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OPTIMAL ORDER CONVERGENCE IMPLIES NUMERICAL SMOOTHNESS II: THE PULLBACK POLYNOMIAL CASE

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Abstract. A piecewise smooth numerical approximation should be in some sense as smooth as its target function in order to have the optimal order of approximation measured in Sobolev norms. In the context of discontinuous finite element approximation, that means the shape function needs to be numerically smooth in the interiors as well as across the interfaces of elements. In previous papers [2, 8] we defined the concept of numerical smoothness and stated the principle: numerical smoothness is necessary for optimal order convergence. We proved this principle for discontinuous piecewise polynomials on $\mathbb{R}^n, 1 \leq n \leq 3$. In this paper, we generalize it to include discontinuous piecewise non-polynomial functions, e.g., rational functions, on quadrilateral subdivisions whose pullbacks are polynomials such as bilinears, bicubics and so on.

Key words. Adaptive algorithm, discontinuous Galerkin, numerical smoothness, optimal order convergence.

1. Introduction

Consider the problem of approximating a function u defined on a domain Ω in $\mathbb{R}^n$ by a sequence of numerical solutions $\{u_h\}$ that are defined on subdivisions of Ω parametrized by the maximum mesh size h. The target function u may be the exact solution of a partial differential equation, and the sequence, discontinuous piecewise polynomials from a discontinuous Galerkin or finite volume method [6, 7], or post-processed finite element solutions to achieve superconvergence [11]. Now suppose that u is in $W_s^{p+1}(\Omega)$ (standard notation for Sobolev spaces here, supindex for the order of derivative and subindex for the L^s-based space) and that an optimal order approximation

$$||u - u_h||_{L^s(\Omega)} = \mathcal{O}(h^{p+1}), \tag{1}$$

holds, we would like to know what kind of smoothness u_h must have. For this purpose we defined across the interface smoothness in [2, 8] for $1 \leq n \leq 3$ and in particular for $n = 2$ it is as follows.

Definition 1.1. Interface Numerical Smoothness. *Let $\{\mathcal{Q}_h\}$ be a family of triangulations or quadriangulation (by quadrilaterals) of $\Omega \subset \mathbb{R}^n$. Let W_h be a function space such that*

$$W_h \subset \{v : \Omega \to \mathbb{R} : v|_\kappa \in C^{p+1}(\bar{\kappa}), \kappa \in \mathcal{Q}_h\}, \;\; dim\; W_h < \infty.$$

Let $\{x_i\}_{i=1}^{N^\circ}$ be the set of all midpoints of interior edges. Then, $u_h \in W_h$ is said to be interface $W_s^{p+1}(\Omega)$-smooth, $s \geq 1$, if there is a constant $C_s > 0$, independent of h and u_h, such that

$$\sum_{i=1}^{N^\circ} h^2 \|D_i\|^s \leq C_s, \tag{2}$$

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and interface $W^{p+1}_\infty(\Omega)$-smooth, if there exists a constant $C_\infty > 0$, independent of h and u_h, such that

$$\max_{1\le i\le N^\circ} \|D_i\| \le C_\infty, \tag{3}$$

where the components of D_i are the scaled jumps $J_i^{(\alpha)}$ of partial derivatives at x_i

$$D_i^\alpha = J_i^{(\alpha)}/(h^{p+1-|\alpha|}), \quad J_i^{(\alpha)} := [\![\partial^\alpha u_h]\!]_{x_i}, \quad |\alpha| = k,\ 0\le k\le p. \tag{4}$$

Two important examples of W_h are piecewise polynomial space and space of piecewise continuously differentiable functions whose pre-images under bilinear transformation are polynomial. It is most instructive just by looking at the $n = 1$ case, and see that several natural conditions for optimal convergence are already included, e.g., the scaled functional value $|D_i^0| \le C$ for all i in the case of $k = 0$, and at the other end in the case of $k = p$ that $|D_i^p| \le C$ or (2) with $s = 1$ implies the piecewise constant function $\frac{d^p u_h}{dx^p}$ has bounded variation, when W_h is the space of piecewise polynomials of degree at most p.

Intuitively, the smoothness of a numerical solution $u_h \in W_h$ should be measured by the boundedness of partial derivatives $\partial^\alpha u_h$. On an element $\kappa \in \mathcal{Q}_h$, by Taylor expansion around any point x_m in $\bar\kappa$, e.g., the center of κ or a point on the boundary of κ using one-sided derivatives, we see that boundedness of the quantities $\partial^\alpha u_h(x_m)$ would be sufficient to guarantee the interior smoothness, i.e., there exists a constant $M > 0$, independent of h, such that

$$|\partial^\alpha u_h(x_m)| \le M, \quad \forall\ |\alpha| = k,\ 0\le k\le p. \tag{5}$$

On the other hand, intuitively the smoothness across the interface boundary of an element should be measured by the jumps of partials $J_i^{(\alpha)}$. The crucial part of Definition 1.1 is to point out one should use instead the scaled jump quantities D_i^α in (4). Notice that this definition does not involve any target solution u. Next, to give a corresponding interior numerical smoothness we replace the quantity D_i by F_i, the difference between the derivatives of u_h and the target u at x_m.

Definition 1.2. Interior Numerical Smoothness. *Let $u \in C^{p+1}(\Omega), \Omega \subset \mathbb{R}^2$ and let u_h be as in Def. 1.1 and let $\{x_i\}_{i=1}^{\mathcal{N}}$ be a collection of points $x_i \in \kappa_i \in \mathcal{Q}_h, 1 \le i \le \mathcal{N}$, where $\mathcal{N}$ is the number of elements in $\mathcal{Q}_h$. Then, u_h is said to be interior $W^{p+1}_s(\Omega)$-smooth, $s \ge 1$, if there is a constant C_s, independent of h and u_h, such that*

$$\sum_{i=1}^{\mathcal{N}} h^2 \|F_i\|^s \le C_s, \tag{6}$$

and interior $W^{p+1}_\infty(\Omega)$-smooth, if there exists a constant C_∞ independent of h and u_h such that

$$\max_{1\le i\le N^{\mathcal{T}}} \|F_i\| \le C_\infty, \tag{7}$$

where the components of F_i are the scaled differences between partial derivatives

$$F_i^\alpha = \partial^\alpha(u - u_h)(x_i)/(h^{p+1-|\alpha|}), \quad |\alpha| = k,\ 0\le k\le p.$$

The main result that states optimal order convergence implies numerical smoothness is proved in Theorems 3.1 and 3.2. In particular, as a byproduct we have the following simultaneous approximation result: If

$$\|u - u_h\|_{L^\infty(\Omega)} \le C h^{p+1} |u|_{W^{p+1}_\infty},$$

then all the kth partial derivatives at $x_i \in \kappa \in \mathcal{Q}_h$ satisfy

$$(8) \qquad |\partial^\alpha (u - u_h)(x_i)| \le C h^{p+1-k} |u|_{W^{p+1}_\infty}, \quad |\alpha| = k,\ 0 \le k \le p.$$

While we have proven in [2] all these results in high dimensions and *in the case W_h is piecewise polynomial space*, it is important to point out that the techniques used there cannot handle the case when W_h is a space of piecewise non-polynomial functions. The latter case arises when the finite element space contains functions whose pre-image on the reference element under bilinear transformation is bilinear for example. In the quadrilateral elements we may have piecewise rational functions. In this paper, we take a completely new approach, using the Taylor expansion to make the transition from non-polynomials to polynomials. For readers familiar with [2], it should not be difficult to see that the present approach works for those cases in [2] as well. Let us briefly mention how we proved our main theorems in §3. They all depend on a fact in Thm 2.8 that states for $s = 1, 2, \infty$

$$(9) \qquad \|u - u_h\|_{L^s(\Omega)} \ge C_1 h^{p+1} \left(\left(\sum_{i=1}^{N_e^\circ} h^2 \|D_i\|^s \right)^{1/s} - |u|_{W_s^{p+1}(\Omega)} \right),$$

which combining with (1) validates the statement: optimal order convergence implies numerical smoothness. In proving (9) we use the Taylor expansion and the central Lemma 2.1, which is L_2 based. The reason we are interested in $s = 1, 2, \infty$ has a background in analyzing hyperbolic conservation laws. L_1 norm is theoretically more natural due to the possible L_1 contraction property, but L_2 norm is much easier to use in analysis. This lemma allows us to transfer L_2 analysis to the L_1, L_∞ cases. We emphasize the fact that W_h is finite dimensional is essential here for its success.

We end this section with a few comments on practical use of numerical smoothness so defined. Our results indicate those scaled jump quantities in (4) should be controlled during computation. Indeed, they were included in the construction of smooth indicators in [9, 10] for computing the numerical solution of one dimensional nonlinear conservation laws. Sharp numerical shock wave solutions were successfully captured. We feel that the idea of incorporating numerical smoothness into adaptive schemes has a potential of a very broad scope of applications; e.g. safeguarding divergence or negating optimal order convergence in designing new methods, which we wish to explore in the future.

The organization of the rest of this paper is as follows. In § 2, we first derive basic error estimates without imposing conditions on meshes other than the shape regularity. The main theorem is Theorem 2.8, now under the quasi-uniform condition on the mesh.

2. Basic Estimates for Numerical Smoothness

In this section we give error estimates for approximations using piecewise smooth functions u_h with respect to a given subdivision $\mathcal{Q}_h$ of a domain in $\mathbb{R}^n, n \ge 1$. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \alpha_i \ge 0, 1 \le i \le n$ be a multi-index and $|\alpha| = \sum_i^n \alpha_i$. At a nodal point $x \in \mathbb{R}^n$ of interest such as a midpoint of a common edge ($n = 2$) or a center of a common face $n = 3$, to measure the smoothness of a piecewise function u_h, we will exam all the jumps $[\![\partial^\alpha u_h]\!]_x, |\alpha| = k$ in the partial derivatives of order k for $0 \le k \le p$. In this perspective, we state the next lemma.

Lemma 2.1. *Let $\hat{\Omega}_{\pm}$ be two open sets in $\mathbb{R}^n$ and let $\mathbb{P}_p$ be the space of all polynomials of total degree at most p. Define*

$$Q(\Delta) = \min_{\hat{v}\in\mathcal{P}} \left(\left\| \hat{v} + \frac{1}{2}\sum_{\alpha\in\mathcal{I}} \frac{\Delta_\alpha}{\alpha!}\xi^\alpha \right\|^2_{L^2(\hat{\Omega}_-)} + \left\| \hat{v} - \frac{1}{2}\sum_{\alpha\in\mathcal{I}} \frac{\Delta_\alpha}{\alpha!}\xi^\alpha \right\|^2_{L^2(\hat{\Omega}_+)} \right),$$

where the minimum is taken over $\mathcal{P} = \mathbb{P}_p$ in ξ. Here the index set

$$\mathcal{I} = \{\alpha : |\alpha| = k,\ 0 \le k \le p\}, \tag{10}$$

$\Delta_\alpha \in \mathbb{R}$, and $\Delta = (\Delta_0, \Delta_1, \cdots, \Delta_p), \Delta_k = \{\Delta_\alpha\}_{|\alpha|=k}, 0 \le k \le p$.

Then $Q(\Delta)$ is a positive definite quadratic form in Δ, and there exists a constant $C_p > 0$ such that

$$Q(\Delta) \ge C_p\|\Delta\|^2 = C_p \sum_{i=0}^{p} \|\Delta_i\|^2, \tag{11}$$

where $\|\Delta_i\|$ is the spectral norm of vector Δ_i.

Remark 2.2. *Note that Δ_α (α, a multi-index) is a number, while Δ_i (i, a scalar index) is a vector. In later application $\Delta_\alpha = \partial^\alpha u$, while Δ_i collects all partials of order i. The two open sets will be the left and right half disks of a closed disk on a reference plane for $n = 2$. This lemma was first given in* [8] *for $n = 1$ and generalized to higher dimensions in* [2]*. To prove it, simply notice that the minimizer $\sum_\alpha V_\alpha \xi^\alpha$ is such that each V_α is a linear combination of Δ_β's. Non-degeneracy comes from the fact that $V_\alpha + \frac{1}{2}\frac{\Delta_\alpha}{\alpha!} = 0$ and $V_\alpha - \frac{1}{2}\frac{\Delta_\alpha}{\alpha!} = 0$ implies $\Delta_\alpha = 0$.*

Let $\mathcal{Q}_h = \{\kappa\}$ be a partition of a polygonal domain Ω into convex quadrilaterals κ with diameters not greater than h. We take the unit square $\hat{\kappa} = [0,1]^2$ in the $\hat{x}\hat{y}$-plane as the reference element and label the four vertices as $\hat{\mathbf{x}}_i, i = 1,2,3,4$, in a counterclockwise order, starting at the origin. Let $\hat{\mathbf{x}} = (\hat{x}, \hat{y})$ and $\mathbf{x} = (x, y)$. For a typical quadrilateral $\kappa \in \mathcal{Q}_h$ with vertices $\mathbf{x}_i, i = 1,2,3,4$ arranged in a counterclockwise order, there exists a unique bilinear bijection F_κ from $\hat{\kappa}$ onto κ defined by

$$\mathbf{x} = F_\kappa(\hat{\mathbf{x}}) = \mathbf{x}_1 + \mathbf{x}_{21}\hat{x} + \mathbf{x}_{41}\hat{y} + \mathbf{g}\hat{x}\hat{y}, \tag{12}$$

where

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \quad \mathbf{g} = \mathbf{x}_{12} + \mathbf{x}_{34}.$$

Thus $\mathbf{x}_i = F_\kappa(\hat{\mathbf{x}}_i), i = 1,2,3,4$. The Jacobian matrix DF_κ of F_κ is given by

$$DF_\kappa = \begin{pmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{y}} \\ \frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{y}} \end{pmatrix} = (\mathbf{x}_{21} + \mathbf{g}\hat{y}, \mathbf{x}_{41} + \mathbf{g}\hat{x}). \tag{13}$$

In addition, the determinant $J_{F_\kappa} = \det DF_\kappa$ is a linear function of $\hat{x}$ and $\hat{y}$:

$$J_{F_\kappa}(\hat{x}, \hat{y}) = \alpha + \beta\hat{x} + \gamma\hat{y}, \tag{14}$$

where

$$\alpha = \det(\mathbf{x}_{21}, \mathbf{x}_{41}), \quad \beta = \det(\mathbf{x}_{21}, \mathbf{g}), \quad \gamma = \det(\mathbf{g}, \mathbf{x}_{41}).$$

Denote by S_i the subtriangle of κ with vertices $\mathbf{x}_{i-1}, \mathbf{x}_i$ and $\mathbf{x}_{i+1}$ ($\mathbf{x}_0 = \mathbf{x}_4$). Let h_κ be the diameter of κ and

$$\rho_\kappa = 2 \min_{1\le i\le 4}\{\text{ diameter of circle inscribed in } S_i\} \tag{15}$$

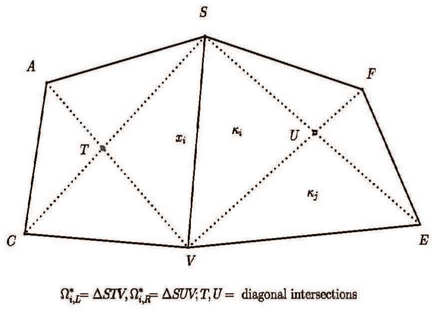

FIGURE 1. Quadrilateral mesh with its covolumes ($STVU$).

We have the following upper bounds:

$$\|DF_\kappa\|_{L^\infty(\hat{\kappa})} \le C_1 h_\kappa, \qquad \|J_{F_\kappa}\|_{L^\infty(\hat{\kappa})} \le C_2 h_\kappa^2$$
$$\|DF_\kappa^{-1}\|_{L^\infty(\hat{\kappa})} \le C_3(h_\kappa/\rho_\kappa^2), \qquad \|J_{F_\kappa^{-1}}\|_{L^\infty(\hat{\kappa})} \le C_4(1/\rho_\kappa^2). \tag{16}$$

A family of quadrilateral partitions $\{\mathcal{Q}_h\}$ is said to be regular [4, 5] if there exists a positive constant σ, independent of h, such that

$$\frac{h_\kappa}{\rho_\kappa} \le \sigma, \quad \forall \kappa \in \mathcal{Q}_h, \forall \mathcal{Q}_h \in \mathcal{Q}. \tag{17}$$

Lemma 2.3. [5, p. 107] *Let κ be a convex quadrilateral. For each integer $m \ge 0$ and real $s \in [1, \infty]$ there exist positive constants C_1, C_2 and C_3 independent of the geometry of κ such that the following upper bounds hold for all $v \in W_s^m(\kappa)$ with $|v|_{l,s,\kappa} := |v|_{W_s^l(\kappa)}$, we have*

$$|v|_{m,s,\kappa} \le C_2 (h_\kappa/\rho_\kappa)^{3m-2}(h_\kappa^{2/s}/\rho_\kappa^m)\left(\sum_{l=1}^{m} |\hat{v}|_{l,s,\hat{\kappa}}^s\right)^{1/s}, \quad m \ge 1 \tag{18}$$

with the obvious extension for the infinity norm. Here $\hat{v}(\hat{\mathbf{x}}) = v(\mathbf{x}), \mathbf{x} = F_\kappa(\hat{\mathbf{x}})$.

Theorem 2.4. *Let $u \in H^{p+1}(\Omega)$, $\Omega \subset \mathbb{R}^2$, and let $\{\mathcal{Q}_h\}$ be a family of regular subdivisions of Ω into quadrilaterals κ. Suppose that u_h is a piecewise defined function from a finite dimensional function space W_h, i.e.,*

H1.

$$u_h \in W_h \subset \{v : \Omega \to \mathbb{R} : v|_\kappa \in C^{p+1}(\bar{\kappa}), \kappa \in \mathcal{Q}_h\}, \ \textit{dim}\ W_h < \infty.$$

Then there exists a positive constant C_1, independent of h, u and u_h, such that

$$\|u - u_h\|_{L^2(\Omega)} \ge C_1 h^{p+1}\left(\sqrt{\sum_{i=1}^{N_e^\circ} h_{\min}^2 \|\tilde{D}_i\|^2} - |u|_{H^{p+1}(\Omega)}\right),$$

where the components of $\tilde{D}_i$ are

$$\tilde{D}_i^\alpha = \frac{J_i^{(\alpha)}}{h^{p+1} h_{\min}^{-|\alpha|}}, \quad J_i^{(\alpha)} = [\![\partial^\alpha u_h]\!]_{x_i}, \ |\alpha| = k, \ 0 \le k \le p. \tag{19}$$

Here N_e° is the number of interior edges, x_i are the midpoints of interior edges, and $h_{\min}$ is the least edge length.

Remark 2.5. *Note that the finite element space W_h can be taken either as piecewise polynomials or rational polynomials due to the bilinear transformation (12).*

Proof. For each $\mathcal{Q}_h$ we define a dual mesh $\mathcal{Q}_h^*$ as follows, cf. [4]. With reference to Figure 1, in each quadrilateral element (e.g. $SFEV$) we connect the vertices (e.g., S, F, V, E) by the two diagonals (e.g., U = intersection point), to create four new triangles. Similarly for the quadrilateral $ASVC$. The two half-covolumes (e.g. $\triangle STV$, $\triangle SUV$) form a single covolume $STVU$ *associated with* the midpoint x_i of the common edge SV. Note that the covolume associated with a boundary edge is a triangle, however, it will not be relevant in the proof since we will not use the boundary edges. All covolumes form a new subdivision called the dual mesh $\mathcal{Q}_h^*$. Let $\mathbb{P}_{p,h}^*$ be the space of all piecewise polynomials of degree at most p with respect to the dual mesh. We can find a $u^I \in \mathbb{P}_{p,h}^*$ so that

$$\|u - u^I\|_{L^2(\Omega)} \leq C_2 h^{p+1} |u|_{H^{p+1}(\Omega)} \tag{20}$$

holds under no regularity conditions on the dual mesh by quadrilaterals. The u^I is the *local* L^2 projection and estimate (20) can be found in [5, p. 108].

Now notice that each $\kappa \in \mathcal{Q}_h$ is split into four triangles $\kappa_j, 1 \leq j \leq 4$, one triangle (associated with a midpoint, x_i) for each edge of κ. We define a piecewise polynomial $T_p u_h$ with respect to the triangulation $\{\mathcal{T}_h\}$ formed by these triangles. Let x_i be a midpoint of an edge in κ, then $T_p u_h$ restricted to κ_i is the Taylor polynomial of degree p evaluated at x_i. Thus using the fact that $u_h \in C^{p+1}(\kappa)$

$$\|u_h - T_p u_h\|_{L^2(\kappa_i)} \leq C h_{\kappa_i}^{p+1} |u_h|_{H^{p+1}(\kappa_i)} \leq C \|u_h\|_{L^2(\kappa_i)}, \tag{21}$$

where the last inequality is obtained by an affine scaling argument over κ_i and the equivalence of H^{p+1} and L^2 norms on the finite dimensional $\hat{W}_h(\hat{\kappa})$, the space of functions on the unit triangle affinely related to $W_h(\kappa_i) := \{v|_{\kappa_i} : v \in W_h\}$. Squaring and summing (21) over κ_i, we have

$$\|u_h - T_p u_h\|_{L^2(\Omega)} \leq C \|u_h\|_{L^2(\Omega)}. \tag{22}$$

Note that we have actually shown that for any function w in W_h

$$\|(I - T_p) w\|_{L^2(\Omega)} \leq C \|w\|_{L^2(\Omega)}. \tag{23}$$

Hence by the triangle inequality T_p is L_2- stable in W_h:

$$\|T_p w\|_{L^2(\Omega)} \leq C_0 \|w\|_{L^2(\Omega)} \quad \forall w \in W_h. \tag{24}$$

We now show that

$$\|u_h - T_p u_h\|_{L^2(\Omega)} \leq C(\|u - u_h\|_{L^2(\Omega)} + \|u - P_h u\|_{L^2(\Omega)}), \tag{25}$$

where P_h is the L^2 projection onto $\mathbb{P}_p^h$, the space of piecewise polynomials of degree $\leq p$ with respect to the triangulation $\mathcal{T}_h$. In fact, using the notation $\|\cdot\|$ for the L_2 norm and (24) we have

$$\begin{aligned}
\|u_h - T_p u_h\| &\leq \|u_h - u\| + \|u - P_h u\| + \|P_h u - T_p u_h\| \\
&= \|u - u_h\| + \|u - P_h u\| + \|T_p P_h u - T_p u_h\| \\
&\leq \|u - u_h\| + \|u - P_h u\| + C_0 \|P_h u - u_h\| \\
&\leq \|u - u_h\| + \|u - P_h u\| + C_0(\|P_h u - u\| + \|u - u_h\|) \\
&\leq C(\|u - u_h\| + \|P_h u - u\|).
\end{aligned}$$

On the other hand, using the triangle inequality and then (25) we have

$$\begin{aligned}
\|u-u_h\|_{L^2(\Omega)} &\geq \|u^I-u_h\|_{L^2(\Omega)} - \|u-u^I\|_{L^2(\Omega)}\\
&\geq \|u^I-T_pu_h\|_{L^2(\Omega)} - \|u_h-T_pu_h\|_{L^2(\Omega)} - \|u-u^I\|_{L^2(\Omega)}\\
&\geq \|u^I-T_pu_h\|_{L^2(\Omega)} - \|u_h-T_pu_h\|_{L^2(\Omega)} - C_2h^{p+1}|u|_{H^{p+1}(\Omega)},\\
&\geq \|u^I-T_pu_h\|_{L^2(\Omega)} - C\|u-u_h\|_{L^2(\Omega)} -\\
&\quad -C\|u-P_hu\|_{L^2(\Omega)} - C_2h^{p+1}|u|_{H^{p+1}(\Omega)},
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
\|u-u_h\|_{L^2(\Omega)} &\geq C_0\|u^I-T_pu_h\|_{L^2(\Omega)} - C_3h^{p+1}|u|_{H^{p+1}(\Omega)}\\
&\geq C_0\sqrt{\sum_{i=1}^{N_e^\circ}\|u^I-T_pu_h\|^2_{L^2(\Omega_i^*)}} - C_3h^{p+1}|u|_{H^{p+1}(\Omega)}, \qquad (26)
\end{aligned}$$

where Ω_i^* is a smaller subset of the covolume associated with x_i. We next show how to choose Ω_i^*.

Suppose that x_i is the midpoint of an interior edge e_i common to two half-covolumes $\Omega_{i,L}^*, \Omega_{i,R}^*$. In Figure 1, $\Omega_{i,L}^* = \triangle STV, \Omega_{i,R}^* = \triangle SUV$. Let us take Ω_i^* to be a closed disk with center x_i and a radius δ small enough so that it is fully contained in the interior of $\bar\Omega_{i,L}^* \cup \bar\Omega_{i,R}^*$. The radius, however, has to work for all midpoints x_j of interior edges. Since the shape regularity is equivalent to the minimal angle condition [3, Theorem 4.1], and consequently there is a constant θ_0 such that all interior angles of $\kappa \in \mathcal{Q}_h$ and all the interior angles of the sub-triangles S_i in (15) are bounded below: there is a constant θ_0 such that $\theta \geq \theta_0$ for all h. Without loss of generality, suppose that the distance from x_i to the boundary of $\bar\Omega_{i,L}^* \cup \bar\Omega_{i_R}^*$ is attained by $|\overline{x_iF_i}|$, where the foot F_i is on $\overline{SU}$. Then

$$|\overline{x_iF_i}| = |\overline{Sx_i}|\sin\angle VSE \geq \frac{1}{2}h_{\min}\sin\theta_0$$

where we have used the fact that the sine function is increasing on $[0,\frac{\pi}{2}]$ and that SU is on the shortest distance side. Thus it suffices to take $\delta = \frac{1}{4}h_{\min}\sin\theta_0$ as the common radius for all midpoints x_i. Let $\{q\}$ denote the average of q^+ and q^- and let $[\![q]\!] = q^+ - q^-$ denote the jump. Then it is trivial that

$$q^+ - \{q\} = \frac{1}{2}[\![q]\!] \quad \text{and } q^- - \{q\} = -\frac{1}{2}[\![q]\!]. \qquad (27)$$

For ease of notation in the rest of proof, we write $\tilde u_h := T_pu_h$. Applying (27) with the (possible) discontinuous $q = \partial^\alpha \tilde u_h(x_i)$ and letting

$$w(x) = \sum_{k=0}^{p}\sum_{|\alpha|=k}\frac{\{q\}}{\alpha!}(x-x_i)^\alpha,$$

we have with $J_i^{(\alpha)} := [\![\partial^\alpha\tilde u_h]\!]_{x_i}$ that

$$\tilde u_h - w = \frac{1}{2}\sum_{k=0}^{p}\sum_{|\alpha|=k}\frac{J_i^{(\alpha)}}{\alpha!}(x-x_i)^\alpha, \quad \forall x \in \Omega_{i,+}^* = \Omega_i^* \cap \{(x-x_i)\cdot n \geq 0\}$$

and

$$\tilde u_h - w = -\frac{1}{2}\sum_{k=0}^{p}\sum_{|\alpha|=k}\frac{J_i^{(\alpha)}}{\alpha!}(x-x_i)^\alpha, \quad \forall x \in \Omega_{i,-}^* := \Omega_i^* \cap \{(x-x_i)\cdot n \leq 0\},$$

where n is a unit normal of the edge containing x_i. Now using the change of variable $\xi = (x - x_i)/h_{\min}$ below, we have

$$
\begin{aligned}
(28) \qquad & \| u^I - \tilde{u}_h\|^2_{L^2(\Omega_i^*)} \geq \min_{v\in\mathbb{P}_p} \|v - \tilde{u}_h\|^2_{L^2(\Omega_i^*)} \\
&= \min_{v\in\mathbb{P}_p} \|v - (\tilde{u}_h - w)\|^2_{L^2(\Omega_i^*)} \\
&= \min_{v\in\mathbb{P}_p} \left(\left\| v + \frac{1}{2}\sum_{k=0}^{p}\sum_{|\alpha|=k} \frac{J_i^{(\alpha)}}{\alpha!}(x-x_i)^\alpha \right\|^2_{L^2(\Omega_{i,-}^*)} \right. \\
&\quad \left. + \left\| v - \frac{1}{2}\sum_{k=0}^{p}\sum_{|\alpha|=k} \frac{J_i^{(\alpha)}}{\alpha!}(x-x_i)^\alpha \right\|^2_{L^2(\Omega_{i,+}^*)} \right) \\
&= h_{\min}^2 \min_{\hat{v}\in\hat{\mathbb{P}}_p} \left(\left\| \hat{v} + \frac{1}{2}\sum_{k=0}^{p}\sum_{|\alpha|=k} \frac{J_i^{(\alpha)}}{\alpha!}(\xi h_{\min})^\alpha \right\|^2_{L^2(\hat{\Omega}_-)} + \right. \\
&\quad \left. \left\| \hat{v} - \frac{1}{2}\sum_{k=0}^{p}\sum_{|\alpha|=k} \frac{J_i^{(\alpha)}}{\alpha!}(\xi h_{\min})^\alpha \right\|^2_{L^2(\hat{\Omega}_+)} \right), \\
&\quad \hat{\Omega}_- = \{\xi : \|\xi\| \leq \frac{1}{4}\sin\theta_0,\ \xi_1 \leq 0\}, \quad \hat{\Omega}_+ = \{\xi : \|\xi\| \leq \frac{1}{4}\sin\theta_0,\ \xi_1 \geq 0\} \\
&= (h_{\min}^2 h^{2p+2}) \min_{\hat{v}\in\hat{\mathbb{P}}_p} \left(\left\| \hat{v} + \frac{1}{2}\sum_{k=0}^{p}\sum_{|\alpha|=k} \frac{\tilde{D}_i^{(\alpha)}}{\alpha!}\xi^\alpha \right\|^2_{L^2(\hat{\Omega}_-)} \right. \\
(29) \qquad &\quad \left. + \left\| \hat{v} - \frac{1}{2}\sum_{k=0}^{p}\sum_{|\alpha|=k} \frac{\tilde{D}_i^{(\alpha)}}{\alpha!}\xi^\alpha \right\|^2_{L^2(\hat{\Omega}_+)} \right), \\
&= h_{\min}^2 h^{2p+2} Q(\tilde{D}_i^0, \tilde{D}_i^1, \cdots, \tilde{D}_i^p).
\end{aligned}
$$

(Note that the minimization–range's change to $\mathbb{P}_p$ in (28) was possible due to the fact that $u^I \in \mathbb{P}_h^*$ is a single piece of a polynomial on the covolume.)

Now invoking (11) on

$$
(30) \qquad \|u^I - \tilde{u}_h\|^2_{L^2(\Omega_i^*)} \geq h_{\min}^2 h^{2p+2} Q(\tilde{D}_i^0, \tilde{D}_i^1, \cdots, \tilde{D}_i^p)
$$

completes the proof. □

Theorem 2.6. *Suppose that*

$$
H0 : u \text{ is in } W_1^{p+1}(\Omega), \Omega \subset \mathbb{R}^2,
$$

and that assumption H1 of Theorem 2.4 holds. Then there are constants $C_1 > 0$ independent of h, u and u_h such that

$$
(31) \qquad \|u - u_h\|_{L^1(\Omega)} \geq C_1 h^{p+1} \left[\sum_{1\leq i\leq N_e^\circ} h_{\min}^2 \|\tilde{D}_i\| - |u|_{W_1^{p+1}(\Omega)} \right],
$$

where $\tilde{D}_i$ has components

$$\tilde{D}_i^{\alpha} = J_i^{(\alpha)}/(h^{p+1}h_{\min}^{-|\alpha|}), \quad J_i^{(\alpha)} = [\![\partial^{\alpha} u_h]\!]_{x_i},\ |\alpha| = k,\ 0 \le k \le p.$$

Proof. As before, for $u \in W_1^{p+1}(\Omega)$, there is a $u^I \in \mathbb{P}_{p,h}^*$ so that

$$\|u - u^I\|_{L^1(\Omega)} \le Ch^{p+1}|u|_{W_1^{p+1}(\Omega)},$$

where $C > 0$ is a constant independent of u and h.

We proceed exactly as in Theorem 2.4. Let x_i be a midpoint of an edge in κ, and recall that $T_p u_h$ is the Taylor polynomial of degree p evaluated at x_i over the triangle κ_i. Then

$$\|u_h - T_p u_h\|_{L^1(\kappa_i)} \le Ch_{\kappa_i}^{p+1}|u_h|_{W_1^{p+1}(\kappa_i)} \le Ch_{\kappa}^{p+1}|u_h|_{W_1^{p+1}(\kappa)}. \tag{32}$$

Applying Lemma 2.3 with $m = p+1$, $s = 1$ and $v = u_h$ and using the regularity condition (17) and equivalence of norms on $\hat{\kappa}$, we have

$$|u_h|_{W_1^{p+1}(\kappa)} \le Ch_{\kappa}^{-(p+1)}\|\hat{u}_h\|_{L^1(\hat{\kappa})}$$

so that by (16)

$$h_{\kappa}^{p+1}|u_h|_{W_1^{p+1}(\kappa)} \le Ch_{\kappa}^2(1/\rho_{\kappa}^2)\|u_h\|_{L^1(\kappa)} \le C\|u_h\|_{L^1(\kappa)}.$$

Summing (32) over κ, we have, using the last inequality,

$$\|u_h - T_p u_h\|_{L^1(\Omega)} \le C\|u_h\|_{L^1(\Omega)}. \tag{33}$$

Hence

$$\|T_p w\|_{L^1(\Omega)} \le C_0\|w\|_{L^1(\Omega)} \quad \forall w \in W_h \tag{34}$$

by the triangle inequality.

We now show that

$$\|u_h - T_p u_h\|_{L^1(\Omega)} \le C(\|u - u_h\|_{L^1(\Omega)} + Ch^{p+1}\|u\|_{W_1^{p+1}(\Omega)}), \tag{35}$$

where P_h is the L^2 projection onto $\mathbb{P}_h$, the space of piecewise polynomials of degree $\le p$ with respect to the triangulation $\mathcal{T}_h$. In fact, using the notation $\|\cdot\|_{0,1}$ for the $L_1(\Omega)$ norms and (34) we have

$$\begin{aligned}
\|u_h - T_p u_h\|_{0,1} &\le \|u_h - u\|_{0,1} + \|u - P_h u\|_{0,1} + \|P_h u - T_p u_h\|_{0,1} \\
&= \|u - u_h\|_{0,1} + \|u - P_h u\|_{0,1} + \|T_p P_h u - T_p u_h\|_{0,1} \\
&\le \|u - u_h\|_{0,1} + \|u - P_h u\|_{0,1} + C_0\|P_h u - u_h\|_{0,1} \\
&\le \|u - u_h\|_{0,1} + \|u - P_h u\|_{0,1} + C_0(\|P_h u - u\|_{0,1} + \|u - u_h\|_{0,1}) \\
&\le C(\|u - u_h\|_{0,1} + \|P_h u - u\|_{0,1}).
\end{aligned}$$

It is known [5, p. 102, Eq. A. 26] that

$$\|P_h u - u\|_{0,1} \le Ch^{p+1}\|u\|_{W_1^{p+1}(\Omega)}$$

and hence we derive (35).

Now on the other hand, using the triangle inequality and then (33) we have

$$\begin{aligned}
\|u - u_h\|_{L^1(\Omega)} &\ge \|u^I - u_h\|_{L^1(\Omega)} - \|u - u^I\|_{L^1(\Omega)} \\
&\ge \|u^I - T_p u_h\|_{L^1(\Omega)} - \|u_h - T_p u_h\|_{L^1(\Omega)} - \|u - u^I\|_{L^1(\Omega)} \\
&\ge \|u^I - T_p u_h\|_{L^1(\Omega)} - \|u_h - T_p u_h\|_{L^1(\Omega)} - C_2 h^{p+1}|u|_{W_1^{p+1}(\Omega)}, \\
&\ge \|u^I - T_p u_h\|_{L^1(\Omega)} - C\|u - u_h\|_{L^1(\Omega)} - \\
&\quad -C_3 h^{p+1}|u|_{W_1^{p+1}(\Omega)},
\end{aligned}$$

from which we conclude that

$$\begin{aligned} \|u-u_h\|_{L^1(\Omega)} &\geq C_0\|u^I - T_p u_h\|_{L^1(\Omega)} - C_3 h^{p+1}|u|_{W_1^{p+1}(\Omega)} \\ &\geq C_0\sqrt{\sum_{i=1}^{N}\|u^I - T_p u_h\|^2_{L^1(\Omega_i^*)}} - C_3 h^{p+1}|u|_{W_1^{p+1}(\Omega)}, \end{aligned} \tag{36}$$

where Ω_i^* is as in the proof of Theorem 2.4. By a standard scaling argument, and (30) or the argument leading to it, we have

$$\|u^I - T_p u_h\|_{L^1(\Omega_i^*)} \geq C_2 h_{\min}\|u^I - T_p u_h\|_{L^1(\Omega_i^*)} \geq C_2 h_{\min}^2 h^{p+1}\sqrt{Q(\tilde{D}_i^0, \tilde{D}_i^1, \cdots, \tilde{D}_i^p)}.$$

This completes the proof. □

Theorem 2.7. *Suppose*

$$H_\infty: \quad u \in W_\infty^{p+1}(\Omega),\ \Omega \subset \mathbb{R}^2,$$

and that H1 of Theorem 2.4 holds. Then there exists a constant $C_\infty > 0$*, independent of* h, u *and* u_h*, such that*

$$\|u-u_h\|_{L^\infty(\Omega)} \geq C_\infty h^{p+1}\left[(\frac{h_{\min}}{h})\max_{1\leq i\leq N_e^\circ}\|\tilde{D}_i\| - |u|_{W_\infty^{p+1}(\Omega)}\right], \tag{37}$$

where $\tilde{D}_i$ *has components*

$$\tilde{D}_i^\alpha = J_i^{(\alpha)}/(h^{p+1}h_{\min}^{-|\alpha|}), \qquad |\alpha| = k,\ 0 \leq k \leq p.$$

Proof. As before, for $u \in W_\infty^{p+1}(\Omega)$, there is a $u^I \in \mathbb{P}_{p,h}^*$ so that

$$\|u - u^I\|_{L^\infty(\Omega)} \leq Ch^{p+1}|u|_{W_\infty^{p+1}(\Omega)},$$

where $C > 0$ is a constant independent of u and h.

We proceed exactly as in Theorem 2.4. Let x_i be a midpoint of an edge in κ, and recall that $T_p u_h$ is the Taylor polynomial of degree p evaluated at x_i over the triangle κ_i. Then

$$\|u_h - T_p u_h\|_{L^\infty(\kappa_i)} \leq Ch_{\kappa_i}^{p+1}|u_h|_{W_\infty^{p+1}(\kappa_i)} \leq Ch_\kappa^{p+1}|u_h|_{W_\infty^{p+1}(\kappa)}. \tag{38}$$

Applying Lemma 2.3 with $m = p+1$, $s = \infty$ and $v = u_h$ and using the regularity condition (17) and equivalence of norms on $\hat{\kappa}$, we have

$$|u_h|_{W_\infty^{p+1}(\kappa)} \leq C\rho_\kappa^{-(p+1)}\|u_h\|_{L^\infty(\hat{\kappa})}$$

so that by (17)

$$h_\kappa^{p+1}|u_h|_{W_\infty^{p+1}(\kappa)} \leq C\|u_h\|_{L^\infty(\kappa)}.$$

Taking the maximum of (38) over κ, we have, using the last inequality,

$$\|u_h - T_p u_h\|_{L^\infty(\Omega)} \leq C\|u_h\|_{L^\infty(\Omega)}. \tag{39}$$

So as before, using the stability of T_p in the L_∞ norm and the approximation property of P_h [5], we conclude that

$$\|u-u_h\|_{L^\infty(\Omega)} \geq C_0\max_i\|u^I - T_p u_h\|_{L^\infty(\Omega_i^*)} - Ch^{p+1}|u|_{W_\infty^{p+1}(\Omega)},$$

where Ω_i^* is as in the proof of Theorem 2.4. Let $U_i = \|u^I - T_p u_h\|_{L^\infty(\Omega_i^*)}$ and use (30) to derive

$$\begin{aligned} U_i^2 &= \frac{1}{|\Omega_i^*|}\int_{\Omega_i^*} U_i^2 dx \geq \frac{1}{|\Omega_i^*|}\int_{\Omega_i^*}(u^I - T_p u_h)^2 dx \\ &\geq Ch^{-2}\|u^I - T_p u_h\|^2_{L^2(\Omega_i^*)} \geq (h_{\min}/h)^2 h^{2p+2} Q(\tilde{D}_i^0, \tilde{D}_i^1, \cdots, \tilde{D}_i^p). \end{aligned}$$

Invoking (11) and taking a common minimum constant completes the proof. □

Now we impose quasi-uniform conditions on the meshes to get the next theorem.

Theorem 2.8. *Let $\{\mathcal{Q}_h\}$ be a family of quasi-uniform subdivisions of Ω into quadrilaterals κ. Suppose the following two assumptions hold.*

H0. $u \in W_s^{p+1}(\Omega)$, $\Omega \subset \mathbb{R}^2$.

H1. $u_h \in W_h \subset \{v : \Omega \to \mathbb{R} : v|_\kappa \in C^{p+1}(\bar{\kappa}), \kappa \in \mathcal{Q}_h\}$, *dim* $W_h < \infty$.

Then

(i) *in case $s = 1, 2$, there exists a positive constant C_1 independent of h, u and u_h such that*

$$\|u - u_h\|_{L^s(\Omega)} \geq C_1 h^{p+1} \left(\left(\sum_{i=1}^{N_e^\circ} h^2 \|D_i\|^s \right)^{1/s} - |u|_{W_s^{p+1}(\Omega)} \right), \tag{40}$$

(ii) *in case $s = \infty$, there exists a constant $C_\infty > 0$, independent of h, u and u_h, such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \geq C_\infty h^{p+1} \left[\max_{1 \leq i \leq N_e^\circ} \|D_i\| - |u|_{W_\infty^{p+1}(\Omega)} \right],$$

where the components of D_i are

$$D_i^\alpha = J_i^{(\alpha)} / (h^{p+1-|\alpha|}), \quad J_i^{(\alpha)} = [\![\partial^\alpha u_h]\!]_{x_i}, \quad |\alpha| = k, \ 0 \leq k \leq p.$$

Proof. Note that since $\tilde{D}_i^\alpha = D_i^\alpha (\frac{h_{\min}}{h})^k$, $\|\tilde{D}_i\|^2 = \sum_{k=0}^p (D_i^\alpha)^2 (\frac{h_{\min}}{h})^{2k}$. By quasi-uniformness, $h/h_{\min}$ is uniformly bounded above and we can replace all occurrences of $h_{\min}$ by Ch in all the previous theorems. This completes the proof. □

3. Optimal order convergence implies numerical smoothness

In this section we can mathematically justify that "a numerical approximate solution ought to be as smooth as its targeted exact solution."

Theorem 3.1. *Suppose that $u \in W_s^{p+1}(\Omega), s = 1, 2, \infty, \Omega \subset \mathbb{R}^n$ and that u_h is in W_h on a quasi-uniform family of meshes on Ω into quadrilaterals. Then a necessary condition for*

$$\|u - u_h\|_{L^s(\Omega)} = \mathcal{O}(h^{p+1})$$

is for u_h to be W_s^{p+1} smooth. In particular, for

$$\|u - u_h\|_{L^\infty(\Omega)} = \mathcal{O}(h^{p+1})$$

a necessary condition is that all jumps in the k^{th} partial derivatives at midpoints x_i satisfy

$$[\![\partial^\alpha u_h]\!]_{x_i} = \mathcal{O}(h^{p+1-k}), \quad |\alpha| = k, \ 0 \leq k \leq p.$$

Here all smoothness refers to interface smoothness.

Proof. Suppose $\|u - u_h\|_{L^s(\Omega)} \leq Ch^{p+1+\sigma}, \sigma \geq 0$. Applying this to inequality (40) deduces the result. Other assertions follow in a similar way. □

Note that all D_i^α need to be bounded for convergence as a consequence of this theorem.

Theorem 3.2. *Suppose that $u \in C^{p+1}(\Omega), s = 1, 2, \infty, \Omega \subset \mathbb{R}^2$ and that $u_h \in W_h$ on a quasi-uniform family $\{\mathcal{Q}_h\}$ of meshes on Ω into quadrilaterals. Then a necessary condition for*

$$\|u - u_h\|_{L^s(\Omega)} = \mathcal{O}(h^{p+1})$$

is for u_h to be W_s^{p+1} smooth. In particular, for

$$\|u - u_h\|_{L^\infty(\Omega)} = \mathcal{O}(h^{p+1})$$

a necessary condition is that all the k^{th} partial derivatives at $x_i \in T$ satisfy

$$\partial^\alpha(u - u_h)(x_i) = \mathcal{O}(h^{p+1-k}), \quad |\alpha| = k, \ 0 \leq k \leq p. \tag{41}$$

In other words, we have a simultaneous approximation result. Here all smoothness refers to interior smoothness and $\{x_i\}$ is any collection of points, one from each element.

Proof. Let $\mathcal{Q}_h$ be a quasi-uniform subdivision on Ω in $\mathbb{R}^2$, and let $u \in W_\infty^{p+1}(\Omega)$ and $u^I \in \mathbb{P}_p^h$ be such that

$$\|u - u^I\|_{L^\infty(\Omega)} \leq Ch^{p+1}|u|_{W_\infty^{p+1}(\Omega)}.$$

Let $u_h \in \mathbb{Q}_p^h$ be given and to simplify the presentation, we will use shorthand notations: let $|\alpha| = k$ and since we will treat one kth derivative at a time, there is no ambiguity in setting $u_h^{(k)} = \partial^\alpha u_h$, $u_I^{(k)} = \partial^\alpha u^I$, and $u^{(k)} = \partial^\alpha u$. At a typical point $x_m \in \kappa \in \mathcal{Q}_h$, we denote by $T_p u_h$ the Taylor polynomial of degree p evaluated at x_m so that $T_p u_h \in \mathbb{P}_p(\kappa)$. Now we have the difference in derivatives

$$\begin{aligned} |\tilde{F}_i^{(k)}| &:= |u_h^{(k)}(x_m) - u^{(k)}(x_m)| \\ &\leq |u_h^{(k)}(x_m) - (T_p u_h)^{(k)}(x_m)| + |(T_p u_h)^{(k)}(x_m) - u_I^{(k)}(x_m)| \\ &\quad + |u_I^{(k)}(x_m) - u^{(k)}(x_m)| \\ &= I_1 + I_2 + I_3. \end{aligned}$$

On the one hand

$$I_1 + I_3 \leq Ch^{p+1-k}|u|_{W_\infty^{p+1}(\kappa)}, \tag{42}$$

and on the other hand

$$I_2 \leq Ch^{-k}\|T_p u_h - u^I\|_{L^\infty(\kappa)}, \tag{43}$$

where we have used quasi-uniformness of the mesh. In addition

$$\|T_p u_h - u^I\|_{L^\infty(\kappa)} \leq \|T_p u_h - u_h\|_{L^\infty(\kappa)} + \|u_h - u\|_{L^\infty(\kappa)} + \|u - u^I\|_{L^\infty(\kappa)}.$$

Combining all the related estimates, we have

$$|\tilde{F}_i^{(k)}| \leq Ch^{-k}\left(h^{p+1}|u|_{W_\infty^{p+1}(\Omega)} + \|u_h - u\|_{L^\infty(\Omega)}\right), \tag{44}$$

which stated in a more practical manner is (41). □

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