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EXISTENCE AND ERGODICITY FOR THE TWO-DIMENSIONAL STOCHASTIC BOUSSINESQ EQUATION

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(Communicated by A. Labovsky)

This paper is dedicated to our friend and mentor William Layton.

Abstract. The existence of solutions to the Boussinesq system driven by random exterior forcing terms both in the velocity field and the temperature is proven using a semigroup approach. We also obtain the existence and uniqueness of an invariant measure via coupling methods.

Key words. Stochastic Boussinesq equation, invariant measure, coupling, ergodicity.

1. Introduction

We study the existence and ergodicity of the stochastic Boussinesq equation

$$
\begin{aligned}
&du = (\nu\Delta u - (u\cdot\nabla)u + \sigma\theta - \nabla p)dt + \sqrt{Q_1}dW_1(t),\\
&d\theta = (\chi\Delta\theta - (u\cdot\nabla)\theta)dt + \sqrt{Q_2}dW_2(t),\\
&\nabla\cdot u = 0 \quad \text{in } (0,+\infty)\times\mathcal{O},\\
&u = 0, \quad \theta = 0 \quad \text{on } (0,+\infty)\times\partial\mathcal{O},\\
&u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x) \quad \text{in } \mathcal{O},
\end{aligned}
\tag{1}
$$

which models the interactions between an incompressible fluid flow coupled with thermal dynamics in two dimensions, in the presence of random perturbations. Here $\mathcal{O}\subset\mathbb{R}^2$ is a bounded, open and simply connected domain with smooth boundary $\partial\mathcal{O}$, and $u=(u_1,u_2)$ denotes the fluid velocity field, θ is the temperature of the fluid, p stands for the pressure, ν is the kinematic viscosity and χ is the thermal diffusivity, σ is a constant two component-vector. Also W_1 and W_2 represent two independent cylindrical Wiener processes [18, 21] defined, respectively, on a filtered space $(\Omega,\mathcal{F}_t,\mathbb{P})$ taking values in

$$
H = \left\{ v\in \left(L^2(\mathcal{O})\right)^2 : \nabla\cdot v = 0 \text{ in } \mathcal{O}, \quad v\cdot n = 0 \text{ on } \partial\mathcal{O} \right\}, \quad H_1 = L^2(\mathcal{O}).
$$

Finally, Q_1 and Q_2 are linear continuous, positive and symmetric operators on H and H_1, respectively, of trace class (see Definition A.1 in the Appendix A), i.e., $Tr\,Q_i<\infty$, $i=1,2$, satisfying the following condition:

$$
Q_1 = A^{-\gamma}, \qquad Q_2 = A_1^{-\gamma}, \tag{2}
$$

where $1/2<\gamma<1$, A and A_1 are as defined in (3).

Herein we prove the existence and uniqueness of a solution $(u(t,u_0,\theta_0),\theta(t,u_0,\theta_0))$ of the stochastic Boussinesq system (1), and of the corresponding invariant measure in the space $H\times H_1$. The deterministic version of the Boussinesq system (1) was comprehensively studied in the literature (see, e.g. [1, 16, 25] and the references therein). In the 19th century, Boussinesq conjectured that turbulent flow cannot be described solely with deterministic methods, and indicated that a

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stochastic framework should be used [23]. In the case of two-dimensional Navier-Stokes equations, the existence and uniqueness of a solution, the uniqueness of the invariant measure and properties of the corresponding Kolmogorov operators were studied in [3, 7, 6, 12, 11]. For the two-dimensional magnetohydrodynamics system, see [2, 24, 5, 22, 15, 14]. Recently, many authors have studied ergodicity for the solutions of the stochastic Boussinesq equations [10, 26, 27, 4, 9, 13, 17]. Notwithstanding the physical differences between the Navier-Stokes equations, magnetohydrodynamics and the Boussinesq equations (different conserved quantities, unitless physical parameters, energy cascades, fine scale structure of flows, complex pattern formation, mean heat transport, Alfvén waves, wavepackets), from a functional analysis viewpoint, similar results hold. There is an increasing set of standard tools which can be often adapted to prove deterministic and statistical properties for all these flows. Inhere we follow closely [2] by adjusting most of the proofs to the fact that the temperature field is non-solenoidal.

The paper is organized as follows. In Section 2 we formulate problem (1) in an appropriate functional setting (see [25, 8, 21, 18]) and in Section 3 we give the main existence and uniqueness result for (1) which is proved via an approximating regularizing scheme. In Section 4 we prove the existence of an invariant measure μ corresponding to the stochastic flow $t \mapsto (u(t), \theta(t))$, and its uniqueness via coupling methods, following [19, 2]. In particular, the uniqueness of the invariant measure implies that the flow is ergodic, i.e.,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t),\theta(t))\,dt = \int_{H\times H} \phi\,d\mu \qquad \forall \phi \in L^2(H\times H;\mu)$$

which agrees with some physical hypothesis on the Boussinesq flow. In the concluding Section 5 we summarize our results, in the Appendix A we provide some definitions used throughout the report, while Section 6 contains acknowledgements.

2. Functional setting and formulation of the problem

We introduce the functional spaces to represent the coupled Navier-Stokes and heat equations (1) as infinite dimensional differential equation

$$V = \left\{ v \in \left(H_0^1(\mathcal{O})\right)^2 : \nabla\cdot v = 0 \text{ in } \mathcal{O} \right\}, \quad V_1 = H_0^1(\mathcal{O}).$$

The norms of V and V_1 are denoted by the same symbol $\|\cdot\|$:

$$\|v\|^2 = \sum_{i=1}^{2}\int_{\mathcal{O}} |\nabla v_i|^2 dx, \quad v = (v_1, v_2) \in V,$$
$$\|\eta\|^2 = \int_{\mathcal{O}} |\nabla \eta|^2 dx, \quad \eta \in V_1.$$

Let denote by V' and $V_1' = H^{-1}(\mathcal{O})$ the duals of V and V_1 respectively, endowed with the dual norms. Denote again $(\cdot,\cdot)$ the scalar product on H and the pairing between V and V', V_1 and V_1'. The norm on H and $L^2(\mathcal{O})$ will both be denoted by $|\cdot|$. Identifying H with its own dual we have $V \subset H \subset V'$. Let $A \in L(V, V')$, $A_1 \in$

$L(V_1, V_1'), b : V \times V \times V \to \mathbb{R}$ be defined by

$$
\begin{aligned}
(Av, w) &= \int_{\mathcal{O}} \nabla v \cdot \nabla w \, dx, \quad v, w \in V, \\
(A_1\alpha, \beta) &= \int_{\mathcal{O}} \nabla \alpha \cdot \nabla \beta dx, \quad \alpha, \beta \in V_1, \\
b(u, v, w) &= \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i D_i v_j w_j dx, \quad u, v, w \in V,
\end{aligned} \tag{3}
$$

and $B : V \to V'$ given by

$$(Bv, w) = b(v, v, w), \quad v, w \in V.$$

Then system (1) can be written as

$$
\begin{aligned}
&du + (\nu Au + B(u) - \sigma\theta)dt = \sqrt{Q_1} dW_1(t), \\
&d\theta + (\chi A_1\theta + (u \cdot \nabla)\theta)dt = \sqrt{Q_2} dW_2(t), \\
&u(0) = u_0, \quad \theta(0) = \theta_0.
\end{aligned} \tag{4}
$$

The cylindrical Wiener process $W = (W_1, W_2)$ on $H \times H$ is defined [18] by

$$W_i(t) = \sum_{j=1}^{\infty} \beta_j^i(t) e_j^i, \qquad i = 1, 2,$$

where $\{e_j^1\}, \{e_j^2\}$ are two complete orthonormal bases of eigenfunctions of A, respectively A_1, and $\{\beta_j^i\}, i = 1, 2$ are two independent sequences of mutually independent Brownian motions on the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We denote by $C_W(0, T; H \times H_1)$ the space of all continuous functions $Z : [0, T] \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; H \times H_1)$ which are adapted to the filtration $\mathcal{F}_t$. The spaces $L^2_W(0, T; V \times V)$ and $L^2_W(0, T; V' \times V_1')$ are similarly defined.

Consider the stochastic convolution that is the mild solution of the problem

$$
\begin{aligned}
&dW_{\mathcal{A}}(t) + \mathcal{A} W_{\mathcal{A}}(t) dt = \sqrt{Q} dW(t), \\
&W_{\mathcal{A}}(0) = 0,
\end{aligned} \tag{5}
$$

given by

$$W_{\mathcal{A}}(t) = \int_0^t e^{-\mathcal{A}(t-s)} \sqrt{Q} dW(s) := (W_A^1(t), W_A^2(t)),$$

where

$$\mathcal{A} = \begin{pmatrix} \nu A & 0 \\ 0 & \chi A_1 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Under our assumptions it follows that [6]

$$W_{\mathcal{A}} \in C_W(0, T; H \times H) \cap (L^4_W([0, T] \times \mathcal{O}))^2,$$

and by Theorem 2.13 of [6] we have that

$$\mathbb{E} \left(\sup_{(t,x) \in [0,T] \times \mathcal{O}} |W_A^i(t, x)|^4 \right) < +\infty.$$

3. Existence and uniqueness result

Our main theorem is as follows.

Theorem 3.1. *For all* $(u_0, \theta_0) \in H \times H_1$ *and* $T > 0$ *problem* (4) *has a unique solution* $(u, \theta) \in L^2_W(0, T; V \times V_1)$.

To prove Theorem 3.1 we reduce (4) to a deterministic equation with random coefficients, via the substitution

$$u(t) = v(t) + W^1_A(t), \quad \theta(t) = \eta(t) + W^2_A(t).$$

Then (4) reduces to

$$\begin{aligned} & v' + \nu Av + B(v) + v \cdot \nabla W^1_A + W^1_A \cdot \nabla v - \sigma\theta - \sigma W^2_A = -B(W^1_A), \\ & \eta' + \chi A_1\eta + v \cdot \nabla\eta + v \cdot \nabla W^2_A + W^1_A \cdot \nabla\eta = -W^1_A \cdot \nabla W^2_A, \\ & v(0) = u_0, \quad \eta(0) = \theta_0. \end{aligned} \tag{6}$$

We recall the following standard estimates, which will be used in the sequel:

$$\begin{aligned} & |(B(v), w)| \le C|v|\|v\|\|w\| \quad \Rightarrow \quad \|B(v)\|_{V'} \le C|v|\|v\|, \\ & \|v \cdot \nabla\eta\|_{V_1'} \le C|v|^{1/2}\|v\|^{1/2}|\eta|^{1/2}\|\eta\|^{1/2}, \\ & \|W^1_A \cdot \nabla v\|_{V'} + \|v \cdot \nabla W^1_A\|_{V'} \le C|W^1_A|^4|v|^{1/2}\|v\|^{1/2}, \\ & \|v \cdot \nabla W^2_A\|_{V_1'} \le C|W^2_A|^4|v|^{1/2}\|v\|^{1/2}, \\ & \|W^2_A \cdot \nabla\eta\|_{V_1'} \le C|W^1_A|^4|\eta|^{1/2}\|\eta\|^{1/2}. \end{aligned}$$

Proposition 3.1. *Let* $(u_0, \theta_0) \in H \times H_1$. *Then there is a unique solution* $(v, \eta) \in L^2_W(0, T; V \times V_1)$ *to* (6) *such that* $\mathbb{P}$*-a.s.* $(v, \eta) : [0, T] \to V' \times V_1'$ *is absolutely continuous on* $[0, T]$ *and*

(i) $v' \in L^2(0, T; V')$, $\eta' \in L^2(0, T; V_1')$, $\mathbb{P}$*-a.s.*
(ii) $v \in C([0, T], H)$ *and* $\eta \in C([0, T], H_1)$, $\mathbb{P}$*-a.s.*

Proof. We consider the approximating equation

$$\begin{aligned} & v_\varepsilon' + \nu Av_\varepsilon + \Phi^\varepsilon_1(v_\varepsilon) + v_\varepsilon \cdot \nabla W^1_A + W^1_A \cdot \nabla v_\varepsilon - \sigma\theta_\varepsilon - \sigma W^2_A = -B(W^1_A), \\ & \eta_\varepsilon' + \chi A_1\eta_\varepsilon + \Phi^\varepsilon_2(v_\varepsilon, \eta_\varepsilon) + v_\varepsilon \cdot \nabla W^2_A + W^1_A \cdot \nabla\eta_\varepsilon = -W^1_A \cdot \nabla W^2_A, \\ & v(0) = u_0, \quad \eta(0) = \theta_0, \end{aligned} \tag{7}$$

where

$$\Phi^\varepsilon_1(v) = \begin{cases} B(v) & \text{if } \|v\| \le \frac{1}{\varepsilon}, \\ \frac{B(v)}{\varepsilon^2\|v\|^2} & \text{if } \|v\| > \frac{1}{\varepsilon}. \end{cases}$$

and

$$\Phi^\varepsilon_2(v, \eta) = \begin{cases} v \cdot \nabla\eta & \text{if } \|v\| + \|\eta\| \le \frac{1}{\varepsilon}, \\ \frac{v \cdot \nabla\eta}{\varepsilon^2(\|v\| + \|\eta\|)^2} & \text{if } \|v\| + \|\eta\| > \frac{1}{\varepsilon}. \end{cases}$$

It is easy to see that $u_\varepsilon = v_\varepsilon + W^1_A$ and $\theta_\varepsilon = \eta_\varepsilon + W^2_A$ satisfy

$$\begin{aligned} & du_\varepsilon + \big(\nu Au_\varepsilon + \Phi^\varepsilon_1(u_\varepsilon) - \sigma\theta_\varepsilon\big)dt = \sqrt{Q_1}dW_1(t), \\ & d\theta_\varepsilon + \big(\chi A_1\theta_\varepsilon + \Phi^\varepsilon_2(u_\varepsilon, \theta_\varepsilon)\big)dt = \sqrt{Q_2}dW_2(t), \\ & u(0) = u_0, \quad \theta(0) = \theta_0. \end{aligned} \tag{8}$$

Multiplying the first and second equations of (7) by v_ε and θ_ε respectively, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(|v_\varepsilon|^2+|\eta_\varepsilon|^2\right)+\nu\|v_\varepsilon\|^2+\chi\|\eta_\varepsilon\|^2+b(v_\varepsilon,W_A^1,v_\varepsilon)+b(v_\varepsilon,W_A^2,\eta_\varepsilon)\\&=-b(W_A^1,\eta_\varepsilon,\eta_\varepsilon)+(\sigma\eta_\varepsilon,v_\varepsilon)+(\sigma W_A^2,v_\varepsilon)-b(W_A^1,W_A^1,v_\varepsilon)-b(W_A^1,W_A^2,\eta_\varepsilon).\end{aligned}$$

By use of Young's inequality we obtain

$$\begin{aligned}&b(v_\varepsilon,W_A^1,v_\varepsilon)\le C|v_\varepsilon|^{1/2}\|v_\varepsilon\|^{3/2}|W_A^1|_4\le\frac{\nu}{6}\|v_\varepsilon\|^2+C|v_\varepsilon|^2|W_A^1|_4^4,\\&b(v_\varepsilon,W_A^2,\eta_\varepsilon)\le C|v_\varepsilon|^{1/2}\|v_\varepsilon\|^{1/2}|\|\eta_\varepsilon\||W_A^2|_4\\&\quad\le C|v_\varepsilon|\|v_\varepsilon\||W_A^2|_4^2+\frac{\chi}{6}\|\eta_\varepsilon\|^2\le\frac{\nu}{6}\|v_\varepsilon\|^2+C|v_\varepsilon|^2|W_A^2|_4^4+\frac{\chi}{6}\|\eta_\varepsilon\|^2,\\&b(W_A^1,W_A^1,v_\varepsilon)\le C|W_A^1|_4^2\|v_\varepsilon\|\le\frac{\nu}{6}\|v_\varepsilon\|^2+C|W_A^1|_4^4,\\&b(W_A^1,\eta_\varepsilon,\eta_\varepsilon)\le\frac{\chi}{6}\|\eta_\varepsilon\|^2+C|\eta_\varepsilon|^2|W_A^1|_4^4,\\&b(W_A^1,W_A^2,\eta_\varepsilon)\le C|W_A^1|_4|W_A^2|_4\|\eta_\varepsilon\|\le\frac{\chi}{6}\|\eta_\varepsilon\|^2+C|W_A^1|_4^2|W_A^2|_4^2\\&(\sigma\eta_\varepsilon,v_\varepsilon)\le C(|\eta_\varepsilon|^2+|v_\varepsilon|^2),\end{aligned}$$

which by the above energy equality yields

(9)
$$\frac{d}{dt}\left(|v_\varepsilon|^2+|\eta_\varepsilon|^2\right)+\nu\|v_\varepsilon\|^2+\chi\|\eta_\varepsilon\|^2\le C(|W_A^1|_4^4+|W_A^2|_4^4+1)(|\eta_\varepsilon|^2+|v_\varepsilon|^2+1).$$

Integrating (9) with respect to $t\in[0,T]$ and using Gronwall's inequality, we have

$$\begin{aligned}&|v_\varepsilon(t)|^2+|\eta_\varepsilon(t)|^2+\int_0^T(\|v_\varepsilon(s)\|^2+\|\eta_\varepsilon(s)\|^2)ds\\&\le C(|u_0|^2+|\theta_0|^2)\exp\left(C\int_0^T(|W_A^1|_4^4+|W_A^2|_4^4+C)ds\right)+C,\quad t\in[0,T],\end{aligned}\tag{10}$$

where C is independent of ε and ω.

Now we fix $\omega\in\Omega$ and select a sub-sequence $\varepsilon=\varepsilon(\omega)$ such that

$$\begin{aligned}&v_\varepsilon(t)\to v(t)\quad\text{weakly in }L^2(0,T;V),\ \text{weak star in }L^\infty(0,T;H),\\&\eta_\varepsilon(t)\to\eta(t)\quad\text{weakly in }L^2(0,T;V_1),\ \text{weak star in }L^\infty(0,T;L^2(\Omega)),\\&Av_\varepsilon(t)\to Av(t)\quad\text{weakly in }L^2(0,T;V'),\\&A\eta_\varepsilon(t)\to A\eta(t)\quad\text{weakly in }L^2(0,T;V_1'),\end{aligned}$$

and similarly

$$\begin{aligned}&\Phi_1^\varepsilon(v_\varepsilon(t))\to\varphi_1(t)\quad\text{weakly in }L^2(0,T;V')\\&\Phi_2^\varepsilon(v_\varepsilon(t),\eta_\varepsilon(t))\to\varphi_2(t)\quad\text{weakly in }L^2(0,T;V_1')\\&v_\varepsilon(t)\cdot\nabla W_A^1\to v(t)\cdot W_A^1\quad\text{weakly in }L^2(0,T;V')\\&W_A^1\cdot\nabla v_\varepsilon(t)\to W_A^1\cdot\nabla v(t)\quad\text{weakly in }L^2(0,T;V')\\&\sigma\eta_\varepsilon(t)\to\sigma\eta(t)\quad\text{weakly in }L^2(0,T;V_1')\\&v_\varepsilon(t)\cdot\nabla W_A^2\to v(t)\cdot W_A^2\quad\text{weakly in }L^2(0,T;V_1')\\&W_A^1\cdot\nabla\eta_\varepsilon(t)\to W_A^1\cdot\nabla\eta(t)\quad\text{weakly in }L^2(0,T;V_1').\end{aligned}$$

Thus, we have

$$
(11)\quad
\begin{aligned}
&v' + \nu Av + \varphi_1 + v \cdot \nabla W_A^1 + W_A^1 \cdot \nabla v = -B(W_A^1) + \sigma\theta + \sigma W_A^2, \text{ a.e. } t \in [0,T],\\
&\eta' + \chi A_1 \eta + \varphi_2 + v \cdot \nabla W_A^2 + W_A^1 \cdot \nabla \eta = -W_A^1 \cdot \nabla W_A^2, \text{ a.e. } t \in [0,T],\\
&v(0) = u_0, \quad \eta(0) = \theta_0.
\end{aligned}
$$

On the other hand, since v'_ε and η'_ε are bounded in $L^2(0,T;V')$ and $L^2(0,T;V'_1)$ respectively, we also have that for $\varepsilon \to 0$

$$
(12)\quad
\begin{aligned}
&v_\varepsilon \to v \text{ strongly in } L^2(0,T;H),\\
&\eta_\varepsilon \to \eta \text{ strongly in } L^2(0,T;L^2(\mathcal{O})).
\end{aligned}
$$

Moreover,

$$
(13)\qquad \int_0^T (\varphi_1(t), \psi(t))dt \to \int_0^T b(v,v,\psi)dt, \quad \forall \psi \in C([0,T], D(A)),
$$

and the reason is as follows.

$$
\begin{aligned}
&\int_0^T (\varphi_1(t), \psi(t))dt\\
&= \int_{t\in[0,T]:\|v_\varepsilon\|\le 1/\varepsilon} b(v_\varepsilon, v_\varepsilon, \psi)dt + \int_{t\in[0,T]:\|v_\varepsilon\|>1/\varepsilon} \frac{b(v_\varepsilon, v_\varepsilon, \psi)}{\varepsilon^2\|v_\varepsilon^2\|}dt\\
&= I_\varepsilon^1 + I_\varepsilon^2.
\end{aligned}
$$

Using (12) it is straightforward to obtain that

$$
b(v_\varepsilon, v_\varepsilon, \psi) \to b(v,v,\psi), \quad \text{a.e. } t \in [0,T].
$$

Also, since

$$
|b(v_\varepsilon, v_\varepsilon, \psi)| \le C|v_\varepsilon|\|v_\varepsilon\|,
$$

we deduce by Lebesgue's dominated convergence theorem that

$$
I_\varepsilon^1 \to \int_0^T b(v,v,\psi)dt \quad \text{as } \varepsilon \to 0.
$$

On the other hand, by (10) we infer that

$$
\sup_{t\in[0,T]} \{\|v_\varepsilon(t)\| > 1/\varepsilon\} \le C\varepsilon^2.
$$

Therefore,

$$
|I_\varepsilon^2| \le C\varepsilon^2 \frac{|v_\varepsilon|\|v_\varepsilon\|\|\psi\|}{\varepsilon^2\|v_\varepsilon\|^2} \le C\frac{1}{\|v_\varepsilon\|} \le C\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.
$$

Thus, it follows that $\varphi_1(t) = B(v(t))$, a.e. $t \in [0,T]$. Similarly, we have $\varphi_2(t) = v \cdot \nabla\eta$.

This means that the pair (v,η) is a solution to (6) for every fixed $\omega \in \Omega$. On the other hand, it is readily seen that for each $\omega \in \Omega$, (11) with $\varphi_1 = B(v)$ and $\varphi_2 = v \cdot \nabla\eta$ has at most one solution (v,η) with the above properties. This implies that, for $\varepsilon \to 0$,

$$
v_\varepsilon(t) \to v(t), \quad \eta_\varepsilon(t) \to \eta(t),
$$

weakly in $L^2(0,T;V)$ and $L^2(0,T;V_1)$, respectively, $\mathbb{P}$-a.s. This indicates that v and η (and v' and η') are adapted to the filtration $\mathcal{F}_t$ and therefore $(v,\eta) \in L_W^2(0,T;V \times V_1)$ and $(v',\eta') \in L_W^2(0,T;V' \times V_1')$. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. For the first equation of (8), we have by Itô's formula

$$\frac{1}{2}\mathbb{E}|u_\varepsilon(t)|^2 + \nu\mathbb{E}\int_0^t \|u_\varepsilon(s)\|^2 ds = \frac{1}{2}|u_0|^2 + \frac{1}{2}t\,Tr\,Q_1 + \mathbb{E}\int_0^t (\sigma\theta_\varepsilon, u_\varepsilon)ds. \tag{14}$$

Similarly, the second equation in (8) yields

$$\frac{1}{2}\mathbb{E}|\theta_\varepsilon(t)|^2 + \chi\mathbb{E}\int_0^t \|\theta_\varepsilon(s)\|^2 ds = \frac{1}{2}|\theta_0|^2 + \frac{1}{2}t\,Tr\,Q_2. \tag{15}$$

Combining (14) and (15) we obtain

$$\begin{aligned} &\mathbb{E}(|u_\varepsilon(t)|^2 + |\theta_\varepsilon(t)|^2) + 2\mathbb{E}\int_0^t (\nu\|u_\varepsilon(s)\|^2 + \chi\|\theta_\varepsilon(s)\|^2)ds \\ &\quad = |u_0|^2 + |\theta_0|^2 + t\,Tr\,(Q_1+Q_2) + 2\mathbb{E}\int_0^t (\sigma\theta_\varepsilon(s), u_\varepsilon(s))ds, \quad \forall t\in[0,T]. \end{aligned} \tag{16}$$

By Gronwall's inequality, we deduce from (16) that

$$\mathbb{E}(|u_\varepsilon(t)|^2 + |\theta_\varepsilon(t)|^2) + \mathbb{E}\int_0^t (\|u_\varepsilon(s)\|^2 + \|\theta_\varepsilon(s)\|^2)ds \le C. \tag{17}$$

This implies that, for $\varepsilon \to 0$,

$$\begin{aligned} u_\varepsilon \to u = v + W_A^1 \quad &\text{weakly in } L_W^2(0,T;V), \\ \theta_\varepsilon \to \theta = \eta + W_A^2 \quad &\text{weakly in } L_W^2(0,T;V_1), \end{aligned}$$

where (u,θ) is a solution to (1).

As for uniqueness, if $(\tilde{u}(t), \tilde{\theta}(t))$ is a solution with initial data (u_1,θ_1) we have by (4) that

$$\begin{aligned} &\frac{1}{2}d(|u(t)-\tilde{u}(t)|^2 + |\theta(t)-\tilde{\theta}(t)|^2) + \nu\|u(t)-\tilde{u}(t)\|^2 + \chi\|\theta(t)-\tilde{\theta}(t)\|^2 \\ &\le |b(u-\tilde{u},\tilde{u},u-\tilde{u})| + |((u-\tilde{u})\cdot\nabla\tilde{\theta}, \theta-\tilde{\theta})| + |(\sigma(\theta-\tilde{\theta}), u-\tilde{u})| \\ &\le C|u-\tilde{u}|\|u-\tilde{u}\|\|\tilde{u}\| + C|u-\tilde{u}|^{1/2}\|u-\tilde{u}\|^{1/2}|\tilde{\theta}|^{1/2}\|\tilde{\theta}\|^{1/2}\|\theta-\tilde{\theta}\| + C|\theta-\tilde{\theta}||u-\tilde{u}| \\ &\le C|u-\tilde{u}|^2\|\tilde{u}\|^2 + \frac{\nu}{4}\|u-\tilde{u}\|^2 + C|u-\tilde{u}|^2|\tilde{\theta}|^2\|\tilde{\theta}\|^2 + \frac{\nu}{4}\|u-\tilde{u}\|^2 \\ &\qquad + \frac{\chi}{2}\|\theta-\tilde{\theta}\|^2 + C(|\theta-\tilde{\theta}|^2 + |u-\tilde{u}|^2) \\ &\le C(|\theta-\tilde{\theta}|^2 + |u-\tilde{u}|^2)(1+\|\tilde{u}\|^2 + |\tilde{\theta}|^2\|\tilde{\theta}\|^2) + \frac{\nu}{2}\|u-\tilde{u}\|^2 + \frac{\chi}{2}\|\theta-\tilde{\theta}\|^2. \end{aligned}$$

Using Gronwall's inequality there holds

$$\begin{aligned} &|u(t)-\tilde{u}(t)|^2 + |\theta(t)-\tilde{\theta}(t)|^2 \\ &\le C(|u_0-u_1|^2 + |\theta_0-\theta_1|^2) \times \exp\Big(C\int_0^t (1+\|\tilde{u}\|^2 + |\tilde{\theta}|^2\|\tilde{\theta}\|^2)ds\Big), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

This completes the uniqueness of (u,θ) as well as the continuity of $(u_0,\theta_0) \to (u(t),\theta(t))$. □

4. Ergodicity

4.1. Existence of invariant measure. Let $(u(t,u_0,\theta_0), \theta(t,u_0,\theta_0)) \in L_W^2(0,T;V\times V_1)$ be the solution of (1) with initial data (u_0,θ_0). Set

$$P_t\phi(u_0,\theta_0) = \mathbb{E}[\phi(u(t,u_0,\theta_0),\theta(t,u_0,\theta_0))], \quad \forall (u_0,\theta_0)\in H\times H_1,\ \phi\in C_b(H\times H_1).$$

Recall that a Borel probability measure μ in $H \times H_1$ is invariant (Definition A.3) for the transition semigroup P_t if

$$\int_{H\times H_1} P_t\phi d\mu = \int_{H\times H_1} \phi d\mu, \quad \forall \phi \in C_b(H\times H_1).$$

Theorem 4.1. *There exists at least one invariant measure μ for P_t.*

Proof. From (16) we have that

$$\mathbb{E}(|u(t,u_0,\theta_0)|^2 + |\theta(t,u_0,\theta_0)|^2) + \mathbb{E}\int_0^t (\|u(s,u_0,\theta_0)\|^2 + \|\theta(s,u_0,\theta_0)\|^2)ds$$
$$\le C\left(|u_0|^2 + |\theta_0|^2 + tTr(Q_1+Q_2)\right), \quad t\ge 0. \tag{18}$$

Let $\pi_t(u_0,\theta_0,\cdot)$ be the law of process $(u(t),\theta(t))$. Then

$$P_t\phi(u_0,\theta_0) = \int_0^t \phi(u_1,\theta_1)\pi_t(u_0,\theta_0,du_1,d\theta_1).$$

In order to prove the existence of an invariant measure, it is enough to show that the set of measures

$$\mu_T := \frac{1}{T}\int_0^T \pi_t(u_0,\theta_0,\cdot)dt, \quad T>1,$$

is tight in $\mathcal{P}(H\times H_1)$ (see the definition A.4 in the Appendix A). With fixed $(u_0,\theta_0)\in H\times H_1$ we have that

$$\frac{1}{t}\mathbb{E}\int_0^t (\|u(s,u_0,\theta_0)\|^2 + \|\theta(s,u_0,\theta_0)\|^2)ds \le C(|u_0|^2+|\theta_0|^2+Tr(Q)).$$

Let B_R denote the ball of radius R in $V\times V_1$. Then $\forall R>0$, we have

$$\begin{aligned}\mu_T(B_R^c) &= \frac{1}{T}\int_0^T \pi_t(u_0,\theta_0,B_R^c)dt\\ &\le \frac{1}{TR^2}\int_0^T \mathbb{E}(\|u(s,u_0,\theta_0)\|^2 + \|\theta(s,u_0,\theta_0)\|^2)ds\\ &\le \frac{1}{R^2}C(|u_0|^2+|\theta_0|^2+Tr(Q)),\end{aligned}$$

which yields that $\{\mu_T\}_{T\ge 1}$ is tight. □

4.2. Uniqueness of invariant measure. In this section we follow the approach in [2, 19] to prove the uniqueness of the invariant measure μ, using the coupling method (see, e.g., [2, 19, 11, 12]). The main steps in the proof are comprised in Lemmas 4.1-4.3. With these a priori estimates, the main result, Theorem 4.2, follows exactly the same framework as in [2]. Therefore, we only present Lemmas 4.1-4.3 in this section. For a detailed proof of Theorem 4.2 we refer to [2].

Lemma 4.1. *The following estimate holds:*

$$\nu^*\mathbb{E}\int_0^t (\|u(s,u_0,\theta_0)\|^2 + \|\theta(s,u_0,\theta_0)\|^2)ds \le |u_0|^2+|\theta_0|^2+\frac{t}{2}Tr(Q), \tag{19}$$

where $\nu^* = \min\{\nu,\chi\}$.

Proof. This is a direct consequence of (18). □

Lemma 4.2. *Let $\rho_0,\rho_1>0$. Then there exist $\alpha=\alpha(\rho_0,\rho_1)$ and $T=T(\rho_0,\rho_1)>0$ such that for any $t\in[T,2T]$, $|u_0|\le\rho_0, |\theta_0|\le\rho_0$, we have*

$$\mathbb{P}\big(|u(t,u_0,\theta_0)|\le\rho_1, |\theta(t,u_0,\theta_0)|\le\rho_1\big)\ge\alpha(\rho_0,\rho_1). \tag{20}$$

Proof. Let $v = u - W_A^1$, $\eta = \theta - W_A^2$, where W_A^1 and W_A^2 are mild solutions to (5). Multiplying the second equation (6) with η yields $\mathbb{P}$-a.s.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta(t)|^2 + \chi\|\eta(t)\|^2 &\le |b(v(t), W_A^2(t), \eta(t))| + |b(W_A^1(t), W_A^2(t), \eta(t))\\ &\le C\big(\|v(t)\|^2|W_A^2(t)|_4^2 + |W_A^1(t)|_4^2|W_A^2(t)|_4^2\big) + \frac{\chi}{2}\|\eta(t)\|^2.\end{aligned}$$

Thus,

$$\frac{d}{dt}|\eta(t)|^2 + \chi\|\eta(t)\|^2 \le C|W_A^2(t)|_4^2\big(\|v(t)\|^2 + |W_A^1(t)|_4^2\big),$$

hence, with δ independent of t, we have

$$\frac{d}{dt}\left(e^{\delta t}|\eta(t)|^2\right) \le C|W_A^2(t)|_4^2\big(\|v(t)\|^2 + |W_A^1(t)|_4^2\big)e^{\delta t}. \tag{21}$$

Note that W_A^1 and W_A^2 are independent Gaussian processes in $L^4(\mathcal{O})$, and following the argument in [6], for every $\epsilon > 0$ we have

$$\mathbb{P}(S_\epsilon) > 0, \quad S_\epsilon = \{\omega \in \Omega : |W_A^1(t)|_4^2 + |W_A^2(t)|_4^2 \le \epsilon,\ \forall t \in [0,\, 2T]\}.$$

Integrating and rearranging (21), for ϵ small enough, we deduce

$$\begin{aligned}|\eta(t)|^2 &\le e^{-\delta t}|\eta(0)|^2 + Ce^{-\delta t}\epsilon\int_0^t e^{\delta s}(\|v(s)\|^2 + \epsilon)ds\\ &\le e^{-\delta t}|\eta(0)|^2 + C\epsilon, \qquad \text{for a.e. } t \in [0, 2T],\ \mathbb{P}\text{-a.s. on } S_\epsilon,\end{aligned} \tag{22}$$

where we used the a priori bound on v from (10). Now multiply equation (6) with v and η respectively, to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|v(t)|^2 + |\eta(t)|^2) + \nu\|v(t)\|^2 + \chi\|\eta(t)\|^2\\ \le& C\big(|W_A^1(t)|_4^4 + |W_A^2(t)|_4^4\big)(|v(t)|^2 + 1)\\ &+ \frac{\nu}{4}\|v(t)\|^2 + C_\nu|\eta(t)|^2 + \frac{\nu}{4}\|v(t)\|^2 + \frac{\chi}{2}\|\eta(t)\|^2.\end{aligned}$$

Applying again (10) for the bound of v, and the estimate (22), the above relation yields

$$\frac{d}{dt}(|v(t)|^2 + |\eta(t)|^2) + \alpha(\|v(t)\|^2 + \|\eta(t)\|^2) \le C(|\eta(0)|^2 e^{-\delta t} + \epsilon). \tag{23}$$

Multiply by $e^{\alpha t}$ both sides of (23) and integrate from 0 to t, then we have

$$|v(t)|^2 + |\eta(t)|^2 \le e^{-\alpha t}(|v(0)|^2 + |\eta(0)|^2) + C|\eta(0)|^2 e^{-\min(\alpha,\delta)t} + C\epsilon t, \tag{24}$$

$\mathbb{P}$-a.s. on S_ϵ and for all $t \in [0, 2T]$, therefore also on $[T, 2T]$. The right-hand side will be small by choosing T large enough first, and then letting ϵ small enough. □

For notational simplicity, in the sequel we denote by $(x, y) \in H \times H_1$ the initial values of (u, θ), the solution of (1).

Lemma 4.3. *Let $g \in C_b(H \times H_1)$ be such that $\|g\|_0 \le 1$. Then for any $t > 0$ there exists $\delta > 0$ such that*

$$|P_t g(x, y) - P_t g(x_1, y_1)| \le \frac{1}{2},$$

for all $(x, y), (x_1, y_1) \in H \times H_1$, $x, y, x_1, y_1 \in B_\delta(0)$, where $B_\delta(0)$ denotes the ball centered at the origin of δ radius.

Proof. Let $Z=(u,\theta)$ be the solution of (1) with initial value $(x,y)\in H\times H_1$ and by DZ the Gâteaux derivative of Z. Denote

$$\xi_1 = D_x u, \quad \xi_2 = D_x\theta, \quad \xi_3 = D_y u, \quad \xi_4 = D_y\theta,$$

where D_x and D_y are Gâteaux derivatives with respect to x and y. Then

$$\begin{aligned} &\xi_1' + \nu A\xi_1 + B'(u)\xi_1 - \sigma\xi_2 = 0,\\ &\xi_2' + \chi A_1\xi_2 + (\xi_1\cdot\nabla)\theta + (u\cdot\nabla)\xi_2 = 0,\\ &\xi_1(0) = 1, \qquad \xi_2(0) = 0, \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\xi_3' + \nu A\xi_3 + B'(u)\xi_3 - \sigma\xi_4 = 0,\\ &\xi_4' + \chi A_1\xi_4 + (\xi_3\cdot\nabla)\theta + (u\cdot\nabla)\xi_4 = 0,\\ &\xi_3(0) = 0, \qquad \xi_4(1) = 1. \end{aligned} \tag{26}$$

By multiplying (25) by ξ_1 and ξ_2, respectively, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\xi_1|^2+|\xi_2|^2) + \nu\|\xi_1\|^2 + \chi\|\xi_2\|^2\\ =&-b(\xi_1,u,\xi_1) + (\sigma\xi_2,\xi_1) - b(\xi_1,\theta,\xi_2)\\ \le& C|\xi_1|\|\xi_1\|\|u\| + \frac{\sigma}{2}(|\xi_2|^2+|\xi_1|^2) + C\|\xi_1\|^{\frac12}\|\xi_2\|^{\frac12}|\xi_1|^{\frac12}|\xi_2|^{\frac12}\|\theta\|\\ \le&\epsilon\|\xi_1\|^2 + C|\xi_1|^2\|u\|^2 + \frac{\sigma}{2}(|\xi_2|^2+|\xi_1|^2) + \frac{\epsilon}{2}(\|\xi_1\|^2+\|\xi_2\|^2) + C(|\xi_1|^2+|\xi_2|^2)\|\theta\|^2. \end{aligned}$$

For properly chosen ϵ, there exists $\gamma>0$ such that

$$\frac{d}{dt}(|\xi_1|^2+|\xi_2|^2) + \gamma(\|\xi_1\|^2+\|\xi_2\|^2) \le C(|\xi_1|^2+|\xi_2|^2)(\|u\|^2+\|\theta\|^2+C),$$

and by Gronwall's inequality, we obtain

$$|\xi_1|^2+|\xi_2|^2+\gamma\int_0^t(\|\xi_1\|^2+\|\xi_2\|^2)ds \le C\exp\left(C\int_0^t(\|u\|^2+\|\theta\|^2+C)ds\right), \tag{27}$$

for all $t\in[0,T]$. Similarly,

$$|\xi_3|^2+|\xi_4|^2+\gamma\int_0^t(\|\xi_3\|^2+\|\xi_4\|^2)ds \le C\exp\left(C\int_0^t(\|u\|^2+\|\theta\|^2+C)ds\right). \tag{28}$$

The next step is to estimate

$$\mathbb{E}\left[g\left(u(t,x,y),\theta(t,x,y)\right) - g\left(u(t,x_1,y_1),\theta(t,x_1,y_1)\right)\right]$$

by following the argument in [19]. First we recall the definition of a real cut-off function

$$\Phi_K(r) = \begin{cases} 1 & \text{if } r\in[0,K]\\ 0 & \text{if } r\in[2K,\infty]\\ \in[0,1] & \text{if } r\in[K,2K]. \end{cases}$$

Then

$$
\begin{aligned}
&\mathbb{E}\left[g\left(u(t,x,y),\theta(t,x,y)\right)-g\left(u(t,x_1,y_1),\theta(t,x_1,y_1)\right)\right]\\
&=\underbrace{\mathbb{E}\left[g\left(u(t,x,y),\theta(t,x,y)\right)\cdot\Phi_K\left(\int_0^t(\|u(s,x,y)\|^2+\|\theta(s,x,y)\|^2)ds\right)\right]-\mathbb{E}\left[g\left(u(t,x_1,y_1),\theta(t,x_1,y_1)\right)\cdot\Phi_K\left(\int_0^t(\|u(s,x_1,y_1)\|^2+\|\theta(s,x_1,y_1)\|^2)ds\right)\right]}_{H_1}\\
&\underbrace{+\mathbb{E}\Big[g(u(t,x,y),\theta(t,x,y))\cdot\Big(1-\Phi_K\big(\int_0^t(\|u(s,x,y)\|^2+\|\theta(s,x,y)\|^2)ds\big)\Big)\Big]}_{H_2}\\
&\underbrace{-\mathbb{E}\Big[g(u(t,x_1,y_1),\theta(t,x_1,y_1))\cdot\Big(1-\Phi_K\big(\int_0^t(\|u(s,x_1,y_1)\|^2+\|\theta(s,x_1,y_1)\|^2)ds\big)\Big)\Big]}_{H_3}\\
&=H_1(t)+H_2(t)+H_3(t).
\end{aligned}
$$

Using Markov's inequality and Lemma 4.1 we have

$$
\begin{aligned}
|H_2(t)| &\le \mathbb{P}\left(\int_0^t(\|u(s,x,y)\|^2+\|\theta(s,x,y)\|^2)ds\ge K\right)\|g\|_0\\
&\le \frac{\|g\|_0}{\nu^* K}\Big(|x|^2+|y|^2+\frac{t}{2}Tr(Q_1+Q_2)\Big).
\end{aligned}
\tag{29}
$$

Similarly,

$$
|H_3(t)|\le \frac{\|g\|_0}{\nu^* K}\Big(|x_1|^2+|y_1|^2+\frac{t}{2}Tr(Q_1+Q_2)\Big).
\tag{30}
$$

In order to estimate $H_1(t)$, we write it as follows.

$$
\begin{aligned}
H_1(t)=\int_0^1\frac{d}{d\lambda}\mathbb{E}\Bigg[&g\left(u(t,x_\lambda,y_\lambda),\theta(t,x_\lambda,y_\lambda)\right)\\
&\times\Phi_K\left(\int_0^t(\|u(s,x_\lambda,y_\lambda)\|^2+\|\theta(s,x_\lambda,y_\lambda)\|^2)ds\right)\Bigg]d\lambda,
\end{aligned}
$$

where

$$
x_\lambda=\lambda x+(1-\lambda)x_1,\qquad y_\lambda=\lambda y+(1-\lambda)y_1,\quad \lambda\in[0,1].
$$

Denoting by $h=(x-x_1,y-y_1)$, $\mathscr{A}:V\times V_1\to V'\times V_1'$ the canonical isomorphism of $V\times V_1$ onto $V'\times V_1'$, and

$$
\tau_\lambda=\inf\left\{t>0:\int_0^t(\|u(s,x_\lambda,y_\lambda)\|^2+\|\theta(s,x_\lambda,y_\lambda)\|^2)ds\ge 2K\right\},
$$

the Bismut-Elworthy [20] formula yields

$$H_1(t) = \int_0^1 \frac{1}{t}\mathbb{E}\Bigg[g\left(Z(t,x_\lambda,y_\lambda)\right)\cdot \Phi_K\left(\int_0^t (\|u(s,x_\lambda,y_\lambda)\|^2+\|\theta(s,x_\lambda,y_\lambda)\|^2)ds\right) \cdot \int_0^t (Q^{-1/2}DZ(s,x_\lambda,y_\lambda)h, dW(s))\Bigg]d\lambda$$
$$+ 2\int_0^1 \mathbb{E}\Bigg[g\left(Z(t,x_\lambda,y_\lambda)\right)\cdot \Phi_K'\left(\int_0^t (\|u(s,x_\lambda,y_\lambda)\|^2+\|\theta(s,x_\lambda,y_\lambda)\|^2)ds\right) \cdot \int_0^t (1-\frac{s}{t})(\mathscr{A}Z(s,x_\lambda,y_\lambda), DZ(s,x_\lambda,y_\lambda)h)\Bigg]d\lambda.$$

Then we have

$$|H_1(t)| \le C\|g\|_0 \int_0^1 d\lambda \Bigg[\frac{1}{t}\mathbb{E}\left[\int_0^{t\wedge\tau_\lambda} |Q^{-1/2}DZ(s,x_\lambda,y_\lambda)h|^2 ds\right]^{1/2} + 2\|\Phi_K'\|_0\mathbb{E}\left[\left(\int_0^{t\wedge\tau_\lambda} \|\xi^h(s,x_\lambda,y_\lambda)\|^2_{V\times V_1}ds\right)^{1/2}\left(\int_0^t \|Z(s,x_\lambda,y_\lambda)\|^2\right)^{1/2}\right]\Bigg],$$

where $\xi^h = DZ\cdot h$. By estimates (27) and (28), as well as the condition (2), we have that

$$\int_0^t |Q^{-1/2}DZ(s,x_\lambda,y_\lambda)h|^2 ds \le C|h|^2.$$

Finally, using the estimates (19) and (27)-(30), we obtain

$$|\mathbb{E}\left[g(Z(t,x,y)) - g(Z(t,x_1,y_1))\right]| \le C\|g\|_0\delta\Big(\frac{\delta}{K}+2e^{\delta K}(1+t^{-1/2})\Big) \le \frac{1}{2}, \tag{31}$$

for all $x, y, x_1, y_1 \in B_\delta(0)$ when K is appropriately chosen and δ is small enough. □

With the a priori estimates in Lemmas 4.1-4.3, the next theorem can be obtained by following the same (coupling method) approach in [2].

Theorem 4.2. *There is a unique invariant measure μ for semigroup P_t.*

5. Summary

We have proved an existence and uniqueness result for the Boussinesq system with random exterior forcing both in the velocity and temperature fields, using a semigroup approach and an approximating regularizing scheme. The ergodicity of the stochastic Boussinesq flow $t \mapsto (u(t),\theta(t))$ is a consequence of the existence and uniqueness of an invariant measure, which was derived by coupling methods.

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Appendix A.

Definition A.1. *Suppose H is a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. A linear continuous operator Q is of trace class if it satisfies,*

- *positivity: $(Qx, x) \geq 0$, $x \in H$,*
- *symmetry: $(Qx, y) = (x, Qy)$, $x, y \in H$,*
- *bounded trace: Tr $Q := \sum_{k=1}^{\infty}(Qe_k, e_k) < +\infty$ for one (and consequently for all) complete orthonormal system (e_k) in H.*

Definition A.2. *A Markov semigroup P_t on $B_b(H)$ is a mapping*

$$[0, +\infty) \to L(B_b(H)), \quad t \mapsto P_t,$$

such that

(i) $P_0 = 1$, $P_{t+s} = P_t P_s$ *for all* $t, s \geq 0$.
(ii) For any $t \geq 0$ and $x \in H$ there exists a probability measure $\pi_t(x,\cdot) \in \mathcal{P}(H)$ such that

$$P_t\varphi(x) = \int_H \varphi(y)\pi_t(x, dy) \quad \textit{for all } \varphi \in B_b(H).$$

(iii) For any $\varphi \in C_b(H)$ (resp. $B_b(H)$) and $x \in H$, the mapping $t \mapsto P_t\varphi(x)$ is continuous (resp. Borel).

Definition A.3. *Assume P_t represents a Markov semigroup A.2 on a Hilbert space H. A probability measure $\mu \in \mathcal{P}(H)$ is said to be invariant for P_t if*

$$\int_H P_t\varphi d\mu = \int_H \varphi d\mu, \quad \textit{for all } \varphi \in B_b(H) \textit{ and } t \geq 0,$$

where $B_b(H)$ is the Banach space of all real-valued Borel bounded mappings defined on H with the norm

$$\|\varphi\|_0 = \sup_{x\in H} |\varphi(x)|.$$

Definition A.4. *A subset $\Lambda \subset \mathcal{P}(H)$ is said to be tight if there exists an increasing sequence (K_n) of compact sets of H such that*

$$\lim_{n\to\infty} \mu(K_n) = 1 \quad \textit{uniformly on } \Lambda,$$

or, equivalently, if for any $\varepsilon > 0$ there exists a compact set K_ε such that

$$\mu(K_\varepsilon) \geq 1 - \varepsilon, \quad \mu \in \Lambda.$$

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