), (50) and (52), we get

$$\|\widehat{\mathbf{H}}^{(1)}\|^2 + \|\widehat{\mathbf{E}}^{(1)}\|^2 = \|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2. \tag{60}$$

Adding equations (53), (55) and (57), we obtain

$$\|\mathbf{E}^{(2)}\|^2 + \|\mathbf{H}^{(2)}\|^2 = \|\mathbf{E}^{(1)}\|^2 + \|\mathbf{H}^{(1)}\|^2, \tag{61}$$

and finally adding equations (54), (56) and (58), we get

$$\|\widehat{\mathbf{H}}^{(2)}\|^2 + \|\widehat{\mathbf{E}}^{(2)}\|^2 = \|\widehat{\mathbf{H}}^{(1)}\|^2 + \|\widehat{\mathbf{E}}^{(1)}\|^2.$$

Since

$$\mathbf{E}_\theta^{n+1} = \theta\mathbf{E}^{(2)} + (1-\theta)\widehat{\mathbf{E}}^{(2)} \quad \text{and} \quad \mathbf{H}_\theta^{n+1} = \theta\mathbf{H}^{(2)} + (1-\theta)\widehat{\mathbf{H}}^{(2)},$$

we have

$$
\begin{aligned}
\left(\mathcal{E}_\theta^{n+1}\right)^2 &= \|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2 \\
&= \theta^2\left(\|\mathbf{E}^{(2)}\|^2 + \|\mathbf{H}^{(2)}\|^2\right) + (1-\theta)^2\left(\|\widehat{\mathbf{E}}^{(2)}\|^2 + \|\widehat{\mathbf{H}}^{(2)}\|^2\right) \\
&\qquad + 2\theta(1-\theta)\left[\left\langle \mathbf{E}^{(2)}, \widehat{\mathbf{E}}^{(2)}\right\rangle + \left\langle \mathbf{H}^{(2)}, \widehat{\mathbf{H}}^{(2)}\right\rangle\right] \\
&= \left(\theta^2 + (1-\theta)^2\right)\left(\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2\right) + 2\theta(1-\theta)\left[\left\langle \mathbf{E}^{(2)}, \widehat{\mathbf{E}}^{(2)}\right\rangle + \left\langle \mathbf{H}^{(2)}, \widehat{\mathbf{H}}^{(2)}\right\rangle\right]. \qquad (62)
\end{aligned}
$$

From (62) we can see that for the case $\theta = 0$ or $\theta = 1$, the `Maxwell WSS-FDTD` scheme satisfies the energy conservation. By using Cauchy-Schwartz inequality and AM-GM inequality, we get that

$$
\begin{aligned}
\left\langle \mathbf{E}^{(2)}, \widehat{\mathbf{E}}^{(2)}\right\rangle + \left\langle \mathbf{H}^{(2)}, \widehat{\mathbf{H}}^{(2)}\right\rangle &\leq \|\mathbf{E}^{(2)}\|\|\widehat{\mathbf{E}}^{(2)}\| + \|\mathbf{H}^{(2)}\|\|\widehat{\mathbf{H}}^{(2)}\| \\
&\leq \frac{1}{2}\left(\|\mathbf{E}^{(2)}\|^2 + \|\widehat{\mathbf{E}}^{(2)}\|^2 + \|\mathbf{H}^{(2)}\|^2 + \|\widehat{\mathbf{H}}^{(2)}\|^2\right) \\
&= \|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2. \qquad (63)
\end{aligned}
$$

So from the equation (62), the energy of `Maxwell WSS` scheme decays:

$$
\begin{aligned}
\left(\mathcal{E}_\theta^{n+1}\right)^2 &= \|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2 \leq \left(\theta^2 + (1-\theta)^2 + 2\theta(1-\theta)\right)\left(\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2\right) \\
&= \|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2 = \left(\mathcal{E}_\theta^n\right)^2.
\end{aligned}
$$

□

Since the energy conservation or decay is unconditional on the time step Δt or the mesh step sizes $\Delta x, \Delta y$ or Δz, we have the following corollary:

Corollary 1. *The weighted sequential splitting methods,* `Maxwell WSS-FDTD`*, for the three dimensional Maxwell's equations with weight parameters $\theta \in [0,1]$, and PEC boundary conditions* (5) *are unconditionally stable.*

6.1. Analysis of Discrete Energy Decay in the `Maxwell WSS-FDTD` Methods. In this section, we further analyze the energy decay of the `Maxwell WSS-FDTD` schemes for $\theta \in (0,1)$. We obtain a lower bound on the energy ratio at time $n+1$ and n. We also compute an energy decay rate that we prove to be third order in Δt.

For a lower bound on the energy ratio, we rewrite equation (62) as

$$
\begin{aligned}
&\|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2 \\
&\quad = \|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2 + 2\theta(1-\theta)\left[\left\langle \mathbf{E}^{(2)}, \widehat{\mathbf{E}}^{(2)}\right\rangle + \left\langle \mathbf{H}^{(2)}, \widehat{\mathbf{H}}^{(2)}\right\rangle - \left(\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2\right)\right].
\end{aligned}
$$

Dividing both sides by $\|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2$ we have

$$
\frac{\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2}{\|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2} = \frac{1}{1-2\theta(1-\theta)}\left[1 - 2\theta(1-\theta)\frac{\left\langle \mathbf{E}^{(2)}, \widehat{\mathbf{E}}^{(2)}\right\rangle + \left\langle \mathbf{H}^{(2)}, \widehat{\mathbf{H}}^{(2)}\right\rangle}{\|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2}\right].
$$

Using the fact that energy decays and (63), we get the inequality

$$
\begin{aligned}
\frac{\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2}{\|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2} &\leq \frac{1}{1-2\theta(1-\theta)}\left[1 - 2\theta(1-\theta)\frac{\left\langle \mathbf{E}^{(2)}, \widehat{\mathbf{E}}^{(2)}\right\rangle + \left\langle \mathbf{H}^{(2)}, \widehat{\mathbf{H}}^{(2)}\right\rangle}{\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2}\right] \\
&\leq \frac{1+2\theta(1-\theta)}{1-2\theta(1-\theta)}.
\end{aligned}
$$

Thus, the energy ratio satisfies the lower bound

$$
\frac{\|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2}{\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2} \geq \frac{1-2\theta(1-\theta)}{1+2\theta(1-\theta)},
$$

and we obtain

$$
(64)\qquad \frac{1-2\theta(1-\theta)}{1+2\theta(1-\theta)} \le \frac{\|\mathbf{E}_\theta^{n+1}\|^2+\|\mathbf{H}_\theta^{n+1}\|^2}{\|\mathbf{E}_\theta^{n}\|^2+\|\mathbf{H}_\theta^{n}\|^2} = \Big(\frac{\mathcal{E}_\theta^{n+1}}{\mathcal{E}_\theta^{n}}\Big)^2 \le 1,
$$

for $\theta \in (0,1)$ and $n = 1,2,...,N$. We note that the minimum value of the lower bound on the left hand side in the inequality above is at $\theta = 0.5$. Thus, we expect that the `Maxwell SWSS-FDTD` scheme is the most dissipative scheme in the family of `Maxwell WSS-FDTD` schemes. However, as we will see below, this dissipation is of higher order in time than the convergence rate of the schemes.

6.1.1. Computation of the Energy Decay Rate. In this section, we will show that the discrete energy decay rate at time $t^{n+\frac{1}{2}}$, defined as

$$
(65)\qquad \delta_t \mathcal{E}_\theta^{n+\frac{1}{2}} := \frac{\mathcal{E}_\theta^{n+1}-\mathcal{E}_\theta^{n}}{\Delta t},
$$

is third order in the time step Δt. For convenience, we introduce the temporary variables

$$
\mathbf{W}^\alpha = \begin{pmatrix}\mathbf{E}^{(\alpha)}\\ \mathbf{H}^{(\alpha)}\end{pmatrix}, \quad \widehat{\mathbf{W}}^\alpha = \begin{pmatrix}\widehat{\mathbf{E}}^{(\alpha)}\\ \widehat{\mathbf{H}}^{(\alpha)}\end{pmatrix}, \quad \text{and} \quad \mathbf{W}_\theta^\beta = \begin{pmatrix}\mathbf{E}_\theta^{\beta}\\ \mathbf{H}_\theta^{\beta}\end{pmatrix},
$$

where $\alpha \in \{1,2\}$ and $\beta \in \{n, n+1\}$. From (62), we have

$$
(66)\qquad \Big(\mathcal{E}_\theta^{n+1}\Big)^2 = \Big(\mathcal{E}_\theta^{n}\Big)^2 + 2\theta(1-\theta)\left[\Big\langle \mathbf{W}^2, \widehat{\mathbf{W}}^2\Big\rangle - \Big(\mathcal{E}_\theta^{n}\Big)^2\right].
$$

Defining the average $\bar{\mathcal{E}}_\theta^{\,n+1/2} := \frac{1}{2}(\mathcal{E}_\theta^{n+1} + \mathcal{E}_\theta^{n})$, equation (66) becomes

$$
(67)\qquad \mathcal{E}_\theta^{n+1} = \mathcal{E}_\theta^{n} + \frac{\theta(1-\theta)}{\bar{\mathcal{E}}_\theta^{\,n+1/2}}\left(\Big\langle \mathbf{W}^2, \widehat{\mathbf{W}}^2\Big\rangle - \Big(\mathcal{E}_\theta^{n}\Big)^2\right).
$$

From the first equation of Stage 1 (28a) of the `Maxwell WSS-FDTD` method we have

$$
\mathbf{W}^1 = \mathbf{W}_\theta^n + \frac{\triangle t}{2}\mathbf{A}_h^+(\mathbf{W}^1 + \mathbf{W}_\theta^n).
$$

Recursively substituting $\mathbf{W}^1$ on the right side several times, we get that

$$
\begin{aligned}
\mathbf{W}^1 = \mathbf{W}_\theta^n + \triangle t\mathbf{A}_h^+\mathbf{W}_\theta^n + \frac{\triangle t^2}{2}(\mathbf{A}_h^+)^2\mathbf{W}_\theta^n + ... + \frac{\triangle t^k}{2^{k-1}}(\mathbf{A}_h^+)^k\mathbf{W}_\theta^n \\
+ \Big(\frac{\triangle t}{2}\Big)^{k+1}(\mathbf{A}_h^+)^{k+1}(\mathbf{W}_\theta^n + \mathbf{W}^1).
\end{aligned}
$$

Similarly, from the other equations in Stage 1 and Stage 2 (29a) of the `Maxwell WSS-FDTD` method we obtain the identities

$$
\begin{aligned}
\widehat{\mathbf{W}}^1 &= \mathbf{W}_\theta^n + \triangle t\mathbf{A}_h^-\mathbf{W}_\theta^n + \frac{\triangle t^2}{2}(\mathbf{A}_h^-)^2\mathbf{W}_\theta^n + ... + \frac{\triangle t^k}{2^{k-1}}(\mathbf{A}_h^-)^k\mathbf{W}_\theta^n \\
&\quad + \Big(\frac{\triangle t}{2}\Big)^{k+1}(\mathbf{A}_h^-)^{k+1}(\mathbf{W}_\theta^n + \widehat{\mathbf{W}}^1), \\
\mathbf{W}^2 &= \mathbf{W}^1 + \triangle t\mathbf{A}_h^-\mathbf{W}^1 + \frac{\triangle t^2}{2}(\mathbf{A}_h^-)^2\mathbf{W}^1 + ... + \frac{\triangle t^k}{2^{k-1}}(\mathbf{A}_h^-)^k\mathbf{W}^1 \\
&\quad + \Big(\frac{\triangle t}{2}\Big)^{k+1}(\mathbf{A}_h^-)^{k+1}(\mathbf{W}^1 + \mathbf{W}^2), \\
\widehat{\mathbf{W}}^2 &= \widehat{\mathbf{W}}^1 + \triangle t\mathbf{A}_h^+\widehat{\mathbf{W}}^1 + \frac{\triangle t^2}{2}(\mathbf{A}_h^+)^2\widehat{\mathbf{W}}^1 + ... + \frac{\triangle t^k}{2^{k-1}}(\mathbf{A}_h^+)^k\widehat{\mathbf{W}}^1 \\
&\quad + \Big(\frac{\triangle t}{2}\Big)^{k+1}(\mathbf{A}_h^+)^{k+1}(\widehat{\mathbf{W}}^1 + \widehat{\mathbf{W}}^2),
\end{aligned}
$$

Thus, both $\mathbf{W}^2$ and $\widehat{\mathbf{W}}^2$ can be written in terms of $\mathbf{W}_\theta^n$ as

$$\begin{aligned}
\mathbf{W}^2 &= \mathbf{W}_\theta^n + \triangle t\mathbf{A}_h\mathbf{W}_\theta^n + \frac{\triangle t^2}{2}\Big((\mathbf{A}_h^+)^2 + 2\mathbf{A}_h^-\mathbf{A}_h^+ + (\mathbf{A}_h^-)^2\Big)\mathbf{W}_\theta^n\\
&+ \frac{\triangle t^3}{2^2}\Big((\mathbf{A}_h^+)^3 + 2\mathbf{A}_h^-(\mathbf{A}_h^+)^2 + 2(\mathbf{A}_h^-)^2\mathbf{A}_h^+ + (\mathbf{A}_h^-)^3\Big)\mathbf{W}_\theta^n\\
&+ \frac{\triangle t^4}{2^3}\Big((\mathbf{A}_h^+)^4 + 2\mathbf{A}_h^-(\mathbf{A}_h^+)^3 + 2(\mathbf{A}_h^-)^2(\mathbf{A}_h^+)^2 + 2(\mathbf{A}_h^-)^3\mathbf{A}_h^+ + (\mathbf{A}_h^-)^4\Big)\mathbf{W}_\theta^n + \mathcal{O}(\triangle t^5)\\
\widehat{\mathbf{W}}^2 &= \mathbf{W}_\theta^n + \triangle t\mathbf{A}_h\mathbf{W}_\theta^n + \frac{\triangle t^2}{2}\Big((\mathbf{A}_h^+)^2 + 2\mathbf{A}_h^+\mathbf{A}_h^- + (\mathbf{A}_h^-)^2\Big)\mathbf{W}_\theta^n\\
&+ \frac{\triangle t^3}{2^2}\Big((\mathbf{A}_h^+)^3 + 2\mathbf{A}_h^+(\mathbf{A}_h^-)^2 + 2(\mathbf{A}_h^+)^2\mathbf{A}_h^- + (\mathbf{A}_h^-)^3\Big)\mathbf{W}_\theta^n\\
&+ \frac{\triangle t^4}{2^3}\Big((\mathbf{A}_h^+)^4 + 2\mathbf{A}_h^+(\mathbf{A}_h^-)^3 + 2(\mathbf{A}_h^+)^2(\mathbf{A}_h^-)^2 + 2(\mathbf{A}_h^+)^3\mathbf{A}_h^- + (\mathbf{A}_h^-)^4\Big)\mathbf{W}_\theta^n + \mathcal{O}(\triangle t^5)
\end{aligned}$$

Hence the inner product $\left\langle \mathbf{W}^2, \widehat{\mathbf{W}}^2\right\rangle$ can be computed as

(68)

$$\begin{aligned}
&\left\langle \mathbf{W}^2, \widehat{\mathbf{W}}^2\right\rangle = \langle \mathbf{W}_\theta^n, \mathbf{W}_\theta^n\rangle + 2\triangle t\,\langle \mathbf{W}_\theta^n, \mathbf{A}_h\mathbf{W}_\theta^n\rangle\\
&+ \triangle t^2\Big(\langle \mathbf{A}_h\mathbf{W}_\theta^n, \mathbf{A}_h\mathbf{W}_\theta^n\rangle + \langle \mathbf{A}_h^2\mathbf{W}_\theta^n, \mathbf{W}_\theta^n\rangle\Big)\\
&+ \frac{\triangle t^3}{2}\Big[\Big\langle\Big(\mathbf{A}_h^3 - \mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^+ - \mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^-\Big)\mathbf{W}_\theta^n, \mathbf{W}_\theta^n\Big\rangle + 2\left\langle \mathbf{A}_h^2\mathbf{W}_\theta^n, \mathbf{A}_h\mathbf{W}_\theta^n\right\rangle\Big]\\
&+ \frac{\triangle t^4}{2^2}\Big[\Big\langle\Big(\mathbf{A}_h^2 + [\mathbf{A}_h^+, \mathbf{A}_h^-]\Big)\mathbf{W}_\theta^n, \Big(\mathbf{A}_h^2 - [\mathbf{A}_h^+, \mathbf{A}_h^-]\Big)\mathbf{W}_\theta^n\Big\rangle\\
&\quad + \Big\langle\Big(\mathbf{A}_h^4 - \mathbf{A}_h^-(\mathbf{A}_h^+)^2\mathbf{A}_h^- - (\mathbf{A}_h^-)^2\mathbf{A}_h^+\mathbf{A}_h^- - (\mathbf{A}_h^+)^2\mathbf{A}_h^-\mathbf{A}_h^+ - \mathbf{A}_h^+(\mathbf{A}_h^-)^2\mathbf{A}_h^+\Big)\mathbf{W}_\theta^n, \mathbf{W}_\theta^n\Big\rangle\\
&\quad + 2\Big\langle\Big(\mathbf{A}_h^3 - \mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^+ - \mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^-\Big)\mathbf{W}_\theta^n, \mathbf{A}_h\mathbf{W}_\theta^n\Big\rangle\Big] + \mathcal{O}(\triangle t^5).
\end{aligned}$$

Finally, substituting (68) into equation (67) we get

(69)

$$\begin{aligned}
\mathcal{E}_\theta^{n+1} =\ & \mathcal{E}_\theta^n + 2\frac{\theta(1-\theta)\triangle t}{\bar{\mathcal{E}}_\theta^{\,n+1/2}}\langle \mathbf{W}_\theta^n, \mathbf{A}_h\mathbf{W}_\theta^n\rangle\\
&+ \frac{\theta(1-\theta)\triangle t^2}{\bar{\mathcal{E}}_\theta^{\,n+1/2}}\Big(\langle \mathbf{A}_h\mathbf{W}_\theta^n, \mathbf{A}_h\mathbf{W}_\theta^n\rangle + \langle \mathbf{A}_h^2\mathbf{W}_\theta^n, \mathbf{W}_\theta^n\rangle\Big)\\
&+ \frac{\theta(1-\theta)\triangle t^3}{2\ \bar{\mathcal{E}}_\theta^{\,n+1/2}}\Big[\Big\langle\Big(\mathbf{A}_h^3 - \mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^+ - \mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^-\Big)\mathbf{W}_\theta^n, \mathbf{W}_\theta^n\Big\rangle + 2\left\langle \mathbf{A}_h^2\mathbf{W}_\theta^n, \mathbf{A}_h\mathbf{W}_\theta^n\right\rangle\Big]\\
&+ \frac{\theta(1-\theta)\triangle t^4}{2^2\ \bar{\mathcal{E}}_\theta^{\,n+1/2}}\Big[\Big\langle\Big(\mathbf{A}_h^2 + [\mathbf{A}_h^+, \mathbf{A}_h^-]\Big)\mathbf{W}_\theta^n, \Big(\mathbf{A}_h^2 - [\mathbf{A}_h^+, \mathbf{A}_h^-]\Big)\mathbf{W}_\theta^n\Big\rangle\\
&+ \Big\langle\Big(\mathbf{A}_h^4 - \mathbf{A}_h^-(\mathbf{A}_h^+)^2\mathbf{A}_h^- - (\mathbf{A}_h^-)^2\mathbf{A}_h^+\mathbf{A}_h^- - (\mathbf{A}_h^+)^2\mathbf{A}_h^-\mathbf{A}_h^+ - \mathbf{A}_h^+(\mathbf{A}_h^-)^2\mathbf{A}_h^+\Big)\mathbf{W}_\theta^n, \mathbf{W}_\theta^n\Big\rangle\\
&+ 2\Big\langle\Big(\mathbf{A}_h^3 - \mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^+ - \mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^-\Big)\mathbf{W}_\theta^n, \mathbf{A}_h\mathbf{W}_\theta^n\Big\rangle\Big] + \mathcal{O}(\triangle t^5).
\end{aligned}$$

We now show that the coefficients of Δt, Δt^2, Δt^3 in the energy identity (69) are all zero.

Proposition 1. *For $k \in \mathbb{N}$, let $\mathbf{H}^k$ be a discrete magnetic field at time t^k, and $\mathbf{E}^k$ be a discrete electric field at time t^k, satisfying the PEC boundary conditions* (5)*. Then the following identity holds:*

$$\left\langle \mathbf{E}^k, \mathbf{curl}_h\mathbf{H}^k\right\rangle = \left\langle \mathbf{H}^k, \mathbf{curl}_h\mathbf{E}^k\right\rangle.$$

Proof. This proposition essentially says that the discrete curl operator, $\mathbf{curl}_h$, is self-adjoint. This self-adjointness of the $\mathbf{curl}_h$ operator follows from the SBP property given in Lemma 3. □

Lemma 4. *For $k \in \mathbb{N}$, and discrete vector functions $\mathbf{W}^k := (\mathbf{E}^k, \mathbf{H}^k)^T$ at time t^k, and $\mathbf{W}^m := (\mathbf{E}^m, \mathbf{H}^m)^T$ at time t^m, where $\mathbf{E}^k$ and $\mathbf{E}^m$ satisfy PEC boundary conditions* (5), *we have*

$$\left\langle \mathbf{A}_h \mathbf{W}^k, \mathbf{W}^m \right\rangle + \left\langle \mathbf{W}^k, \mathbf{A}_h \mathbf{W}^m \right\rangle = 0.$$

Proof. Apply Proposition 1. □

Theorem 3. *For all $n \in \mathbb{N}$, and $\mathbf{W}_\theta^n = (\mathbf{E}_\theta^n, \mathbf{H}_\theta^n)^T$, solution to the* `Maxwell WSS-FDTD` *method for $\theta \in [0, 1]$, the following identities hold*

$$\langle \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \rangle = 0, \tag{70}$$

$$\langle \mathbf{A}_h \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \rangle + \left\langle \mathbf{A}_h^2 \mathbf{W}_\theta^n, \mathbf{W}_\theta^n \right\rangle = 0, \tag{71}$$

$$\left\langle \mathbf{A}_h^2 \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \right\rangle = 0, \tag{72}$$

$$\left\langle \left(\mathbf{A}_h^3 - \mathbf{A}_h^+ \mathbf{A}_h^- \mathbf{A}_h^+ - \mathbf{A}_h^- \mathbf{A}_h^+ \mathbf{A}_h^- \right) \mathbf{W}_\theta^n, \mathbf{W}_\theta^n \right\rangle + \left\langle \mathbf{A}_h^2 \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \right\rangle = 0. \tag{73}$$

Proof. From Algorithm 3 we have

$$\begin{aligned}
\left\langle \mathbf{W}_\theta^{n+1}, \mathbf{A}_h \mathbf{W}_\theta^{n+1} \right\rangle &= \left\langle \mathbf{E}_\theta^{n+1}, \mathbf{curl}_h \mathbf{H}_\theta^{n+1} \right\rangle - \left\langle \mathbf{H}_\theta^{n+1}, \mathbf{curl}_h \mathbf{E}_\theta^{n+1} \right\rangle \\
&= \theta^2 \Big[\left\langle \mathbf{E}^{(2)}, \mathbf{curl}_h \mathbf{H}^{(2)} \right\rangle - \left\langle \mathbf{H}^{(2)}, \mathbf{curl}_h \mathbf{E}^{(2)} \right\rangle \Big] \\
&\quad + \theta(1-\theta) \Big[\left\langle \widehat{\mathbf{E}}^{(2)}, \mathbf{curl}_h \mathbf{H}^{(2)} \right\rangle - \left\langle \mathbf{H}^{(2)}, \mathbf{curl}_h \widehat{\mathbf{E}}^{(2)} \right\rangle \Big] \\
&\qquad + \theta(1-\theta) \Big[\left\langle \mathbf{E}^{(2)}, \mathbf{curl}_h \widehat{\mathbf{H}}^{(2)} \right\rangle - \left\langle \widehat{\mathbf{H}}^{(2)}, \mathbf{curl}_h \mathbf{E}^{(2)} \right\rangle \Big] + \\
&\qquad\quad (1-\theta)^2 \Big[\left\langle \widehat{\mathbf{E}}^{(2)}, \mathbf{curl}_h \widehat{\mathbf{H}}^{(2)} \right\rangle - \left\langle \widehat{\mathbf{H}}^{(2)}, \mathbf{curl}_h \widehat{\mathbf{E}}^{(2)} \right\rangle \Big].
\end{aligned}$$

Applying Proposition 1, we get that $\left\langle \mathbf{W}_\theta^{n+1}, \mathbf{A}_h \mathbf{W}_\theta^{n+1} \right\rangle = 0$.

Next, we have

$$\begin{aligned}
\left\langle \mathbf{A}_h^2 \mathbf{W}_\theta^n, \mathbf{W}_\theta^n \right\rangle &= - \langle \mathbf{curl}_h (\mathbf{curl}_h \mathbf{E}_\theta^n), \mathbf{E}_\theta^n \rangle - \langle \mathbf{curl}_h (\mathbf{curl}_h \mathbf{H}_\theta^n), \mathbf{H}_\theta^n \rangle \\
&= - \langle \mathbf{curl}_h \mathbf{E}_\theta^n, \mathbf{curl}_h \mathbf{E}_\theta^n \rangle - \langle \mathbf{curl}_h \mathbf{H}_\theta^n, \mathbf{curl}_h \mathbf{H}_\theta^n \rangle \\
&= - \langle \mathbf{A}_h \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \rangle,
\end{aligned}$$

and the second identity holds.

Next, applying Lemma 4, we have

$$\begin{aligned}
\left\langle \mathbf{A}_h^2 \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \right\rangle &= \langle \mathbf{A}_h (\mathbf{A}_h \mathbf{W}_\theta^n), \mathbf{A}_h \mathbf{W}_\theta^n \rangle = - \langle \mathbf{A}_h \mathbf{W}_\theta^n, \mathbf{A}_h (\mathbf{A}_h \mathbf{W}_\theta^n) \rangle \\
&= - \left\langle \mathbf{A}_h \mathbf{W}_\theta^n, \mathbf{A}_h^2 \mathbf{W}_\theta^n \right\rangle.
\end{aligned}$$

Thus, $\left\langle \mathbf{A}_h^2 \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \right\rangle = 0$, and the third identity is proved.

Finally, based on the fact that the discrete differential operators δ_x, δ_y and δ_z commute, we can show that $\mathbf{A}_h^+ \mathbf{A}_h^- \mathbf{A}_h^+ + \mathbf{A}_h^- \mathbf{A}_h^+ \mathbf{A}_h^- = \mathbf{0}$. Then, applying Lemma 4, we get

$$\left\langle \mathbf{A}_h^3 \mathbf{W}_\theta^n, \mathbf{W}_\theta^n \right\rangle = \left\langle \mathbf{A}_h (\mathbf{A}_h^2 \mathbf{W}_\theta^n), \mathbf{W}_\theta^n \right\rangle = - \left\langle \mathbf{A}_h^2 \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \right\rangle$$

Thus $\left\langle \mathbf{A}_h^3 \mathbf{W}_\theta^n, \mathbf{W}_\theta^n \right\rangle + \left\langle \mathbf{A}_h^2 \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \right\rangle = 0$. □

Applying Theorem 3, we get that the coefficients of $\triangle t, \triangle t^2$ and $\triangle t^3$ in the energy equation (69) are all zero. Thus, from (69) we have

$$
\begin{aligned}
\delta_t \mathcal{E}_\theta^{n+\frac{1}{2}} &= \frac{\mathcal{E}_\theta^{n+1} - \ \mathcal{E}_\theta^{n}}{\Delta t} \\
&= \frac{\theta(1-\theta)}{2^2 \ \bar{\mathcal{E}}_\theta^{\,n+1/2}} \triangle t^3 \Big[\left\langle \mathcal{B}_h \mathbf{W}_\theta^n, \mathbf{W}_\theta^n \right\rangle + 2 \left\langle \mathbf{A}_h^3 \mathbf{W}_\theta^n, \mathbf{A}_h \mathbf{W}_\theta^n \right\rangle \qquad (74) \\
&+ \Big\langle \Big(\mathbf{A}_h^2 + [\mathbf{A}_h^+, \mathbf{A}_h^-] \Big) \mathbf{W}_\theta^n, \Big(\mathbf{A}_h^2 - [\mathbf{A}_h^+, \mathbf{A}_h^-] \Big) \mathbf{W}_\theta^n \Big\rangle \Big] + \mathcal{O}(\triangle t^4).
\end{aligned}
$$

where

$$
\mathcal{B}_h := \Big(\mathbf{A}_h^4 - \mathbf{A}_h^- (\mathbf{A}_h^+)^2 \mathbf{A}_h^- - (\mathbf{A}_h^-)^2 \mathbf{A}_h^+ \mathbf{A}_h^- - (\mathbf{A}_h^+)^2 \mathbf{A}_h^- \mathbf{A}_h^+ - \mathbf{A}_h^+ (\mathbf{A}_h^-)^2 \mathbf{A}_h^+ \Big). \qquad (75)
$$

The coefficient of $\triangle t^3$ in the energy decay rate is not zero. Thus, the energy decay rate is of third order in $\triangle t$ for all values of $\theta \in (0, 1)$.

7. Convergence Analysis of the Maxwell WSS-FDTD Methods

In this section we prove the convergence of the fully discrete `Maxwell WSS-FDTD` scheme for the case $\theta \in (0, 1)$. The proof of convergence for the case of $\theta \in \{0, 1\}$, for which the `Maxwell WSS-FDTD` scheme reduces to the standard sequential splitting, has been carried out in [7]. The proof of convergence is based on the stability analysis in Section 6, and the analysis of truncation errors. In order to prove convergence we need to define new intermediate variables. We define the intermediate variables

(76)

$$
\begin{pmatrix} \widetilde{\mathbf{E}}^{(1)} \\ \widetilde{\mathbf{H}}^{(1)} \end{pmatrix}(t^n) := \Big[I + \Delta t \mathbf{A}_h^+ + \frac{(\Delta t)^2}{2} (\mathbf{A}_h^+)^2 \Big] \begin{pmatrix} \mathbf{E}(t^n) \\ \mathbf{H}(t^n) \end{pmatrix} = \mathrm{e}^{\Delta t \mathbf{A}_h^+} \begin{pmatrix} \mathbf{E}(t^n) \\ \mathbf{H}(t^n) \end{pmatrix} + \mathcal{O}(\triangle t^3),
$$

(77)

$$
\begin{pmatrix} \widetilde{\widetilde{\mathbf{E}}}^{(1)} \\ \widetilde{\widetilde{\mathbf{H}}}^{(1)} \end{pmatrix}(t^n) := \Big[I + \Delta t \mathbf{A}_h^- + \frac{(\Delta t)^2}{2} (\mathbf{A}_h^-)^2 \Big] \begin{pmatrix} \mathbf{E}(t^n) \\ \mathbf{H}(t^n) \end{pmatrix} = \mathrm{e}^{\Delta t \mathbf{A}_h^-} \begin{pmatrix} \mathbf{E}(t^n) \\ \mathbf{H}(t^n) \end{pmatrix} + \mathcal{O}(\triangle t^3),
$$

$$
\begin{aligned}
\begin{pmatrix} \widetilde{\widetilde{\mathbf{E}}}^{(2)} \\ \widetilde{\widetilde{\mathbf{H}}}^{(2)} \end{pmatrix}(t^n) &:= \Big[I + \Delta t \mathbf{A}_h + \frac{(\Delta t)^2}{2} \left((\mathbf{A}_h^+)^2 + 2 \mathbf{A}_h^+ \mathbf{A}_h^- + (\mathbf{A}_h^-)^2 \right) \Big] \begin{pmatrix} \mathbf{E}(t^n) \\ \mathbf{H}(t^n) \end{pmatrix} \\
&= \mathrm{e}^{\Delta t \mathbf{A}_h^+} \mathrm{e}^{\Delta t \mathbf{A}_h^-} \begin{pmatrix} \mathbf{E}(t^n) \\ \mathbf{H}(t^n) \end{pmatrix} + \mathcal{O}(\triangle t^3). \qquad (78)
\end{aligned}
$$

We also define one more intermediate variable as

$$
\begin{pmatrix} \widetilde{\mathbf{E}}^{(2)} \\ \widetilde{\mathbf{H}}^{(2)} \end{pmatrix}(t^n) = \frac{1}{\theta} \begin{pmatrix} \mathbf{E}(t^{n+1}) \\ \mathbf{H}(t^{n+1}) \end{pmatrix} - \frac{1-\theta}{\theta} \begin{pmatrix} \widetilde{\widetilde{\mathbf{E}}}^{(2)} \\ \widetilde{\widetilde{\mathbf{H}}}^{(2)} \end{pmatrix}(t^n)
$$

(79)

$$
= \frac{1}{\theta} \begin{pmatrix} \mathbf{E}(t^{n+1}) \\ \mathbf{H}(t^{n+1}) \end{pmatrix} - \frac{1-\theta}{\theta} \left[I + \Delta t \mathbf{A}_h + \frac{(\Delta t)^2}{2} \left((\mathbf{A}_h^+)^2 + 2 \mathbf{A}_h^+ \mathbf{A}_h^- + (\mathbf{A}_h^-)^2 \right) \right] \begin{pmatrix} \mathbf{E}(t^n) \\ \mathbf{H}(t^n) \end{pmatrix}.
$$

Substituting the intermediate variables (76)-(79) and exact solutions $\mathbf{E}, \mathbf{H}$ into the `Maxwell WSS-FDTD` scheme (28a)-(29a), and denoting $\vec{\xi}, \widehat{\vec{\xi}}, \vec{\eta}, \widehat{\vec{\eta}}$ to be the corresponding error functions we have

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\widetilde{\mathbf{E}}^{(1)}\\ \widetilde{\mathbf{H}}^{(1)}\end{pmatrix}-\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}\right]=\frac{1}{2}\mathbf{A}_h^{+}\left[\begin{pmatrix}\widetilde{\mathbf{E}}^{(1)}\\ \widetilde{\mathbf{H}}^{(1)}\end{pmatrix}+\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}\right]+\vec{\xi}, \tag{80}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\widetilde{\widetilde{\mathbf{E}}}^{(1)}\\ \widetilde{\widetilde{\mathbf{H}}}^{(1)}\end{pmatrix}-\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}\right]=\frac{1}{2}\mathbf{A}_h^{-}\left[\begin{pmatrix}\widetilde{\widetilde{\mathbf{E}}}^{(1)}\\ \widetilde{\widetilde{\mathbf{H}}}^{(1)}\end{pmatrix}+\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}\right]+\widehat{\vec{\xi}}, \tag{81}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\widetilde{\mathbf{E}}^{(2)}\\ \widetilde{\mathbf{H}}^{(2)}\end{pmatrix}-\begin{pmatrix}\widetilde{\mathbf{E}}^{(1)}\\ \widetilde{\mathbf{H}}^{(1)}\end{pmatrix}\right]=\frac{1}{2}\mathbf{A}_h^{-}\left[\begin{pmatrix}\widetilde{\mathbf{E}}^{(2)}\\ \widetilde{\mathbf{H}}^{(2)}\end{pmatrix}+\begin{pmatrix}\widetilde{\mathbf{E}}^{(1)}\\ \widetilde{\mathbf{H}}^{(1)}\end{pmatrix}\right]+\vec{\eta}, \tag{82}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\widetilde{\widetilde{\mathbf{E}}}^{(2)}\\ \widetilde{\widetilde{\mathbf{H}}}^{(2)}\end{pmatrix}-\begin{pmatrix}\widetilde{\widetilde{\mathbf{E}}}^{(1)}\\ \widetilde{\widetilde{\mathbf{H}}}^{(1)}\end{pmatrix}\right]=\frac{1}{2}\mathbf{A}_h^{+}\left[\begin{pmatrix}\widetilde{\widetilde{\mathbf{E}}}^{(2)}\\ \widetilde{\widetilde{\mathbf{H}}}^{(2)}\end{pmatrix}+\begin{pmatrix}\widetilde{\widetilde{\mathbf{E}}}^{(1)}\\ \widetilde{\widetilde{\mathbf{H}}}^{(1)}\end{pmatrix}\right]+\widehat{\vec{\eta}}, \tag{83}$$

where the vector $\vec{\xi}=(\xi_1,\xi_2,...,\xi_6)^T$ and $\widehat{\vec{\xi}}=(\widehat{\xi}_1,\widehat{\xi}_2,...,\widehat{\xi}_6)^T$. We similarly define $\vec{\eta}$ and $\widehat{\vec{\eta}}$. Based on equations (80)-(83) we have the following proposition.

Proposition 2 (**Truncation Errors**). *Suppose that the exact solutions* $\mathbf{E}, \mathbf{H}$ *satisfy*

$$\mathbf{E}\in C^3\left([0,T]:\left[C^3(\overline{\Omega})\right]^3\right), \quad \textit{and} \quad \mathbf{H}\in C^3\left([0,T]:\left[C^3(\overline{\Omega})\right]^3\right).$$

Let $\theta\in(0,1)$. *Then the error functions* $\vec{\xi}, \widehat{\vec{\xi}}, \vec{\eta}$ *and* $\widehat{\vec{\eta}}$ *are bounded, i.e. there exists a constant* $C_\theta>0$ *independent of* $\Delta t, \Delta x, \Delta y$ *and* Δz *(but dependent on* θ*) such that for the case* $\theta\neq 0.5$

$$\max_{0\le n\le N}\left\{\|\vec{\xi}\|_\infty, \|\widehat{\vec{\xi}}\|_\infty, \|\vec{\eta}\|_\infty, \|\widehat{\vec{\eta}}\|_\infty\right\}\le C_\theta\left(\Delta t+\Delta x^2+\Delta y^2+\Delta z^2\right),$$

and for the case $\theta=0.5$, *there exists a constant* $C>0$ *independent of* $\Delta t, \Delta x, \Delta y$ *and* Δz *such that*

$$\max_{0\le n\le N}\left\{\|\vec{\xi}\|_\infty, \|\widehat{\vec{\xi}}\|_\infty, \|\vec{\eta}\|_\infty, \|\widehat{\vec{\eta}}\|_\infty\right\}\le C\left(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2\right).$$

Proof. Substituting definition (76) into equation (80), we get

$$\frac{1}{\Delta t}\Big(\Delta t\mathbf{A}_h^{+}+\frac{(\Delta t)^2}{2}(\mathbf{A}_h^{+})^2\Big)\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}=\frac{1}{2}\mathbf{A}_h^{+}\Big(2I+\Delta t\mathbf{A}_h^{+}+\frac{(\Delta t)^2}{2}(\mathbf{A}_h^{+})^2\Big)\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}+\vec{\xi},$$

which simplifies to

$$\Big(\mathbf{A}_h^{+}+\frac{(\Delta t)}{2}(\mathbf{A}_h^{+})^2\Big)\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}=\mathbf{A}_h^{+}\Big[\Big(I+\frac{\Delta t}{2}\mathbf{A}_h^{+}+\frac{(\Delta t)^2}{4}(\mathbf{A}_h^{+})^2\Big)\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}\Big]+\vec{\xi}.$$

From the above we get

$$\vec{\xi}=-\frac{(\Delta t)^2}{4}(\mathbf{A}_h^{+})^3\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}.$$

Using the smoothness assumptions on the exact solutions and Taylor expansions around the point t^n, we get

$$\|\vec{\xi}\|_\infty\le C\left(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2\right). \tag{84}$$

Similarly, by substituting (77) into (81) we can show that $\widehat{\vec{\xi}}=-\frac{(\Delta t)^2}{4}(\mathbf{A}_h^{-})^3\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}$ and

$$\|\widehat{\vec{\xi}}\|_\infty\le C\left(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2\right). \tag{85}$$

Next, substituting the definitions (76) and (78) into equation (82) and rearranging terms, we get

(86)
$$\vec{\eta} = \frac{1}{\Delta t}\left[\frac{1}{\theta}\begin{pmatrix}\mathbf{E}(t^{n+1})\\ \mathbf{H}(t^{n+1})\end{pmatrix} - \frac{1-\theta}{\theta}\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} - \begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}\right] + \left[-\frac{1-\theta}{\theta}\mathbf{A}_h\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}\right.$$
$$\left. - \mathbf{A}_h^+\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} - \frac{1}{2\theta}\mathbf{A}_h^-\begin{pmatrix}\mathbf{E}(t^{n+1})\\ \mathbf{H}(t^{n+1})\end{pmatrix} + \frac{1-\theta}{2\theta}\mathbf{A}_h^-\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} - \frac{1}{2}\mathbf{A}_h^-\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}\right]$$
$$+ \frac{\Delta t}{2}\left[-\frac{1-\theta}{\theta}\left((\mathbf{A}_h^+)^2 + 2\mathbf{A}_h^+\mathbf{A}_h^- + (\mathbf{A}_h^-)^2\right)\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} - (\mathbf{A}_h^+)^2\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}\right.$$
$$\left. + \frac{1-\theta}{\theta}\mathbf{A}_h^-\mathbf{A}_h\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} - \mathbf{A}_h^-\mathbf{A}_h^+\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}\right] + \mathcal{O}(\Delta t^2).$$

Simplifying, the first term on the right hand side of (86) can be rewritten as

$$\frac{1}{\Delta t}\left[\frac{1}{\theta}\begin{pmatrix}\mathbf{E}(t^{n+1})\\ \mathbf{H}(t^{n+1})\end{pmatrix} - \frac{1-\theta}{\theta}\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} - \begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}\right] = \frac{1}{\theta}\frac{1}{\Delta t}\left[\begin{pmatrix}\mathbf{E}(t^{n+1})\\ \mathbf{H}(t^{n+1})\end{pmatrix} - \begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}\right]$$
$$= \frac{1}{\theta}\delta_t\begin{pmatrix}\mathbf{E}(t^{n+\frac{1}{2}})\\ \mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix}. \tag{87}$$

The second term on the right hand side of (86) can be rewritten as

$$-\frac{1-\theta}{\theta}\mathbf{A}_h\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} - \mathbf{A}_h^+\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} - \frac{1}{2\theta}\mathbf{A}_h^-\begin{pmatrix}\mathbf{E}(t^{n+1})\\ \mathbf{H}(t^{n+1})\end{pmatrix} + \frac{1-\theta}{2\theta}\mathbf{A}_h^-\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}$$
$$-\frac{1}{2}\mathbf{A}_h^-\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} = -\frac{1}{\theta}\mathbf{A}_h\begin{pmatrix}\overline{\mathbf{E}}(t^{n+\frac{1}{2}})\\ \overline{\mathbf{H}}(t^{n+\frac{1}{2}})\end{pmatrix} + \frac{\Delta t}{2\theta}\mathbf{A}_h^+\delta_t\begin{pmatrix}\mathbf{E}(t^{n+\frac{1}{2}})\\ \mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix}, \tag{88}$$

where, the average values of the electric and magnetic fields at $t^{n+\frac{1}{2}}$ are defined as

$$\overline{\mathbf{E}}(t^{n+\frac{1}{2}}) := \frac{1}{2}\left(\mathbf{E}(t^{n+1}) + \mathbf{E}(t^n)\right) \quad \text{and} \quad \overline{\mathbf{H}}(t^{n+\frac{1}{2}}) := \frac{1}{2}\left(\mathbf{H}(t^{n+1}) + \mathbf{H}(t^n)\right).$$

Finally, the third term on the right of (86) can be written as

(89)
$$\frac{\Delta t}{2}\left[-\frac{1-\theta}{\theta}\left((\mathbf{A}_h^+)^2 + 2\mathbf{A}_h^+\mathbf{A}_h^- + (\mathbf{A}_h^-)^2 - \mathbf{A}_h^-\mathbf{A}_h\right) - \left((\mathbf{A}_h^+)^2 + \mathbf{A}_h^-\mathbf{A}_h^+\right)\right]\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}$$
$$= \frac{\Delta t}{2}\left[-\frac{1}{\theta}(\mathbf{A}_h^+)^2 - 2\frac{1-\theta}{\theta}\mathbf{A}_h^+\mathbf{A}_h^- + \frac{1-2\theta}{\theta}\mathbf{A}_h^-\mathbf{A}_h^+\right]\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix}.$$

Thus, substituting equations (87), (88) and (89) into equation (86), we get

$$\vec{\eta} = \frac{1}{\theta}\delta_t\begin{pmatrix}\mathbf{E}(t^{n+\frac{1}{2}})\\ \mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix} - \frac{1}{\theta}\mathbf{A}_h\begin{pmatrix}\overline{\mathbf{E}}(t^{n+\frac{1}{2}})\\ \overline{\mathbf{H}}(t^{n+\frac{1}{2}})\end{pmatrix} + \frac{\Delta t}{2\theta}\mathbf{A}_h^+\delta_t\begin{pmatrix}\mathbf{E}(t^{n+\frac{1}{2}})\\ \mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix}$$
$$+ \frac{\Delta t}{2}\left[-\frac{1}{\theta}(\mathbf{A}_h^+)^2 - 2\frac{1-\theta}{\theta}\mathbf{A}_h^+\mathbf{A}_h^- + \frac{1-2\theta}{\theta}\mathbf{A}_h^-\mathbf{A}_h^+\right]\begin{pmatrix}\mathbf{E}(t^{n})\\ \mathbf{H}(t^{n})\end{pmatrix} + \mathcal{O}(\Delta t^2). \tag{90}$$

We can show that the first two terms on the right of (90) are bounded by $\frac{1}{\theta}\mathcal{O}(\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2)$, since

$$\delta_t\begin{pmatrix}\mathbf{E}(t^{n+\frac{1}{2}})\\ \mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix} - \mathbf{A}_h\begin{pmatrix}\overline{\mathbf{E}}(t^{n+\frac{1}{2}})\\ \overline{\mathbf{H}}(t^{n+\frac{1}{2}})\end{pmatrix} = \mathcal{O}(\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2). \tag{91}$$

So equation (90) becomes

$$\vec{\eta} = \frac{\Delta t}{2\theta}\mathbf{A}_h^+\delta_t\begin{pmatrix}\mathbf{E}(t^{n+\frac{1}{2}})\\ \mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix} + \frac{\Delta t}{2}\left[-\frac{1}{\theta}(\mathbf{A}_h^+)^2 - 2\frac{1-\theta}{\theta}\mathbf{A}_h^+\mathbf{A}_h^- + \frac{1-2\theta}{\theta}\mathbf{A}_h^-\mathbf{A}_h^+\right]\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}$$
$$+\frac{1}{\theta}\mathcal{O}(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2)$$

(92)

$$= \frac{\Delta t}{2\theta}\left[\mathbf{A}_h^+\mathbf{A}_h\begin{pmatrix}\overline{\mathbf{E}}(t^{n+\frac{1}{2}})\\ \overline{\mathbf{H}}(t^{n+\frac{1}{2}})\end{pmatrix} - \mathbf{A}_h^+\left(\mathbf{A}_h + (1-2\theta)\mathbf{A}_h^-\right)\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}\right]$$
$$+\frac{\Delta t}{2\theta}(1-2\theta)\mathbf{A}_h^-\mathbf{A}_h^+\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix} + \frac{1}{\theta}\mathcal{O}(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2).$$

Thus,

$$\vec{\eta} = \frac{\Delta t}{2\theta}\left[\mathbf{A}_h^+\mathbf{A}_h\begin{pmatrix}\overline{\mathbf{E}}(t^{n+\frac{1}{2}})-\mathbf{E}(t^n)\\ \overline{\mathbf{H}}(t^{n+\frac{1}{2}})-\mathbf{H}(t^n)\end{pmatrix}\right] - \frac{\Delta t}{2\theta}(1-2\theta)\mathbf{A}_h^+\mathbf{A}_h^-\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}$$
$$+\frac{\Delta t}{2\theta}(1-2\theta)\mathbf{A}_h^-\mathbf{A}_h^+\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix} + \frac{1}{\theta}\mathcal{O}(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2)$$

$$= \frac{\Delta t}{4\theta}\left[\mathbf{A}_h^+\mathbf{A}_h\begin{pmatrix}\mathbf{E}(t^{n+1})-\mathbf{E}(t^n)\\ \mathbf{H}(t^{n+1})-\mathbf{H}(t^n)\end{pmatrix}\right] + \frac{\Delta t}{2\theta}(1-2\theta)\left[\mathbf{A}_h^-\mathbf{A}_h^+ - \mathbf{A}_h^+\mathbf{A}_h^-\right]\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix} \tag{93}$$
$$+\frac{1}{\theta}\mathcal{O}(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2).$$

Next, we can show that the first term on the right of (93) is bounded by $\frac{1}{\theta}\mathcal{O}(\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2)$ by using appropriate Taylor expansions. Moreover by the regularity assumptions on the exact solutions $\mathbf{E}$ and $\mathbf{H}$, we can bound $\vec{\eta}$ as

$$\|\vec{\eta}\|_\infty \le \frac{C_1}{\theta}\left(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2\right) + C_2\Big(\frac{1-2\theta}{2\theta}\Big)\left(\Delta t+\Delta x^2+\Delta y^2+\Delta z^2\right). \tag{94}$$

Lastly we will find a bound for $\widehat{\vec{\eta}}$. From the definition of the error function (83) and the intermediate variable (78), we get

$$\widehat{\vec{\eta}} = \frac{1}{\Delta t}\begin{pmatrix}\widetilde{\widehat{\mathbf{E}}}^{(2)}\\ \widetilde{\widehat{\mathbf{H}}}^{(2)}\end{pmatrix} - \frac{1}{\Delta t}\begin{pmatrix}\widetilde{\widehat{\mathbf{E}}}^{(1)}\\ \widetilde{\widehat{\mathbf{H}}}^{(1)}\end{pmatrix} - \frac{1}{2}\mathbf{A}_h^+\begin{pmatrix}\widetilde{\widehat{\mathbf{E}}}^{(2)}\\ \widetilde{\widehat{\mathbf{H}}}^{(2)}\end{pmatrix} - \frac{1}{2}\mathbf{A}_h^+\begin{pmatrix}\widetilde{\widehat{\mathbf{E}}}^{(1)}\\ \widetilde{\widehat{\mathbf{H}}}^{(1)}\end{pmatrix}$$
$$= \left[\mathbf{A}_h - \mathbf{A}_h^- - \frac{1}{2}\mathbf{A}_h^+ - \frac{1}{2}\mathbf{A}_h^+\right]\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix} + \mathcal{O}(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2)$$
$$+\frac{\Delta t}{2}\left[\left((\mathbf{A}_h^+)^2 + 2\mathbf{A}_h^+\mathbf{A}_h^- + (\mathbf{A}_h^-)^2\right) - (\mathbf{A}_h^-)^2 - \mathbf{A}_h^+\mathbf{A}_h - \mathbf{A}_h^+\mathbf{A}_h^-\right]\begin{pmatrix}\mathbf{E}(t^n)\\ \mathbf{H}(t^n)\end{pmatrix}.$$

In the identity above for $\widehat{\vec{\eta}}$, the first and third term on the right hand side are identically zero. Thus, we have that

$$\|\widehat{\vec{\eta}}\|_\infty \le C_1\left(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2\right). \tag{95}$$

Combining the bounds (84), (85), (94) and (95) for the case $\theta \neq 0.5$, we get

$$\max_{0\le n\le N}\left\{\|\vec{\xi}\|_\infty, \|\widehat{\vec{\xi}}\|_\infty, \|\vec{\eta}\|_\infty, \|\widehat{\vec{\eta}}\|_\infty\right\} \le C_\theta\left(\Delta t+\Delta x^2+\Delta y^2+\Delta z^2\right),$$

and for the case $\theta = 0.5$ we obtain

$$\max_{0\le n\le N}\left\{\|\vec{\xi}\|_\infty, \|\widehat{\vec{\xi}}\|_\infty, \|\vec{\eta}\|_\infty, \|\widehat{\vec{\eta}}\|_\infty\right\} \le C\left(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2\right).$$

$\square$

Theorem 4 (**Convergence of** `Maxwell WSS-FDTD`). *Suppose that the exact solutions* $\mathbf{E},\mathbf{H}$ *satisfy*

$$\mathbf{E}\in C^3\left([0,T]:\left[C^3(\overline{\Omega})\right]^3\right),\quad \textit{and}\quad \mathbf{H}\in C^3\left([0,T]:\left[C^3(\overline{\Omega})\right]^3\right).$$

Let $\mathbf{E}_\theta^n$ *and* $\mathbf{H}_\theta^n$ *be the solution of the* `Maxwell WSS-FDTD` *method. Then for fixed* $T=N\Delta t>0$*, there exists a constant* K_θ *independent of* $\Delta t,\Delta x,\Delta y$ *and* Δz*, (but depending on* θ*) such that*

$$\max_{0\le n\le N}\left(\|\mathbf{E}(t^n)-\mathbf{E}_\theta^n\|^2+\|\mathbf{H}(t^n)-\mathbf{H}_\theta^n\|^2\right)^{\frac{1}{2}}$$
$$\le\left(\|\mathbf{E}(t^0)-\mathbf{E}_\theta^0\|^2+\|\mathbf{H}(t^0)-\mathbf{H}_\theta^0\|^2\right)^{\frac{1}{2}}+K_\theta T(\Delta t+\Delta x^2+\Delta y^2+\Delta z^2),$$

when $\theta\neq 0.5$*. Otherwise, for* $\theta=0.5$ *we have*

$$\max_{0\le n\le N}\left(\|\mathbf{E}(t^n)-\mathbf{E}_\theta^n\|^2+\|\mathbf{H}(t^n)-\mathbf{H}_\theta^n\|^2\right)^{\frac{1}{2}}$$
$$\le\left(\|\mathbf{E}(t^0)-\mathbf{E}_\theta^0\|^2+\|\mathbf{H}(t^0)-\mathbf{H}_\theta^0\|^2\right)^{\frac{1}{2}}+KT(\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2).$$

Proof. We define the error functions $\vec{\mathcal{E}}_\theta^n$, and $\vec{\mathcal{H}}_\theta^n$ as

$$\vec{\mathcal{E}}_\theta^n=\mathbf{E}(t^n)-\mathbf{E}_\theta^n,$$
$$\vec{\mathcal{H}}_\theta^n=\mathbf{H}(t^n)-\mathbf{H}_\theta^n,$$

and for $i=1,2$,

$$\vec{\mathcal{E}}^{(i)}=\widetilde{\mathbf{E}}^{(i)}(t^n)-\mathbf{E}^{(i)},\qquad \widehat{\vec{\mathcal{E}}}^{(i)}=\widehat{\widetilde{\mathbf{E}}}^{(i)}(t^n)-\widehat{\mathbf{E}}^{(i)},$$
$$\vec{\mathcal{H}}^{(i)}=\widetilde{\mathbf{H}}^{(i)}(t^n)-\mathbf{H}^{(i)},\qquad \widehat{\vec{\mathcal{H}}}^{(i)}=\widehat{\widetilde{\mathbf{H}}}^{(i)}(t^n)-\widehat{\mathbf{H}}^{(i)},$$

where the intermediate variables in the errors above are defined in (76)-(79). From equation (80) and the first equation of (28a), we get

$$\text{(96)}\qquad \frac{1}{\Delta t}\left[\begin{pmatrix}\vec{\mathcal{E}}^{(1)}\\ \vec{\mathcal{H}}^{(1)}\end{pmatrix}-\begin{pmatrix}\vec{\mathcal{E}}_\theta^n\\ \vec{\mathcal{H}}_\theta^n\end{pmatrix}\right]=\frac{1}{2}\mathbf{A}_h^+\left[\begin{pmatrix}\vec{\mathcal{E}}^{(1)}\\ \vec{\mathcal{H}}^{(1)}\end{pmatrix}+\begin{pmatrix}\vec{\mathcal{E}}_\theta^n\\ \vec{\mathcal{H}}_\theta^n\end{pmatrix}\right]+\vec{\xi}.$$

Similarly we get

$$\text{(97)}\qquad \frac{1}{\Delta t}\left[\begin{pmatrix}\widehat{\vec{\mathcal{E}}}^{(1)}\\ \widehat{\vec{\mathcal{H}}}^{(1)}\end{pmatrix}-\begin{pmatrix}\vec{\mathcal{E}}_\theta^n\\ \vec{\mathcal{H}}_\theta^n\end{pmatrix}\right]=\frac{1}{2}\mathbf{A}_h^-\left[\begin{pmatrix}\widehat{\vec{\mathcal{E}}}^{(1)}\\ \widehat{\vec{\mathcal{H}}}^{(1)}\end{pmatrix}+\begin{pmatrix}\vec{\mathcal{E}}_\theta^n\\ \vec{\mathcal{H}}_\theta^n\end{pmatrix}\right]+\widehat{\vec{\xi}}$$

$$\text{(98)}\qquad \frac{1}{\Delta t}\left[\begin{pmatrix}\vec{\mathcal{E}}^{(2)}\\ \vec{\mathcal{H}}^{(2)}\end{pmatrix}-\begin{pmatrix}\vec{\mathcal{E}}^{(1)}\\ \vec{\mathcal{H}}^{(1)}\end{pmatrix}\right]=\frac{1}{2}\mathbf{A}_h^-\left[\begin{pmatrix}\vec{\mathcal{E}}^{(2)}\\ \vec{\mathcal{H}}^{(2)}\end{pmatrix}+\begin{pmatrix}\vec{\mathcal{E}}^{(1)}\\ \vec{\mathcal{H}}^{(1)}\end{pmatrix}\right]+\vec{\eta}$$

$$\text{(99)}\qquad \frac{1}{\Delta t}\left[\begin{pmatrix}\widehat{\vec{\mathcal{E}}}^{(2)}\\ \widehat{\vec{\mathcal{H}}}^{(2)}\end{pmatrix}-\begin{pmatrix}\widehat{\vec{\mathcal{E}}}^{(1)}\\ \widehat{\vec{\mathcal{H}}}^{(1)}\end{pmatrix}\right]=\frac{1}{2}\mathbf{A}_h^+\left[\begin{pmatrix}\widehat{\vec{\mathcal{E}}}^{(2)}\\ \widehat{\vec{\mathcal{H}}}^{(2)}\end{pmatrix}+\begin{pmatrix}\widehat{\vec{\mathcal{E}}}^{(1)}\\ \widehat{\vec{\mathcal{H}}}^{(1)}\end{pmatrix}\right]+\widehat{\vec{\eta}}.$$

For convenience, we use the following notation:

- $\vec{\xi}_E=(\xi_1,\xi_2,\xi_3)^T$, the first three components of $\vec{\xi}$, and $\vec{\xi}_H=(\xi_4,\xi_5,\xi_6)^T$, the last three components of $\vec{\xi}$, i.e. $\vec{\xi}=(\vec{\xi}_E,\vec{\xi}_H)^T$.
- $\widehat{\vec{\xi}}_E=(\widehat{\xi}_1,\widehat{\xi}_2,\widehat{\xi}_3)^T$, the first three components of $\widehat{\vec{\xi}}$, and $\widehat{\vec{\xi}}_H=(\widehat{\xi}_4,\widehat{\xi}_5,\widehat{\xi}_6)^T$, the last three components of $\widehat{\vec{\xi}}$, i.e. $\widehat{\vec{\xi}}=(\widehat{\vec{\xi}}_E,\widehat{\vec{\xi}}_H)^T$.

The scalar form of equation (96) can be written as the system of equations

(100)
$$\begin{cases} \dfrac{1}{\Delta t}\left[\mathcal{E}^{(1)}_{x_{i+\frac12,j,k}} - \mathcal{E}^{n}_{\theta,x_{i+\frac12,j,k}}\right] = \dfrac{1}{2}\left[(\delta_y \mathcal{H}^{(1)}_z)_{i+\frac12,j,k} + (\delta_y \mathcal{H}^{n}_{\theta,z})_{i+\frac12,j,k}\right] + \xi_{1_{i+\frac12,j,k}}, \\ \dfrac{1}{\Delta t}\left[\mathcal{H}^{(1)}_{z_{i+\frac12,j+\frac12,k}} - \mathcal{H}^{n}_{\theta,z_{i+\frac12,j+\frac12,k}}\right] = \dfrac{1}{2}\left[(\delta_y \mathcal{E}^{(1)}_x)_{i+\frac12,j+\frac12,k} + (\delta_y \mathcal{E}^{n}_{\theta,x})_{i+\frac12,j+\frac12,k}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad +\xi_{6_{i+\frac12,j+\frac12,k}}, \end{cases}$$

(101)
$$\begin{cases} \dfrac{1}{\Delta t}\left[\mathcal{E}^{(1)}_{y_{i,j+\frac12,k}} - \mathcal{E}^{n}_{\theta,y_{i,j+\frac12,k}}\right] = \dfrac{1}{2}\left[(\delta_z \mathcal{H}^{(1)}_x)_{i,j+\frac12,k} + (\delta_z \mathcal{H}^{n}_{\theta,x})_{i,j+\frac12,k}\right] + \xi_{2_{i,j+\frac12,k}}, \\ \dfrac{1}{\Delta t}\left[\mathcal{H}^{(1)}_{x_{i,j+\frac12,k+\frac12}} - \mathcal{H}^{n}_{\theta,x_{i,j+\frac12,k+\frac12}}\right] = \dfrac{1}{2}\left[(\delta_z \mathcal{E}^{(1)}_y)_{i,j+\frac12,k+\frac12} + (\delta_z \mathcal{E}^{n}_{\theta,y})_{i,j+\frac12,k+\frac12}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad +\xi_{4_{i,j+\frac12,k+\frac12}}, \end{cases}$$

(102)
$$\begin{cases} \dfrac{1}{\Delta t}\left[\mathcal{E}^{(1)}_{z_{i,j,k+\frac12}} - \mathcal{E}^{n}_{\theta,z_{i,j,k+\frac12}}\right] = \dfrac{1}{2}\left[(\delta_x H^{(1)}_y)_{i,j,k+\frac12} + (\delta_x H^{n}_{\theta,y})_{i,j,k+\frac12}\right] + \xi_{3_{i,j,k+\frac12}}, \\ \dfrac{1}{\Delta t}\left[\mathcal{H}^{(1)}_{y_{i+\frac12,j,k+\frac12}} - \mathcal{H}^{n}_{\theta,y_{i+\frac12,j,k+\frac12}}\right] = \dfrac{1}{2}\left[(\delta_x \mathcal{E}^{(1)}_z)_{i+\frac12,j,k+\frac12} + (\delta_x \mathcal{E}^{n}_{\theta,z})_{i+\frac12,j,k+\frac12}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad +\xi_{5_{i+\frac12,j,k+\frac12}}. \end{cases}$$

Using the energy method (similar to the stability analysis), from equation (100), we get

$$\begin{aligned} \|\mathcal{E}^{(1)}_x\|^2 + \|\mathcal{H}^{(1)}_z\|^2 - \|\mathcal{E}^{n}_{\theta,x}\|^2 - \|\mathcal{H}^{n}_{\theta,z}\|^2 &= \Delta \sum_{i,j,k} \xi_{1_{i+\frac12,j,k}} \left(\mathcal{E}^{(1)}_{x_{i+\frac12,j,k}} + \mathcal{E}^{n}_{\theta,x_{i+\frac12,j,k}}\right) \\ &\quad + \Delta \sum_{i,j,k} \xi_{4_{i+\frac12,j+\frac12,k}} \left(\mathcal{H}^{(1)}_{z_{i+\frac12,j+\frac12,k}} + \mathcal{H}^{n}_{\theta,z_{i+\frac12,j+\frac12,k}}\right), \end{aligned}$$

where $\Delta = \Delta t \Delta x \Delta y \Delta z$. By completing squares, we get that

$$\|\mathcal{E}^{(1)}_x - \frac{1}{2}\Delta t \xi_1\|^2 + \|\mathcal{H}^{(1)}_z - \frac{1}{2}\Delta t \xi_6\|^2 = \|\mathcal{E}^{n}_{\theta,x} + \frac{1}{2}\Delta t \xi_1\|^2 + \|\mathcal{H}^{n}_{\theta,z} + \frac{1}{2}\Delta t \xi_6\|^2.$$

Similarly for the systems (101) and (102), we get

$$\|\mathcal{E}^{(1)}_y - \frac{1}{2}\Delta t \xi_2\|^2 + \|\mathcal{H}^{(1)}_x - \frac{1}{2}\Delta t \xi_4\|^2 = \|\mathcal{E}^{n}_{\theta,y} + \frac{1}{2}\Delta t \xi_2\|^2 + \|\mathcal{H}^{n}_{\theta,x} + \frac{1}{2}\Delta t \xi_4\|^2,$$

and

$$\|\mathcal{E}^{(1)}_z - \frac{1}{2}\Delta t \xi_3\|^2 + \|\mathcal{H}^{(1)}_y - \frac{1}{2}\Delta t \xi_5\|^2 = \|\mathcal{E}^{n}_{\theta,z} + \frac{1}{2}\Delta t \xi_3\|^2 + \|\mathcal{H}^{n}_{\theta,y} + \frac{1}{2}\Delta t \xi_5\|^2.$$

Combining the above three equations, we have

(103) $$\|\vec{\mathcal{E}}^{(1)} - \frac{1}{2}\Delta t \vec{\xi}_E\|^2 + \|\vec{\mathcal{H}}^{(1)} - \frac{1}{2}\Delta t \vec{\xi}_H\|^2 = \|\vec{\mathcal{E}}^{n}_{\theta} + \frac{1}{2}\Delta t \vec{\xi}_E\|^2 + \|\vec{\mathcal{H}}^{n}_{\theta} + \frac{1}{2}\Delta t \vec{\xi}_H\|^2.$$

Similarly, we apply the energy method to equations (97), (98) and (99) to get

(104) $$\|\widehat{\vec{\mathcal{E}}}^{(1)} - \frac{1}{2}\Delta t \widehat{\vec{\xi}}_E\|^2 + \|\widehat{\vec{\mathcal{H}}}^{(1)} - \frac{1}{2}\Delta t \widehat{\vec{\xi}}_H\|^2 = \|\vec{\mathcal{E}}^{n}_{\theta} + \frac{1}{2}\Delta t \widehat{\vec{\xi}}_E\|^2 + \|\vec{\mathcal{H}}^{n}_{\theta} + \frac{1}{2}\Delta t \widehat{\vec{\xi}}_H\|^2,$$

(105) $$\|\vec{\mathcal{E}}^{(2)} - \frac{1}{2}\Delta t \vec{\eta}_E\|^2 + \|\vec{\mathcal{H}}^{(2)} - \frac{1}{2}\Delta t \vec{\eta}_H\|^2 = \|\vec{\mathcal{E}}^{(1)} + \frac{1}{2}\Delta t \vec{\eta}_E\|^2 + \|\vec{\mathcal{H}}^{(1)} + \frac{1}{2}\Delta t \vec{\eta}_H\|^2,$$

and

(106) $$\|\widehat{\vec{\mathcal{E}}}^{(2)} - \frac{1}{2}\Delta t \widehat{\vec{\eta}}_E\|^2 + \|\widehat{\vec{\mathcal{H}}}^{(2)} - \frac{1}{2}\Delta t \widehat{\vec{\eta}}_H\|^2 = \|\widehat{\vec{\mathcal{E}}}^{(1)} + \frac{1}{2}\Delta t \widehat{\vec{\eta}}_E\|^2 + \|\widehat{\vec{\mathcal{H}}}^{(1)} + \frac{1}{2}\Delta t \widehat{\vec{\eta}}_H\|^2.$$

The difference between the exact and WSS solutions for the electric field at time t^{n+1} is computed as

(107)
$$
\begin{aligned}
\mathbf{E}(t^{n+1}) - \mathbf{E}_\theta^{n+1} &= \theta\left(\mathbf{E}(t^{n+1}) - \mathbf{E}^{(2)}\right) + (1-\theta)\left(\mathbf{E}(t^{n+1}) - \widehat{\mathbf{E}}^{(2)}\right) \\
&= \theta\left(\mathbf{E}(t^{n+1}) - \widetilde{\mathbf{E}}^{(2)} + \widetilde{\mathbf{E}}^{(2)} - \mathbf{E}^{(2)}\right) + (1-\theta)\left(\mathbf{E}(t^{n+1}) - \widetilde{\widehat{\mathbf{E}}}^{(2)} + \widetilde{\widehat{\mathbf{E}}}^{(2)} - \widehat{\mathbf{E}}^{(2)}\right) \\
&= \theta\left(\mathbf{E}(t^{n+1}) - \widetilde{\mathbf{E}}^{(2)} + \vec{\mathcal{E}}^{(2)}\right) + (1-\theta)\left(\mathbf{E}(t^{n+1}) - \widetilde{\widehat{\mathbf{E}}}^{(2)} + \widehat{\vec{\mathcal{E}}}^{(2)}\right) \\
&= \mathbf{E}(t^{n+1}) - \left(\theta\widetilde{\mathbf{E}}^{(2)} + (1-\theta)\widetilde{\widehat{\mathbf{E}}}^{(2)}\right) + \theta\vec{\mathcal{E}}^{(2)} + (1-\theta)\widehat{\vec{\mathcal{E}}}^{(2)}.
\end{aligned}
$$

Similarly, for the magnetic field we have

$$
\mathbf{H}(t^{n+1}) - \mathbf{H}_\theta^{n+1} = \mathbf{H}(t^{n+1}) - \left(\theta\widetilde{\mathbf{H}}^{(2)} + (1-\theta)\widetilde{\widehat{\mathbf{H}}}^{(2)}\right) + \theta\vec{\mathcal{H}}^{(2)} + (1-\theta)\widehat{\vec{\mathcal{H}}}^{(2)}. \tag{108}
$$

Using the triangle inequality, and identities (107) and (108) we have

(109)
$$
\begin{aligned}
&\left(\|\mathbf{E}(t^{n+1}) - \mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}(t^{n+1}) - \mathbf{H}_\theta^{n+1}\|^2\right)^{\frac{1}{2}} \\
&\quad \le \left[\|\mathbf{E}(t^{n+1}) - \left(\theta\widetilde{\mathbf{E}}^{(2)} + (1-\theta)\widetilde{\widehat{\mathbf{E}}}^{(2)}\right)\|^2 + \|\mathbf{H}(t^{n+1}) - \left(\theta\widetilde{\mathbf{H}}^{(2)} + (1-\theta)\widetilde{\widehat{\mathbf{H}}}^{(2)}\right)\|^2\right]^{\frac{1}{2}} \\
&\quad + \left[\|\theta\vec{\mathcal{E}}^{(2)}\|^2 + \|\theta\vec{\mathcal{H}}^{(2)}\|^2\right]^{\frac{1}{2}} + \left[\|(1-\theta)\widehat{\vec{\mathcal{E}}}^{(2)}\|^2 + \|(1-\theta)\widehat{\vec{\mathcal{H}}}^{(2)}\|^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

From the definition of the intermediate variable (79), we have

$$
\begin{pmatrix} \mathbf{E}(t^{n+1}) \\ \mathbf{H}(t^{n+1}) \end{pmatrix} = \theta\begin{pmatrix} \widetilde{\mathbf{E}}^{(2)} \\ \widetilde{\mathbf{H}}^{(2)} \end{pmatrix} + (1-\theta)\begin{pmatrix} \widetilde{\widehat{\mathbf{E}}}^{(2)} \\ \widetilde{\widehat{\mathbf{H}}}^{(2)} \end{pmatrix}.
$$

So the first term on the right hand side in the inequality (109) is zero. Thus the inequality (109) becomes

$$
\begin{aligned}
&\left(\left\|\mathbf{E}(t^{n+1}) - \mathbf{E}_\theta^{n+1}\right\|^2 + \left\|\mathbf{H}(t^{n+1}) - \mathbf{H}_\theta^{n+1}\right\|^2\right)^{\frac{1}{2}} \\
&\qquad \le \theta\left[\|\vec{\mathcal{E}}^{(2)}\|^2 + \|\vec{\mathcal{H}}^{(2)}\|^2\right]^{\frac{1}{2}} + (1-\theta)\left[\|\widehat{\vec{\mathcal{E}}}^{(2)}\|^2 + \|\widehat{\vec{\mathcal{H}}}^{(2)}\|^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{110}
$$

Consider the first term on the right hand side of the inequality (110). By using (105) we get

$$
\begin{aligned}
\left[\|\vec{\mathcal{E}}^{(2)}\|^2 + \|\vec{\mathcal{H}}^{(2)}\|^2\right]^{\frac{1}{2}} &\le \left(\|\vec{\mathcal{E}}^{(2)} - \frac{\Delta t}{2}\vec{\eta}_E\|^2 + \|\vec{\mathcal{H}}^{(2)} - \frac{\Delta t}{2}\vec{\eta}_H\|^2\right)^{\frac{1}{2}} + \frac{\Delta t}{2}\left(\|\vec{\eta}_E\|^2 + \|\vec{\eta}_H\|^2\right)^{\frac{1}{2}} \\
&\le \left(\|\vec{\mathcal{E}}^{(1)} + \frac{\Delta t}{2}\vec{\eta}_E\|^2 + \|\vec{\mathcal{H}}^{(1)} + \frac{\Delta t}{2}\vec{\eta}_H\|^2\right)^{\frac{1}{2}} + \frac{\Delta t}{2}\left(\|\vec{\eta}_E\|^2 + \|\vec{\eta}_H\|^2\right)^{\frac{1}{2}} \\
&\le \left(\|\vec{\mathcal{E}}^{(1)}\|^2 + \|\vec{\mathcal{H}}^{(1)}\|^2\right)^{\frac{1}{2}} + \Delta t\left(\|\vec{\eta}_E\|^2 + \|\vec{\eta}_H\|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{111}
$$

Similarly, we can show that

$$
\left(\|\vec{\mathcal{E}}^{(1)}\|^2 + \|\vec{\mathcal{H}}^{(1)}\|^2\right)^{\frac{1}{2}} \le \left(\|\vec{\mathcal{E}}_\theta^n\|^2 + \|\vec{\mathcal{H}}_\theta^n\|^2\right)^{\frac{1}{2}} + \Delta t\left(\|\vec{\xi}_E\|^2 + \|\vec{\xi}_H\|^2\right)^{\frac{1}{2}}. \tag{112}
$$

So substituting (112) into (111) we get

(113)
$$
\begin{aligned}
\left[\|\vec{\mathcal{E}}^{(2)}\|^2 + \|\vec{\mathcal{H}}^{(2)}\|^2\right]^{\frac{1}{2}} &\le \left(\|\vec{\mathcal{E}}_\theta^n\|^2 + \|\vec{\mathcal{H}}_\theta^n\|^2\right)^{\frac{1}{2}} \\
&\qquad + \Delta t\left(\|\vec{\xi}_E\|^2 + \|\vec{\xi}_H\|^2\right)^{\frac{1}{2}} + \Delta t\left(\|\vec{\eta}_E\|^2 + \|\vec{\eta}_H\|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Consider the case $\theta \neq 0.5$. From the smoothness assumption on $\mathbf{E}$ and $\mathbf{H}$ in Proposition 2, $\|\vec{\eta}_E\|, \|\vec{\eta}_H\|, \|\vec{\xi}_E\|, \|\vec{\xi}_H\|$ are bounded by

$$C_{1,\theta}(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2),$$

where $C_{1,\theta}$ is a constant independent of $\Delta t, \Delta x, \Delta y, \Delta z$. Thus the inequality (113) can be rewritten as

$$\left[\|\vec{\mathcal{E}}^{(2)}\|^2 + \|\vec{\mathcal{H}}^{(2)}\|^2\right]^{\frac{1}{2}} \leq \left(\|\vec{\mathcal{E}}_\theta^n\|^2 + \|\vec{\mathcal{H}}_\theta^n\|^2\right)^{\frac{1}{2}} + C_{1,\theta}\Delta t\left(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2\right). \tag{114}$$

Similarly, $\|\widehat{\vec{\eta}}_E\|, \|\widehat{\vec{\eta}}_H\|, \|\widehat{\vec{\xi}}_E\|, \|\widehat{\vec{\xi}}_H\|$ are bounded by

$$C_{2,\theta}(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2),$$

where $C_{2,\theta}$ is a constant independent of $\Delta t, \Delta x, \Delta y, \Delta z$. So we get that the second term on the right hand side of (110) is bounded as

$$\left[\|\widehat{\vec{\mathcal{E}}}^{(2)}\|^2 + \|\widehat{\vec{\mathcal{H}}}^{(2)}\|^2\right]^{\frac{1}{2}} \leq \left(\|\vec{\mathcal{E}}_\theta^n\|^2 + \|\vec{\mathcal{H}}_\theta^n\|^2\right)^{\frac{1}{2}} + C_{2,\theta}\Delta t\left(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2\right). \tag{115}$$

Substituting (114) and (115) into (110), we have

$$\begin{aligned}
&\left(\|\mathbf{E}(t^{n+1}) - \mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}(t^{n+1}) - \mathbf{H}_\theta^{n+1}\|^2\right)^{\frac{1}{2}} \\
&\quad \leq \theta\Big[\left(\|\vec{\mathcal{E}}_\theta^n\|^2 + \|\vec{\mathcal{H}}_\theta^n\|^2\right)^{\frac{1}{2}} + C_{1,\theta}\Delta t\left(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2\right)\Big] \\
&\qquad + (1-\theta)\Big[\left(\|\vec{\mathcal{E}}_\theta^n\|^2 + \|\vec{\mathcal{H}}_\theta^n\|^2\right)^{\frac{1}{2}} + C_{2,\theta}\Delta t\left(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2\right)\Big] \\
&\quad \leq \left(\|\vec{\mathcal{E}}_\theta^n\|^2 + \|\vec{\mathcal{H}}_\theta^n\|^2\right)^{\frac{1}{2}} + K_\theta\Delta t(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2) \\
&\quad \leq \left(\|\mathbf{E}(t^n) - \mathbf{E}_\theta^n\|^2 + \|\mathbf{H}(t^n) - \mathbf{H}_\theta^n\|^2\right)^{\frac{1}{2}} + K_\theta\Delta t(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2),
\end{aligned} \tag{116}$$

where the constant K_θ is independent of $\Delta t, \Delta x, \Delta y$ and Δz. Applying (116) recursively from time level n to 0 and using the fact that $N\Delta t = T$, we get

$$\begin{aligned}
&\left(\|\mathbf{E}(t^n) - \mathbf{E}_\theta^n\|^2 + \|\mathbf{H}(t^n) - \mathbf{H}_\theta^n\|^2\right)^{\frac{1}{2}} \\
&\quad \leq \left(\left\|\mathbf{E}(t^0) - \mathbf{E}_\theta^0\right\|^2 + \left\|\mathbf{H}(t^0) - \mathbf{H}_\theta^0\right\|^2\right)^{\frac{1}{2}} + nK_\theta\Delta t(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2) \\
&\quad \leq \left(\left\|\mathbf{E}(t^0) - \mathbf{E}_\theta^0\right\|^2 + \left\|\mathbf{H}(t^0) - \mathbf{H}_\theta^0\right\|^2\right)^{\frac{1}{2}} + K_\theta N\Delta t(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2) \\
&\quad \leq \left(\left\|\mathbf{E}(t^0) - \mathbf{E}_\theta^0\right\|^2 + \left\|\mathbf{H}(t^0) - \mathbf{H}_\theta^0\right\|^2\right)^{\frac{1}{2}} + K_\theta T(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2).
\end{aligned}$$

So we have

$$\begin{aligned}
&\max_{0\leq n\leq N}\left(\|\mathbf{E}(t^n) - \mathbf{E}_\theta^n\|^2 + \|\mathbf{H}(t^n) - \mathbf{H}_\theta^n\|^2\right)^{\frac{1}{2}} \\
&\qquad \leq \left(\left\|\mathbf{E}(t^0) - \mathbf{E}_\theta^0\right\|^2 + \left\|\mathbf{H}(t^0) - \mathbf{H}_\theta^0\right\|^2\right)^{\frac{1}{2}} + K_\theta T(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2).
\end{aligned}$$

Similarly, for the case $\theta = 0.5$ we obtain

$$\begin{aligned}
&\max_{0\leq n\leq N}\left(\|\mathbf{E}(t^n) - \mathbf{E}_\theta^n\|^2 + \|\mathbf{H}(t^n) - \mathbf{H}_\theta^n\|^2\right)^{\frac{1}{2}} \\
&\qquad \leq \left(\left\|\mathbf{E}(t^0) - \mathbf{E}_\theta^0\right\|^2 + \left\|\mathbf{H}(t^0) - \mathbf{H}_\theta^0\right\|^2\right)^{\frac{1}{2}} + KT(\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2),
\end{aligned}$$

where K is independent of $\Delta t, \Delta x, \Delta y$ and Δz. $\square$

8. Discrete Divergence of Solutions of `Maxwell WSS-FDTD`

In this section, we will define the discrete divergence operator and analyze the discrete divergence of the solution of the `Maxwell WSS-FDTD` scheme for any given value of $\theta \in [0,1]$. We define the vertex based and cell-centered based meshes, respectively, as

$$\Omega_h^{\mathrm{div}\mathbf{E}} := \{(x_i, y_j, z_k) \mid 0 \le i \le I, 0 \le j \le J, 0 \le k \le K\},$$

and

$$\Omega_h^{\mathrm{div}\mathbf{H}} := \left\{\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}}\right) \mid 0 \le i \le I-1, 0 \le j \le J-1, 0 \le k \le K-1\right\},$$

Let $\mathbf{F}$ be one of the the field variables $\mathbf{E}$ or $\mathbf{H}$. We define the discrete divergence operator div_h as

$$\mathrm{div}_h := [\delta_x \;\; \delta_y \;\; \delta_z].$$

Then the discrete divergence of electric fields gives us a new scalar field defined on the vertex based mesh $\Omega_h^{\mathrm{div}\mathbf{E}}$, while the discrete divergence of magnetic fields produces a new scalar field defined on the cell-centered mesh $\Omega_h^{\mathrm{div}\mathbf{H}}$. It is easy to see that the following properties hold for div_h in either case. It is a linear operator, and

$$\mathrm{div}_h \mathbf{curl}_h = [0 \;\; 0 \;\; 0], \quad \text{and} \quad \mathrm{div}_h(\mathbf{curl}_h)^T = [0 \;\; 0 \;\; 0].$$

We also define the discrete vector divergence operator Div_h by

$$\mathrm{Div}_h = \begin{pmatrix} \mathrm{div}_h & \mathbf{0} \\ \mathbf{0} & \mathrm{div}_h \end{pmatrix},$$

operating on vector fields of unknowns of the form $(\mathbf{E}, \mathbf{H})^T$, and producing a vector field defined on the mesh $\Omega_h^{\mathrm{div}\mathbf{E}} \times \Omega_h^{\mathrm{div}\mathbf{H}}$. The size of the matrix $\mathbf{0}$ in Div_h is 1×3. The operator Div_h is linear and $\mathrm{Div}_h \mathbf{A}_h = \mathbf{0}$, a 2×6 zero matrix.

Next, we develop some identities to analyze the discrete divergence of the solution of the `Maxwell WSS-FDTD` method for given $\theta \in [0,1]$. For convenience, we introduce the notation

$$\mathbf{W}^\alpha = \begin{pmatrix} \mathbf{E}^{(\alpha)} \\ \mathbf{H}^{(\alpha)} \end{pmatrix}, \quad \widehat{\mathbf{W}}^\alpha = \begin{pmatrix} \widehat{\mathbf{E}}^{(\alpha)} \\ \widehat{\mathbf{H}}^{(\alpha)} \end{pmatrix}, \quad \text{and} \quad \mathbf{W}_\theta^\beta = \begin{pmatrix} \mathbf{E}_\theta^\beta \\ \mathbf{H}_\theta^\beta \end{pmatrix},$$

where $\alpha \in \{1,2\}$ and $\beta \in \{n, n+1\}$. We can then rewrite equation (29a) as

$$\begin{cases} \mathbf{W}^2 - \mathbf{W}^1 &= \dfrac{\triangle t}{2} \mathbf{A}_h^-(\mathbf{W}^2 + \mathbf{W}^1), \\[2ex] \widehat{\mathbf{W}}^2 - \widehat{\mathbf{W}}^1 &= \dfrac{\triangle t}{2} \mathbf{A}_h^+(\widehat{\mathbf{W}}^2 + \widehat{\mathbf{W}}^1). \end{cases} \tag{117}$$

Combining the above equations, we get that

$$\mathbf{W}_\theta^{n+1} - \left(\theta \mathbf{W}^1 + (1-\theta)\widehat{\mathbf{W}}^1\right) = \frac{\triangle t}{2}\left(\theta \mathbf{A}_h^-(\mathbf{W}^2 + \mathbf{W}^1) + (1-\theta)\mathbf{A}_h^+(\widehat{\mathbf{W}}^2 + \widehat{\mathbf{W}}^1)\right). \tag{118}$$

Similarly, we can rewrite equation (28a) as

$$\begin{cases} \mathbf{W}^1 - \mathbf{W}_\theta^n &= \dfrac{\triangle t}{2} \mathbf{A}_h^+(\mathbf{W}^1 + \mathbf{W}_\theta^n), \\[2ex] \widehat{\mathbf{W}}^1 - \mathbf{W}_\theta^n &= \dfrac{\triangle t}{2} \mathbf{A}_h^-(\widehat{\mathbf{W}}^1 + \mathbf{W}_\theta^n), \end{cases} \tag{119}$$

from which we get

$$\left(\theta \mathbf{W}^1 + (1-\theta)\widehat{\mathbf{W}}^1\right) - \mathbf{W}_\theta^n = \frac{\triangle t}{2}\left(\theta \mathbf{A}_h^+(\mathbf{W}^1 + \mathbf{W}_\theta^n) + (1-\theta)\mathbf{A}_h^-(\widehat{\mathbf{W}}^1 + \mathbf{W}_\theta^n)\right). \tag{120}$$

Adding equation (118) and equation(120), we obtain

$$\mathbf{W}_\theta^{n+1} - \mathbf{W}_\theta^n = \frac{\triangle t}{2}\left(\theta \mathbf{A}_h^+ + (1-\theta)\mathbf{A}_h^-\right)\mathbf{W}_\theta^n + \frac{\triangle t}{2}\mathbf{A}_h\left(\theta \mathbf{W}^1 + (1-\theta)\widehat{\mathbf{W}}^1\right)$$
$$+ \frac{\triangle t}{2}\left(\theta \mathbf{A}_h^- \mathbf{W}^2 + (1-\theta)\mathbf{A}_h^+\widehat{\mathbf{W}}^2\right). \tag{121}$$

Multiplying the first equation in system (119) by $\theta\mathbf{A}_h^-$ and the second equation by $(1-\theta)\mathbf{A}_h^+$, we obtain

$$\theta \mathbf{A}_h^- \mathbf{W}^1 = \theta \mathbf{A}_h^- \mathbf{W}_\theta^n + \theta\frac{\triangle t}{2}\mathbf{A}_h^-\mathbf{A}_h^+(\mathbf{W}^1 + \mathbf{W}_\theta^n),$$

and

$$(1-\theta)\mathbf{A}_h^+\widehat{\mathbf{W}}^1 = (1-\theta)\mathbf{A}_h^+\mathbf{W}_\theta^n + (1-\theta)\frac{\triangle t}{2}\mathbf{A}_h^+\mathbf{A}_h^-(\widehat{\mathbf{W}}^1 + \mathbf{W}_\theta^n).$$

Adding the above two equations together, we get

$$\theta \mathbf{A}_h^- \mathbf{W}^1 + (1-\theta)\mathbf{A}_h^+\widehat{\mathbf{W}}^1$$
$$= \theta \mathbf{A}_h^- \mathbf{W}_\theta^n + (1-\theta)\mathbf{A}_h^+\mathbf{W}_\theta^n + \frac{\triangle t}{2}\left(\theta \mathbf{A}_h^-\mathbf{A}_h^+(\mathbf{W}^1 + \mathbf{W}_\theta^n) + (1-\theta)\mathbf{A}_h^+\mathbf{A}_h^-(\widehat{\mathbf{W}}^1 + \mathbf{W}_\theta^n)\right). \tag{122}$$

Similarly, from the system (117), we produce the identity

$$\theta \mathbf{A}_h^- \mathbf{W}^2 + (1-\theta)\mathbf{A}_h^+\widehat{\mathbf{W}}^2$$
$$= \theta \mathbf{A}_h^- \mathbf{W}^1 + (1-\theta)\mathbf{A}_h^+\widehat{\mathbf{W}}^1 + \frac{\triangle t}{2}\left(\theta(\mathbf{A}_h^-)^2(\mathbf{W}^2 + \mathbf{W}^1) + (1-\theta)(\mathbf{A}_h^+)^2(\widehat{\mathbf{W}}^2 + \widehat{\mathbf{W}}^1)\right). \tag{123}$$

Substituting (122) into (123), we obtain

$$\theta \mathbf{A}_h^- \mathbf{W}^2 + (1-\theta)\mathbf{A}_h^+\widehat{\mathbf{W}}^2 = \theta \mathbf{A}_h^- \mathbf{W}_\theta^n + (1-\theta)\mathbf{A}_h^+\mathbf{W}_\theta^n$$
$$+ \frac{\triangle t}{2}\left(\theta \mathbf{A}_h^-\mathbf{A}_h^+(\mathbf{W}^1 + \mathbf{W}_\theta^n) + (1-\theta)\mathbf{A}_h^+\mathbf{A}_h^-(\widehat{\mathbf{W}}^1 + \mathbf{W}_\theta^n)\right)$$
$$+ \frac{\triangle t}{2}\left(\theta(\mathbf{A}_h^-)^2(\mathbf{W}^2 + \mathbf{W}^1) + (1-\theta)(\mathbf{A}_h^+)^2(\widehat{\mathbf{W}}^2 + \widehat{\mathbf{W}}^1)\right). \tag{124}$$

Substituting (124) into (121), we get

$$\mathbf{W}_\theta^{n+1} - \mathbf{W}_\theta^n = \frac{\triangle t}{2}\mathbf{A}_h\left(\mathbf{W}_\theta^n + \theta \mathbf{W}^1 + (1-\theta)\widehat{\mathbf{W}}^1\right)$$
$$+ \frac{\triangle t^2}{4}\left(\theta \mathbf{A}_h^-\mathbf{A}_h^+(\mathbf{W}^1 + \mathbf{W}_\theta^n) + (1-\theta)\mathbf{A}_h^+\mathbf{A}_h^-(\widehat{\mathbf{W}}^1 + \mathbf{W}_\theta^n)\right)$$
$$+ \frac{\triangle t^2}{4}\left(\theta(\mathbf{A}_h^-)^2(\mathbf{W}^2 + \mathbf{W}^1) + (1-\theta)(\mathbf{A}_h^+)^2(\widehat{\mathbf{W}}^2 + \widehat{\mathbf{W}}^1)\right). \tag{125}$$

From equation (118), we have

$$\theta \mathbf{W}^1 + (1-\theta)\widehat{\mathbf{W}}^1 = \mathbf{W}_\theta^{n+1} - \frac{\triangle t}{2}\left(\theta \mathbf{A}_h^-(\mathbf{W}^2 + \mathbf{W}^1) + (1-\theta)\mathbf{A}_h^+(\widehat{\mathbf{W}}^2 + \widehat{\mathbf{W}}^1)\right).$$

Substituting this equation into (125), we get

$$\mathbf{W}_\theta^{n+1} - \mathbf{W}_\theta^n = \frac{\triangle t}{2}\mathbf{A}_h\left(\mathbf{W}_\theta^{n+1} + \mathbf{W}_\theta^n\right)$$
$$- \frac{\triangle t^2}{4}\left(\theta \mathbf{A}_h\mathbf{A}_h^-(\mathbf{W}^2 + \mathbf{W}^1) + (1-\theta)\mathbf{A}_h\mathbf{A}_h^+(\widehat{\mathbf{W}}^2 + \widehat{\mathbf{W}}^1)\right)$$
$$+ \frac{\triangle t^2}{4}\left(\theta \mathbf{A}_h^-\mathbf{A}_h^+(\mathbf{W}^1 + \mathbf{W}_\theta^n) + (1-\theta)\mathbf{A}_h^+\mathbf{A}_h^-(\widehat{\mathbf{W}}^1 + \mathbf{W}_\theta^n)\right)$$
$$+ \frac{\triangle t^2}{4}\left(\theta(\mathbf{A}_h^-)^2(\mathbf{W}^2 + \mathbf{W}^1) + (1-\theta)(\mathbf{A}_h^+)^2(\widehat{\mathbf{W}}^2 + \widehat{\mathbf{W}}^1)\right). \tag{126}$$

Using the results from (117) and (119) and rearranging terms, equation (126) becomes
(127)
$$\frac{\mathbf{W}_\theta^{n+1}-\mathbf{W}_\theta^n}{\triangle t} = \frac{\mathbf{A}_h}{2}\left(\mathbf{W}_\theta^{n+1}+\mathbf{W}_\theta^n\right) + \frac{\triangle t}{2}\mathcal{B}_\theta \mathbf{W}_\theta^n + \frac{\triangle t^2}{4}\mathcal{C}_\theta \mathbf{W}_\theta^n + \frac{\triangle t^3}{8}\mathcal{D}_\theta \mathbf{W}_\theta^n + \mathcal{O}(\triangle t^4)$$
where
$$\begin{aligned}
\mathcal{B}_\theta &= (1-2\theta)[\mathbf{A}_h^+, \mathbf{A}_h^-]\\
\mathcal{C}_\theta &= (1-2\theta)\left(\mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^- - \mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^+\right) - 2\left(\theta \mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^+ + (1-\theta)\mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^-\right)\\
\mathcal{D}_\theta &= (1-2\theta)\left(\mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^-\mathbf{A}_h^- - \mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^+\mathbf{A}_h^+\right) - 2\theta\left(\mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^+ + \mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^-\mathbf{A}_h^+\right)\\
&\quad - 2(1-\theta)\left(\mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^- + \mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^+\mathbf{A}_h^-\right).
\end{aligned}$$
Finally, using the identity (127), we have the following result.

Proposition 3. *Suppose that the exact solutions* $\mathbf{E}, \mathbf{H}$ *satisfy*
$$\mathbf{E}, \in C^3\left([0,T] : \left[C^3(\overline{\Omega})\right]^3\right) \quad \textit{and} \quad \mathbf{H}, \in C^3\left([0,T] : \left[C^3(\overline{\Omega})\right]^3\right),$$
then for fixed $T = N\triangle t > 0$*, the discrete divergence of the solutions to the* `Maxwell WSS-FDTD` *schemes for* $\theta \in [0,1]$*, given in Algorithm 3, satisfy the identity*

(128)
$$\begin{aligned}
\mathrm{Div}_h \mathbf{W}_\theta^N &= \mathrm{Div}_h \mathbf{W}_\theta^0 + T\frac{\triangle t}{2}\mathrm{Div}_h\left(\mathcal{B}_\theta \mathbf{W}_\theta^0\right)\\
&\quad + T\frac{\triangle t^2}{4}\left[\mathrm{Div}_h\left(\mathcal{C}_\theta \mathbf{W}_\theta^0\right) + \frac{1}{N}\mathrm{Div}_h\left(\mathcal{B}_\theta \mathbf{A}_h \sum_{k=1}^{N-1}\mathcal{S}_\theta^k\right)\right]\\
&\quad + T\frac{\triangle t^3}{8}\left[\mathrm{Div}_h\left(\mathcal{D}_\theta \mathbf{W}_\theta^0 + (N-1)\mathcal{B}_\theta^2 \mathbf{W}_\theta^0\right) + \frac{1}{N}\mathrm{Div}_h\left(\mathcal{C}_\theta \mathbf{A}_h \sum_{k=1}^{N-1}\mathcal{S}_\theta^k\right)\right] + \mathcal{O}(\triangle t^5),
\end{aligned}$$
for $\theta \neq 0.5$*, and for* $\theta = 0.5$

(129)
$$\mathrm{Div}_h \mathbf{W}_\theta^N = \mathrm{Div}_h \mathbf{W}_\theta^0 + T\frac{\triangle t^3}{8}\mathrm{Div}_h\left(\mathcal{D}_{\frac{1}{2}} \mathbf{W}_\theta^0\right) + \mathcal{O}(\triangle t^5),$$
where
$$\mathcal{S}_\theta^M = \begin{cases} \mathbf{W}_\theta^M + 2\sum_{k=1}^{M-1}\mathbf{W}_\theta^k + \mathbf{W}_\theta^0, & M \geq 2\\ \mathbf{W}_\theta^1 + \mathbf{W}_\theta^0, & M = 1\end{cases}$$

Proof. From equation (127), we have

(130)
$$\mathbf{W}_\theta^{n+1} = \mathbf{W}_\theta^n + \mathbf{A}_h\frac{\triangle t}{2}\left(\mathbf{W}_\theta^{n+1}+\mathbf{W}_\theta^n\right) + \frac{\triangle t^2}{2}\mathcal{B}_\theta \mathbf{W}_\theta^n + \frac{\triangle t^3}{4}\mathcal{C}_\theta \mathbf{W}_\theta^n + \frac{\triangle t^4}{8}\mathcal{D}_\theta \mathbf{W}_\theta^n + \mathcal{O}(\triangle t^5).$$

Recursively applying (130), we obtain

$$\begin{aligned}
\mathbf{W}_\theta^N &= \mathbf{W}_\theta^0 + \mathbf{A}_h\frac{\triangle t}{2}\mathcal{S}_\theta^N + T\frac{\triangle t}{2}\mathcal{B}_\theta \mathbf{W}_\theta^0 + T\frac{\triangle t^2}{4}\left(\mathcal{C}_\theta \mathbf{W}_\theta^0 + \frac{1}{N}\mathcal{B}_\theta \mathbf{A}\sum_{k=1}^{N-1}\mathcal{S}_\theta^k\right)\\
&\quad + T\frac{\triangle t^3}{8}\left(\left(\mathcal{D}_\theta + (N-1)\mathcal{B}_\theta^2\right)\mathbf{W}_\theta^0 + \frac{1}{N}\mathcal{C}_\theta \mathbf{A}\sum_{k=1}^{N-1}\mathcal{S}_\theta^k\right) + \mathcal{O}(\triangle t^5).
\end{aligned} \tag{131}$$

For the case $\theta \neq 0.5$, the first identity (128) can be proved by applying Div_h to both sides in (131), and using the fact that $\mathrm{Div}_h \mathbf{A}_h = \mathbf{0}$. In the case $\theta = 0.5$, the operators $\mathcal{B}_{\frac{1}{2}}$ and $\mathcal{C}_{\frac{1}{2}}$ are zero while the operator
$$\mathcal{D}_{\frac{1}{2}} = -\left(\mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^+ + \mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^-\mathbf{A}_h^- + \mathbf{A}_h^-\mathbf{A}_h^+\mathbf{A}_h^+\mathbf{A}_h^-\right) \neq \mathbf{0}.$$

Thus, applying Div_h to both sides in (131), and using the fact that $\mathrm{Div}_h \mathbf{A}_h = \mathbf{0}$, the second identity (129) holds for $\theta = 0.5$.

□

Our analysis shows that the discrete divergence identities (128) and (129) give us accuracy of first order in the discrete divergence of solutions of the `Maxwell WSS-FDTD` method when $\theta \neq 0.5$ and the *third order* of accuracy when $\theta = 0.5$. In Section 9 we will illustrate this theoretical result with numerical simulations.

9. Numerical Results for the Maxwell WSS-FDTD Methods

In this section, we perform numerical simulations to illustrate our theoretical results for the `Maxwell WSS-FDTD` schemes. We choose the domain $\Omega = [0,1] \times [0,1] \times [0,1]$. We also assume a uniform mesh with $\triangle x = \triangle y = \triangle z = \triangle t$. We consider the exact solution [7] of the three dimensional Maxwell's equations, $\mathbf{E} = (E_x, E_y, E_z)^T$ and $\mathbf{H} = (H_x, H_y, H_z)^T$ given by

$$
\begin{aligned}
E_x &= \frac{k_y - k_z}{\epsilon\sqrt{\mu}\omega} \cos(\omega\pi t)\cos(k_x\pi x)\sin(k_y\pi y)\sin(k_z\pi z),\\
E_y &= \frac{k_z - k_x}{\epsilon\sqrt{\mu}\omega} \cos(\omega\pi t)\sin(k_x\pi x)\cos(k_y\pi y)\sin(k_z\pi z),\\
E_z &= \frac{k_x - k_y}{\epsilon\sqrt{\mu}\omega} \cos(\omega\pi t)\sin(k_x\pi x)\sin(k_y\pi y)\cos(k_z\pi z),\\
H_x &= \sin(\omega\pi t)\sin(k_x\pi x)\cos(k_y\pi y)\cos(k_z\pi z),\\
H_y &= \sin(\omega\pi t)\cos(k_x\pi x)\sin(k_y\pi y)\cos(k_z\pi z),\\
H_z &= \sin(\omega\pi t)\cos(k_x\pi x)\cos(k_y\pi y)\sin(k_z\pi z),
\end{aligned}
\tag{132}
$$

where $\mathbf{k} = (k_x, k_y, k_z)^T$ is the wave vector satisfying the dispersion relation $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2)$. In our simulation, we use the wave vector $\mathbf{k} = k_x(1, 2, -3)^T$, for different values of k_x, with the base case $k_x = 1$, and we assume that $\epsilon = \mu_0 = c = 1$.

9.1. Numerical Computation of the Discrete Energy. The exact solution (132) satisfies the PEC boundary conditions (5), $\mathbf{E} \times \vec{\mathbf{n}} = \mathbf{0}$ on the boundary of Ω. The energy of the exact solution can be computed as

$$
\mathcal{E}(t) := \left(\int_\Omega |\mathbf{E}(x,t)|^2 \, dx + \int_\Omega |\mathbf{H}(x,t)|^2 \, dx \right)^{\frac{1}{2}} = \sqrt{\frac{3}{8}},
\tag{133}
$$

where $\mathcal{E}(t)$, as defined above, is the energy associated to the electromagnetic field. To analyze how well the discrete energy of the `Maxwell WSS-FDTD` solutions approximates the continuous exact energy function, we compute several measures of the discrete energy below.

We first define the *Relative Energy Error* in the energy of the `Maxwell WSS-FDTD` methods as

$$
\text{Relative Energy Error} := \max_{0 \le n \le N} \frac{|\mathcal{E}_\theta^n - \mathcal{E}(t)|}{|\mathcal{E}(t)|},
\tag{134}
$$

where the energy of the `Maxwell WSS-FDTD` solution for given $\theta \in [0,1]$, is defined as

$$
\mathcal{E}_\theta^n := \left(\|\mathbf{E}_\theta^n\|^2 + \|\mathbf{H}_\theta^n\|^2 \right)^{\frac{1}{2}}.
\tag{135}
$$

In Table (9.1) we present the relative energy error (134)-(135) of the `Maxwell WSS-FDTD` schemes for various values of θ, using $\Delta t = 0.02$ (recall that mesh and time steps are chosen to be identical). As seen in this table when $\theta = 0$ or 1, the relative error is essentially zero (machine epsilon), which illustrates the energy conservation of the solution of the sequential splitting schemes. When $\theta \in (0,1)$ the discrete energy decays. Thus, the relative energy error is nonzero in this case, with the error being the largest for the `Maxwell WSS-FDTD` scheme with $\theta = 0.5$, for which the energy decay is the most significant.

TABLE 1. The Relative Energy Error (134) for the `Maxwell WSS-FDTD` schemes with wave-vectors $k_x(1, 2, -3)^T$.

Relative Energy Error in `Maxwell WSS-FDTD`			
θ	$k_x = 1$	$k_x = 2$	$k_x = 5$
0	2.5382×10^{-15}	2.5382×10^{-15}	2.1756×10^{-15}
0.3	0.0029	0.0428	0.7015
0.5	0.0034	0.0508	0.7653
1	2.5382×10^{-15}	2.5382×10^{-15}	2.1756×10^{-15}

From equation (64), we have the inequalities

$$\frac{1-2\theta(1-\theta)}{1+2\theta(1-\theta)} \leq \left(\frac{\mathcal{E}_\theta^{n+1}}{\mathcal{E}_\theta^n}\right)^2 := \frac{\|\mathbf{E}_\theta^{n+1}\|^2 + \|\mathbf{H}_\theta^{n+1}\|^2}{\|\mathbf{E}_\theta^{n}\|^2 + \|\mathbf{H}_\theta^{n}\|^2} \leq 1.$$

In Figure 2, we demonstrate the boundedness of the discrete energy ratio $\frac{\mathcal{E}_\theta^{n+1}}{\mathcal{E}_\theta^n}$ of the `Maxwell WSS-FDTD` schemes for different values of the parameter $\theta \in [0, 1]$ chosen equally spaced with two step sizes $\triangle\theta = 0.1$, and 0.008. We choose the time step to be $\triangle t = h$, and space step $h = \triangle x = \triangle y = \triangle z = 0.2, 0.1, 0.04$, and 0.02. We find that the discrete energy ratio is less than 1 and bigger than $\dfrac{1-2\theta(1-\theta)}{1+2\theta(1-\theta)}$ for any value of θ as shown in Figure 2. We also see that for $\Delta t = h = 0.02$, the ratio of the discrete energies at times t^{n+1} and t^n is essentially 1, indicating very low order of energy decay.

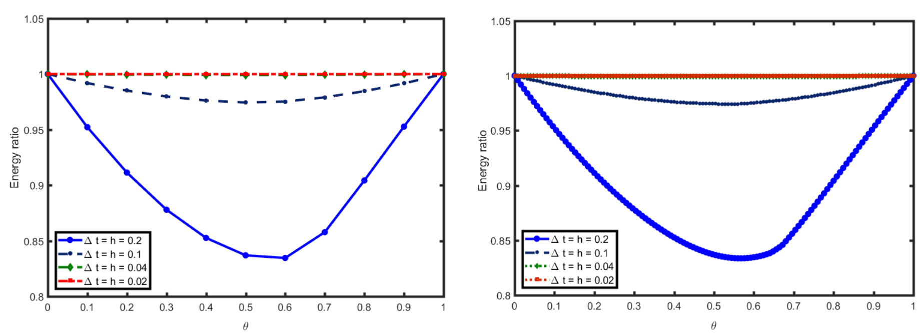


FIGURE 2. The discrete energy ratio $\frac{\mathcal{E}_\theta^{n+1}}{\mathcal{E}_\theta^n}$ of the `Maxwell WSS-FDTD` schemes with (Left Plot) $\triangle\theta = 0.1$, and (Right Plot) $\triangle\theta = 0.008$. In these plots $\mathbf{k} = (1, 2, -3)^T$.

We next demonstrate through our numerical simulations that the *discrete energy decay rate*, defined in (65) is third order in time, and the *relative change in discrete energy* defined below is fourth order in time. These rates of decay are higher than the rate of convergence of the `Maxwell WSS-FDTD` solution (either first order in time for $\theta \neq 0.5$, or second order in time for $\theta = 0.5$), as demonstrated in the next section. Thus, due to the high order of the energy decay rate and relative changes in discrete energy, we do not expect the discrete energy decay to significantly affect the convergence rate of the scheme.

We plot the relative change in discrete energy, defined as

$$\text{(136)} \qquad \text{Relative Change in Energy : RE}(\theta) := \max_{0 \leq n \leq N} \left| \frac{\mathcal{E}_\theta^{n+1} - \mathcal{E}_\theta^n}{\mathcal{E}_\theta^n} \right|,$$

and the maximum energy decay rate defined as

$$\text{Energy Decay Rate}: \mathrm{ED}(\theta) := \max_{0\le n\le N} \delta_t \mathcal{E}_\theta(t^{n+\frac{1}{2}}) = \max_{0\le n\le N} \left| \frac{\mathcal{E}_\theta^{n+1} - \mathcal{E}_\theta^n}{\Delta t} \right|, \tag{137}$$

in Figure 3 for different values of the weight parameters $\theta \in (0,1)$. The largest mesh and time step sizes are taken to be $h = \triangle x = \triangle y = \triangle z = \triangle t = 0.25$, and these values are successively decreased by half to obtain the corresponding rates in six simulation runs. These plots demonstrate that the discrete energy decay rate is third order in time, (Right plot in Figure 3, see equation (74)), while the relative change in discrete energy is fourth order in time (Left plot, Figure 3).

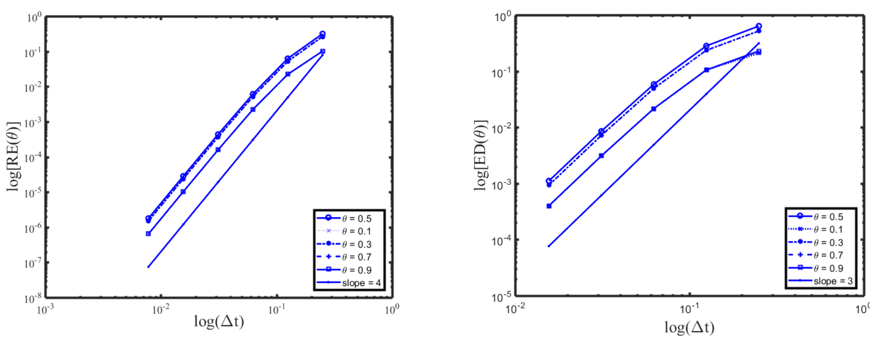


FIGURE 3. (Left Plot) Log of the relative change in discrete energy of the `Maxwell WSS-FDTD` solution defined in (136). (Right Plot) The discrete energy decay rate over all time steps defined in (137). In both plots $k_x = 1$.

9.2. Computation of Convergence Rates. In this section, we compute the convergence rates of the `Maxwell WSS-FDTD` method for different values of θ using the energy norm. We define the error in the energy norm between the exact solution given in (132) and the numerical solution obtained from the `Maxwell WSS-FDTD` method for $\theta \in [0,1]$ as

$$\mathrm{Err}_h(\theta) := \max_{0\le n\le N} \frac{\left(\|\mathbf{E}(t^n) - \mathbf{E}_\theta^n\|^2 + \|\mathbf{H}(t^n) - \mathbf{H}_\theta^n\|^2\right)^{\frac{1}{2}}}{\mathcal{E}(t^n)}, \tag{138}$$

where $\mathbf{E}(t^n)$ and $\mathbf{H}(t^n)$ denotes the exact solutions (132) at time t^n, $\mathbf{E}_\theta^n$ and $\mathbf{H}_\theta^n$ denote the `Maxwell WSS-FDTD` numerical solutions at t^n with weight parameter $\theta \in [0,1]$ and mesh step size $h = \Delta t$, and $\mathcal{E}(t^n)$ is the exact energy (133) at time t^n.
The convergence rate of the `Maxwell WSS-FDTD` method for given $\theta \in [0,1]$ is defined as

$$\mathrm{Rate}_h(\theta) = \left| \frac{\log \mathrm{Err}_{2h}^n(\theta) - \log \mathrm{Err}_h^n(\theta)}{\log 2h - \log h} \right|. \tag{139}$$

In Tables 2, and 3, we present convergence results for the `Maxwell WSS-FDTD` schemes for varying $\theta \in [0,1]$, and wave vectors $\mathbf{k} = (1,2,-3)^T$, and $\mathbf{k} = (2,4,-6)^T$, respectively. The `Maxwell WSS-FDTD` schemes achieve first order convergence when the weighted parameter $\theta \neq 0.5$. The scheme achieves second order of convergence in space when $\theta = 0.5$. Tables 2-3 and Figure 4 demonstrate our analytical results presented in Theorem 4.

9.3. Numerical Convergence of the Discrete Divergence. We define the discrete infinity norm of the divergence of the electric field solution $\{\mathbf{E}_\theta^n\}_{n=0}^N$, and the magnetic

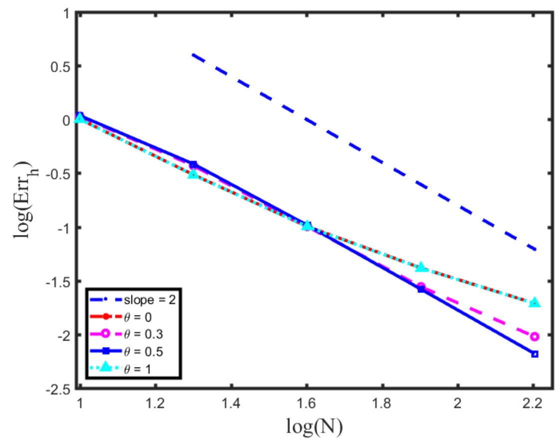


FIGURE 4. Errors (Err_h) defined in (138) displaying convergence rates of the `Maxwell WSS-FDTD` solutions for different values of θ when $k_x = 1$.

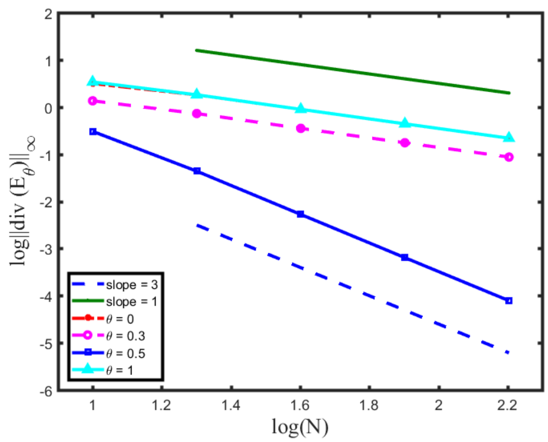
TABLE 2. Errors (defined in (138)) and convergence rates (defined in (139)) for the `Maxwell WSS-FDTD` schemes for different θ. In all cases $k_x = 1$.

		Numbers of time steps (N)				
θ		10	20	40	80	160
0	Err_h	1.001	0.305	0.101	0.041	0.019
	Rate_h		1.722	1.594	1.286	1.095
0.3	Err_h	1.072	0.371	0.101	0.028	0.010
	Rate_h		1.532	1.879	1.863	1.543
0.5	Err_h	1.085	0.384	0.104	0.026	0.007
	Rate_h		1.498	1.892	1.977	1.995
1	Err_h	1.004	0.304	0.101	0.041	0.019
	Rate_h		1.725	1.596	1.285	1.093

TABLE 3. Errors (defined in (138)) and convergence rates (defined in (139)) for the `Maxwell WSS-FDTD` schemes for different θ. In all cases $k_x = 2$.

		Numbers of time steps (N)				
θ		10	20	40	80	160
0	Err_h	1.934	1.707	0.579	0.164	0.053
	Rate_h		0.180	1.559	1.820	1.641
0.3	Err_h	1.446	1.545	0.712	0.199	0.052
	Rate_h		0.095	1.118	1.839	1.928
0.5	Err_h	1.375	1.513	0.736	0.206	0.053
	Rate_h		0.138	1.039	1.836	1.971
1	Err_h	1.923	1.704	0.578	0.164	0.052
	Rate_h		0.174	1.559	1.820	1.642

field solution $\{\mathbf{H}_\theta^n\}_{n=0}^N$, of the `Maxwell WSS-FDTD` method for given $\theta \in [0,1]$, on their



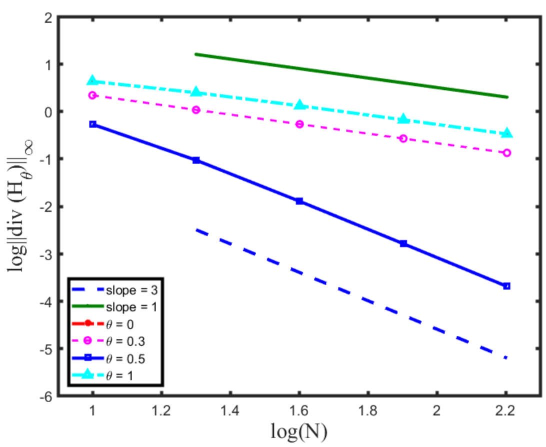


FIGURE 5. Rates of convergence of the discrete divergence (defined in (140)) of the electric field, $\mathrm{div}(\mathbf{E}_\theta)$ (Left plot), and discrete divergence of the magnetic field, $\mathrm{div}(\mathbf{H}_\theta)$ (Right plot).

respective discrete spatial meshes as

$$\|\mathrm{div}(\mathbf{E}_\theta)\|_\infty = \max_{\substack{1\le i,j,k\le N-1\\ 0\le n\le N}} \left|(\delta_x E_x)^n_{i,j,k} + (\delta_y E_y)^n_{i,j,k} + (\delta_z E_z)^n_{i,j,k}\right|, \tag{140a}$$

$$\|\mathrm{div}(\mathbf{H}_\theta)\|_\infty = \max_{\substack{0\le i,j,k\le N-1\\ 0\le n\le N}} \left|(\delta_x H_x)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + (\delta_y H_y)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}\right. \tag{140b}$$

$$\left. + (\delta_z H_z)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}\right|. \tag{140c}$$

From Table 4 and Table 5 we see that the numerical divergence of the solution of the `Maxwell WSS-FDTD` scheme has the first order of accuracy when the weight parameter $\theta \neq 0.5$. Moreover for the weight parameter $\theta = 0.5$, the numerical divergence of the `Maxwell WSS-FDTD` solution has the third order of accuracy, which illustrates our theoretical results on the discrete divergence of these solutions. Note that in these tables Rate_h denotes the rate of convergence of the discrete divergence of discrete electric or magnetic fields. In Figure 5, the convergence rates are displayed graphically for different values of θ.

TABLE 4. Convergence rates of the discrete divergence of the electric field $\mathbf{E}_\theta$, defined in (140).

		Numbers of time steps (N)				
θ		10	20	40	80	160
0	$\|\|\mathrm{div}(\mathbf{E}_\theta)\|\|_\infty$	3.1475	1.8077	0.8911	0.4393	0.2179
	Rate$_h$		0.8001	1.0206	1.0204	1.0116
0.3	$\|\|\mathrm{div}(\mathbf{E}_\theta)\|\|_\infty$	1.3509	0.7261	0.3521	0.1743	0.0867
	Rate$_h$		0.8957	1.0441	1.0148	1.0067
$\theta = 0.5$	$\|\|\mathrm{div}(\mathbf{E}_\theta)\|\|_\infty$	0.3041	0.0436	0.0053	0.0006	0.0001
	Rate$_h$		2.8031	3.0477	3.0436	3.0301
$\theta = 1$	$\|\|\mathrm{div}(\mathbf{E}_\theta)\|\|_\infty$	3.3876	1.8118	0.8911	0.4394	0.2179
	Rate$_h$		0.9029	1.0238	1.0200	1.0120

TABLE 5. Convergence rates of the divergence of the electric field $\mathbf{H}_\theta$, defined in (140).

		Numbers of time steps (N)				
θ		10	20	40	80	160
$\theta = 0$	$\|\|\mathrm{div}(\mathbf{H}_\theta)\|\|_\infty$	4.2543	2.4613	1.2994	0.6570	0.3294
	Rate$_h$		0.7895	0.9216	0.9839	0.9960
$\theta = 0.3$	$\|\|\mathrm{div}(\mathbf{H}_\theta)\|\|_\infty$	2.1682	1.0663	0.5321	0.2643	0.1320
	Rate$_h$		1.0240	1.0029	1.0096	1.0018
$\theta = 0.5$	$\|\|\mathrm{div}(\mathbf{H}_\theta)\|\|_\infty$	0.5291	0.0925	0.0125	0.0016	0.0002
	Rate$_h$		2.5166	2.8868	2.9735	2.9932
$\theta = 1$	$\|\|\mathrm{div}(\mathbf{H}_\theta)\|\|_\infty$	4.2637	2.4590	1.2995	0.6570	0.3294
	Rate$_h$		0.7940	0.9202	0.9840	0.9960

10. Conclusion

In this paper, we have constructed a family of Weighted Sequential Splitting (WSS) methods, `Maxwell WSS`, for Maxwell's equations in a linear non-dispersive and non-dissipative dielectric by extending the sequential splitting method constructed and analyzed in [7]. We have designed and analyzed fully discretized versions of the `Maxwell WSS` splitting method, called `Maxwell WSS-FDTD`, by staggering the electric and magnetic components in space as in the Yee scheme, while for the time discretization we apply the implicit Crank-Nicolson method. Our theoretical analysis shows that the fully discrete `Maxwell WSS-FDTD` scheme is of the first order of accuracy in time when the weight parameter $\theta \neq 0.5$ and is second order accurate in time for $\theta = 0.5$ (`Maxwell SWSS-FDTD`). For all values of θ, the splitting schemes are second order accurate in space. We have proved that the fully discrete splitting schemes, `Maxwell WSS-FDTD`, are unconditionally stable for all values of θ. The `Maxwell WSS-FDTD` methods satisfy a discrete energy conservation for the weighted parameter $\theta = 0, 1$ (proved in [7]), but the discrete energy decays for $\theta \in (0, 1)$ (result of this paper). While energy decay is undesirable for this non-dispersive and non-dissipative wave propagation problem, we prove that the decay in the energy is of fourth order in time for any value of $\theta \in (0, 1)$ (or cubic in time if we consider the energy decay rate), and thus quite small for a method that is overall first or second order accurate in time. With such a small energy decay, we do not expect dissipation to overwhelm the discrete solution over long time integration.

We also analyze the (discrete) divergence of the solutions of the `Maxwell WSS-FDTD` methods and prove that the divergence is approximated to first order for $\theta \neq 0.5$, but for $\theta = 0.5$ (SWSS), we are able to get a third order approximation to the exact divergence. The Strang symmetrized splitting FDTD method analyzed in [7] has a discrete solution with a second order accurate approximation to the exact divergence. Thus, the `Maxwell`

`SWSS-FDTD` method for $\theta = 0.5$ has a distinct advantage over the Strang symmetrized splitting method analyzed in [7]. In addition, we note that the implementation of the `Maxwell SWSS-FDTD` method for the case $\theta = 0.5$ can be parallelized to give a more efficient splitting scheme than the Strang symmetrized scheme, since the WSS scheme has two sequential stages with several decoupled 1D Maxwell systems in each stage, while the Strang symmetrized scheme in [7] has three sequential stages with several decoupled 1D Maxwell systems in each stage. However, the Strang symmetrized splitting method conserves the EM energy, while the `Maxwell SWSS-FDTD` method does not. Given the fact that the Strang symmetrized splitting and the `Maxwell SWSS-FDTD` splitting for $\theta = 0.5$ are both second order accurate in space and time, the above remarks indicate that our new scheme has some advantages as well as disadvantages (energy decay) to standard operator splitting schemes in the literature.

We have provided numerical experiments that confirm our theoretical results on stability and order of accuracy of all our `Maxwell WSS-FDTD` schemes, energy decay and approximation of the discrete divergence.

Acknowledgments

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