INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 15, Number 3, Pages 428–451

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Computing and Information

A NEW METHOD FOR SIMULTANEOUSLY RECONSTRUCTING THE SPACE-TIME DEPENDENT ROBIN COEFFICIENT AND HEAT FLUX IN A PARABOLIC SYSTEM

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This paper is dedicated to the memory of Professor Mahmoud Hashem Farag

Abstract. This paper studies a regularization approach for simultaneously reconstructing space-time dependent Robin coefficient $\gamma(\mathbf{x}, t)$ and heat flux $q(\mathbf{x}, t)$. The differentiability results and adjoint systems are established. A standard finite element method (FEM) is employed to discretize the constrained optimization problem which is reduced to a sequence of unconstrained optimization problem by adding regularization terms. We propose an improved algorithm for the introduced problem based on modified conjugate gradient method (MCGM) for quadratic minimization. Numerical experiments present the efficiency, accuracy, and robustness of the proposed algorithm.

Key words. Simultaneous identification, Numerical reconstruction, Robin coefficient and heat flux, Tikhonov regularization, FEM, MCGM.

1. Introduction

The inverse problem arising in reconstructing the heat transfer coefficient, called Robin coefficient $\gamma(\mathbf{x}, t)$, which represents the convection between the conducting body and the ambient environment from the boundary measurements of the solution and space-time dependent heat flux $q(\mathbf{x}, t)$. Háo [8] determined the space or time dependent heat transfer coefficient using nonlinear conjugate gradient method combined with a boundary element direct solver. In many distributed parameter identification problems, the Robin inverse problem suffers from ill-posedness, such as, small error in the data which leads to large deviations in the solution. Therefore, specialized techniques are necessary to keep the stability in the solution. Numerically, an engineering approach has been applied to estimate the time-dependent Robin coefficient from the measured temperature data for quenching process [1], using the sequential function specification method [2]. However, the approach is generally influenced by the noise in the data and then the accuracy of the solutions. Another popular engineering approach is the variational method [14, 23]. Marián [19] studied the recovery of a time-dependent Robin coefficient in a semilinear parabolic equation from an over-specified nonlocal boundary conditions and proposed a temporal discretization based on Rothes method with some convergence analysis. However, the spatial discretization that is necessary for practical computations was not considered.

Slodička, et al. [18] introduced a mathematical analysis for the estimation of the time-dependent Robin coefficient in a nonlinear boundary condition for one-dimensional heat equation, and showed the existence and uniqueness of the solution. Deng, et al. [3, 4] introduced a two-level space-time domain decomposition method for solving an inverse source problem associated with the time-dependent convection-diffusion equation in three dimensions. Jiang, et al. [11] proposed an efficient overlapping domain decomposition method for solving some typical linear

Received by the editors November 25, 2016 and, in revised form, March 23, 2017.
2010 *Mathematics Subject Classification.* 65M30, 65M32, 65M12, 65M60, 35K10.

inverse problems, including identification of the flux, source strength, and initial temperature in second order elliptic and parabolic systems. Jin and Lu [13] studied the space-time dependent Robin coefficient in one and two dimensional cases using nonlinear conjugate gradient method. Xie and Zou [21] introduced the mathematical and numerical justification of regularization approaches for reconstructing the heat flux in both space and time with investigation of a FEM.

Recently, Jiang and Abdelhamid [12] focused on the simultaneous reconstruction of the spatially-dependent Robin coefficient and heat flux using Levenberg-Marquardt and surrogate functional method. Abdelhamid, et al. [26] studied the simultaneous reconstruction of the time-dependent Robin coefficient and heat flux in the heat conduction problem. Abdelhamid, T. [27] introduced the numerical reconstruction for the space-time dependent heat transfer coefficient and spatial-dependent heat flux in the parabolic systems. The present study focuses on reconstructing the space-time dependent Robin coefficient and heat flux, simultaneously using MCGM for solving the nonlinear inverse problem.

The rest of this paper is organized as follows: Section 2 describes the mathematical and variational formulation of the problem. Section 3 investigates the Tikhonov regularization approach in the case that $\gamma(\mathbf{x},t)$ and $q(\mathbf{x},t)$ are unknowns parameters. Section 4 derives the differentiability results to find the gradient formula with respect to $\gamma(\mathbf{x},t)$ and $q(\mathbf{x},t)$. Also, introduces the adjoint problems to find explicit relations that simplify computing the step lengths. Section 5 introduces the FEM and its convergence analysis. Section 6 introduces the numerical algorithm based on the MCGM. Section 7 presents the numerical experiments to investigate the efficiency, accuracy, and robustness of the proposed algorithm. Finally, we draw the conclusion and the future work in section 8.

2. Mathematical formulation

Let $\Omega \subset R^d$, $d \geq 1$ be a bounded, connected, and polyhedral domain. Consider the following parabolic system with the Robin and Neumann boundary conditions

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot (\alpha(\mathbf{x})\nabla u) = f(\mathbf{x},t) & \text{in } \Omega \times (0,T), \\[2ex] \alpha(\mathbf{x})\dfrac{\partial u}{\partial n} + \gamma(\mathbf{x},t)u(\mathbf{x},t) = g(\mathbf{x},t) & \text{on } \Gamma_i \times (0,T), \\[2ex] \alpha(\mathbf{x})\dfrac{\partial u}{\partial n} = q(\mathbf{x},t) & \text{on } \Gamma_c \times (0,T), \\[2ex] u(\mathbf{x},0) = 0 & \mathbf{x} \in \Omega. \end{cases}$$

We assume that the boundary $\partial\Omega$ consists of two parts, i.e. $\partial\Omega = \Gamma_i \cup \Gamma_c$, and $\Gamma_c \equiv \Gamma_{c_1} \cup \Gamma_{c_2} \cup \Gamma_{c_3}$ is a finite collection of disjoint, smooth $(d-1)$-dimensional polyhedral domain. Also, $\gamma(\mathbf{x},t)$ and $q(\mathbf{x},t)$ are the heat transfer coefficient (Robin) and heat flux respectively which are contained in the following sets

$$K_1 = \{\gamma(\mathbf{x},t) : 0 < \gamma_1 \leq \gamma(\mathbf{x},t) \leq \gamma_2 < \infty \text{ a.e. in } \Gamma_i \times (0,T)\},$$

$$K_2 = \{q(\mathbf{x},t) : 0 < q_1 \leq q(\mathbf{x},t) \leq q_2 < \infty \text{ a.e. in } \Gamma_c \times (0,T)\},$$

where γ_1, γ_2, q_1, and q_2 are positive given constants. In this problem, the Robin boundary condition is specified on Γ_i and the Neumann boundary condition on Γ_c. We refer readers to [9] and the references therein for more physical backgrounds and [16, 25] for the analytical and numerical methods to solve the inverse problems.

Let $u(\mathbf{x},t)$ solve the forward problem (1) and $z^\delta \in L^2(0,T;L^2(\Gamma_c))$ represent the measured data on Γ_c over $t \in (0,T)$. The parameter δ is used to emphasize the existence of noise in the data. We recall the following lemma for setting the inverse problem to achieve the uniqueness results (see [5]).

proposition 2.1. *Suppose that Ω is an open, bounded, and connected domain with the boundary $\partial\Omega$. The given source strength $f(\mathbf{x},t) \in L^2(0,T;L^2(\Omega))$, $g(\mathbf{x},t) \in L^2(0,T;L^2(\Gamma_i))$, and $q(\mathbf{x},t) \in L^2(0,T;L^2(\Gamma_c))$. The thermal conductivity $\alpha(\mathbf{x}) \in C(\overline{\Omega})$ such that $0 < \alpha_0 < \alpha(\mathbf{x}) < \alpha_1 < \infty$, and α_0 and α_1 are positive constants.*

Lemma 2.1. *Suppose that the norm $\|\cdot\|_{H^1(\Gamma_i)}$ is defined by*

$$\|u\|^2_{H^1(\Gamma_i)} = \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Gamma_i)},$$

this is equivalent to the standard norm $\|u\|^2_{H^1(\Omega)}$ and

$$c_1\|u\|^2_{H^1(\Gamma_i)} \leq \|u\|^2_{H^1(\Omega)} \leq c_2\|u\|^2_{H^1(\Gamma_i)},$$

c_1 and c_2 are positive constants (see Sec. 7.1 [5]).

Lemma 2.2. *([13]) For any $(\gamma,q) \in K_1 \times K_2$ there exists a unique solution $u(\gamma,q)(\mathbf{x},t) \equiv u(\gamma,q) \in L^2(0,T;H^1(\Omega))$ to the forward problem (1). Furthermore, it has the following regularities*

$$u(\gamma,q) \in L^2(0,T;H^1(\Omega)),\ u(\gamma,q) \in L^2(0,T;L^2(\Omega)),\ u(\gamma,q) \in C(0,T;L^2(\Omega)).$$

Then, it satisfies the following priori estimate

$$\|u(\gamma,q)\|_{L^2(0,T;H^1(\Omega))} \leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(0,T;L^2(\Gamma_i))} + \|q\|_{L^2(0,T;L^2(\Gamma_c))}\right).$$

Then, from Lemmas 2.1 and 2.2 we deduce that the solution $u(\gamma,q)$ of the forward problem (1) is well-defined on $L^2(0,T;H^1(\Omega))$.

3. Tikhonov regularization and the existence of minimizers

In this section, we investigate the Tikhonov regularization approach to the considered problem (1). The ill-posedness in the considered problem lies in the following aspects:
(1) The solution of the problem $u(\gamma,q)$ does not necessarily exist for the given measured data.
(2) The solution may not be unique, and/or does not continuously depend on the cauchy data which are collected experimentally using devices such as thermal sensors. To cope with the numerical instability of system (1), we introduce the following Tikhonov regularization:

$$\begin{aligned}\min_{(\gamma,q)\in K_1\times K_2} J(\gamma,q) &= \frac{1}{2}\int_0^T \|u(\gamma,q)-z^\delta\|^2_{L^2(\Gamma_c)}dt \\ &+\frac{\beta}{2}\|\gamma\|^2_{L^2(0,T;L^2(\Gamma_i))} + \frac{\eta}{2}\|q\|^2_{L^2(0,T;L^2(\Gamma_c))},\end{aligned} \tag{2}$$

where $u \equiv u(\gamma,q)(\mathbf{x},t) \in L^2(0,T;H^1(\Omega))$ satisfies

$$u(\mathbf{x},0) = 0 \text{ in } \Omega, \tag{3}$$

and

$$\int_0^T\int_\Omega \partial_t uvdxdt + \int_0^T\int_\Omega \alpha\nabla u\cdot\nabla vdxdt + \int_0^T\int_{\Gamma_i}\gamma uvdsdt = \int_0^T\int_\Omega fvdxdt$$
$$+\int_0^T\int_{\Gamma_i} gvdsdt + \int_0^T\int_{\Gamma_c} qvdsdt\ \forall\ v\in L^2(0,T;H^1(\Omega)). \tag{4}$$

The constrained optimization problem with $\beta=\eta=0$ denoted by J_0, is equivalent to the inverse problem when the cauchy data is achievable.

Theorem 3.1. *([12]) There exists at least one minimizer to system (2)-(4).*

Proof. Clearly the functional $J(\gamma,q)$ is bounded from below by zero in $K_1\times K_2$; thus there exists a minimizing sequence $\{(\gamma^n,q^n)\}$ such that

$$\lim_{n\to\infty} J(\gamma^n,q^n) = \inf_{(\gamma,q)\in K_1\times K_2} J(\gamma,q). \tag{5}$$

From the uniform boundedness of the admissible set $K_1\times K_2$, the sequence $\{(\gamma^n,q^n)\}$ is uniformly bounded in $K_1\times K_2$. Thus the existence of such subsequence as $\{(\gamma^n,q^n)\}$, and $\{(\gamma^n,q^n)\}\rightharpoonup(\gamma^*,q^*)$ weakly in $K_1\times K_2$ (refers to Theorem 1.19 in [7]). Then from Theorem 3.1 [12] it can be shown that there exists at least one minimizer (γ^*,q^*) for (2)-(4) such that

$$\begin{aligned}
&J(\gamma^*,q^*)\\
=\ &\frac{1}{2}\int_{T_0}^T \|u(\gamma^*,q^*)-z^\delta\|^2_{L^2(\Gamma_c)}dt\\
&+\frac{\beta}{2}\|\gamma^*\|^2_{L^2(0,T;L^2(\Gamma_i))}+\frac{\eta}{2}\|q^*\|^2_{L^2(0,T;L^2(\Gamma_c))}\\
=\ &\frac{1}{2}\lim_{n\to\infty}\int_{T_0}^T \|u(\gamma^n,h^n)-z^\delta\|^2_{L^2(\Gamma_c)}dt\\
&+\frac{\beta}{2}\|\gamma^*\|^2_{L^2(0,T;L^2(\Gamma_i))}+\frac{\eta}{2}\|q^*\|^2_{L^2(0,T;L^2(\Gamma_c))}\\
\leq\ &\frac{1}{2}\lim_{n\to\infty}\int_{T_0}^T \|u(\gamma^n,q^n)-z^\delta\|^2_{L^2(\Gamma_c)}dt+\frac{\beta}{2}\lim_{n\to\infty}\inf\|\gamma^n\|^2_{L^2(0,T;L^2(\Gamma_i))}\\
+\ &\frac{\eta}{2}\lim_{n\to\infty}\inf\|q^n\|^2_{L^2(0,T;L^2(\Gamma_c))}\\
\leq\ &\lim_{n\to\infty}\inf J(\gamma^n,q^n)=\inf_{(\gamma,q)\in K\times L^2(\Gamma_c)} J(\gamma,q),
\end{aligned} \tag{6}$$

which implies that (γ^*,q^*) is a minimizer of (2)-(4). □

4. Differentiability results and adjoint problem

In this section, we show the partial Fréchet derivatives of the forward solution $u(\gamma,q)$ and their adjoint problems. Denote $u^1_\gamma\equiv u'_\gamma(\gamma,q)d$ and $u^1_q\equiv u'_q(\gamma,q)p$, as the partial Fréchet derivatives of the forward solution with respect to the Robin coefficient $\gamma(\mathbf{x},t)$ and heat flux $q(\mathbf{x},t)$ in any directions $d\in L^2(0,T;L^2(\Gamma_i))$ and $p\in L^2(0,T;L^2(\Gamma_c))$, respectively. u^1_γ and u^1_q can be obtained by solving the following systems

$$\tag{7} \begin{cases} \dfrac{\partial u_\gamma^1}{\partial t} - \Delta u_\gamma^1 = 0 & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u_\gamma^1}{\partial n} + \gamma u_\gamma^1 = -du(\gamma,q) & \text{on } \Gamma_i \times (0,T), \\ \dfrac{\partial u_\gamma^1}{\partial n} = 0 & \text{on } \Gamma_c \times (0,T), \\ u_\gamma^1(\mathbf{x},0) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\tag{8} \begin{cases} \dfrac{\partial u_q^1}{\partial t} - \Delta u_q^1 = 0 & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u_q^1}{\partial n} + \gamma u_q^1 = 0 & \text{on } \Gamma_i \times (0,T), \\ \dfrac{\partial u_q^1}{\partial n} = p & \text{on } \Gamma_c \times (0,T), \\ u_q^1(\mathbf{x},0) = 0 & \text{in } \Omega, \end{cases}$$

which are linear with respect to d and p, respectively.

Theorem 4.1. *([13]) For any $\gamma, \gamma + d \in K_1$ and $q, q + p \in K_2$, the mapping $(\gamma, q) \mapsto u(\gamma, q)$ from $K_1 \times K_2 \mapsto L^2(0,T;H^1(\Omega))$ is Lipschitz continuity and Fréchet differentiable, then*

$$\tag{9} \lim_{\|d\|_{L^2(0,T;L^\infty(\Gamma_i))}\to 0} \frac{\|u(\gamma+d,q) - u(\gamma,q) - u_\gamma^1\|_{L^2(0,T;H^1(\Omega))}}{\|d\|_{L^2(0,T;L^\infty(\Gamma_i))}} = 0,$$

and

$$\tag{10} \lim_{\|p\|_{L^2(0,T;L^\infty(\Gamma_c))}\to 0} \frac{\|u(\gamma,q+p) - u(\gamma,q) - u_q^1\|_{H^1(\Omega)}}{\|p\|_{L^2(0,T;L^\infty(\Gamma_c))}} = 0.$$

Next, we derive the gradient formula of the functional $J(\gamma, q)$. From Theorem 4.1 we have the following expansions

$$u(\gamma+d,q) = u(\gamma,q) + u_\gamma^{'}(\gamma,q)d + \mathcal{O}(\|d\|^2_{L^2(0,T;L^\infty(\Gamma_i))}),$$

and

$$u(\gamma,q+p) = u(\gamma,q) + u_q^{'}(\gamma,q)p + \mathcal{O}(\|p\|^2_{L^2(0,T;L^\infty(\Gamma_c))}).$$

Now, we define the adjoint equations for the partial Gáteaux derivatives $\omega_\gamma^* \equiv u_\gamma^{'}(\gamma,q)^*\nu$ and $\omega_q^* \equiv u_q^{'}(\gamma,q)^*\zeta$ for any directions ν and ζ, respectively as

$$\tag{11} \begin{cases} \dfrac{\partial \omega_\gamma^*}{\partial t} + \Delta \omega_\gamma^* = 0 & \text{in } \Omega \times (0,T), \\ \dfrac{\partial \omega_\gamma^*}{\partial n} + \gamma \omega_\gamma^* = 0 & \text{on } \Gamma_i \times (0,T), \\ \dfrac{\partial \omega_\gamma^*}{\partial n} = -\nu & \text{on } \Gamma_c \times (0,T), \\ \omega_\gamma^*(\mathbf{x},T) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} \frac{\partial \omega_q^*}{\partial t} + \Delta \omega_q^* = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial \omega_q^*}{\partial n} + \gamma \omega_q^* = 0 & \text{on } \Gamma_i \times (0,T), \\ \frac{\partial \omega_q^*}{\partial n} = \zeta & \text{on } \Gamma_c \times (0,T), \\ \omega_q^*(\mathbf{x},T) = 0 & \text{in } \Omega. \end{cases} \tag{12}$$

The next theorem shows the differentiability of the functional $J(\gamma,q)$ with respect to $\gamma(\mathbf{x},t)$ and $q(\mathbf{x},t)$.

Theorem 4.2. *Suppose that the functional $J(\gamma,q)$ is Fréchet differentiable, then its Fréchet derivatives $J_\gamma^{'}(\gamma,q)$ and $J_q^{'}(\gamma,q)$ in the directions d and p are respectively given by*

$$J_\gamma^{'}[d] = \int_0^T \int_{\Gamma_i} d\left[-u(\gamma,q)\omega_\gamma^* + \beta\gamma\right] dsdt, \tag{13}$$

and

$$J_q^{'}[p] = \int_0^T \int_{\Gamma_c} p\left[\omega_q^* + \eta q\right] dsdt. \tag{14}$$

Proof. From Theorem 4.1 and noting that
$\|u_\gamma^1\|_{L^2(0,T;L^2(\Gamma_i))} \le C\|d\|_{L^2(0,T;L^\infty(\Gamma_i))}$ and $\|u_q^1\|_{L^2(0,T;L^2(\Gamma_c))} \le C\|p\|_{L^2(0,T;L^\infty(\Gamma_c))}$.
We have

$$\min_{(\gamma,q)\in K_1\times K_2} J_0(\gamma,q) = \int_0^T \int_{\Gamma_c} (u(\gamma,q)-z^\delta)^2 dsdt. \tag{15}$$

Then, the difference of the functional $J(\gamma,q)$ with respect to the Robin coefficient is computed as follows

$$\begin{aligned} R_\gamma &\equiv J_0(\gamma+d,q) - J_0(\gamma,q), \\ &= \frac{1}{2}\int_0^T \int_{\Gamma_c} (u(\gamma+d,q)-z^\delta)^2 dsdt - \int_0^T \int_{\Gamma_c} (u(\gamma,q)-z^\delta)^2 dsdt, \\ &= \frac{1}{2}\int_0^T \int_{\Gamma_c} \Big\{(u(\gamma,q)+u_\gamma^1 + \mathcal{O}(\|d\|^2_{L^2(0,T;L^\infty(\Gamma_i))}) - z^\delta)^2 \\ &\quad -(u(\gamma,q)-z^\delta)^2\Big\} dsdt, \\ &= \int_0^T \int_{\Gamma_c} \Big\{(u(\gamma,q)-z^\delta)u_\gamma^1 + \mathcal{O}(\|d\|^2_{L^2(0,T;L^\infty(\Gamma_i))}) \\ &\quad + (u_\gamma^1 + \mathcal{O}(\|d\|^2_{L^2(0,T;L^\infty(\Gamma_i))}))^2\Big\} dsdt, \\ &= \int_0^T \int_{\Gamma_c} (u(\gamma,q)-z^\delta)u_\gamma^1 dsdt + \mathcal{O}(\|d\|^2_{L^2(0,T;L^\infty(\Gamma_i))}) dsdt. \end{aligned}$$

Dividing the above equation by $\|d\|^2_{L^2(0,T;L^\infty(\Gamma_i))}$ and taking the limit with respect to $\|d\|^2_{L^2(0,T;L^\infty(\Gamma_i))}$, we obtain

$$J_{\gamma,0}^{'}[d] = \int_0^T \int_{\Gamma_c} (u(\gamma,q)-z^\delta)u_\gamma^1 dsdt. \tag{16}$$

Similarly, the Fréchet derivative $J_{q,0}^{'}$ in the direction p is defined as

$$J_{q,0}^{'}[p] = \int_0^T \int_{\Gamma_c} (u(\gamma, q) - z^\delta) u_q^1 dsdt. \tag{17}$$

Taking $\varphi = u_\gamma^1$ and $\psi = \omega_\gamma^*$, multiplying (7) by ψ, multiplying (11) by φ, and applying Green's second identity, we obtain

$$\begin{aligned} &\int_0^T \int_\Omega \{\psi \nabla \cdot \nabla \varphi - \varphi \nabla \cdot \nabla \psi\} \, dxdt \\ = &\int_0^T \int_{\partial\Omega} \left(\frac{\partial \varphi}{\partial n} \psi - \frac{\partial \psi}{\partial n} \varphi \right) dsdt = 0. \end{aligned} \tag{18}$$

Substituting the boundary conditions for φ and ψ, we obtain

$$-\int_0^T \int_{\Gamma_i} du(\gamma, q) \psi dsdt = \int_0^T \int_{\Gamma_c} \nu \varphi dsdt. \tag{19}$$

Taking $\tilde{\varphi} = u_q^1$, $\tilde{\psi} = \omega_q^*$, multiplying (8) by $\tilde{\psi}$ and multiplying (12) by $\tilde{\varphi}$. Then, we apply Green's second identity

$$\begin{aligned} &\int_0^T \int_\Omega \left(\tilde{\psi} \nabla \cdot \nabla \tilde{\varphi} - \tilde{\varphi} \nabla \cdot \nabla \tilde{\psi} \right) dxdt \\ = &\int_0^T \int_{\partial\Omega} \left(\frac{\partial \tilde{\varphi}}{\partial n} \tilde{\psi} - \frac{\partial \tilde{\psi}}{\partial n} \tilde{\varphi} \right) dsdt = 0. \end{aligned} \tag{20}$$

Substituting the boundary conditions for $\tilde{\varphi}$ and $\tilde{\psi}$, we get

$$\int_0^T \int_{\Gamma_c} p \tilde{\psi} dsdt = \int_0^T \int_{\Gamma_c} \zeta \tilde{\varphi} dsdt. \tag{21}$$

Substituting (19) into (16), we deduce

$$J_\gamma^{'}[d] = \int_0^T \int_{\Gamma_i} d \left[-u(\gamma, q) \omega_\gamma^* + \beta \gamma \right] dsdt, \tag{22}$$

and

$$J_{\gamma,0}^{'}[d] = -\int_0^T \int_{\Gamma_i} d \left[u(\gamma, q) \omega_\gamma^* \right] dsdt.$$

Similarly, the Fréchet derivative of J_0 with respect to $q(\mathbf{x}, t)$ in the direction p has the form

$$J_q^{'}[p] = \int_0^T \int_{\Gamma_c} p \left[\omega_q^* + \eta q \right] dsdt, \tag{23}$$

and

$$J_{q,0}^{'}[p] = \int_0^T \int_{\Gamma_c} p \omega_q^* dsdt.$$

This completes the proof of Theorem 4.2. □

5. Finite element method and its convergence

In this section we investigate the FEM for solving the continuous minimization problem (2)-(4). We define the finite element space V^h using the continuous and piecewise linear space over the triangulation $\mathcal{T}^h$ as

$$V^h = \left\{\varphi_h : \varphi_h \in C(\bar{\Omega}) : \varphi_h|_{\mathcal{T}^i} \in p(\mathcal{T}^i) \ \forall \ \mathcal{T}^i \in \mathcal{T}^h\right\},$$

where $p(\mathcal{T}^i)$ denotes the space of linear polynomials on the element $\mathcal{T}^i$. Also, we define the spaces $V^h_{\Gamma_i}$ and $V^h_{\Gamma_c}$ as the restrictions of V^h on Γ_i and Γ_c, respectively. Let $\{x_i\}_{i=1}^N$ be the set of all nodal points of triangulation $\mathcal{T}^h$, then the constrained subsets K_1 and K_2 are approximated by

$$K_1^{h,\tau} = \{\gamma_h \in V^h_{\Gamma_i} : \gamma_1 \le \gamma_h \le \gamma_2 \text{ a.e. } (x,t) \in (0,T)\times\Gamma_i\},$$
$$K_2^{h,\tau} = \{q_h \in V^h_{\Gamma_c} : q_1 \le q_h \le q_2 \text{ a.e. } (x,t) \in (0,T)\times\Gamma_c\}.$$

To fully discretize the optimization problem (2)-(4), for the time discretization, the time interval $(0,T)$ is divided into M equally spaced subintervals using nodal points

$$\Delta_t : 0 = t_0 < t_1 < \cdots < t_M = T,$$

with $t_n = n\tau, \tau = T/M$. For $u : (0,T) \to L^2(\Omega)$, we define $u^n = u(\cdot, t_n)$, $0 < n < M$. For a given sequence $\{u^n\}_{n=0}^M \subset L^2(\Omega)$ we have

$$\partial_t u^n = \frac{u^n - u^{n-1}}{\tau}, \ \bar{u}^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} u(\cdot,t)dt, \text{ and } \bar{u}^0 = u(\cdot,0) \text{ at } n=0. \tag{24}$$

For the convergence analysis, we use the projection operator Q_h from $H^1(\Omega)$ to V^h (cf. [22, 21])

$$\begin{aligned} \lim_{h\to 0}\|v - Q_h v\|_{H^1(\Omega)} &= 0 \ \forall v \in H^1(\Omega) \\ \|Q_h v\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega)}, \ \|Q_h v\|_{H^1(\Omega)} &\le C\|v\|_{H^1(\Omega)}, \ \forall v \in H^1(\Omega) \\ \|v - Q_h v\|_{L^2(\Omega)} &\le Ch\|v\|_{H^1(\Omega)}, \ \forall v \in H^1(\Omega). \end{aligned} \tag{25}$$

We formulate the finite element problem corresponding to (2)-(4) as follows

$$\begin{aligned} \min_{(\gamma_h^n, q_h^n)\in K_1^{h,\tau}\times K_2^{h,\tau}} J_h^M(\gamma_h, q_h) = \frac{\tau}{2}\sum_{n=0}^M \rho_n \|u_h^n(\gamma_h, q_h) - z^\delta\|^2_{L^2(\Gamma_c)} + \\ \frac{\tau}{2}\beta\sum_{n=0}^M \rho_n\|\gamma_h^n\|^2_{L^2(\Gamma_i)} + \frac{\tau}{2}\eta\sum_{n=0}^M \rho_n\|q_h^n\|^2_{L^2(\Gamma_c)}, \end{aligned} \tag{26}$$

where $u_h^n \equiv u_h^n(\gamma_h, q_h)(\mathbf{x}, t) \in V^h$ and $u_h^0 = 0$, satisfies

$$\begin{aligned} \int_\Omega \partial_\tau u_h^n v_h dx + \int_\Omega \alpha^n \nabla u_h^n \cdot \nabla v_h dx + \int_{\Gamma_i} \gamma_h^n u_h^n v_h ds = \int_\Omega \bar{f}^n v_h dx \\ + \int_{\Gamma_i} \bar{g}^n v_h ds + \int_{\Gamma_c} q_h^n v_h ds \ \forall \ v_h \in V^h. \end{aligned} \tag{27}$$

Here, $\{\rho_n\}$ are the coefficients of the trapezoidal rule, i.e., $\rho_0 = \rho_M = \frac{1}{2}$ and $\rho_n = 1$ for $n = 1, \ldots, M-1$.

Lemma 5.1. *For any sequence $\{(\gamma_h^n, q_h^n)\}$ of system (26)-(27) in $K_1^{h,\tau} \times K_2^{h,\tau}$ converges weakly to some $(\gamma, q) \in K_1 \times K_2$ as $n \to \infty$, we have*

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_c} |u_h^n(\gamma_h^n, q_h^n) - z^n|^2 dxdt = \int_0^T\int_{\Gamma_c} |u(\gamma^*, q^*) - z|^2 dxdt.$$

Proof. Let $\{z^n\}$ be a sequence such that $z^n \to z$ strongly in $L^2(0,T;L^2(\Gamma_c))$. By the definition of $\{(\gamma_h^n, q_h^n)\}$, we have

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_c}|u_h^n(\gamma_h^n,q_h^n)-z^n|^2dxdt \leq \lim_{n\to\infty}\int_0^T\int_{\Gamma_c}|u(\gamma,q)-z|^2dxdt \quad \forall\ (\gamma,q)\in K_1\times K_2.$$

This implies that $\{(\gamma_h^n, q_h^n)\}$ is bounded in $K_1^{h,\tau}\times K_2^{h,\tau}$. Therefore, there exists a subsequence denoted by $\{(\gamma_h^n, q_h^n)\}$, such that

$$(\gamma_h^n, q_h^n) \rightharpoonup (\gamma^*, q^*) \text{ weakly in } K_1\times K_2.$$

Since, there exists a minimizer for the optimization problem (26)-(27) (Theorem 3.1), we deduce

$$u_h^n(\gamma_h^n, q_h^n) \to u(\gamma^*, q^*) \text{ srtongly in } K_1\times K_2$$

By applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_0^T\int_{\Gamma_c}|u_h^n(\gamma_h^n,q_h^n)-z^n|^2dxdt-\int_0^T\int_{\Gamma_c}|u(\gamma^*,q^*)-z|^2dxdt\\
=\ &\int_0^T\int_{\Gamma_c}\big(|u_h^n(\gamma_h^n,q_h^n)-z^n|^2+|u_h^n(\gamma_h^n,q_h^n)-z|^2-|u_h^n(\gamma_h^n,q_h^n)-z|^2\\
&-|u(\gamma^*,q^*)-z|^2\big)\,dxdt,\\
=\ &\int_0^T\int_{\Gamma_c}\{(z-z^n)(2u(\gamma_h^n,q_h^n)-z-z^n)+(u(\gamma_h^n,q_h^n)-u(\gamma^*,q^*))\\
&(u(\gamma_h^n,q_h^n)+u(\gamma^*,q^*)-2z)\}\,dxdt,\\
\leq\ &(\int_0^T\int_{\Gamma_c}|z-z^n|^2dxdt)^{\frac12}(\int_0^T\int_{\Gamma_c}|2u(\gamma_h^n,q_h^n)-z-z^n|^2dxdt)^{\frac12}\\
&+(\int_0^T\int_{\Gamma_c}|u(\gamma_h^n,q_h^n)-u(\gamma^*,q^*)|^2dxdt)^{\frac12}\\
&\times(\int_0^T\int_{\Gamma_c}|u(\gamma_h^n,q_h^n)+u(\gamma^*,q^*)-2z|^2dxdt)^{\frac12}.
\end{aligned}\tag{28}$$

Taking the limit of (28) at $n\to\infty$, which concludes the proof. □

Next, we investigate the convergence analysis of the discrete minimizer to the minimizer of the continuous problem (2)-(4). We introduce a step function approximation of a function $f(x,t)\in C((0,T);X)$ for a Banach space X

$$S_\Delta f(\mathbf{x},t)=\sum_{n=1}^{M}\chi_n(t)f(\mathbf{x},t_n).\tag{29}$$

Then, the following limit holds [24]

$$\lim_{\tau\to 0}\int_0^T\|S_\Delta f(\cdot,t)-f(\cdot,t)\|_X^2dt=0.\tag{30}$$

We approximate the Robin coefficient $\gamma(\mathbf{x},t)$ and heat flux $q(\mathbf{x},t)$ by a piecewise constant functions $\gamma_{h,\tau}(\mathbf{x},t)$ and $q_{h,\tau}(\mathbf{x},t)$ over the interval Δ_t as follows:

$$\gamma_{h,\tau}(\mathbf{x},t)=\sum_{n=1}^{M}\chi_n(t)\gamma_h^n(\mathbf{x}) \text{ and } q_{h,\tau}(\mathbf{x},t)=\sum_{n=1}^{M}\chi_n(t)q_h^n(\mathbf{x}),$$

where $\gamma_h^n(\mathbf{x}) \in V_{\Gamma_i}^h$, $q_h^n(\mathbf{x}) \in V_{\Gamma_c}^h$, and $\chi_n(t)$ is the characteristic function in the interval (t_{n-1}, t_n), which we will use in the rest of the paper. To prove the existence of the minimizer to the finite element problem (26)-(27), we show the following continuity of $u_h^n(\gamma_h, q_h)$ in (26)-(27) with respect to (γ_h, q_h).

Lemma 5.2. *For any sequence $\{\gamma_{h,\tau}^k, q_{h,\tau}^k\}$ in $K_1^{h,\tau} \times K_2^{h,\tau}$ converges weakly to some $(\gamma, q) \in K_1^{h,\tau} \times K_2^{h,\tau}$ in a certain norm as $k \to \infty$ we have, for $n = 1, 2, \ldots, M$,*

$$u_h^n(\gamma_{h,\tau}^k, q_{h,\tau}^k) \to u_h^n(\gamma, q) \text{ in } H^1(\Omega) \text{ as } k \to \infty.$$

Proof. By the definition of $u_h^n(\gamma_{h,\tau}, q_{h,\tau})$, (for simplicity, we use the notations $u_h^n = u_h^n(\gamma_{h,\tau}, q_{h,\tau})$ and $u^n = u(\gamma, q)(\cdot, t^n)$ in the following proof), we have

$$\begin{aligned} &\int_\Omega \partial_t u_h^n v_h dx + \int_\Omega \alpha^n \nabla u_h^n \cdot \nabla v_h dx + \int_{\Gamma_i} \gamma_{h,\tau}^k u_h^n v_h ds = \int_\Omega \bar{f}^n v_h dx \\ &\qquad + \int_{\Gamma_i} \bar{g}^n v_h ds + \int_{\Gamma_c} q_{h,\tau}^k v_h ds \ \forall\ v_h \in V^h, \end{aligned} \tag{31}$$

$$\begin{aligned} &\int_\Omega \partial_t u_h^n v_h dx + \int_\Omega \alpha^n \nabla u_h^n \cdot \nabla v_h dx + \int_{\Gamma_i} \gamma_h u_h^n v_h ds = \int_\Omega \bar{f}^n v_h dx \\ &\qquad + \int_{\Gamma_i} \bar{g}^n v_h ds + \int_{\Gamma_c} q_h v_h ds \ \forall\ v_h \in V^h. \end{aligned} \tag{32}$$

Taking $v_h = \tau u_h^n$ in (31), we obtain

$$\begin{aligned} &\frac{1}{2}\|u_h^n\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u_h^{n-1}\|_{L^2(\Omega)}^2 + \tau\alpha^n\|\nabla u_h^n\|_{L^2(\Omega)}^2 + \tau\gamma_1\|u_h^n\|_{L^2(\Gamma_i)}^2 \\ \leq\ & \tau\|\bar{f}^n\|_{L^2(\Omega)}\|u_h^n\|_{L^2(\Omega)} + \tau\|\bar{g}^n\|_{L^2(\Gamma_i)}\|u_h^n\|_{L^2(\Gamma_i)} + \tau\|q_h^k\|_{L^2(\Gamma_c)}\|u_h^n\|_{L^2(\Gamma_c)} \\ \leq\ & A_1 + A_2 + A_3. \end{aligned} \tag{33}$$

Using Young's inequality and Sobolev trace theorem, we derive

$$A_1 \leq \frac{\tau}{4\varepsilon}\|\bar{f}^n\|_{L^2(\Omega)}^2 + \tau\varepsilon\|u_h^n\|_{L^2(\Omega)}^2, \tag{34}$$

$$\begin{aligned} A_2 &\leq \frac{\tau}{4\varepsilon}\|\bar{g}^n\|_{L^2(\Gamma_i)}^2 + \tau\varepsilon\|u_h^n\|_{L^2(\Gamma_i)}^2 \\ &\leq \frac{\tau}{4\varepsilon}\|\bar{g}^n\|_{L^2(\Gamma_i)}^2 + \tau\varepsilon c_1\|u_h^n\|_{H^1(\Omega)}^2, \end{aligned} \tag{35}$$

$$A_3 \leq \frac{\tau}{4\varepsilon}\|q_h^n\|_{L^2(\Gamma_c)}^2 + \tau\varepsilon c_2\|u_h^n\|_{H^1(\Omega)}^2. \tag{36}$$

Substituting (34)-(36) into (33) and summing both sides over $n = 1, 2, \ldots, k \leq M$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u_h^k\|_{L^2(\Omega)}^2 + \tau c_3\sum_{n=1}^k \|\nabla u_h^n\|_{L^2(\Omega)}^2 + \tau\gamma_1\sum_{n=1}^k \|u_h^n\|_{L^2(\Gamma_i)}^2 \\ \leq\ & \frac{\tau}{4\varepsilon}\sum_{n=1}^k [\|\bar{f}^n\|_{L^2(\Omega)}^2 + \|\bar{g}^n\|_{L^2(\Gamma_i)}^2 + \|q_h^n\|_{L^2(\Gamma_c)}^2] + \tau c_4\sum_{n=1}^k \|u_h^n\|_{L^2(\Omega)}^2. \end{aligned} \tag{37}$$

Here, $c_i (i = 1, 2, 3, 4)$ are constants that do not dependent on h, τ, and k, and ε is a small constant. We have the initial $u_h^0 = 0$, $c_3 = 1 - \varepsilon c_1 - \varepsilon c_2$, and $c_4 = \varepsilon(1 + c_1 + c_2)$.

Then, using Gronwall's inequality [5], we obtain

$$\max_{1\leq n\leq M}\|u_h^n\|_{L^2(\Omega)}\leq C,\ \tau\sum_{n=1}^{M}\|\nabla u_h^n\|_{L^2(\Omega)}\leq C,\ \tau\sum_{n=1}^{M}\|u_h^n\|_{L^2(\Gamma_i)}\leq C. \tag{38}$$

And C is independent of h,τ, and k. By subtracting (31) from (32), and let $w_h^n(k)=u_{h,\tau}^n(\gamma_h^k,q_{h,\tau}^k)-u_h^n(\gamma_h,q_h)$, we get

$$\begin{aligned}&\int_\Omega \partial_t w_h^n(k)v_h dx+\int_\Omega \alpha^n\nabla w_h^n(k)\cdot\nabla v_h dx+\int_{\Gamma_i}\gamma_{h,\tau}^k w_h^n(k)v_h ds\\ &=\int_{\Gamma_i}(\gamma_h-\gamma_{h,\tau}^k)u_h^n(k)v_h ds+\int_{\Gamma_c}(q_{h,\tau}^k-q_h)v_h ds\ \forall\ v_h\in V^h.\end{aligned} \tag{39}$$

Then, by taking $v_h=\tau w_h^n(k)$, it yields

$$\begin{aligned}&\frac{1}{2}\|w_h^n(k)\|_{L^2(\Omega)}^2-\frac{1}{2}\|w_h^{n-1}(k)\|_{L^2(\Omega)}^2\\ &+\tau\alpha^n\|\nabla w_h^n(k)\|_{L^2(\Omega)}^2+\tau\gamma_1\|w_h^n(k)\|_{L^2(\Gamma_i)}^2\\ \leq\ &\tau\max_{x\in\Gamma_i}|\gamma_h-\gamma_{h,\tau}^k|\|u_h^n(k)\|_{L^2(\Gamma_i)}\|w_h^n(k)\|_{L^2(\Gamma_i)}\\ &+\tau\max_{x\in\Gamma_c}|q_{h,\tau}^k-q_h|\|w_h^n(k)\|_{L^2(\Gamma_c)}.\end{aligned} \tag{40}$$

By summing up the above equation over $n=1,2,\ldots,k\leq M$ and using Young's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\|w_h^k(k)\|_{L^2(\Omega)}^2-\frac{1}{2}\|w_h^0(k)\|_{L^2(\Omega)}^2\\ &+\tau\sum_{n=1}^{k}\alpha_0\|\nabla w_h^n(k)\|_{L^2(\Omega)}^2+\tau\sum_{n=1}^{k}\gamma_1\|w_h^n(k)\|_{L^2(\Gamma_i)}^2\\ \leq\ &\tau\sum_{n=1}^{k}\max_{x\in\Gamma_i}|\gamma_h-\gamma_{h,\tau}^k|\|u_h^n(k)\|_{L^2(\Gamma_i)}\|w_h^n(k)\|_{L^2(\Gamma_i)}\\ &+\tau\sum_{n=1}^{k}\max_{x\in\Gamma_c}|q_{h,\tau}^k-q_h|\|w_h^n(k)\|_{L^2(\Gamma_c)}.\end{aligned} \tag{41}$$

Then, we deduce $w_h^n(k)\to 0$ in $H^1(\Omega)$, by applying Gronwall's inequality [15, 21]. □

The following theorem shows the existence of minimizers to system (26)-(27). The proof of the following theorem is similar to Theorem 3.1.

Theorem 5.1. *There exists at least one minimizer to the finite element problem (26)-(27).*

The following two lemmas are important for convergence analysis of the finite element approximation

Lemma 5.3. *Let $u_h^n(\gamma_h,q_h)$ be the solution of system (26)-(27) corresponding to $(\gamma_h,q_h)\in K_1^{h,\tau}\times K_2^{h,\tau}$, then the following stability estimates are hold:*

$$\begin{aligned}&\max_{1\leq n\leq M}\|u_h^n\|_{L^2(\Omega)}^2+\gamma_1\sum_{n=1}^{M}\|u_h^n\|_{L^2(\Gamma_i)}^2+\tau\alpha\sum_{n=1}^{M}\|\nabla u_h^n\|_{L^2(\Omega)}^2\\ &\leq C\left[\|f\|_{L^2(0,T;L^2(\Omega))}^2+\|g\|_{L^2(0,T;L^2(\Gamma_i))}^2+\|q_h\|_{L^2(0,T;L^2(\Gamma_c))}\right]\end{aligned} \tag{42}$$

$$\max_{1\le n\le M}\|\nabla u_h^n\|^2_{L^2(\Omega)}+\gamma_1\max_{1\le n\le M}\|u_h^n\|^2_{L^2(\Gamma_i)}+\tau\sum_{n=1}^{M}\|\partial_\tau u_h^n\|^2_{L^2(\Omega)}$$

$$\le C\left[\|f\|^2_{L^2(0,T;L^2(\Omega))}+\|g\|^2_{L^2(0,T;L^2(\Gamma_i))}+\|q_h\|_{L^2(0,T;L^2(\Gamma_c))}\right]. \tag{43}$$

Proof. Take $v_h^n=\tau u_h^n(\gamma_h^k,q_h^k)$ in (31) and follow the same proof of Lemma 5.2, then we obtain (42). The stability estimate (43) can be proved similarly by taking $v_h=\tau\partial_\tau v_h^n$ in (27); this concludes the proof of this lemma. □

Lemma 5.4. *For any sequence* $\{\gamma_h,q_h\}\in K_1^{h,\tau}\times K_2^{h,\tau}$ *and* (γ_h,q_h) *converges weakly to some* (γ,q) *in* $K_1\times K_2$ *as* h,τ *tend to 0, then*

$$\tau\sum_{n=0}^{M}\int_{\Gamma_c}|u_h^n(\gamma_h,q_h)-z^\delta|^2ds\to\int_0^T\int_{\Gamma_c}|u(\gamma,q)-z^\delta|^2dxdt \text{ strongly.} \tag{44}$$

Proof. We use u_h^n and u to denote $u_h^n(\gamma_h,q_h)$ and $u(\gamma,q)$, respectively and

$$\begin{aligned} u^n=u(\gamma,q;t_n)&\equiv u(\gamma,q;n\tau)\quad \text{for } 0\le n\le M,\\ \bar u^n\equiv\bar u^n(\gamma,q)&=\frac{1}{\tau}\int_{t^{n-1}}^{t^n}u(\gamma,q;t)dt\quad\text{for } 1\le n\le M,\\ \bar u^0&=\bar u^0(\gamma,q)=0. \end{aligned}\tag{45}$$

By taking $v=\tau^{-1}v_h$ in (4), then integrating over (t^{n-1},t^n) and subtracting it from (27), it yields

$$\begin{aligned} &\int_\Omega\partial_\tau(u_h^n-u^n)v_hdx+\frac{1}{\tau}\int_{t^{n-1}}^{t^n}\int_\Omega\alpha^n\nabla(u_h^n-u)\cdot\nabla v_hdxdt\\ &+\frac{1}{\tau}\int_{t^{n-1}}^{t^n}\int_{\Gamma_i}\gamma_h^n(u_h^n-u)v_hdsdt\\ =\ &\frac{1}{\tau}\int_{t^{n-1}}^{t^n}\int_{\Gamma_i}(\gamma_h^n-\gamma)uv_hdsdt+\frac{1}{\tau}\int_{t^{n-1}}^{t^n}\int_{\Gamma_c}(q_h^n-q)v_hdsdt\ \forall\ v_h\in V^h. \end{aligned}\tag{46}$$

Taking $v_h=\tau\eta_h^n$ in the above equation and letting $\eta_h^n=u_h^n-Q_h\bar u^n$, we have

$$\begin{aligned} &\frac12\|\eta_h^n\|^2_{L^2(\Omega)}-\frac12\|\eta_h^{n-1}\|^2_{L^2(\Omega)}+\tau\alpha_0\|\nabla\eta_h^n\|^2_{L^2(\Omega)}+\tau\gamma_1\|\eta_h^n\|^2_{L^2(\Omega)}\\ \le\ &\tau\int_\Omega\partial_\tau\eta_h^n\eta_h^ndx+\int_{t^{n-1}}^{t^n}\int_{\Gamma_i}(\gamma_h^n-\gamma)u\eta_h^ndsdt\\ &+\int_{t^{n-1}}^{t^n}\int_{\Gamma_c}(q_h^n-q)\eta_h^ndsdt\ \forall\ v_h\in V^h,\\ =\ &(I)_1+(I)_2+(I)_3. \end{aligned}\tag{47}$$

Summing the above equation over $n=1,2,\dots,k\le M$, we obtain

$$\begin{aligned} &\frac12\|\eta_h^k\|^2_{L^2(\Omega)}-\frac12\|\eta_h^0\|^2_{L^2(\Omega)}+\tau\alpha_0\sum_{n=1}^{k}\|\nabla\eta_h^n\|^2_{L^2(\Omega)}+\tau\gamma_1\sum_{n=1}^{k}\|\eta_h^n\|^2_{L^2(\Omega)}\\ \le&\sum_{n=1}^{k}(I)_1+\sum_{n=1}^{k}(I)_2+\sum_{n=1}^{k}(I)_3. \end{aligned}\tag{48}$$

We next estimate $(I)_1, (I)_2$, and $(I)_3$. For any sequence $\{a_n\}$ and $\{b_n\}$, the following formula is satisfied

$$\sum_{n=1}^{k}(a_n - a_{n-1})b_n = a_k b_k - a_0 b_0 - \sum_{n=1}^{k} a_{n-1}(b_n - b_{n-1}). \tag{49}$$

We have

$$\begin{aligned} \sum_{n=1}^{k}(I)_1 &= \tau\sum_{n=1}^{k}\int_\Omega \partial_\tau(u^n - \bar{u}^n)\eta_h^n dx, \\ &= \int_\Omega (u^k - \bar{u}^k)\eta_h^k dx - \tau\sum_{n=1}^{k}\int_\Omega (u^{n-1} - \bar{u}^{n-1})\partial_\tau \eta_h^n dx, \\ &\le \sqrt{\tau}\left\{\int_{t^{k-1}}^{t^k} \|u_t\|^2_{L^2(\Omega)} dt\right\}^{\frac{1}{2}} \|\eta_h^k\|_{L^2(\Omega)} \\ &\quad +\tau\left\{\int_0^T \|u_t\|^2_{L^2(\Omega)} dt\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{k}\|\eta_h^n\|_{L^2(\Omega)}\right\}^{\frac{1}{2}}, \\ &\le C\sqrt{\tau}. \end{aligned} \tag{50}$$

Here, we have used the stability estimates (42)-(43) and the property of Q_h

$$\begin{aligned} \sum_{n=1}^{k}(I)_2 &= \sum_{n=1}^{k}\int_{t^{n-1}}^{t^n}\int_{\Gamma_i}(\gamma_h^n - \gamma)u\eta_h^n ds dt, \\ &\le \frac{\gamma_1}{4\gamma_2}\sum_{n=1}^{k}\int_{t^{n-1}}^{t^n}\int_{\Gamma_i}|\gamma_h^n - \gamma||\eta_h^n|^2 ds dt \\ &\quad +\frac{\gamma_2}{\gamma_1}\sum_{n=1}^{k}\int_{t^{n-1}}^{t^n}\int_{\Gamma_i}|\gamma_h^n - \gamma||u|^2 ds dt, \\ &\le \frac{\gamma_1}{2}\tau\sum_{n=1}^{k}\|\eta_h^n\|^2_{L^2(\Gamma_i)} + \frac{\gamma_2}{\gamma_1}\int_0^T\int_{\Gamma_i}|\gamma_h^n - \gamma||u|^2 ds dt, \end{aligned} \tag{51}$$

and

$$\begin{aligned} \sum_{n=1}^{k}(I)_3 &= \sum_{n=1}^{k}\int_{t^{n-1}}^{t^n}\int_{\Gamma_c}(q_h^n - q)\eta_h^n ds dt, \\ &\le C\tau\sum_{n=1}^{k}\|\eta_h^n\|^2_{L^2(\Gamma_c)}. \end{aligned} \tag{52}$$

Then

$$\max_{1\le n\le M}\|\eta_h^n\|^2_{L^2(\Omega)} \le \sum_{n=1}^{k}(I)_1 + \sum_{n=1}^{k}(I)_2 + \sum_{n=1}^{k}(I)_3, \tag{53}$$

$$\tau\sum_{n=1}^{k}\|\nabla\eta_h^n\|^2_{L^2(\Omega)} \le \sum_{n=1}^{k}(I)_1 + \sum_{n=1}^{k}(I)_2 + \sum_{n=1}^{k}(I)_3, \tag{54}$$

and

$$\tau \sum_{n=1}^{k} \|\eta_h^n\|^2_{L^2(\Omega)} \leq \sum_{n=1}^{k} (I)_1 + \sum_{n=1}^{k} (I)_2 + \sum_{n=1}^{k} (I)_3. \tag{55}$$

Now, using the above estimates $(I)_1$, $(I)_2$, and $(I)_3$ as τ and $h \to 0$, we deduce $\eta_h^n \to 0$ and $u_h^n \to Q_h \bar{u}^n$ as $n \to \infty$ (see Lebesgue dominant convergence theorem [17]). Using the following relation

$$u_h^n - \bar{u}^n = (u_h^n - Q_h \bar{u}^n) + (Q_h \bar{u}^n - \bar{u}^n),$$

we obtain

$$\max_{1 \leq n \leq M} \|u_h^n - \bar{u}^n\|^2_{L^2(\Omega)} \to 0,\ \tau \sum_{n=1}^{k} \|\nabla(u_h^n - \bar{u}^n)\|^2_{L^2(\Omega)} \to 0,$$

$$\text{and } \tau \sum_{n=1}^{k} \|u_h^n - \bar{u}^n\|^2_{L^2(\Omega)} \to 0 \text{ as } \tau, h \to 0. \tag{56}$$

Now, we can verify Lemma 5.4. Using the boundedness of u_h^n, γ_h, q_h, and property of Q_h, it suffices to prove that

$$I_h^M - I_h^\sigma \equiv \tau \sum_{n=0}^{M} \int_{\Gamma_c} |u_h^n(\gamma_h, q_h) - z_h^\delta|^2 ds - \int_0^T \int_{\Gamma_c} |u(\gamma, q) - z_h^\delta|^2 dx dt \to 0$$
$$\text{as } \tau, h \to 0, \tag{57}$$

where $z_h^\delta = Q_h z^\delta$. By rewriting (57) as follows

$$I_h^M - I_h^\sigma \equiv \tau \sum_{n=0}^{M} \int_{t_{n-1}}^{t_n} \int_{\Gamma_c} \left(|u_h^n(\gamma_h, q_h) - z_h^\delta|^2 - |u(\gamma, q) - z_h^\delta|^2 \right) dx dt. \tag{58}$$

Hence, by using the Cauchy-Schwarz inequality and stability estimates (42), we get

$$\begin{aligned} I_h^M - I_h^\sigma &\leq \tau \left(\sum_{n=0}^{M} \int_{t_{n-1}}^{t_n} \|\nabla(u_h^n - u)\|^2_{L^2(\Gamma_c)} dt \right)^{1/2} \\ &\quad \times \left(\sum_{n=0}^{M} \int_{t_{n-1}}^{t_n} \|\nabla(u_h^n + u - 2z_h^\delta)\|^2_{L^2(\Gamma_c)} dt \right)^{1/2}, \\ &\leq C \left(\sum_{n=0}^{M} \int_{t_{n-1}}^{t_n} \|\nabla(u_h^n - u)\|^2_{L^2(\Gamma_c)} dt \right)^{1/2}. \end{aligned} \tag{59}$$

Moreover, the following identity is true

$$\|\nabla(u_h^n - u)\|^2_{L^2(\Gamma_c)} \leq \|\nabla(u_h^n - \bar{u}^n)\|^2_{L^2(\Gamma_c)} + \|\nabla(u - \bar{u}^n)\|^2_{L^2(\Gamma_c)}.$$

Using the obtained results of (53)-(55) and the approximation property of the averaging function, we deduce $I_h^M \to I_h^\sigma$ as τ and h tend to 0. The proof is complete. □

Next, we prove the convergence analysis of the finite element problem (26)-(27).

Theorem 5.2. *Let $\{(\gamma^*_{h,\tau}, q^*_{h,\tau})\}_{h,\tau>0}$ be a sequence of a minimizer to the finite element minimization problem (26)-(27). Then each subsequence $\{(\gamma^*_{h,\tau}, q^*_{h,\tau})\}_{h,\tau>0}$ has a subsequence converges weakly to a minimizer of the continuous problem (2)-(4).*

Proof. First, we take $\gamma_{h,\tau} = \gamma_1$ and $q_{h,\tau} = q_1$ in (26) and $u_h^n(\gamma_{h,\tau}, q_{h,\tau})$ be the corresponding solution to (26)-(27), we easily see that

$$J_{h,\tau}^M(\gamma_{h,\tau}^*, q_{h,\tau}^*) \leq J_{h,\tau}^M(\gamma_1, q_1) \leq C.$$

Using the stability estimates (42)-(43), with C not depend on h and τ. Hence, by the definition of $J_{h,\tau}^M$ we have $\|\gamma_{h,\tau}^*\|_{L^2(0,T;L^2(\Gamma_i))} \leq C$, and $\|q_h^*\|_{L^2(0,T;L^2(\Gamma_c))} \leq C$. Subsequently, there exists a subsequence of $\{(\gamma_{h,\tau}^*, q_{h,\tau}^*)\}_{h,\tau>0}$ still denoted by $(\gamma_{h,\tau}^*, q_{h,\tau}^*)$ converges weakly to some $(\gamma^*, q^*) \in K_1 \times K_2$ as $h, \tau \to 0$ (see [7]).
Now, for any $(\gamma, q) \in K_1 \times K_2$ and $\varepsilon_1, \varepsilon_2 > 0$, there exists a function $(\gamma_\varepsilon, q_\varepsilon) \in H^1(0,T; H^{\frac{1}{2}}(\Gamma_i)) \times H^1(0,T; H^{\frac{1}{2}}(\Gamma_c))$; (see [6]) such that

$$\|\gamma_\varepsilon - \gamma\|_{L^2(0,T;L^2(\Gamma_i))} \leq \varepsilon_1 \text{ and } \|q_\varepsilon - q\|_{L^2(0,T;L^2(\Gamma_c))} \leq \varepsilon_2.$$

Define an extension $(\hat{\gamma}_\varepsilon,\ \hat{q}_\varepsilon)$ of $(\gamma_\varepsilon,\ q_\varepsilon)$ such that $(\hat{\gamma}_\varepsilon, \hat{q}_\varepsilon) \in H^1(0,T; H^1(\Omega)) \times H^1(0,T; H^1(\Omega))$ (see [20]),

$$\|\hat{\gamma}_\varepsilon\|_{H^1(0,T;H^1(\Omega))} \leq C\|\gamma_\varepsilon\|_{H^1(0,T;H^{\frac{1}{2}}(\Gamma_i))}$$

and

$$\|\hat{q}_\varepsilon\|_{H^1(0,T;H^1(\Omega))} \leq C\|q_\varepsilon\|_{H^1(0,T;H^{\frac{1}{2}}(\Gamma_c))}.$$

Noting that $(\gamma_{h,\tau}^*, q_{h,\tau}^*)$ is a minimizer of $J_{h,\tau}^M$ over $K_1^{h,\tau} \times K_2^{h,\tau}$, we define

$$\hat{\gamma}_\varepsilon^{h,\tau}(\mathbf{x}, t) = \sum_{n=1}^{M} \chi_n(t) Q_h \hat{\gamma}_\varepsilon(\mathbf{x}, t_n) \text{ and } \hat{q}_\varepsilon^{h,\tau}(\mathbf{x}, t) = \sum_{n=1}^{M} \chi_n(t) Q_h \hat{q}_\varepsilon(\mathbf{x}, t_n).$$

Suppose that $\gamma_\varepsilon^{h,\tau}$ and $q_\varepsilon^{h,\tau}$ are restrictions of $\hat{\gamma}_\varepsilon^{h,\tau}$ and $\hat{q}_\varepsilon^{h,\tau}$ on Γ_i and Γ_c, respectively. Then, $(\gamma_\varepsilon^{h,\tau}, q_\varepsilon^{h,\tau}) \in K_1^{h,\tau} \times K_2^{h,\tau}$ and for any $\varepsilon > 0$, we have

$$\gamma_\varepsilon^{h,\tau}(\mathbf{x}, t) = \begin{cases} \hat{\gamma}_\varepsilon(\mathbf{x}, t) & \text{if } \gamma_1 \leq \hat{\gamma}_\varepsilon(\mathbf{x}, t) \leq \gamma_2, \\ \gamma_1 & \text{if } \hat{\gamma}_\varepsilon(\mathbf{x}, t) < \gamma_1, \\ \gamma_2 & \text{if } \hat{\gamma}_\varepsilon(\mathbf{x}, t) > \gamma_2, \end{cases}$$

and

$$q_\varepsilon^{h,\tau}(\mathbf{x}, t) = \begin{cases} \hat{q}_\varepsilon(\mathbf{x}, t) & \text{if } q_1 \leq \hat{q}_\varepsilon(\mathbf{x}, t) \leq q_2, \\ q_1 & \text{if } \hat{q}_\varepsilon(x) < q_1, \\ q_2 & \text{if } \hat{q}_\varepsilon(x) > q_2. \end{cases}$$

Then, using the trace theorem, we have

$$\begin{aligned} & \|\gamma_\varepsilon^{h,\tau} - \gamma_\varepsilon\|_{L^2(0,T;L^2(\Gamma_i))}^2 \leq \|\gamma_\varepsilon^{h,\tau} - \gamma_\varepsilon\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma_i))}^2 \\ \leq\ & C\|\hat{\gamma}_\varepsilon^{h,\tau} - \hat{\gamma}_\varepsilon\|_{L^2(0,T;H^1(\Omega))} = C\|Q_h \hat{\gamma}_\varepsilon - \hat{\gamma}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}, \end{aligned} \tag{60}$$

and

$$\begin{aligned} & \|q_\varepsilon^{h,\tau} - q_\varepsilon\|_{L^2(0,T;L^2(\Gamma_c))}^2 \leq \|q_\varepsilon^{h,\tau} - q_\varepsilon\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma_c))}^2 \\ \leq\ & C\|\hat{q}_\varepsilon^{h,\tau} - \hat{q}_\varepsilon\|_{L^2(0,T;H^1(\Omega))} = C\|Q_h \hat{q}_\varepsilon - \hat{q}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}. \end{aligned} \tag{61}$$

Thus, $(\gamma_\varepsilon^{h,\tau}, q_\varepsilon^{h,\tau}) \to (\gamma_\varepsilon, q_\varepsilon)$ in $K_1 \times K_2$ as $h, \tau \to 0$.
By using this, the lower simi-continuity of L^2-norm, and Lemma 5.4, we deduce

$$\begin{aligned} J(\gamma^*, q^*) &= \frac{1}{2}\int_0^T \|u(\gamma^*, q^*) - z^\delta\|^2_{L^2(\Gamma_c)} dt \\ &\quad + \frac{1}{2}\beta\|\gamma^*\|^2_{L^2(0,T;L^2(\Gamma_i))} + \frac{1}{2}\eta\|q^*\|^2_{L^2(0,T;L^2(\Gamma_c))}, \\ &\le \lim_{h,\tau\to 0} \inf J^M_{h,\tau}(\gamma^*_{h,\tau}, q^*_{h,\tau}), \\ &\le \lim_{h,\tau\to 0} \inf J^M_{h,\tau}(\gamma_\varepsilon^{h,\tau}, q_\varepsilon^{h,\tau}), \\ &= \frac{1}{2}\int_0^T \|u(\gamma_\varepsilon, q_\varepsilon) - z^\delta\|^2_{L^2(\Gamma_c)} dt \\ &\quad + \frac{1}{2}\beta\|\gamma_\varepsilon\|^2_{L^2(0,T;L^2(\Gamma_i))} + \frac{1}{2}\eta\|q_\varepsilon\|^2_{L^2(0,T;L^2(\Gamma_c))}, \\ &= J(\gamma_\varepsilon, q_\varepsilon). \end{aligned}$$

Taking $\varepsilon \to 0$ and using Lemma 2.1, we deduce

$$J(\gamma^*, q^*) \le J(\gamma, q) \ \forall(\gamma, q) \in K_1 \times K_2.$$

Then, (γ^*, q^*) is a minimizer to system (2)-(4). This completes the proof. □

6. Numerical algorithm

In this section, we describe the MCGM for the numerical solution of the optimization problem. Every iteration of the algorithm requires solving two auxiliary equations (adjoint and sensitivity) which are needed for computing the gradient and step lengths, respectively. The modification consists of simultaneously reconstructing the space-time dependent parameters Robin coefficient and heat flux. The detailed steps of the proposed algorithm are given below.

Algorithm 6.1.

(1) *Choose the initial guess* (γ^0, q^0), *set* $(d_q^0, d_\gamma^0) = (-J'_{q^0}, -J'_{\gamma^0})$, *and* $k := 0$.

(2) *Solve (1) for* $u(\gamma^k, q^k)$ *and compute residual at* k_{th} *step*

$$r_q^k = u(\gamma^k, q^k) - z^\delta \text{ on } \Gamma_c \times (0, T).$$

(3) *Solve (12) for* $\omega_q^*(\gamma^k, q^k)$ *with the boundary conditions and compute the gradient:*

$$J'_q(\gamma^k, q^k) = \omega_q^*(\gamma^k, q^k) + \eta q^k.$$

(4) *Determine the conjugate coefficient:*

$$\beta_q^k = \frac{\|J'_q(\gamma^k, q^k)\|^2_{L^2(0,T;L^2(\Gamma_c))}}{\|J'_q(\gamma^{k-1}, q^{k-1})\|^2_{L^2(0,T;L^2(\Gamma_c))}}.$$

(5) *Compute the descent direction for* $q(\mathbf{x}, t)$

$$d_q^{k+1} = -J'_q(\gamma^k, q^k) + \beta_q^k d_q^k.$$

(6) *Solve (8) for* $u_q^1(\gamma^k, q^k)$.

(7) *Compute the step length*

$$\alpha_q^k = -\frac{\langle r_k, u_q^1(\gamma^k, q^k, d_q^{k+1})\rangle_{L^2(0,T,L^2(\Gamma_c))} + \eta\langle q^k, d_q^{k+1}\rangle_{L^2(0,T,L^2(\Gamma_c))}}{\|u_q^1(\gamma^k, q^k, d_q^{k+1})\|^2_{L^2(0,T,L^2(\Gamma_c))} + \eta\|d_q^{k+1}\|^2_{L^2(0,T,L^2(\Gamma_c))}}.$$

(8) *Update* $q(\mathbf{x},t)$ *by*

$$q^{k+1} = q^k + \alpha_k d_q^{k+1}.$$

(9) *Solve (1) for* $u(\gamma^k, q^{k+1})$ *and compute the residual for* γ

$$r_\gamma^k = u(\gamma^k, q^{k+1}) - z^\delta \text{ on } \Gamma_c \times (0,T).$$

(10) *Solve (11) for* $\omega_\gamma^*(\gamma^k, q^{k+1})$ *and compute the gradient:*

$$J_\gamma^{'}(\gamma^k, q^{k+1}) = -u(\gamma^k, q^{k+1})\omega_\gamma^*(\gamma^k, q^{k+1}) + \beta\gamma^k.$$

(11) *Determine the conjugate coefficient:*

$$\beta_\gamma^k = \frac{\|J_\gamma^{'}(\gamma^k, q^{k+1})\|^2_{L^2(0,T;L^2(\Gamma_i))}}{\|J_\gamma^{'}(\gamma^{k-1}, q^{k+1})\|^2_{L^2(0,T;L^2(\Gamma_i))}}.$$

(12) *Compute the descent direction for* $\gamma(\mathbf{x},t)$

$$d_\gamma^{k+1} = -J_\gamma^{'}(\gamma^k, q^{k+1}) + \beta_q^k d_q^k.$$

(13) *Solve (7) for* $u_\gamma^1(\gamma^k, q^{k+1})$.

(14) *Compute the step length*

$$\alpha_\gamma^k = -\frac{\langle r_k, u_\gamma^1(\gamma^k, q^{k+1}, d_\gamma^{k+1})\rangle_{L^2(0,T,L^2(\Gamma_c))} + \beta\langle\gamma^k, d_\gamma^{k+1}\rangle_{L^2(0,T;L^2(\Gamma_i))}}{\|u_\gamma^{'}(\gamma^k, q^{k+1}, d_\gamma^{k+1})\|^2_{L^2(0,T,L^2(\Gamma_c))} + \beta\|d_\gamma^{k+1}\|^2_{L^2(0,T;L^2(\Gamma_i))}}.$$

(15) *Update* $\gamma(\mathbf{x},t)$ *by*

$$\gamma^{k+1} = \gamma^k + \alpha_\gamma^k d_\gamma^{k+1}.$$

(16) *If* $\dfrac{\|q^{k+1} - q^k\|_{L^2(0,T;L^2(\Gamma_c))}}{\|q^k\|_{L^2(0,T;L^2(\Gamma_c))}} \leq \varepsilon_1$ *and* $\dfrac{\|\gamma^{k+1} - \gamma^k\|_{L^2(0,T;L^2(\Gamma_i))}}{\|\gamma^k\|_{L^2(0,T;L^2(\Gamma_i))}} \leq \varepsilon_2$ *stop; otherwise* $k := k+1$ *and go to 2.*

The quadratic approximation of the cost functional determines the step lengths α_q^k and α_γ^k. The numerical experiments indicate that the step lengths work very well. There are many successful applications in various scientific fields that provide the numerical procedure for Hilbert space gradient. Jameson [10] observed that, the aerodynamic shapes are generally smooth while the gradients are not. In addition, recommended for an explicit smoothing in the optimization process which is now regarded as one of the most successful implementation of the gradient adjoint method in aerodynamic design and optimization. Also, applications of the inverse problems into other fields such as: electrical impedance tomography, inverse spectral problem, and the Cauchy problem with the boundary conditions have been investigated. From the previous works and their applications, it encourages us to study the performance and the simultaneous reconstruction of the parameters identification γ and q for the optimization inverse problem (2)-(4).

7. Numerical experiments and discussions

The previous section introduced the numerical algorithm 6.1 using the MCGM for solving the nonlinear optimization inverse problem. In this section, we shall execute the proposed algorithm 6.1 for reconstructing the parameters identification heat flux and Robin coefficient, simultaneously in (2)-(4). The considered solution domain Ω is a rectangular as $\Omega = (0,1) \times (0,1)$, discretized using triangular mesh which is generated by dividing each element of the regular rectangular mesh into two triangles. The domain boundaries consist of two parts, $\Gamma_i = \{(x,y) : x = 1,\ 0 \leq y \leq 1\}$, and $\Gamma_c = \partial\Omega/\Gamma_i$. The number of triangular finite elements $\mathcal{NE}$ is equal to $2 \times N \times M$, where N, M are the step sizes of x and y, respectively. We solve

Table 1. The numerical results with 1% noise in the data.

Example	δ	Δt	k	$\mathcal{RE}_\gamma$	$\mathcal{RE}_q$	$J(\gamma, q)$	$\mathcal{NE}$	β	η
7.1	0.01	0.06	5	0.039	0.0352	0.039	512	10^{-3}	10^{-3}
7.2	0.01	0.01	4	0.0382	0.0167	0.0202	288	10^{-4}	10^{-3}
7.3	0.01	0.01	5	0.0392	0.0296	0.0367	864	10^{-3}	10^{-3}
7.4	0.01	0.02	4	0.0379	0.0271	0.024	512	10^{-3}	10^{-3}

the forward problem (1) using linear FEM for space discretization, and backward difference scheme for the time discretization. In our test cases, the simulated noisy data z^δ are generated by adding some uniformly distributed random variable R varying in $[-1, 1]$, where

$$z^\delta = u + \delta R u \text{ on } \Gamma_c \times (0, T). \tag{1}$$

In the following study, the random variable R is realised by using the MATLAB function $rand(\cdot)$ and the regularization parameters β and η and the tolerance parameters ε_1 and ε_2 are set to be 10^{-3}. In all numerical experiments the time varies in the interval $[0, 1]$, step size Δt is optional, and thermal conductivity α is fixed to 1.

Next, we illustrate the numerical experiments for reconstructing the unknown parameters and discuss their performances with the exact solutions.

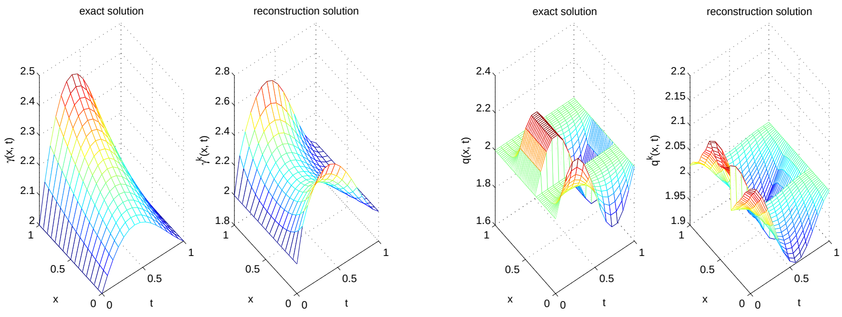
Remark 7.1. *The relative error of the Robin coefficient is defined by* $\mathcal{RE}_\gamma = \dfrac{\|\gamma^k - \gamma\|_{L^2(0,T;L^2(\Gamma_i))}}{\|\gamma\|_{L^2(0,T;L^2(\Gamma_i))}}$ *and the relative error of the heat flux is defined by* $\mathcal{RE}_q = \dfrac{\|q^k - q\|_{L^2(0,T;L^2(\Gamma_c))}}{\|q\|_{L^2(0,T;L^2(\Gamma_c))}}$.

Example 7.1. *Take the exact Robin coefficient* $\gamma(\mathbf{x}, t) = \frac{1}{4}e^{(y-t)} \times \sin(\pi t) + 2$ *on* Γ_i *and the exact heat flux* $q(\mathbf{x}, t) = \frac{1}{4}(x-1)^4 \times \sin(2\pi t) + 3$ *on* Γ_c*. Consider the analytical solution of the forward problem to be* $u(\mathbf{x}, t) = \frac{1}{2}(x^2 + y^2)\sin(\pi t)$ *and then, the source function is given by* $f(\mathbf{x}, t) = \frac{\pi}{2}(x^2 + y^2)\cos(\pi t) - 2\sin(\pi t)$*, and the ambient temperature is given by* $g(\mathbf{x}, t) = x\sin(\pi t) + \frac{1}{2}\gamma(\mathbf{x}, t)(x^2 + y^2)\sin(\pi t)$.

Example 7.2. *Take the exact Robin coefficient* $\gamma(\mathbf{x}, t) = \frac{1}{4}(y-1)e^{y-t} + 3$ *on* Γ_i *and the exact heat flux* $q(\mathbf{x}, t) = \frac{1}{4}(t-1)(x-1)^4 + 3$ *on* Γ_c*. Consider the analytical solution* $u(\mathbf{x}, t) = x^2 + y^2 + t\cos(xy)$ *and then, the source function given by* $f(\mathbf{x}, t) = \cos(xy) - (4 - t\cos(xy)(x^2 + y^2))$ *and ambient temperature* $g(\mathbf{x}, t) = 2x - ty\sin(xy) + \gamma(\mathbf{x}, t)(x^2 + y^2 + t\cos(xy))$ *on* Γ_i.

Example 7.3. *Take the exact Robin coefficient* $\gamma(\mathbf{x}, t) = \frac{1}{4}(y-1)e^{(y-t)} + 3$ *on* Γ_i *and the exact heat flux* $q(\mathbf{x}, t) = \frac{1}{4}(x-1)^2 e^{(y-t)} + 3$ *on* Γ_c*. Consider the exact solution* $u(\mathbf{x}, t) = x^2 + y^2 + t\cos(xy)$*, the source function* $f(\mathbf{x}, t) = \cos(xy) - (4 - t(x^2 + y^2)\cos(xy))$*, and the ambient temperature* $g(\mathbf{x}, t) = 2x - ty\sin(xy) + \gamma(\mathbf{x}, t)(x^2 + y^2 + t\cos(xy))$.

Example 7.4. *Consider the exact Robin coefficient* $\gamma(\mathbf{x}, t) = \frac{1}{4}e^{(y-t)} \times \sin(\pi t) + 3$ *on* Γ_i *and the exact heat flux* $q(\mathbf{x}, t) = \frac{1}{4}(x-1)^4 \times \sin(\pi t) + 3$*, on* Γ_c*,* $u(\mathbf{x}, t) = \frac{1}{2}(x^2 + y^2)\sin(\pi t)$*,* $f(\mathbf{x}, t) = \frac{\pi}{2}(x^2 + y^2)\cos(\pi t) - 2\sin(\pi t)$*, and the ambient temperature function* $g(\mathbf{x}, t) = x\sin(\pi t) + \frac{1}{2}\gamma(\mathbf{x}, t)(x^2 + y^2)\sin(\pi t)$ *on* Γ_i.

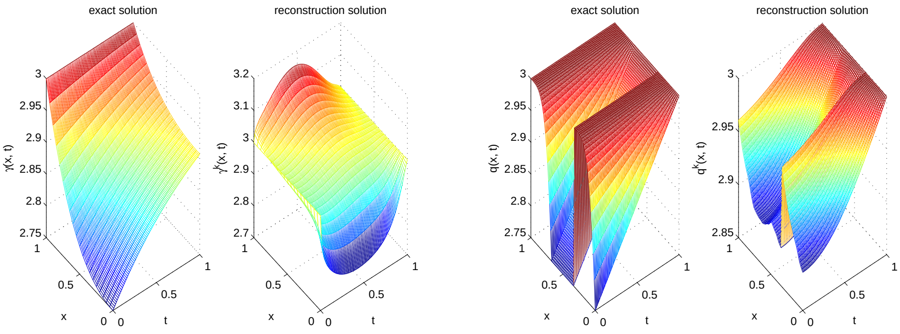


(a) (b)

FIGURE 1. Exact and reconstruction Robin coefficient (a) and heat flux (b) for Example 7.1 and $(\gamma^0, q^0) = (2, 3)$ at 1% noise in the data.

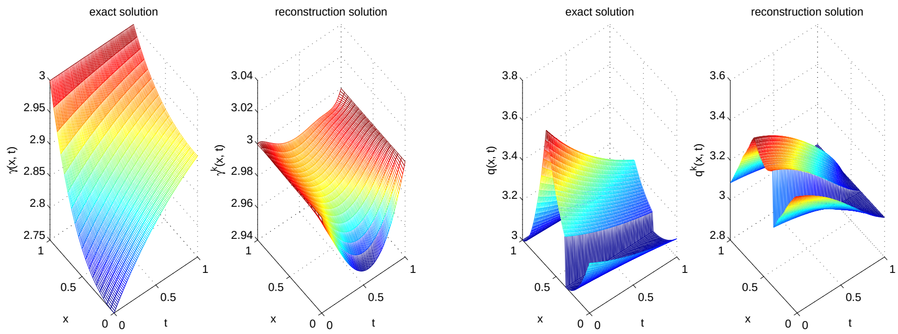
In this work, we study the numerical experiments using variant functions to show the accuracy and efficiency of the proposed algorithm. For the proposed inverse problem (2)-(4), there are some input parameters of great influence on accuracy of the obtained results. Such parameters includes, step sizes N, M, Δt and the noise level in the data, the regularization parameters η and β, as well as the initial guess γ^0 and q^0. Moreover, in case of initial guess taken to be some constants with 1% noise in the data, the numerical results appear quite satisfactory. The numerical experiments show that the reconstructed results are reasonable. In inverse problems, we can not expect the error rate to be very small along with the behaviour because of the nature of ill-posed inverse problems. We focused on the behavior of solutions in addition to the accuracy errors which were defined by the relative errors.

Table 1 presents the relative error, residual error, the number of iterations, and the cost functional in Examples 7.1-7.4. For all examples, 1% noise was added to the measured data and the initial guess was taken to be constants. It can be clearly observed that both plots in each of Figures 1-4 are satisfactory. Hence, the proposed method has a reasonable reconstruction accuracy and low errors. On the other hand, Figures 1(b), 2(a), 3(a), 4(b) apparently, look undesirable. This is because, it is difficult to expect the exact reconstruction of the inverse problem especially, for cases where multiple space-time dependent parameters are required. In addition, the heat flux represented on three parts of the domain boundaries causes discontinuity problem. Also, the Robin inverse problem is highly ill-posedness. In particular, small change in the input parameters makes large deviations in the solutions of the problem. As a test case, we considered the initial guess is close to the exact solution of the Robin coefficient and heat flux. It can be observed that the exact and reconstructed Robin coefficient and heat flux shown in Figures 5-8 are quite satisfactory. This can be attributed to the reduced amount of noise in the data. Figure 9 shows the exact and noisy data obtained by Algorithm 6.1 for Examples 7.2 and 7.4. One of the important advantages of the MCGM is being fast and accurate as shown in Fig. 10. The accuracy errors are defined by residual errors $\|u(\gamma, q) - z^\delta\|_{L^2(0,T;L^2(\Gamma_i))}$ and $\|u(\gamma, q) - z^\delta\|_{L^2(0,T;L^2(\Gamma_c))}$ with respect to γ and q. The relative errors of γ and q are reduced gradually with the iteration number k.



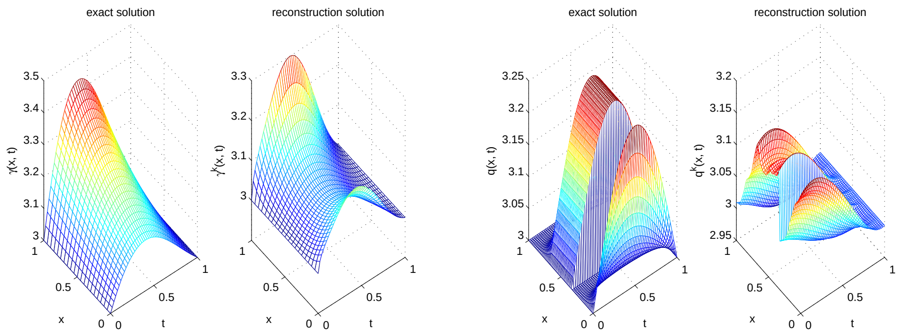
(a) (b)

FIGURE 2. Exact and reconstruction $\gamma(\mathbf{x}, t)$ (a) and $q(\mathbf{x}, t)$ (b) for Example 7.2 and $(\gamma^0, q^0) = (3, 3)$ at 1% noise in the data.



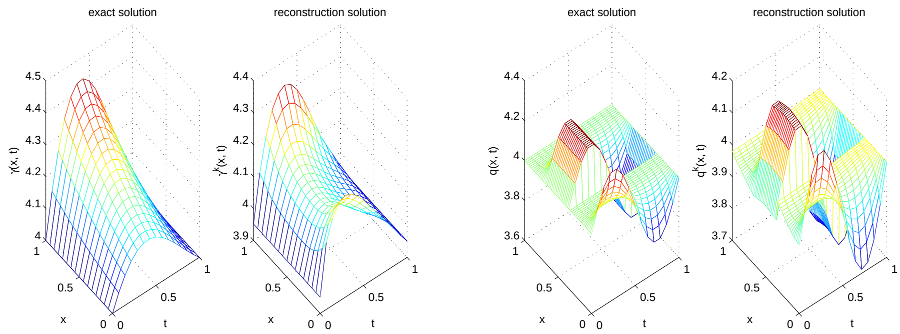
(a) (b)

FIGURE 3. Exact and reconstruction Robin coefficient (a) and heat flux (b) for Example 7.3 and $(\gamma^0, q^0) = (3, 3)$ at 1% noise in the data.



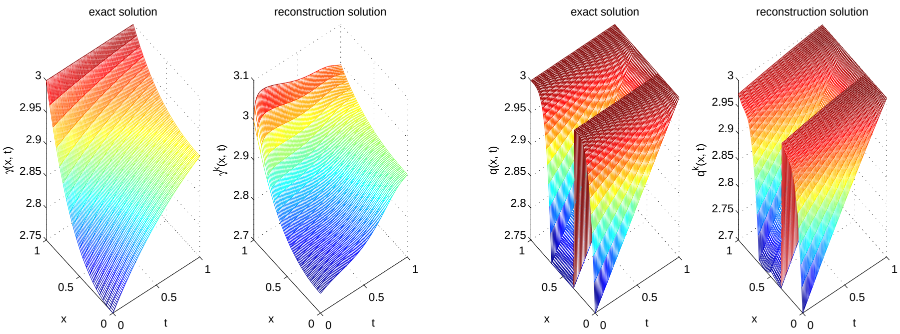
(a) (b)

FIGURE 4. Exact and reconstruction Robin coefficient (a) and heat flux (b) for Example 7.4 and $(\gamma^0, q^0) = (2, 3)$ at 1% noise in the data.



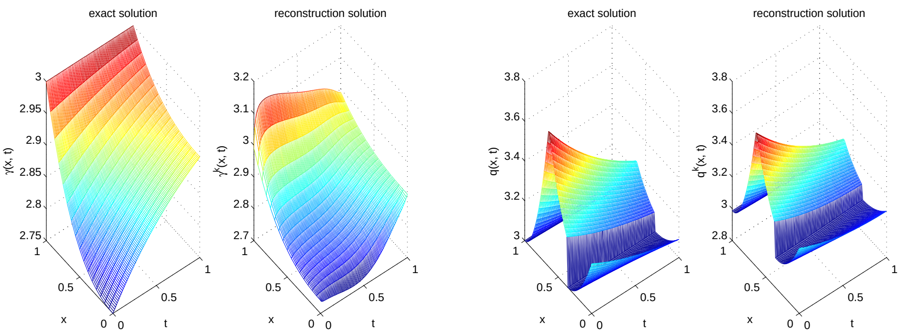
(a) (b)

FIGURE 5. Exact and reconstruction Robin coefficient (a) and heat flux (b) by Algorithm 6.1 for Example 7.1.



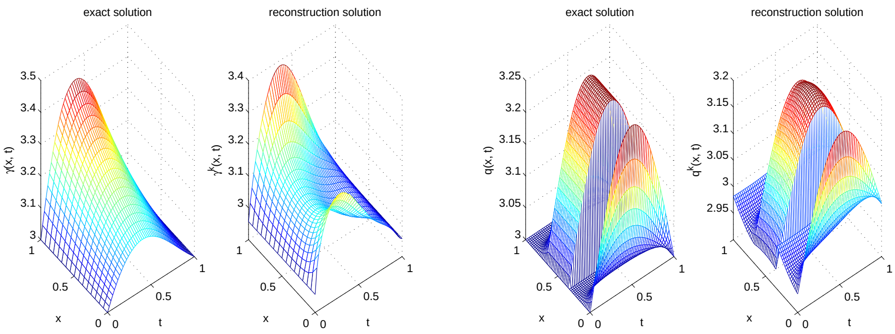
(a) (b)

FIGURE 6. Exact and reconstruction $\gamma(\mathbf{x}, t)$ (a) and $q(\mathbf{x}, t)$ (b) by Algorithm 6.1 for Example 7.2.



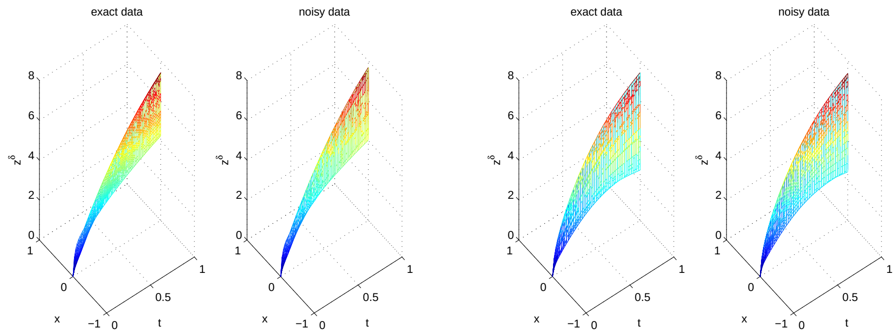
(a) (b)

FIGURE 7. Exact and reconstruction Robin coefficient (a) and heat flux (b) for Example 7.3.



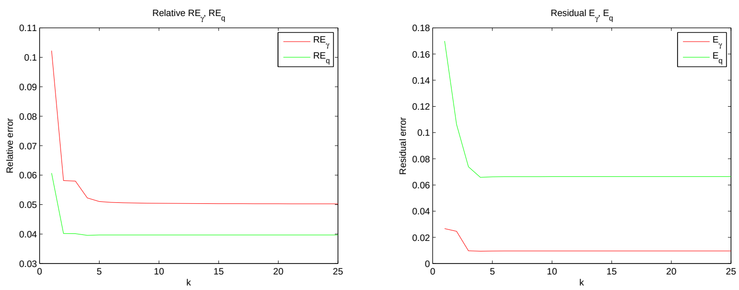
(a) (b)

FIGURE 8. Exact and reconstruction Robin coefficient (a) and heat flux (b) for Example 7.4.



(a) (b)

FIGURE 9. Exact and noisy data (a) and (b) for Examples 7.2 and 7.4, respectively by Algorithm 6.1.



(a) (b)

FIGURE 10. Convergence history of the method: the relative errors (a) and residual errors (b) of γ and q by Algorithm 6.1, with $\delta = 1\%$ noise in the data for Example 7.1.

8. Conclusions and outlook

In this paper, the variational formulation of the parabolic inverse problem is derived and the Tikhonov regularization approach is investigated. We derived the differentiability results to obtain the gradient of the Robin coefficient and heat flux. Also, introduced the adjoint equations to simplify computing the minimizer. We proposed a fully discrete FEM to solve the minimization problem and proved its convergence. A numerical algorithm is proposed for simultaneously reconstructing the unknown parameters using a MCGM. The numerical results show that the proposed algorithm is accurate, efficient, and robust. Furthermore, the proposed method introduced a convergent and stable numerical results for the parameters identification. Several experiments are introduced to verify the efficiency and accuracy of the MCGM.

As a future work, we plan to extend the modified conjugate gradient method to simultaneously reconstruct different parameters in the nonlinear inverse problems, such as reconstructing the space-time dependent conductivity, diffusivity coefficient and others.

Acknowledgments

We are grateful to the reviewers for their careful reading and suggestions which brought substantial improvements to our work. The first author would like to particularly express his sincere gratitude to Prof. Daijun Jiang for his invaluable guidance. Rongliang Chen is supported by the 863 Program of China grant 2015AA01A302 and the NSF of China grant 11401564.

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