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A UNIFIED PARALLEL DEA MODEL AND EFFICIENCY MODELING OF MULTI-ACTIVITY AND/OR NON-HOMOGENEOUS ACTIVITY

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Abstract. Data envelopment analysis(DEA), as originally proposed, is a methodology for evaluating the relative efficiencies of peer decision making units (DMUs) under some general assumptions. DEA models with non-homogeneous DMUs and multi-activity structures are two different subjects referring to relaxing various assumptions. In this paper, we show that these two formulations are both derived by embedding the corresponding process into a general parallel DEA model. Furthermore, following the parallel DEA framework, general DEA models for multi-activity and non-homogeneity are proposed, which are able to handle many situations where different aspects of non-homogeneity or multi-activities exist. This study provides important insights into the existing DEA models for non-homogeneity and multi-activity.

Key words. Data envelopment analysis, parallel model, multi-activity, non-homogeneous.

1. Introduction

Data Envelopment Analysis (DEA) has been a standard methodology for evaluating the relative performance of Decision Making Units (DMUs) since the paper of Charnes et al. [4]. Some underlying assumptions are common to traditional DEA models. DMUs are treated as black boxes since the internal structures of DMUs are ignored in traditional DEA models. Furthermore, all DMUs are considered to be homogeneous, that is they all utilize the same types of inputs to produce the same types of outputs. In the last four decades, thousands of articles and extensive work have appeared to relax the above assumptions, see [11, 29].

In some contexts, the knowledge of the internal structure of DMU can give further insights for the performance evaluation. Extensive studies have been done to model internal structures and networks of the operation, in e.g., [5, 12, 21, 23, 25, 30] and so on. Comprehensive discussions on network DEA have been showed in the handbook of Cook and Zhu [15]. A basic type of network structure is parallel system where a production system is consisted of several subsystems. In the case that a production system with parallel production units, there are currently two fundamental researches: YMK model[32] and Kao's parallel DEA model[22], and we will first show that they are both special cases of a general parallel DEA model.

Multi-activity problem is a special case concerning the internal structure of DMUs, where there exist shared inputs and outputs allocated to various activities. Furthermore, the allocation of resources remains to be determined. This problem was first studied by Beasley[1]. Molinero[28] extend the problem by concentrating on the dual of Beasley's model. Although the problem was first known as the joint determination of efficiency within a DEA context, it is often referred to as a multi-activity or multi-component model now. Subsequently, Cook et al.[10], Jahanshahloo et al. [19] and Tsai and Mar Molinero[31] have revised the model. Many authors have introduced variants of multi-activity models. Castelli et al. [2] have

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reviewed some extensions of multi-activity models, which include considering weight restrictions (Beasley[1]), variable returns to scale (Tsai and Molinero[31]), different weights on shared inputs (Cook et al.[10]), additive objective function (Cook and Hababou[8]) and various forms of inputs/outputs (e.g., Cook and Green[9]; Jahanshahloo et al.[19, 20]). Moreover, the production process of a DMU may contain several stages in which some outputs produced by a former stage are used as inputs to a later stage of production. Färe and Grosskopf[16, 17]thus proposed a network DEA model for measuring efficiency for those DMUs with multiple production stages. Subsequently, Chen [6], Chen et al.[7], Yu and Lin[33], Yu and Fan[34], and Wang et al.[35] studied multi-activity network DEA models which incorporated multiple activities and multiple processes into a unified framework. In this paper we show that multi-activity DEA models can be derived from the general parallel DEA model.

Besides the black-box assumption, in the traditional DEA models, all DMUs are known as homogeneous in the sense that each has the same type of inputs and outputs. However, in some applications such as plants and universities, this assumption may be violated. The situation that the inputs and outputs of a set of DMUs or their input-to-output relations are not exactly the same is known as non-homogeneity. The DEA model with non-homogeneous DMUs is first studied by Molinero[27] with a specific university example, and then a systematic study has been presented by Cook et al. [13, 14]. Following the work of Cook et al. [13, 14], a few extensions around the non-homogeneity phenomenon have been carried out. Imanirad et al. [18] introduced a methodology to allow for efficiency measurement in situations where some DMUs have different input-to-output relations. And in the latest paper, Liang et al.[24] examine non-homogeneity on the input side. In the current paper, by looking inside the non-homogeneity phenomenon, we will show that Cook's model can also be transformed into the form of the general parallel DEA model.

Hence following the parallel DEA framework, general DEA models for multi-activity and non-homogeneity are proposed, which are able to handle many situations where different aspects of non-homogeneity or multi-activities exist.

The paper is organized as follows. After introduction, two classic parallel DEA models are introduced and a general parallel DEA model is proposed in Section 2. Multi-activity models are presented in Section 3, in addition, how to apply the parallel DEA model is explained. Section 4 describes the specific situations of multi-activity models which exist in the literatures. Following the formulation of the general parallel DEA model, a general DEA model with non-homogeneous DMUs is proposed in Section 5. Furthermore, in Section 6 we apply the general models into specific situations. Discussions around the general model and some future directions are presented in Section 7.

2. DEA models for systems with parallel structure

A basic type of internal structure is parallel system. For a system composed of several processes connected in parallel, there are at least two fundamental researches. One is YMK model proposed by Yang et al.[32] in measuring the efficiencies of the production system with independent subsystems, the other is proposed by Kao[22] dealing with the case where all parallel subunits consume the same set of inputs to produce the same set of outputs. We think that the key difference between the YMK model and Kao's parallel DEA model lies in that the corresponding inputs and outputs across subunits are perceived to be (equally) compensable or not

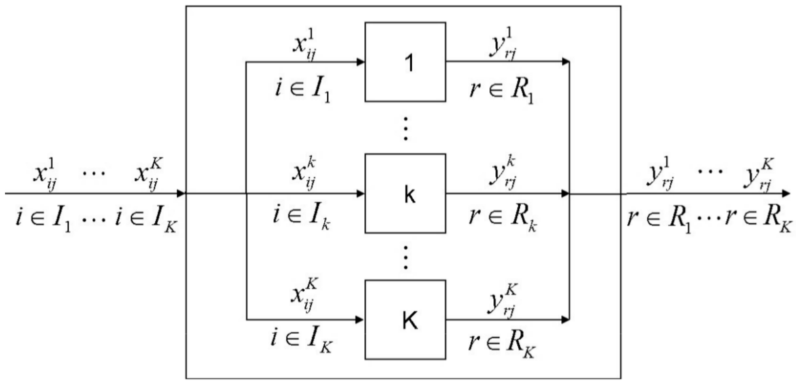

FIGURE 1. The parallel production system of YMK model.

in efficiency evaluation, and this is often a managerial decision. Essentially for two sub-processes that both produce, say, rice, its outputs can be evaluated as two independent outputs or, incorporated into an overall total measure, and this is decided by the decision maker. The notion of (equally) "substitutable" or "compensable" has been investigated in the DEA literature (Liu et al.[26]), where orders or preferences are introduced as a basic element into DEA model building since DEA is to compare performance of different DMUs. Logically one has firstly to know what is the underlying meaning of "better" in a particular DEA model. Here two different preferences of Pareto and Average are involved: Let (Y_1, Z_1) and (Y_2, Z_2) be the outputs of two DMUs, then the former is better than the latter in the sense of Pareto if $Y_1 > Y_2$ and $Z_1 > Z_2$, while in Average sense if $Y_1 + Z_1 > Y_2 + Z_2$. In this sense YMK model has adapted the Pareto order while Kao's uses the Average order for overall-system aggregation, for more details one can refer to Liu et al. [26] and Zhang et al.[36].

After introducing the two classic parallel models, we will further investigate the connection between these two models. Subsequently, a general parallel DEA model is proposed, which can encompass the feature of the both models.

Before introducing the models, some notations are given as follows. For each $DMU_j (j = 1, 2, \cdots, n)$, it acts as a production system consisting of K subsystems, each subsystem k represented by its own input/output bundle. Each subsystem k is formed in such a way that all inputs impact all outputs. For DMU_j , x_{ij}^k and y_{rj}^k represent the input i and output r of subsystem k, respectively.

2.1. YMK model. In the model proposed by Yang et al.[32], the operation of each subsystem is independent. As the YMK production system shown in Fig.1 below, the input/output bundle of subsystem k is denoted by (I_k, R_k). Note that each input/output subset can vary from one another. The numbers of inputs and outputs for subsystem k are m_k and s_k, respectively. In this model, we assume the inputs and outputs are evaluated via Pareto order across subunits, the input and output of the overall production system is constituted of all inputs and outputs of K subsystem. Hence the numbers of inputs and outputs for the system will be $\sum_k m_k$ and $\sum_k s_k$, respectively. Let us emphasize that the corresponding inputs and outputs in different subsystems are compared in the Pareto order, but they may be in fact of either the same type or different types. They are linked only through the radial measurement and objective functions.

The YMK model and its dual are formulated as follows (Yang et al.[32]).

$$
\begin{aligned}
\max\ & \sum_{k=1}^{K}\sum_{r=1}^{s_k} u_r^k y_{rj_0}^k \\
s.t.\ & \sum_{k=1}^{K}\sum_{i=1}^{m_k} v_i^k x_{ij_0}^k = 1, \\
& \sum_{i=1}^{m_k} v_i^k x_{ij}^k - \sum_{r=1}^{s_k} u_r^k y_{rj}^k \leqslant 0, \forall k, \forall j, \\
& u_r^k, v_i^k \geqslant 0, \forall k, \forall j.
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\min\ & \theta \\
s.t.\ & \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \leqslant \theta x_{ij_0}^k, \forall k, \forall i, \\
& \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \geqslant y_{rj_0}^k, \forall k, \forall r, \\
& \lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned} \tag{2}
$$

2.2. Kao's model. In the model proposed by Kao[22], all parallel subsystems are operated independently by utilizing the same set of inputs to produce the same set of outputs which is different from YMK model. Furthermore, the principle underlying this model is that the corresponding inputs and outputs across subunits are evaluated via Average order, hence it is rational for them to be summed up in the evaluation of the system efficiency. The production system of Kao's model is shown in Fig. 2. Let (I_0, R_0) denotes the input/output bundle. There are m inputs and s outputs in every subsystem as well as in the overall system. $x_{ij}(i = 1, 2, \cdots, m)$ and $y_{rj}(r = 1, 2, \cdots, s)$ constitute the overall inputs and outputs of the production system, where $x_{ij} = \sum_{k=1}^{K} x_{ij}^k$ and $y_{rj} = \sum_{k=1}^{K} y_{rj}^k$.

The Kao's model and its dual are as follows (Kao[22]).

$$
\begin{aligned}
\max\ & \sum_{r=1}^{s} u_r y_{rj_0} \\
s.t.\ & \sum_{i=1}^{m} v_i x_{ij_0} = 1, \\
& \sum_{i=1}^{m} v_i x_{ij}^k - \sum_{r=1}^{s} u_r y_{rj}^k \leqslant 0, \forall k, \forall j, \\
& u_r, v_i \geqslant 0.
\end{aligned} \tag{3}
$$

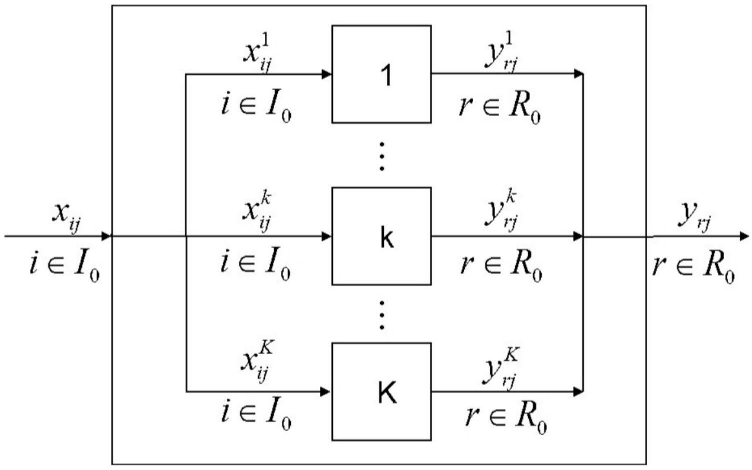

FIGURE 2. The parallel production system of Kao s parallel model.

$$
\begin{aligned}
& \min \ \sigma \\
& \text{s.t.} \quad \sum_{j=1}^{n}\sum_{k=1}^{K} \lambda_j^k x_{ij}^k \leqslant \sigma x_{ij_0}, \ \forall i, \\
& \qquad \sum_{j=1}^{n}\sum_{k=1}^{K} \lambda_j^k y_{rj}^k \geqslant y_{rj_0}, \ \forall r, \\
& \qquad \lambda_j^k \geqslant 0.
\end{aligned} \tag{4}
$$

2.3. General parallel model combining Pareto and Average orders. In more general cases, inputs and outputs can be divided into those equally important but non-compensable and compensable. In the notation of preference Pareto is applied for some of the components of the input/outputs while Average order is used for the others. In the following, we will combine the above two models and derive a general parallel DEA model which can deal with the more general cases.

Our approach is based on observation that actually YMK model can be transformed into the form of Kao's model. Take the YMK production system shown in Fig.3 as a simple example and each subsystem has two inputs and two outputs. To help transfer YMK model into the form of Kao's parallel DEA model, extensions of the inputs and outputs sets are done as follows: in the first subsystem, assume $\overline{x}_1^1$ and $\overline{x}_2^1$ are the same as x_1^1 and x_2^1 in YMK, while letting $\overline{x}_3^1 = \overline{x}_4^1 = 0$; similarly, in the second subsystem, let $\overline{x}_1^2 = \overline{x}_2^2 = 0$, while assuming $\overline{x}_3^2$ and $\overline{x}_4^2$ are the same as x_3^2 and x_4^2 in YMK. The extensions of the outputs are the same as those of the inputs. The system inputs and outputs are $\overline{x}_i = \sum_k \overline{x}_i^k (i = 1, 2)$ and $\overline{y}_r = \sum_k \overline{y}_r^k (r = 1, 2)$, respectively. In this case, the original production system is transformed into a new system (see Fig. 4) which is the same as that of Kaos parallel DEA model. Note that the model derived from the transformed system is equivalent to the original YMK mod

As indicated by the simple example, YMK model can be transformed into Kao's parallel model after extending the original input and output sets by adding zeros.

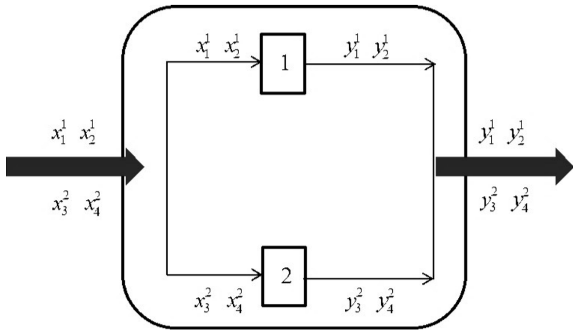

FIGURE 3. A simple YMK production system.

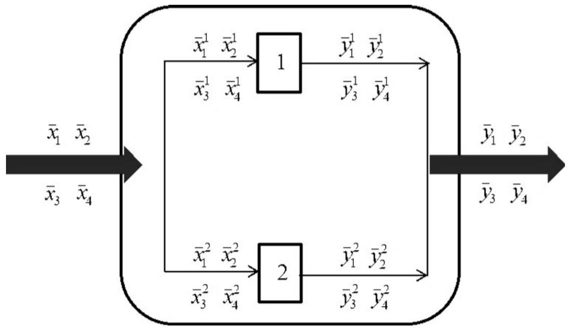

FIGURE 4. A transformed production system.

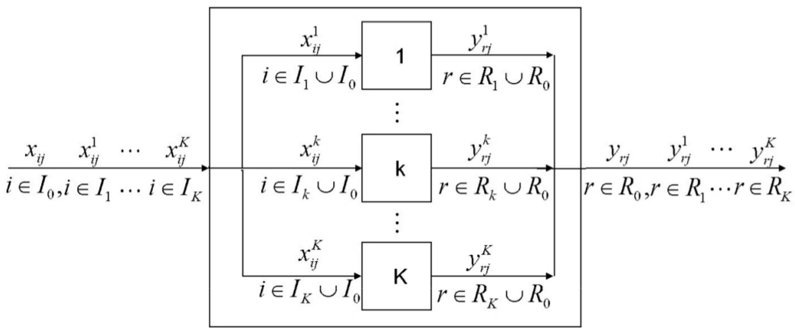

FIGURE 5. The production system of the general parallel DEA model.

With these helps, we now give the general parallel DEA model, where some subsystem inputs/outputs are compensable, while others are perceived to be not. To be more specific, in the evaluation of the system efficiency, some inputs/outputs are addable across subsystems while others are not.

In this model, each subsystem k is represented by its own input/output bundle $(I_k \bigcup I_0, R_k \bigcup R_0)$. There will be $(m_k + m)$ inputs and $(s_k + s)$ outputs for subsystem k. In subsystem k, the inputs belong to I_0 and outputs belong to R_0 are compensable across subsystems as that in Kao's model, while inputs belong to I_k and outputs belong to R_k are the same type as that in YMK model.

The general parallel DEA model and its dual can be formulated as follows:

$$
\begin{aligned}
\max\ & \sum_{r\in R_0} u_r y_{rj_0} + \sum_k \sum_{r\in R_k} u_r^k y_{rj_0}^k \\
s.t.\ & \sum_{i\in I_0} v_i x_{ij_0} + \sum_k \sum_{i\in I_k} v_i^k x_{ij_0}^k = 1, \\
& \sum_{r\in R_0} u_r y_{rj}^k + \sum_{r\in R_k} u_r^k y_{rj}^k \leqslant \sum_{i\in I_0} v_i x_{ij}^k + \sum_{i\in I_k} v_i^k x_{ij}^k, \forall k, \forall j, \\
& u_r \geqslant 0, r = 1, \cdots, s, v_i \geqslant 0, i = 1, \cdots, m, \\
& u_r^k \geqslant 0, r = 1, \cdots, s_k, \forall k, \\
& v_i^k \geqslant 0, i = 1, \cdots, m_k, \forall k.
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\min\ & \theta \\
s.t.\ & \sum_j \sum_k \lambda_j^k x_{ij}^k \leqslant \theta x_{ij_0},\ i \in I_0, \\
& \sum_j \lambda_j^k x_{ij}^k \leqslant \theta x_{ij_0}^k, \forall k, i \in I_k, \\
& \sum_j \sum_k \lambda_j^k y_{rj}^k \geqslant y_{rj_0},\ r \in R_0, \\
& \sum_j \lambda_j^k y_{rj}^k \geqslant y_{rj_0}^k, \forall k, r \in R_k, \\
& \lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned}
\tag{6}
$$

where $x_{ij} = \sum_{k=1}^{K} x_{ij}^k (i \in I_0)$ and $y_{rj} = \sum_{k=1}^{K} y_{rj}^k (r \in R_0)$.

It is clear that when inputs and outputs in different subsystems are compared in Average order, the general model is exactly the same formulation as that of Kao's model. Otherwise, the general model will share the same formulation as that of YMK model. That is, our general model encompasses both the YMK model and Kao's parallel DEA model.

Take the production system shown in Fig. 6 as a simple example of the general parallel system. Assume there are two inputs and two outputs for each subsystem and only the input and output with subscript 2 are perceived to be compensable.

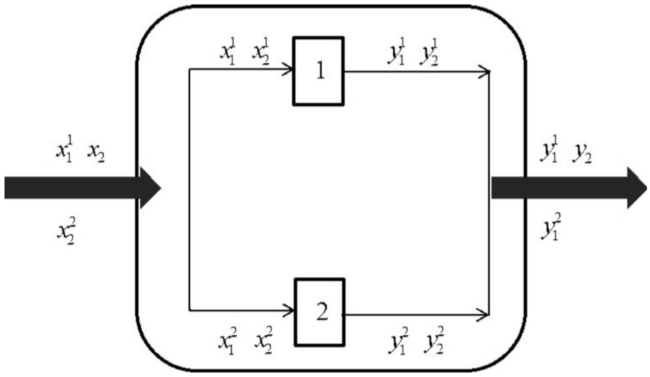

FIGURE 6. A simple general parallel production system.

Following the general parallel DEA model, the parallel DEA models for the above system are formulated as follows.

$$
\begin{aligned}
\max\ & u_1^1 y_{1j_0}^1 + u_1^2 y_{1j_0}^2 + u_2 y_{2j_0} \\
s.t.\ & v_1^1 x_{1j_0}^1 + v_1^2 x_{1j_0}^2 + v_2 x_{2j_0} = 1, \\
& u_1^1 y_{1j}^1 + u_2 y_{2j}^1 \leqslant v_1^1 x_{1j}^1 + v_2 x_{2j}^1, \forall j, \\
& u_1^2 y_{1j}^2 + u_2 y_{2j}^2 \leqslant v_1^2 x_{1j}^2 + v_2 x_{2j}^2, \forall j, \\
& u_1^1, u_1^2 \geqslant 0, \\
& v_1^1, v_1^2 \geqslant 0, \\
& u_2 \geqslant 0, v_2 \geqslant 0.
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\min\ & \theta \\
s.t.\ & \sum_j \lambda_j^1 x_{1j}^1 \leqslant \theta x_{1j_0}^1, \quad \sum_j \lambda_j^2 x_{1j}^2 \leqslant \theta x_{1j_0}^2, \\
& \sum_j \lambda_j^1 x_{2j}^1 + \sum_j \lambda_j^2 x_{2j}^2 \leqslant \theta x_{2j_0}, \\
& \sum_j \lambda_j^1 y_{1j}^1 \geqslant y_{1j_0}^1, \quad \sum_j \lambda_j^2 y_{1j}^2 \geqslant y_{1j_0}^2, \\
& \sum_j \lambda_j^1 y_{2j}^1 + \sum_j \lambda_j^2 y_{2j}^2 \geqslant y_{2j_0}, \\
& \lambda_j^1, \lambda_j^2 \geqslant 0, \forall j.
\end{aligned}
\tag{8}
$$

In the following, we will try to utilize the general parallel DEA model to handle the multi-activity and non-homogeneous input/output processes by embedding the process into a general parallel system.

3. Multi-activity model

In many real situations, the units under evaluation may perform several different functions or may be separated into different subsystems. In such situations, particular resources are often shared among those subsystems. This sharing phenomenon will commonly present the difficulty of how to disaggregate the measure into subsystems. Usually, the amount of shared flow to each subsystem is considered as a decision variable to maximize the DMU efficiencies. This problem is known as multi-activity problem which was first proposed by Beasley[1].

A multi-activity model is formulated after introducing the following notations. Consider n homogeneous DMUs and each engages in K activities. For DMU_j , let $x_{ij}^k(i \in I_k)$ and $y_{rj}^k(r \in R_k)$ be the dedicated input i and output r of subsystem k, respectively. Assume that $x_{ij}(i \in I_0)$ is the shared input and thus all subsystems are shared in x_{ij}. Also, all components play an important role in producing shared output $y_{rj}(r \in R_0)$. Some portion α_i^k of the shared inputs x_{ij} and some portion β_r^k of shared outputs y_{rj} are allocated to subsystem k.

If we know the split proportion of inputs and outputs among activities, that is, $\alpha_i^k x_{ij}(i \in I_0)$ and $\beta_r^k y_{rj}(r \in R_0)$ are known shared inputs and outputs allocated to activity k , we might reasonably apply the general parallel DEA model directly to arrive at an efficiency score for each DMU. That is, $\alpha_i^k x_{ij}(i \in I_0)$ is equivalent to that $x_{ij}^k(i \in I_0)$ in the general parallel model, similarly, $\beta_r^k y_{rj}(r \in R_0)$ is equivalent to $y_{rj}^k(r \in R_0)$ in the general parallel model. However, the precise split of shared inputs and outputs to various activities are not known, α_i^k and β_r^k are variables to be determined. Hence evaluating the efficiency of a given DMU with multi-activity structure is, indeed, a bi-level programming problem which contains two parallel objectives: obtaining the best efficiency score and the most appropriate resource allocation. It may take two steps to solve this problem. First, assuming the split of inputs/outputs is a known quantity, so we can proceed as above, that is to seek a best score for each DMU by directly applying the general parallel DEA model. For this step, the objective function can be either minimal or maximal depending on the orientation chosen. Then the second step is to determine the most appropriate alpha and beta variables. We argue that the best way to allocate the resources is to do so in a way that results in the best overall efficiency for a DMU. For the optimization model with alpha and beta as decision variables, the objective function is supposed to find the maximum efficiency score.

Following the two-steps method shown above, the multi-activity models could be formulated as follows.

Multiplier form:

$$
\begin{aligned}
&\max_{(\alpha,\beta)} \max_{(u,v)} \; e_0(\alpha,\beta,u,v) = \sum_{r\in R_0} u_r y_{rj_0} + \sum_k \sum_{r\in R_k} u_r^k y_{rj_0}^k \\
&s.t. \quad \sum_{i\in I_0} v_i x_{ij_0} + \sum_k \sum_{i\in I_k} v_i^k x_{ij_0}^k = 1, \\
(9) \quad &\qquad \sum_{r\in R_0} u_r \beta_r^k y_{rj} + \sum_{r\in R_k} u_r^k y_{rj}^k \leqslant \sum_{i\in I_0} v_i \alpha_i^k x_{ij} + \sum_{i\in I_k} v_i^k x_{ij}^k, \qquad \forall j, \forall k, \\
&\qquad \sum_k \alpha_i^k = 1, \qquad \forall i \in I_0, \\
&\qquad \sum_k \beta_r^k = 1, \qquad \forall r \in R_0, \\
&\qquad u_r, u_r^k, v_i, v_i^k, \alpha_i^k, \beta_r^k \geqslant 0 \qquad \forall i, r, k.
\end{aligned}
$$

Dual form:

$$
(10) \qquad
\begin{aligned}
&\max_{(\alpha,\beta)} \max_{(\phi,\lambda)} \ e_1(\alpha,\beta,\phi,\lambda) \\
&s.t. \quad \sum_k \sum_j \lambda_j^k \alpha_i^k x_{ij} \leqslant x_{ij_0}, \forall i \in I_0, \\
&\sum_j \lambda_j^k x_{ij}^k \leqslant x_{ij_0}^k, \forall k, \forall i \in I_k, \\
&\sum_k \sum_j \lambda_j^k \beta_r^k y_{rj} \geqslant \phi y_{rj_0}, \forall r \in R_0, \\
&\sum_j \lambda_j^k y_{rj}^k \geqslant \phi y_{rj_0}^k, \forall k, \forall r \in R_k, \\
&\sum_k \alpha_i^k = 1, \forall i \in I_0, \\
&\sum_k \beta_r^k = 1, \forall r \in R_0, \\
&\lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned}
$$

The above bi-level programming problem is linear in each level, but the computational work is enormous. We show below that we only need to compute the objectives $\max\limits_{(\alpha,\beta,u,v)} e_0(\alpha,\beta,u,v)$ and $\max\limits_{(\alpha,\beta,\phi,\lambda)} e_1(\alpha,\beta,\phi,\lambda)$, although they are nonlinear (we will discuss the linearization later).

Lemma 3.1. $\max\limits_x \max\limits_y f(x,y) = \max\limits_{(x,y)} f(x,y)$.

Proof. This lemma appears to be obviously true. However, we cannot find the proof in the literature. Thus we include a proof here for the convenience of readers.

$$\max_x \max_y f(x,y) \leqslant \max_{(x,y)} f(x,y).$$

Let $\max\limits_{(x,y)} f(x,y) = f(x_0,y_0)$, then $\max\limits_x \max\limits_y f(x,y) \geqslant \max\limits_x f(x,y_0) \geqslant f(x_0,y_0)$. Consequently, we can conclude that $\max\limits_x \max\limits_y f(x,y) = \max\limits_{(x,y)} f(x,y)$. □

Theorem 3.2. *The functions* $e_0(\alpha,\beta) = \max\limits_{(u,v)} e_0(\alpha,\beta,u,v)$ *and* $e_1(\alpha,\beta) = \max\limits_{(u,v)} e_1(\alpha,\beta,u,v)$ *are concave with respect to* (α,β)*, so that any local solutions are also global solutions.*

Proof. For any fixed α,β, $e_0(\alpha,\beta,u,v)$ is a linear function, that is, $e_0(\alpha,\beta,u,v)$ is concave with respect to u and v for any fixed α and β , then $\overline{e}_0(\alpha,\beta) = \max\limits_{(u,v)} e_0(\alpha,\beta,u,v)$ is concave with respect to α and β. Similarly, we can show the concavity of $\overline{e}_1(\cdot,\cdot)$. □

With Lemma 3.1, we rewrite model (9) and (10) as follows.

Multiplier form:

$$
\begin{aligned}
& \max \ \sum_{r\in R_0} u_r y_{rj_0} + \sum_k \sum_{r\in R_k} u_r^k y_{rj_0}^k \\
& s.t. \quad \sum_{i\in I_0} v_i x_{ij_0} + \sum_k \sum_{i\in I_k} v_i^k x_{ij_0}^k = 1, \\
& \qquad \sum_{r\in R_0} u_r \beta_r^k y_{rj} + \sum_{r\in R_k} u_r^k y_{rj}^k \leqslant \sum_{i\in I_0} v_i \alpha_i^k x_{ij} + \sum_{i\in I_k} v_i^k x_{ij}^k, && \forall j, \forall k, \\
& \qquad \sum_k \alpha_i^k = 1, && \forall i \in I_0, \\
& \qquad \sum_k \beta_r^k = 1, && \forall r \in R_0, \\
& \qquad u_r, u_r^k, v_i, v_i^k, \alpha_i^k, \beta_r^k \geqslant 0, && \forall i, r, k.
\end{aligned}
\tag{11}
$$

Dual form:

$$
\begin{aligned}
& \max \ \phi \\
& s.t. \quad \sum_k \sum_j \lambda_j^k \alpha_i^k x_{ij} \leqslant x_{ij_0}, \forall i \in I_0, \\
& \qquad \sum_j \lambda_j^k x_{ij}^k \leqslant x_{ij_0}^k, \forall k, \forall i \in I_k, \\
& \qquad \sum_k \sum_j \lambda_j^k \beta_r^k y_{rj} \geqslant \phi y_{rj_0}, \forall r \in R_0, \\
& \qquad \sum_j \lambda_j^k y_{rj}^k \geqslant \phi y_{rj_0}^k, \forall k, \forall r \in R_k, \\
& \qquad \sum_k \alpha_i^k = 1, \forall i \in I_0, \\
& \qquad \sum_k \beta_r^k = 1, \forall r \in R_0, \\
& \qquad \lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned}
\tag{12}
$$

To facilitate linearization, we make the change of variables $\breve{Z}_i^k = v_i \alpha_i^k$ and $\breve{z}_r^k = u_r \beta_r^k$, and note that

$\sum\limits_k \alpha_i^k = 1 \Rightarrow v_i \sum\limits_k \alpha_i^k = v_i \Rightarrow \sum\limits_k \breve{Z}_i^k = v_i,$

$\sum\limits_k \beta_r^k = 1 \Rightarrow u_r \sum\limits_k \beta_r^k = u_r \Rightarrow \sum\limits_k \breve{z}_r^k = u_r.$

Using the usual transformation $t = 1/(\sum\limits_{i\in I_0} \nu_i x_{ij_0} + \sum\limits_k \sum\limits_{i\in I_k} \nu_i^k x_{j_0}^k)$ (see Charnes et al.[4]), and defining $\mu_r = tu_r$, $\mu_r^k = tu_r^k$, $\nu_i = tv_i$, $\nu_i^k = tv_i^k$, $\gamma_i^k = t\breve{Z}_i^k$, $\omega_r^k = t\breve{z}_r^k$, problem (11) reduces to the following form:

$$
\begin{aligned}
\max\ & \sum_{r\in R_0}\mu_r y_{rj_0}+\sum_k\sum_{r\in R_k}\mu_r^k y_{rj_0}^k \\
s.t.\ & \sum_{i\in I_0}\nu_i x_{ij_0}+\sum_k\sum_{i\in I_k}\nu_i^k x_{ij_0}^k=1, \\
& \sum_{r\in R_0}\omega_r^k y_{rj}+\sum_{r\in R_k}\mu_r^k y_{rj}^k\leqslant\sum_{i\in I_0}\gamma_i^k x_{ij}+\sum_{i\in I_k}\nu_i^k x_{ij}^k, && \forall j,\forall k, \\
& \sum_k\gamma_i^k=\nu_i, && \forall i, \\
& \sum_k\omega_r^k=\mu_r, && \forall r, \\
& \mu_r,\mu_r^k,\nu_i,\nu_i^k,\gamma_i^k,\omega_r^k\geqslant 0, && \forall i,r,k.
\end{aligned}
\tag{13}
$$

However, it seems difficult to linearize dual form model (12), which is a key for benchmarking. For the first layer of the bi-level problem, we may take the dual of model (9) and (10), and derive the corresponding multi-activity models.
Multiplier form:

$$
\begin{aligned}
\max_{(\alpha,\beta)}\min_{(u,v)}\ & e_2(\alpha,\beta,u,v)=\sum_{i\in I_0}v_i x_{ij_0}+\sum_k\sum_{i\in I_k}v_i^k x_{ij_0}^k \\
s.t.\ & \sum_{r\in R_0}u_r y_{rj_0}+\sum_k\sum_{r\in R_k}u_r^k y_{rj_0}^k=1, \\
& \sum_{r\in R_0}u_r\beta_r^k y_{rj}+\sum_{r\in R_k}u_r^k y_{rj}^k\leqslant\sum_{i\in I_0}v_i\alpha_i^k x_{ij}+\sum_{i\in I_k}v_i^k x_{ij}^k, && \forall j,\forall k, \\
& \sum_k\alpha_i^k=1, && \forall i\in I_0, \\
& \sum_k\beta_r^k=1, && \forall r\in R_0, \\
& u_r,u_r^k,v_i,v_i^k,\alpha_i^k,\beta_r^k\geqslant 0, && \forall i,r,k.
\end{aligned}
\tag{14}
$$

Dual form:

$$
\begin{aligned}
\max_{(\alpha,\beta)}\min_{(\theta,\lambda)}\ & e_3(\alpha,\beta,\theta,\lambda) \\
s.t.\ & \sum_k\sum_j\lambda_j^k\alpha_i^k x_{ij}\leqslant\theta x_{ij_0},\forall i\in I_0, \\
& \sum_j\lambda_j^k x_{ij}^k\leqslant\theta x_{ij_0}^k,\forall k,\forall i\in I_k, \\
& \sum_k\sum_j\lambda_j^k\beta_r^k y_{rj}\geqslant y_{rj_0},\forall r\in R_0, \\
& \sum_j\lambda_j^k y_{rj}^k\geqslant y_{rj_0}^k,\forall k,\forall r\in R_k, \\
& \sum_k\alpha_i^k=1,\forall i\in I_0, \\
& \sum_k\beta_r^k=1,\forall r\in R_0, \\
& \lambda_j^k\geqslant 0,\forall k,\forall j.
\end{aligned}
\tag{15}
$$

These models are more difficult to solve so to keep the presentation simple, we only consider model (11) and (12) throughout the paper.

4. Models for special situations

The model proposed by Beasley[1] which included shared inputs only can be written as follows.

$$
\begin{aligned}
&\max \sum_k \sum_{r\in R_k} u_r^k y_{rj_0}^k \\
&s.t. \quad \sum_{i\in I_0} v_i x_{ij_0} + \sum_k \sum_{i\in I_k} v_i^k x_{ij_0}^k = 1, \\
&\qquad \sum_{r\in R_k} u_r^k y_{rj}^k \leqslant \sum_{i\in I_0} v_i \alpha_i^k x_{ij} + \sum_{i\in I_k} v_i^k x_{ij}^k, \qquad \forall j, \forall k, \\
&\qquad \sum_k \alpha_i^k = 1, \qquad \forall i \in I_0, \\
&\qquad u_r^k, v_i, v_i^k, \alpha_i^k \geqslant 0, \qquad \forall i, r, k.
\end{aligned} \tag{16}
$$

which is clearly a special case of model (11) where no shared outputs exist. Molinero[28] proposed an approach dual to (16). In addition, the authors included shared inputs. Their output oriented model is:

$$
\begin{aligned}
&\max \sum_k w^k \phi_{j_0}^k \\
&s.t. \quad \sum_k \sum_j \lambda_j^k \alpha_i^k x_{ij} \leqslant x_{ij_0}, \forall i \in I_0, \\
&\qquad \sum_j \lambda_j^k x_{ij}^k \leqslant x_{ij_0}^k, \forall k, \forall i \in I_k, \\
&\qquad \sum_k \sum_j \lambda_j^k \beta_r^k y_{rj} \geqslant \sum_k \phi_{j_0}^k \beta_r^k y_{r0}, \forall r \in R_0, \\
&\qquad \sum_j \lambda_j^k y_{rj}^k \geqslant \phi_{j_0}^k y_{rj_0}^k, \forall k, \forall r \in R_k, \\
&\qquad \sum_k \alpha_i^k = 1, \forall i \in I_0, \\
&\qquad \sum_k \beta_r^k = 1, \forall r \in R_0, \\
&\qquad \lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned} \tag{17}
$$

where w^k are positive weights representing the relative importance of each activity k for DMUj_0 , and ϕ_0^k are inefficiency measures of activity k for DMUj_0. The measures for various activities can be either the same or different, it depends on decision makers, note that the choice will not change the general structure of the multi-activity model. Let w^k and measure ϕ_0^k be the same for all activities, model (17) is a special case of model (12).

As pointed in Introduction, many authors have extended multi-activity models. Castelli et al.[2] has reviewed some extensions of multi-activity models, which includes considering weight restrictions, variable returns to scale, different weights on shared inputs, additive objective function and various form of inputs/outputs. Furthermore, models (11) and (12) can be extended into multi-stage forms, see Chen[6]Yu and Fan [33] and Yu and Lin[34] for example.

5. A general model for non-homogeneity

In some applications such as plants and universities, the inputs and outputs of a set of DMUs are not exactly the same, and note that this is not about missing input or output data for certain DMUs, but a DMU has chosen not to produce certain outputs. Although Molinero et al.[27] first proposed a specific example with university level institutions which engage in different sets of activities, DEA models with non-homogeneous DMUs were systematically studied by Cook et al.[13, 14] assuming to only know the inputs and outputs. With the desire to fairly evaluate a DMU for what it does, the authors proposed a DEA-based model which views the DMU as consisting of a set of subunits. For each DMU, the efficiencies for individual subunits make up the aggregate efficiency.

Cook et al.[14] describes a problem setting involving the evaluation of a set of DMUs with non-homogeneous outputs. Suppose the non-homogeneous DMUs fall into P mutually exclusive (M.E.) groups which we denote by $\{N_p\}_{p=1}^{P}$ such that the outputs are exactly the same to DMUs of a given group. The outputs are non-homogeneous across DMUs, but the inputs are all utilized to produce each output, we may define the inputs are belong to a sharable input set I_0. Then form M.E. output subgroups $R_k, k = 1, 2, \cdots, K$, where R_k denotes the maximum subset of outputs with the property that all its members appear as outputs of the same set of DMUs (same as DMU profile). With R_k properly defined, we may view each DMU as a business unit consisting of a subset of K subunits where each subunit k is represented by its input-to-output bundle (I_0, R_k). The subset can be different, however, from one DMU to another. Some further notations are given below, namely let L_{N_p} denotes those subscript k forming the full specified output set for any DMU in N_p; M_k denotes the set of all DMU groups that has subunit k as a member, that is $M_k = \{N_p \textit{ such that } k \in L_{N_p}\}$. For DMU j, let $y_{rj}^k (r \in R_k)$ be the output of subunit k; $x_{ij} (i \in I_0)$ be the sharable inputs; α_{ip}^k denote the proportion of sharable input i $(i \in I_0)$ to be allocated to subunit k of L_{N_p}. Then the DEA-based model with non-homogeneous DMUs is formulated as follows.

$$
\begin{aligned}
\max \quad & \sum_{k \in L_{N_{p0}}} \sum_{r \in R_k} u_r y_{rj_0} & & \\
s.t. \quad & \sum_i v_i x_{ij_0} = 1, & & \\
& \sum_{r \in R_k} u_r y_{rj} - \sum_i v_i \alpha_{ip}^k x_{ij} \leqslant 0, & & \forall j \in N_p, k \in L_{N_p}, \forall p, \\
& \sum_{k \in L_{N_p}} \alpha_{ip}^k = 1, & & \forall i, \forall p, \\
& a_{ip}^k \leqslant \alpha_{ip}^k \leqslant b_{ip}^k, & & \forall i, k, p, \\
& u_r, v_i, \alpha_{ip}^k \geqslant 0, & & \forall i, k, p.
\end{aligned} \tag{18}
$$

Note that the DEA-based model with non-homogeneous DMUs is a bi-level programming problem as well. But with lemma 1, the bi-level programming problem can be formulated as model (18). Model (18) determines the most appropriate alpha variables and the highest overall efficiency score simultaneously.

However, Cook's model needs to be extended in many applications. Such as a practical application of FHCs, the main business of FHCs can be classified into Banking, Insurance, Security and Others (Chao et al.[3]). The business lines of some given FHCs may be different from others. For example, certain FHC may

TABLE 1. Product lines and DMU groups in Cook et al. [14].

	y_1	y_2	y_3	y_4	y_5	y_6
N_1	×	×	×		×	
N_2		×	×	×	×	×
N_3			×		×	×
N_4	×		×		×	

have Insurance company while others may not. In this case, both inputs and outputs of DMUs are non-homogeneous. Here we extend Cook's model into a general DEA framework which can deal with more general non-homogeneous DMUs following the general parallel DEA model, where some inputs/outputs are shared and some are specialized. Another difference between our general model and Cook's model lies in that the subsystems of our general model will be further defined by the operation properties of DMUs. That is, the subsystems are not purely the virtual units decided by the phenomenon of non-homogeneity as in the Cook's model, but are further defined by the internal structure of DMUs. Here is an example to help understand the difference. Cook et al.[14] considered a set of non-homogeneous manufacturing plants with totally six main product lines. As shown in Table 1, plants with the same product lines have been grouped into four DMU groups. According to the subunits grouping criterion in Cook et al.[14], y_1, y_2, y_4 and y_6 are outputs of different subsystems, because their DMU profiles vary from each other, and the outputs y_3 and y_5 whose DMU profiles are the same should be grouped as one subgroup. Different from the unique grouping result in Cook et al.[14], we argue that the outputs y_3 and y_5 can be grouped into either one or different subsystems according to the real operational structure of DMUs (for example, if the outputs y_3 and y_5 are two different product lines that need to evaluate separately, then it is more appropriate to treat them as two subsystems). Note that our setting up includes the case in Cook et al.[14]. We may argue that without the support of the operation properties, generating uniqueness of maximal output subgroups is only one particular option.

Let us now introduce the more general setting. Suppose the non-homogeneous DMUs with multi-activity structure fall into P mutually exclusive (M.E.) groups which we denote by $\{N_p\}_{p=1}^P$ such that the inputs, outputs and input-to-output relations are all exactly the same to DMUs of a given group. The subsystems are decided by both the multi-activity structure and the non-homogeneity phenomenon. Define the whole production system as a system consisted of K subsystems, each subsystem only engages in one activity and has the property that it appears homogeneous across the DMUs containing this subsystem. To address this problem, we may view each DMU as a business unit consisting of a subset of K subsystems. The subset can be different, however, from one DMU to another. Since the inputs and outputs are extended in our general model, the following notations are further defined. Except the sharable inputs defined above, there are specified inputs belong to I_k that are specially allocated to subsystem k. And similarly, specified outputs in R_k are those specially generated from subsystem k and sharable outputs belong to R_0 are those generated from all subsystems.

One more step before applying the general parallel DEA model is introducing dummy processes. By introducing dummy processes, the non-homogeneous DMU production system can be represented by a homogeneous parallel structure. To be more specific, for any DMU$j \in N_p$, if R_k *not* $\in L_{N_p}$, dummy process k is introduced.

The outputs and inputs of dummy process k are all zeros, this is realized by letting $\beta_{rp}^k = 0(\forall r \in R_0)$, $y_{rj}^k = 0(\forall r \in R_k)$, and $\alpha_{ip}^k = 0(\forall i \in I_0)$, $x_{ij}^k = 0(\forall i \in I_k)$. Note that this step is taking after grouping and deciding subsystems (while missing data approach uses zero as a dummy for blank entries before all these), so this will not affect the non-homogeneous structure. Consequently, applying the general parallel DEA model, we propose the following model with dummy processes to deal with non-homogeneous DMUs.
Multiplier form:
(19)

$$
\begin{aligned}
\max\ & \left(\sum_{r\in R_0} u_r y_{rj_0} + \sum_k \sum_{r\in R_k} u_r^k y_{rj_0}^k\right) \Big/ \left(\sum_{i\in I_0} v_i x_{ij_0} + \sum_k \sum_{i\in I_k} v_i^k x_{ij_0}^k\right) \\
s.t.\ & \sum_{r\in R_0} u_r \beta_{rp}^k y_{rj} + \sum_{r\in R_k} u_r^k y_{rj}^k \leqslant \sum_{i\in I_0} v_i \alpha_{ip}^k x_{ij} + \sum_{i\in I_k} v_i^k x_{ij}^k, && \forall j, \forall k, \forall p, \\
& \sum_k \alpha_{ip}^k = 1, && \forall i \in I_0, \forall p, \\
& \sum_k \beta_{rp}^k = 1, && \forall r \in R_0, \forall p, \\
& u_r, u_r^k, v_i, v_i^k, \alpha_{ip}^k, \beta_{rp}^k \geqslant 0, && \forall i, r, k, \forall p.
\end{aligned}
$$

Dual form:

$$
\begin{aligned}
\max\ & \phi \\
s.t.\ & \sum_k \sum_j \lambda_j^k \alpha_{ip}^k x_{ij} \leqslant x_{ij_0}, \forall i \in I_0, \\
& \sum_j \lambda_j^k x_{ij}^k \leqslant x_{ij_0}^k, \forall k, \forall i \in I_k, \\
& \sum_k \sum_j \lambda_j^k \beta_{rp}^k y_{rj} \geqslant \phi y_{rj_0}, \forall r \in R_0, \\
& \sum_j \lambda_j^k y_{rj}^k \geqslant \phi y_{rj_0}^k, \forall k, \forall r \in R_k, \\
& \sum_k \alpha_{ip}^k = 1, \forall i \in I_0, \forall p, \\
& \sum_k \beta_{rp}^k = 1, \forall r \in R_0, \forall p, \\
& \lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned}
\tag{20}
$$

The objective functions of model (19) and (20) have been transformed with Lemma 3.1, the problem underlying these two models remains a bi-level programming problem. Note that after removing the dummy processes, the above dual model derived from the general parallel DEA model with both shared outputs and dedicated inputs is as follows.

$$
\begin{aligned}
&\max \ \phi \\
&s.t. \ \sum_k \sum_{N_p \in M_k} \sum_{j \in N_p} \lambda_j^k \alpha_{ip}^k x_{ij} \leqslant x_{ij_0}, \forall i \in I_0, \\
&\sum_{N_p \in M_k} \sum_{j \in N_p} \lambda_j^k x_{ij}^k \leqslant x_{ij_0}^k, \forall k, \forall i \in I_k, \\
&\sum_k \sum_{N_p \in M_k} \sum_{j \in N_p} \lambda_j^k \beta_{rp}^k y_{rj} \geqslant \phi y_{rj_0}, \forall r \in R_0, \\
&\sum_{N_p \in M_k} \sum_{j \in N_p} \lambda_j^k y_{rj}^k \geqslant \phi y_{rj_0}^k, \forall k, \forall r \in R_k, \\
&\sum_{k \in L_{N_p}} \alpha_{ip}^k = 1, \forall i \in I_0, \forall p, \\
&\sum_{k \in L_{N_p}} \beta_{rp}^k = 1, \forall r \in R_0, \forall p, \\
&\lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned} \tag{21}
$$

Model (19) is nonlinear in the current form. Similarly, following the linearization procedure shown in Section 3, using the usual transformation $t = 1/(\sum_{i \in I_0} v_i x_{ij_0} + \sum_k \sum_{i \in I_k} v_i^k x_{ij_0}^k)$ (see Charnes et al. [4]), and defining $\mu_r = tu_r$, $\mu_r^k = tu_r^k$, $\nu_i = tv_i$, $\nu_i^k = tv_i^k$, $\gamma_{ip}^k = tv_i \alpha_{ip}^k$, $\omega_{rp}^k = tu_r \beta_{rp}^k$, problem (19) reduces to a linear form.

6. Models for special situations

In the section above, we have developed a general DEA-based model with non-homogeneous DMUs. Below we will show that many models in the literature are special cases of our model.

6.1. Non-homogeneous outputs. Consider the problem discussed by Cook et al.[14] as an example for the situation that non-homogeneity arises due to non-homogeneous outputs. Following our denotation, the case in Cook et al.[14] only considers sharable inputs I_0 and specified outputs R_k. First four M.E. DMU groups are derived. Then,y_1, y_2, y_4 and y_6 are perceived to be outputs of different subsystems. If y_3 and y_5 form the outputs of one subsystem since there is no internal structure given in this case, the envelopment form of this problem can be derived from our general model.

$$
\begin{aligned}
&\max \ \phi \\
&s.t. \ \sum_k \sum_j \lambda_j^k \alpha_{ip}^k x_{ij} \leqslant x_{ij_0}, \forall i \in I_0, \\
&\sum_j \lambda_j^k y_{rj}^k \geqslant \phi y_{rj_0}^k, \forall k, \forall r \in R_k, \\
&\sum_k \alpha_{ip}^k = 1, \forall i \in I_0, \forall p, \\
&\lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned} \tag{22}
$$

TABLE 2. Product lines and DMU groups in Li et al. [24].

	x_1	x_2	x_3	x_4	x_5
N_1	×	×	×		
N_2	×	×	×	×	
N_3	×	×	×		×
N_4	×	×	×	×	×

6.2. Non-homogeneous inputs. Take the situation Li et al. (2016) considered as an example, the non-homogeneity arises when there exists an input configuration in a DMU and is different from the configuration of another DMU.

The methodology proposed in Li et al.[24] can be seen as a special case of our general model, where there exist sharable inputs I_0, specified inputs I_k and overlapping outputs R_0. To be more specific, the outputs y_1 and y_2 are overlapping outputs that can be generated from any subunits, x_1 and x_2 are sharable among all subunits, x_3, x_4 and x_5 are specified inputs allocated to specified production subunits as displayed in Li et al.[24]. The grouping according to our criteria is displayed in Table 2.

Applying our general model, the problem can be formulated as follows.

$$
\begin{aligned}
&\max\ \phi \\
&s.t.\quad \sum_k \sum_j \lambda_j^k \alpha_{ip}^k x_{ij} \leqslant x_{ij_0}, \forall i \in I_0, \\
&\sum_j \lambda_j^k x_{ij}^k \leqslant x_{ij_0}^k, \forall k, \forall i \in I_k, \\
&\sum_k \sum_j \lambda_j^k \beta_{rp}^k y_{rj} \geqslant \phi y_{rj_0}, \forall r \in R_0, \\
&\sum_k \alpha_{ip}^k = 1, \forall i \in I_0, \forall p, \\
&\sum_k \beta_{rp}^k = 1, \forall r \in R_0, \forall p, \\
&\lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned}
\tag{23}
$$

6.3. Non-homogeneous input-output subsystems. When partial input-to-output interactions exist, DMUs can still have a common input/output bundle, however, different input/output structure across the DMU set may give rise to the problem of non-homogeneity. That is, although the inputs and outputs of DMUs are homogeneous, some DMUs have different input-to-output relations than is true of others. Our general model can be successfully applied into this case.

When partial input-to-output interactions exist as shown in Table 3, Imanirad et al.[18] has extend their earlier work to allow for situations that the input/output profiles are non-homogeneous among DMUs. Imanirad et al.[18] first proceed by grouping DMUs into P M.E. groups which are the same with our methodology. The following step is to generate the subunit bundles of inputs and outputs. There is a significant difference in this step. The maximal set of input-to-output bundles denoted by (I_{k_p}, R_{k_p}) in Imanirad et al.[18] is derived within the given DMU group N_p.

TABLE 3. Product lines and DMU groups in Imanirad et al. [18].

	$x_1, x_2, x_3 \to y_1$	$x_1, x_2, x_3 \to y_1, y_2$	$x_1, x_2, x_3 \to y_1, y_3$	$x_1, x_3 \to y_2$	$x_2, x_3 \to y_3$
N_1	×			×	×
N_2			×	×	
N_3		×			×

TABLE 4. Product lines and DMU groups according our methodology.

	$x_1, x_2, x_3 \to y_1$	$x_1, x_2, x_3 \to y_2$	$x_1, x_2, x_3 \to y_3$	$x_1, x_3 \to y_2$	$x_2, x_3 \to y_3$
N_1	×			×	×
N_2	×		×	×	
N_3	×	×			×

In Imanirad et al.[18], N_2(Type 2) has the bundles given by $(I_{1_2}, R_{1_2}) = ((1,2,3), (1,3))$ and $(I_{2_2}, R_{2_2}) = ((1,3),(2))$. Note that there is no comparison among different types of DMUs referring to the common input-to-output impact $(I, R) = ((1,2,3),(1))$. For the reason that it is combined with $(I, R) = ((1,2,3),(3))$ to form the input-to-output bundle $(I_{1_2}, R_{1_2}) = ((1,2,3),(1,3))$ in N_2 , while in N_3, it is included in a different input-to-output bundle $(I_{1_2}, R_{1_2}) = ((1,2,3),(1,2))$. Actually, there is no clear configuration in this case, thus we argue that the combination of outputs such as y_1 and y_2 in N_3 can be relived. To avoid this kind of wrong grouping, the input-to-output bundles would be more appropriate to be derived within the whole DMU set as proposed in our methodology. According to our methodology, the grouping results are shown in Table 4.

The full production system is consisted of five subunits, and each subunit is included by exactly the same DMU profile. The type 1 network displayed in Imanirad et al.[18] is consisted of subunits 1, 4 and 5, the type 2 network is consisted of subunits 1, 3 and 4, finally, the type 3 network has subunit 1, 2 and 5. The model is formulated as follows.

$$
\begin{aligned}
&\max\ \phi \\
&s.t.\quad \sum_k \sum_j \lambda_j^k \alpha_{ip}^k x_{ij} \leqslant x_{ij_0}, \forall i \in I_0, \\
&\qquad \sum_j \lambda_j^k y_{rj}^k \geqslant \phi y_{rj_0}^k, \forall k, \forall r \in R_k, \\
&\qquad \sum_k \alpha_{ip}^k = 1, \forall i \in I_0, \forall p, \\
&\qquad \lambda_j^k \geqslant 0, \forall k, \forall j.
\end{aligned} \tag{24}
$$

7. Discussions and further directions

In this paper, we introduce a general parallel DEA model which encompasses the YMK model and Kao's model in order to provide a unified framework for DEA-based models with non-homogeneity and multi-activity processes. Although these two problems have always been two different subjects referring to relaxing two various assumptions, we show that these formulations are both extended by embedding the corresponding process into the general parallel DEA model. And by applying the general parallel DEA model, we propose a general model which can deal with various aspects of non-homogeneity, including non-homogeneous inputs, outputs and input-to-output relations.

In the current paper, we assume that the sharable inputs and outputs are equally important and compensable across all subunits. This assumption could be relaxed when the internal structures of DMUs are given and decision makers put different significance to the same type of inputs/outputs in different subunits. Hence, an important area for future research is considering the more general form of parallel DEA models, consequently, more general multi-activity models and DEA-based model with non-homogeneous input/output processes can be extended.

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