INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 15, Number 4-5, Pages 520–523

A GLIMPSE ON FOURIER ANALYSIS: THIRD STAGE

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(Communicated by L. Rebholz)

Dedicated to Professor William J. Layton on the occasion of his 60th birthday

Abstract. The third stage of Fourier analysis is considered herein. A generalized Fourier series is considered with real valued, locally integrable functions.

Key words. Fourier analysis, third stage

The **third** stage of Fourier Analysis is concerned with generalized Fourier series of the form

$$\sum_{k=1}^{\infty} a_k \exp\{if_k(t)\}, \tag{1}$$

in which $a_k \in C$, $k \geq 1$, while $f_k(t) : R \to R$, $k \geq 1$, are real valued functions, at least locally integrable on R: $f_k \in L^1_{loc}(R, R)$.

The *first* stage and the *second* correspond to the choice of linear $f_k(t) = \lambda_k t$, $\lambda_k \in R$, leading to the periodic functions when $\lambda_k = k\omega$, $\omega > 0$, $k \geq 0$, and to the Bohr *almost periodic* functions when $\lambda_k \in R$ are arbitrary.

Only for nonlinear $f_k(t)$ one can obtain generalized Fourier series characterizing oscillatory functions, of a more general nature than those in the first of second stages.

A tool helping us to construct series like (1) is the *Poincaré mean value* of a function, on the real line R. The formula used by Poincaré (*Nouvelles Méthodes de la Mécanique Céleste*, 1892-3) is

$$M\{f\} = \lim_{x \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)\mathrm{dt}, \tag{2}$$

with $f : R - C$ a locally integrable function for which the limit exists.

All classes/spaces of almost periodic functions (Bohr, Stepanov, Besicovitch) consist of elements for which the mean value in (2) exists (finite!).

The following formula, as noticed by Poincare, is valued for $\lambda \in R$:

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp\{i\lambda t\}\mathrm{dt} = \begin{cases} 1, & \lambda = 0 \\ 0, & \lambda \neq 0. \end{cases} \tag{3}$$

Formula (3) is sort of an orthogonality condition, since it implies

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp\{(\lambda_k - \lambda_j)t\}\mathrm{dt} = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases} \tag{4}$$

where $\{\lambda_k : k \geq 1\} \subset R$ is a sequence with distinct terms.

Received by the editors December 1, 2016.
2000 *Mathematics Subject Classification.* 42XX.

In order to construct series like (1), it appears possible to obtain solutions, if any, of the functional equation in $\lambda(t)$,

$$\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}\exp\{i\lambda(t)\}\mathrm{dt}=\begin{cases}1, & \lambda(t)=0\\ 0, & \lambda(t)\neq 0.\end{cases}\tag{5}$$

with $\lambda(t)$ real valued and locally integrable on R.

We have, so far, examples of function classes/spaces providing solutions to (5), infinitely many. The *first* space appears to be due to V.F. Osipov, in the book *Almost Periodic Functions of Bohr-Fresnel* (Russian), University of Sankt Petersburg Press, 1992, who has constructed such a space, in which case

$$\lambda_k(t)=\alpha t^2+\mu t$$

α = const. $\in R$, $k\geq 1$. Osipov's construction, according to his statement, has been inspired by a seminal paper of N. Wiener (Acta Mathematica, vol. 55, 1930), to whom the Fresnel waves, $w(t)=\exp\{i(\alpha t^2+\mu t)\}$, are attributed. Using these waves, Osipov constructed this space, called by him the space of a.p. functions of Bohr-Fresnel.

The functions in this space, obviously of oscillatory type, correspond to generalized Fourier series of the form

$$\sum_{k=1}^{\infty}a_k\exp\{i(\alpha t^2+\lambda_k t)\},\tag{6}$$

with α depending on the function to be represented by (6) a real number and $\lambda_k\in R$, $k\geq 1$, distinct.

The Parseval equation holds

$$\sum_{k=1}^{\infty}|f_k|^2=\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}|f(t)|^2\mathrm{dt},\tag{7}$$

where

$$f(t)\sim\sum_{k=1}^{\infty}f_k\exp\{i(\alpha t^2+\lambda_k t)\},\tag{8}$$

and

$$f_k=\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}f(t)\exp\{-i(\alpha t^2+\lambda_k t)\}\mathrm{dt},\ \ k\geq 1,\tag{9}$$

Other properties for the Bohr-Fresnel functions hold true, similar to those encountered for the Bohr almost periodic function: an example is the approximation (uniformly on R) of these functions by generalized trigonometric polynomials with exponents in the class of functions of the form $\alpha t^2+\mu t$, $\mu\in R$ (each taking a finite number of values).

The *second* space of generalized oscillatory functions has been constructed by Ch. Zhang (J. Fourier Analysis, vol. 12(2006); also IEEE Trans. AC, vol. 49(2004)). The construction is reproduced in one of our papers [3] and relies on the properties of a function algebra whose element are generalized polynomials of the form

$$\lambda_k(t)=\sum_{j=1}^{k}c_j t^{\alpha_j},\ \ t\geq 0,\tag{10}$$

where $c_j\in C$, $j=1,2,...,k$, and $\lambda_k(t)$ of the form (10), $c_j\in\mathbb{C}$, $\alpha_1>...>\alpha_k>0$, $k\geq 1$.

The space of Zhang, denoted $SLP(R,C)$ is the closure, under uniform convergence on R, of the set of polynomials like

$$\sum_{k=1}^{m} a_k \exp\{ip_k(t)\},\ \ p_k(t) = \sum_{j=1}^{k} c_j \lambda_j(t). \tag{11}$$

A detailed study is conducted by Zhang, showing that this new space of oscillatory functions satisfies the Bochner property (the relative compactness of the set of "translations").

A more general space than $SLP(R,C)$, denoted by $B^2_\lambda(R,C)$, has been constructed in our paper, mentioned above. It is analogue of the Besicovitch space $B^2(R,C)$, consisting of almost periodic functions, but much larger.

An important step in constructing new spaces in the third stage of Fourier Analysis is to find solutions to equation (5), i.e.,

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} e^{i\lambda(t)} \mathrm{dt} = \begin{cases} 1, & \lambda(t) = 0, \\ 0, & \lambda(t) \neq 0. \end{cases} \tag{12}$$

The problem has been approached in *two different manners.*

First, to check its validity, for functions $\lambda(t)$ of various form (mostly, polynomials), a path which has been followed by Ch. Zhang and his students.

The *second* approach, used by the author in [3], [2], [4], consists in viewing $\lambda(t)$ as the restriction to R of an entire or meromorphic function, and applying the Cauchy's residues formula

$$\int_{-\ell}^{\ell} e^{i\lambda(t)} \mathrm{dt} = 2\pi i \sum_{|z|<\bar{\ell}} \mathrm{res}\lambda(z) + \lim_{\ell\to\infty} [\Lambda(\ell) - \Lambda(-\ell)], \tag{13}$$

where $\Lambda'(z) = \exp\{i\lambda(z)\}$, $\mathrm{Im} z > 0$, and $\bar{\ell}$ sufficiently large.

A detailed discussion will be carried out in a forthcoming paper, condition (13) following from $\Lambda(\ell) - \Lambda(-\ell) = o(\ell)$, $\ell \to \infty$.

Generalized trigonometric series of the form (GTS)

$$\sum_{k=1}^{\infty} a_t \exp\{i\lambda_k(t)\},\ \ t \in R, \tag{14}$$

or just finite sections of them, have been encountered in describing oscillatory motions in various dynamical systems (man made or from nature, society). When $\lambda_k(t) = \lambda_k t$, for some real λ_k, one obtains the almost periodic case we have briefly discussed above. Otherwise, we will generally assume that $\lambda_k(t)$ are some real valued functions depending on $t \in R$ (t-time), at least locally integrable on R. Anyhow, a series like (14), if convergent in some sense, represents an *oscillatory function* (or, an *oscillating function?*).

Under various sets of assumptions, we can organize classes of GTS, of the form (14), in vector spaces endowed with adequate norms (even Banach spaces). A few examples will easily lead to such spaces.

(1) Assume that we impose the absolute convergence of the series

$$\sum_{j=1}^{\infty} |a_j| < \infty \tag{15}$$

which implies the uniform convergence on R of the series (14). It is easy to see that (15) also assures the fact that the class of GTS (14) forms a Banach space, with the norm indicated in the left hand side of (15). If we assume also that $\lambda_j(t)$, $j \geq 1$, are in $L_{loc}(R, C)$, then we obtain as sum of (14) a measurable function. If continuity is assumed on $\lambda_j(t)$, $j \geq 1$, then one obtains continuity of the sum. Further, if the assumption is that of almost periodicity (Bohr) for the generalized exponents $\lambda_j(t)$, $j \geq 1$, then the sum will be almost periodic (Bohr).

(2) Let us assume now that we want to organize (as a Banach space) the class of series (14), which satisfy the weaker condition than (15), namely

$$\sum_{j=1}^{\infty} |a_j|^2 < \infty. \tag{16}$$

In our paper [1], the following result has been proven: *to each series from* (14)*, satisfying condition* (16)*, one can associate a function* $f \in L^2_{loc}(R, C)$*, such that*

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(s)|^2 ds = \sum_{j=1}^{\infty} |a_j|^2. \tag{17}$$

The function f, the limit, is not unique. Indeed, to f one can add any function g, such that

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |g(s)|^2 ds = 0. \tag{18}$$

One can take $f(t) = \exp\{-|t|\}$, $t \in R$. The quantity

$$|f|_{B^2} = \left[\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(s)|^2 ds \right]^{1/2}$$

is only a seminorm on the space of those $f \in L^2_{loc}(R, C)$, such that the right hand side in the above formula is finite. The Banach space associated to the seminormed space is the factor space taken with respect to the zero manifold, i.e., consisting of those elements with zero seminorm (see (18) above). This factor space is a Banach space and it is denoted by $B^2_\lambda(R, C)$. As shown in Corduneanu [2], we assume that the sequence of generalized exponents is satisfying the "orthogonality" conditions

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp\{i(\lambda_k(t) - \lambda_j(t))\} \mathrm{dt} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \tag{19}$$

The main problem/task in constructing new spaces of oscillatory functions is the finding of sequences $\{\lambda_j(t); j \geq 1\}$, satisfying (19) relations. Further approaches are also possible.

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