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APPROXIMATION OF THE LONG-TERM DYNAMICS OF THE DYNAMICAL SYSTEM GENERATED BY A 3D NS-α SYSTEM WITH PHASE TRANSITION

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Abstract. In this article we study an approximate model for a binary fluid flow in a three-dimensional bounded domain. The governing equations consist of the Allen–Cahn equation for the order (phase) parameter ϕ coupled with the Navier–Stokes-α (NS-α) system for the velocity u. We discretize these equations in time using the implicit Euler scheme and we prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero.

Key words. Navier–Stokes-α, phase transition, attractors, implicit Euler scheme, Gronwall Lemma.

1. Introduction

It is well known that the incompressible Navier–Stokes equations govern the motion of single-phase fluids, such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is, fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [23]. For instance, this approach is used in [3] to describe cavitation phenomena in a flowing liquid. The model consists of the Navier–Stokes equations coupled with the phase-field system, [4, 23, 22, 24]. In the isothermal compressible case, the existence of a global weak solution is proved in [18]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity u and the order parameter ϕ. This system can be written as a Navier–Stokes equation coupled with a convective Allen–Cahn equation, [23]. The associated initial and boundary value problem was studied in [23], in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor $\mathcal{A}$. They also established the existence of an exponential attractor $\mathcal{E}$. This entails that $\mathcal{A}$ has a finite fractal dimension, which is estimated in [23] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated, although some important issues remain unresolved, [40]. In the case of binary fluids, the analysis is even more complicated and the mathematical study is still at its infancy, as noted in [23].

In this article we study an approximate model for a binary fluid flow in a three-dimensional bounded domain. The model is derived from the 3D coupled Allen–Cahn–Navier–Stokes system by substituting the 3D Navier–Stokes system with the 3D NS-α equations. This model can be considered as a regularized approximation of the 3D coupled Allen–Cahn–Navier–Stokes system, depending on a small positive parameter $\alpha > 0$, where in some terms, the unknown velocity function v is replaced

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by a smoother function u, solution of the elliptic system $v = u - \alpha^2 \Delta u$. For $\alpha = 0$, the model reduces to the exact 3D coupled Allen–Cahn–Navier–Stokes system.

Since the uniqueness theorem for the global weak solutions (or the global existence of strong solutions) of the initial-value problem of the 3D coupled Allen–Cahn–Navier–Stokes system is not proved yet, the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D coupled Allen–Cahn–Navier-Stokes system. This situation is the same for the 3D Navier–Stokes systems. Using regular approximation equations to study the classical 3D Navier–Stokes systems has become an effective tool both from the numerical and the theoretical point of views. It is well-known that direct numerical simulation of the 3D NSE for many physical applications with high Reynolds number flows is intractable even using state-of-the-art numerical methods on the most advanced supercomputers available nowadays. Recently, many applied mathematicians have developed regularized turbulence models for the 3D NSE as an attempt to overcome this simulation barrier. Their aim is to capture the large, energetic eddies without having to compute the smallest dynamically relevant eddies, by instead modelling the effects of small eddies in terms of the large scales in the 3D NSE. Since 1998, many such regularized models have been proposed, tested and investigated from both the numerical and the mathematical point of views. Among these models, one can find the globally well-posed 3D Navier–Stokes-α (NS-α) equations (also known as the viscous Camassa–Holm equations and Lagrangian averaged Navier–Stokes-α model), the 3D Leray-α models, the modified 3D Leray-α models, the simplified 3D Bardina models, the 3D Navier–Stokes–Voight (NSV) equations, and their inviscid counterparts. As noted in [14], it was demonstrated analytically and numerically that the NS-α model gives a good approximation in the study of many problems related to turbulent flows. In particular, it was found that the explicit steady analytical solution of the NS-α model compares successfully with empirical and numerical experiment data for a wide range of Reynolds numbers in turbulent channel and pipe flows, [14]. Let us recall that the inviscid 3D NS-α equations were first proposed in [19]. As described in [19, 33], the 3D NS-α equations are a systems of partial differential equations for the mean velocity in which a nonlinear dispertive mechanism filters the small scales. As such, the 3D NS-α equations serve as an appropriate model for turbulent flows and a suitable approximation of the 3D Navier–Stokes, as documented in [9, 11, 10, 12, 31, 30, 33, 25, 26, 27, 13, 29]. A successful comparison with data for time-averaged quantities for a wide range of Reynolds numbers in turbulent channel and pipe flows was done in [9, 11]. Further studies of the 3D NS-α models in the context of turbulence modeling appear in [15, 32, 36]. Analytical studies of the global existence, uniqueness and regularity of solutions to the 3D NS-α system are performed in [19] in the case of periodic boundary conditions. Some existence and uniqueness results are also established in [33, 7, 8, 6]. In [33], the authors prove the global well-posedness and regularity of the 3D NS-α equations in a bounded domain with a non-slip boundary condition. A non-autonomous NS-α model is considered in [7], where the authors study the asymptotic behavior of the solutions of a 3D NS-α with delay forces. They prove the existence of a pullback and forward attractors for the model. The stochastic version is also studied in [6].

Motivated by the above works and the fact that a full mathematical theory of the 3D coupled Allen–Cahn–Navier–Stokes system is still lacking, the author in [34] studied an approximate model for a binary fluid flow in a three dimensional bounded domain. The governing equations consist of the Allen–Cahn equations for the order (phase) parameter ϕ coupled with the NS-α system for the velocity u. He analyzed the asymptotic behavior of the solution to the associated initial and boundary value problem. In particular, he proved that the system generates a strongly continuous semigroup on a suitable phase space, which possesses a global attractor $\mathcal{A}$. Then he established the existence of an exponential attractor $\mathcal{E}$, which entails that $\mathcal{A}$ has finite fractal dimension. In [20], the authors study the relations between the long-time dynamics of the 3D Allen–Cahn–NS-α model and the exact 3D Allen–Cahn–Navier–Stokes system. Proceeding in the same spirit as in [14], it is proved in [20] that bounded set of solutions of the Allen–Cahn–NS-α model converge to the trajectory attractor of the 3D Allen–Cahn–Navier–Stokes system as time goes to $+\infty$ and α approaches 0^+. It is also proved that the trajectory attractors of the 3D Allen–Cahn–NS-α model converges to the trajectory attractor of the 3D Allen–Cahn–Navier–Stokes as α approaches 0^+. Moreover, assuming the potential to be real analytic, it is demonstrated in [20] that in absence of external forces, each trajectory converges to a single equilibrium and we obtain a convergence rate estimate.

In this article, we study the numerical approximation of the 3D Allen–Cahn–Navier–Stokes considered in [34, 20]. We discretize the model in time using the implicit Euler scheme and with the aid of the discrete Gronwall lemma and of the discrete uniform Gronwall lemma we prove that the approximate solution is uniformly bounded in a suitable space. Using the theory of the so-called multi-valued attractors, we also prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero.

The article is structured as follows. In the next section, we recall from [34, 20] the 3D NS-α model with phase transition and its mathematical setting, including the weak formulation of the associated initial and boundary value problem. In Subsection 3.1, following some ideas of [23], we prove that the approximate solution is uniformly bounded in a suitable phase space $\mathbb{Y}$. This allows us to prove in Subsection 3.2 that the implicit Euler scheme is uniformly bounded in a Hilbert space $\mathbb{V}$, dense and compactly embedded in $\mathbb{Y}$. Using the tools developed in [16] (see also, [17, 28]), in Section 4 we study the convergence of the discrete (multi-valued) attractors to the continuous (single-valued) attractor. For convenience, we recall those results in Subsection 4.1, and then we apply them to the two-phase flow model in Subsection 4.2.

2. A 3D NS-α model with phase transition and its mathematical setting

2.1. Governing equations. We assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\mathbb{R}^3$. The state of the system is described by a pair (u, ϕ), where $u = (u_1, u_2, u_3)$ is the velocity field of the fluid and ϕ is the order parameter. Let $A_0 u$ denote the Stokes operator. The system of equations for (u, ϕ) reads:

$$
\left\{
\begin{aligned}
&\frac{\partial}{\partial t}(u - \alpha^2 \Delta u) + \nu_1 (A_0 u - \alpha^2 \Delta (A_0 u)) + (u \cdot \nabla)(u - \alpha^2 \Delta u) \\
&\qquad\qquad -\alpha \nabla u^* \cdot \Delta u + \nabla p - \mathcal{K} \mu \nabla \phi = g, \\
&\operatorname{div} u = 0, \\
&\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi + \mu = 0, \\
&\mu = -\nu_2 \Delta \phi + \alpha_0 f(\phi),
\end{aligned}
\right.
\tag{1}
$$

with appropriate initial and boundary conditions.

Let us first note that (1) is obtained by coupling the well-known Allen–Cahn equations

$$\frac{\partial \phi}{\partial t} = -(-\nu_2 \Delta \phi + \alpha_0 f(\phi)) \tag{2}$$

with the 3D NS-α system through convection and order parameter.

In (1), the unknown functions are the velocity $u = (u_1, u_2, u_3)$ of the fluid, its pressure p and the order (phase) parameter ϕ. The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_{\mathcal{M}} \left(\frac{\nu_2}{2} |\nabla \phi|^2 + \alpha_0 F(\phi) \right) ds, \tag{3}$$

where, e.g., $F(r) = \int_0^r f(\zeta) d\zeta$. Here, the constants $\nu_1 > 0$ and $\mathcal{K} > 0$ correspond to the kinematic viscosity of the fluid and the capillarity (stress) coefficient respectively, $\nu_2,\ \alpha_0 > 0$ are two physical parameters describing the interaction between the two phases. In particular, ν_2 is related to the thickness of the interface separating the two fluids. Hereafter, as in [23], we assume that $\nu_2 \leq \alpha_0$. For $\alpha = 0$, the model (1) reduces to the 3D version of coupled Allen–Cahn–Navier–Stokes studied in [23]. Let us recall from [19] that the positive constant α represents the square of the spacial scale at which the fluid motion is filtered.

We endow (1) with the boundary condition

$$u = A_0 u = 0,\ \frac{\partial \phi}{\partial \eta} = 0 \text{ on } \partial\mathcal{M} \times (0, +\infty), \tag{4}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and η is its outward normal. The initial condition is given by

$$(u, \phi)(0) = (u_0, \phi_0) \text{ in } \mathcal{M}. \tag{5}$$

2.2. Mathematical setting. We first present a weak formulation of (1)–(5). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class $\mathcal{C}^2$). As in [23], we assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies

$$\begin{cases} \lim\limits_{|r| \to +\infty} f'(r) > 0, \\ |f'(r)| \leq c_f (1 + |r|^2),\ \forall r \in \mathbb{R}. \end{cases} \tag{6}$$

where c_f is some positive constant. It follows from (6) that

$$|f(r)| \leq c_f (1 + |r|^3),\ \forall r \in \mathbb{R}. \tag{7}$$

Note that from (6), we can find $\gamma > 0$ such that

$$\lim_{|r| \to +\infty} f'(r) > 2\gamma > 0. \tag{8}$$

Let us denote by $L^p(\mathcal{M})$ $(1 \leq p < \infty)$ the classic L^p space with the norm

$$|\Phi|_p = \left(\int_{\mathcal{M}} |\Phi(x, y, z)|^p dx dy dz \right)^{1/p},\ \forall \Phi \in L^p(\mathcal{M}). \tag{9}$$

We also denote by $H^m(\mathcal{M})$ $(m \geq 1)$ the classic Sobolev space of square-integrable functions with square-integrable derivatives up to order m.

We will denote by $(\cdot, \cdot)_{L^2}$ and $|\cdot|_{L^2}$, respectively, the scalar product and associated norm in $(L^2(\mathcal{M}))^3$, and by $(\nabla u, \nabla v)_{L^2}$ the scalar product in $(L^2(\mathcal{M}))^3$ of the gradients of u and v. We consider the scalar product in $(H_0^1(\mathcal{M}))^3$ defined by

$$((u, v)) = (u, v)_{L^2} + \alpha^2 (\nabla u, \nabla v)_{L^2},\ \forall u, v \in (H_0^1(\mathcal{M}))^3, \tag{10}$$

where its associated norm, which is equivalent to the usual gradient norm, will be denoted by $\|\cdot\|$.

Let H be the closure in $(L^2(\mathcal{M}))^3$ of the set $\mathcal{V} = \{v \in (\mathcal{D}(\mathcal{M}))^3 : \nabla\cdot v = 0 \text{ in } \mathcal{M}\}$, and by V the closure of $\mathcal{V}$ in $(H_0^1(\mathcal{M}))^3$.

We denote by A_0 the Stokes operator, with domain $D(A_0) = (H^2(\mathcal{M}))^3 \cap V$, defined by

$$A_0 w = -\mathcal{P}(\Delta w), \ \forall w \in D(A_0),$$

where $\mathcal{P}$ is the Leray operator, i.e., the projection operator from $(L^2(\mathcal{M}))^3$ onto H.

Recall that since $\partial\mathcal{M}$ is Lipschitz, $|A_0 w|_{L^2}$ defines in $D(A_0)$ a norm which is equivalent to the $(H^2(\mathcal{M}))^3$-norm, and thus $D(A_0)$ is a Hilbert space with the scalar product

$$(v, w)_{D(A_0)} = (A_0 v, A_0 w).$$

Hereafter, we set $\mathcal{H} = V$, with the scalar product $(u, v)_{\mathcal{H}} = ((u, v))$, and $\mathcal{U} = D(A_0)$, with the scalar product $((u, v))_{\mathcal{U}} = (A_0 u, A_0 v)$. We also denote by $|\cdot|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{U}}$ the associated norms defined on $\mathcal{H}$ and $\mathcal{U}$ respectively.

Then, $\mathcal{H}$ and $\mathcal{U}$ are two real and separable Hilbert spaces such that $\mathcal{U} \subset \mathcal{H}$, with the injection being compact and dense. We will identify $\mathcal{H}$ with its topological dual $\mathcal{H}^*$, by considering $\mathcal{U}$ as a subspace of $\mathcal{H}^*$, and we identify $v \in \mathcal{U}$ with the element $f_v \in \mathcal{H}^*$ given by

$$f_v(w) = (v, w)_{\mathcal{H}}, \ \forall w \in \mathcal{H}.$$

We will denote by $\|\cdot\|_{\mathcal{U}^*}$ the norm of $\mathcal{U}^*$, and by $\langle\cdot,\cdot\rangle$, the duality product between $\mathcal{U}^*$ and $\mathcal{U}$.

Now, we define the operator A by

$$\langle Au, v\rangle = \langle A_0 u, v\rangle + \alpha^2 \langle A_0 u, A_0 v\rangle, \ \forall u, v \in D(A_0). \tag{11}$$

Then, we have (see [8, 7])

$$A \in \mathcal{L}(\mathcal{U}, \mathcal{U}^*),$$

$$A \text{ is self-adjoint,} \tag{12}$$

$$\text{there exists } \alpha_1 > 0, \text{ such that } \langle Au, u\rangle \geq \alpha_1 \|u\|_{\mathcal{U}}^2, \ \forall u \in \mathcal{U}.$$

Remark 2.1. Owing to the properties of A, we define

$$((u, v))_A = \langle Au, v\rangle, \ \forall u, v \in \mathcal{U}.$$

It is clear that $((\cdot,\cdot))_A$ is a scalar product in $\mathcal{U}$ whose associated norm is equivalent to the usual norm $\|\cdot\|_{\mathcal{U}}$. From now on, without loss of generality, we simply set

$$((u, v))_{\mathcal{U}} = \langle Au, v\rangle, \ \forall u, v \in \mathcal{U} \text{ and } \|u\|_{\mathcal{U}}^2 = \langle Au, u\rangle. \tag{13}$$

It then follows that

$$\lambda_1 |u|_{\mathcal{H}}^2 \leq \|u\|_{\mathcal{U}}^2, \ \forall u \in \mathcal{U}, \tag{14}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator A.

We also note that

$$\langle u + \alpha^2 A_0 u, u\rangle = |u|_{L^2}^2 + \alpha^2 |\nabla u|_{L^2}^2 = |u|_{\mathcal{H}}^2, \ \forall u \in \mathcal{H}, \tag{15}$$

$$\begin{aligned} \langle A_0 u + \alpha^2 A_0(A_0 u), u\rangle &= |\nabla u|_{L^2}^2 + \alpha^2 |A_0 u|_{L^2}^2 = \|u\|_{\mathcal{U}}^2 \\ &= \langle Au, u\rangle, \ \forall u \in \mathcal{U}, \text{ with } A_0 u = 0 \text{ on } \partial\mathcal{M}, \end{aligned} \tag{16}$$

$$\langle u + \alpha^2 A_0 u, A_0 u\rangle = |\nabla u|_{L^2}^2 + \alpha^2 |A_0 u|_{L^2}^2 = \|u\|_{\mathcal{U}}^2, \ \forall u \in \mathcal{U}, \tag{17}$$

$$\begin{aligned}\langle A_0u+\alpha^2A_0(A_0u),A_0u\rangle &= |A_0u|^2_{L^2}+\alpha^2\|A_0u\|^2\\ &= |A_0u|^2_{\mathcal{H}},\ \forall u\in D(A),\ \text{with } A_0u=0 \text{ on } \partial\mathcal{M}.\end{aligned}\tag{18}$$

We define the linear positive unbounded operator A_γ on $L^2(\mathcal{M})$ by:

$$A_\gamma\phi=-\Delta\phi+\gamma\phi,\ \forall\phi\in D(A_\gamma),\tag{19}$$

where

$$D(A_\gamma)=\left\{\rho\in H^2(\mathcal{M});\ \frac{\partial\rho}{\partial\eta}=0 \text{ on } \partial\mathcal{M}\right\}.$$

Note that A_γ^{-1} is a compact linear operator on $L^2(\mathcal{M})$ and $|A_\gamma\cdot|_{L^2}$ is a norm on $D(A_\gamma)$ that is equivalent to the H^2-norm. Hereafter, we will set

$$\|\psi\|^2_\gamma=(A_\gamma\psi,\psi)_{L^2}=|\nabla\psi|^2_{L^2}+\gamma|\psi|^2_{L^2},\ \forall\psi\in D(A_\gamma^{1/2}).\tag{20}$$

2.3. Some properties of the nonlinear term. We first recall from [8, 7] some properties of the nonlinear term

$$(u\cdot\nabla)(u-\alpha^2\Delta u)-\alpha\nabla u^*\cdot\Delta u$$

that appears in (1).

For $u\in D(A_0)$ and $v\in(L^2(\mathcal{M}))^3$, we define $(u\cdot\nabla)v$ as the element of $(H^{-1}(\mathcal{M}))^3$ given by

$$\langle(u\cdot\nabla)v,w\rangle=\sum_{i,j=1}^3\langle\partial_iv_j,u_iw_j\rangle=-\sum_{i,j=1}^3\int_{\mathcal{M}}v_j(w_j\partial_iu_i+u_i\partial_iw_j)dx,\ \forall w\in(H^1_0(\mathcal{M}))^3.\tag{21}$$

We can check that there exists a constant $c_1>0$ depending only on $\mathcal{M}$, such that

$$|\langle(u\cdot\nabla)v,w\rangle|\le c_1|A_0u|_{L^2}|v|_{L^2}\|w\|,\ \forall(u,v,w)\in D(A_0)\times(L^2(\mathcal{M}))^3\times(H^1_0(\mathcal{M}))^3.\tag{22}$$

If $u\in D(A_0)$, then $\nabla u^*\in(H^1(\mathcal{M}))^{3\times3}$ and for $v\in(L^2(\mathcal{M}))^3$, we have $\nabla u^*\cdot v\in H^{-1}(\mathcal{M}))^3$, with

$$\langle\nabla u^*\cdot v,w\rangle=\sum_{i,j=1}^3\int_{\mathcal{M}}(\partial_ju_i)v_iw_jdx,\ \forall w\in(H^1_0(\mathcal{M}))^3.\tag{23}$$

We can check that there exists a constant $c_2>0$ depending only on $\mathcal{M}$, such that

$$|\langle\nabla u^*\cdot v,w\rangle|\le c_2|A_0u|_{L^2}|v|_{L^2}\|w\|,\ \forall(u,v,w)\in D(A_0)\times(L^2(\mathcal{M}))^3\times(H^1_0(\mathcal{M}))^3.\tag{24}$$

We now consider the trilinear form $b^\sharp$ defined by

$$b^\sharp(u,v,w)=\langle(u\cdot\nabla)v,w\rangle+\langle\nabla u^*\cdot v,w\rangle,\ \forall(u,v,w)\in D(A_0)\times(L^2(\mathcal{M})^3)\times(H^1_0(\mathcal{M}))^3.\tag{25}$$

The following result is proved in [8, 7].

Proposition 1. The trilinear form $b^\sharp$ satisfies

$$b^\sharp(u,v,w)=-b^\sharp(w,v,u),\ \forall(u,v,w)\in D(A_0)\times(L^2(\mathcal{M}))^3\times D(A_0),\tag{26}$$

and consequently

$$b^\sharp(u,v,u)=0,\ \forall(u,v)\in D(A_0)\times(L^2(\mathcal{M}))^3.\tag{27}$$

Moreover, there exists a constant $c>0$, depending only on $\mathcal{M}$, such that

$$|b^\sharp(u,v,w)|\le c|A_0u|_{L^2}|v|_{L^2}\|w\|,\ \forall(u,v,w)\in D(A_0)\times(L^2(\mathcal{M}))^3\times(H^1_0(\mathcal{M}))^3,\tag{28}$$

and

(29)
$$|b^\sharp(u,v,w)| \le c\|u\||v|_{L^2}|A_0 w|_{L^2},\ \forall (u,v,w) \in (H_0^1(\mathcal{M}))^3 \times (L^2(\mathcal{M}))^3 \times D(A_0).$$

Thus, in particular, $b^\sharp$ is continuous on $D(A_0) \times (L^2(\mathcal{M}))^3 \times (H_0^1(\mathcal{M}))^3$.

We also define the bilinear operator B_0 and the associated trilinear form b_0 by

$$\begin{aligned} &\langle B_0(u,v),w\rangle = b^\sharp(u, v+\alpha^2 A_0 v, w),\ \forall u,v,w \in D(A_0),\\ &b_0(u,v,w) = \langle B_0(u,v),w\rangle,\ \forall u,v,w \in \mathcal{U}. \end{aligned} \tag{30}$$

Then, the following properties are proved in [7], (see also [6, 8]).

$$b_0(u,v,u) = 0,\ \forall u,v \in \mathcal{U}, \tag{31}$$

$$|b_0(u,v,w)| \le c_b |u|_{\mathcal{H}} \|v\|_{\mathcal{U}} \|w\|_{\mathcal{U}},\ \forall u,v,w \in \mathcal{U}, \tag{32}$$

$$|b_0(u,v,w)| \le c_b \|u\|_{\mathcal{U}} \|v\|_{\mathcal{U}} |w|_{\mathcal{H}},\ \forall u,v,w \in \mathcal{U}. \tag{33}$$

We introduce the bilinear operator B_1 (and its associated trilinear forms b_1) as well as the coupling mapping R_0, which are defined from $D(A_0) \times D(A_\gamma)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times D(A_\gamma^{3/2})$ into H, respectively. More precisely, we set

(34)
$$\begin{aligned} (B_1(u,\phi),\rho)_{L^2} &= \int_{\mathcal{M}} [(u\cdot\nabla)\phi]\rho dx = b_1(u,\phi,\rho),\ \forall u \in D(A_0),\ \phi,\rho \in D(A_\gamma),\\ (R_0(\mu,\phi),w)_{L^2} &= \int_{\mathcal{M}} \mu[\nabla\phi\cdot w]dx\\ &= b_1(w,\phi,\mu),\ \forall w \in D(A_0),\ (\mu,\phi) \in L^2(\mathcal{M}) \times D(A_\gamma^{3/2}). \end{aligned}$$

Note that

$$R_0(\mu,\phi) = \mathcal{P}\mu\nabla\phi,$$

$$b_1(u,\phi,\phi) = 0, \quad b_1(u,\phi,f_\gamma(\phi)) = 0, \tag{35}$$

$$b_1(u,\phi,\psi) = -b_1(u,\psi,\phi), \quad \forall\, u \in D(A), \phi,\psi \in D(A_\gamma). \tag{36}$$

We also have the following continuity properties of the trilinear form b_1:

Lemma 1. There exists a constant $\tilde{c_b} > 0$, depending only on $\mathcal{M}$, such that

$$|b_1(u,\phi,\psi)| \le \tilde{c_b} |u|_{\mathcal{H}}^{1/2} \|u\|_{\mathcal{U}}^{1/2} \|\phi\|^{1/2} |A_\gamma\phi|_{L^2}^{1/2} |\psi|_{L^2}, \tag{37}$$

$$|b_1(u,\phi,\psi)| \le \tilde{c_b} |u|_{\mathcal{H}} \|\phi\|^{1/2} |A_\gamma\phi|_{L^2}^{1/2} |\psi|_{L^2}, \forall u \in D(A_0),\ \phi,\rho \in D(A_\gamma). \tag{38}$$

Proof. Using Holder's inequality, we find

$$|b_1(u,\phi,\psi)| = \left|\int_{\mathcal{M}} [(u\cdot\nabla)\phi]\psi dx\right| \le |u|_{L^6} |\nabla\phi|_{L^3} |\psi|_{L^2}. \tag{39}$$

By the Gagliardo–Nirenberg interpolation inequality, we have

$$|u|_{L^6} \le c|u|_{L^2}^{1/2} \|u\|_{H^2}^{1/2}, \tag{40}$$

$$|\gamma|_{L^3} \le c|\gamma|_{L^2}^{1/2} \|\gamma\|_{H^1}^{1/2}, \tag{41}$$

and thus (39) gives

$$|b_1(u,\phi,\psi)| \le c|u|_{L^2}^{1/2} \|u\|_{H^2}^{1/2} |\nabla\phi|_{L^2}^{1/2} \|\nabla\phi\|_{H^1}^{1/2} |\psi|_{L^2}, \tag{42}$$

from which (37) follows right away.

Inequality (38) follows from (39), (41) and from the Sobolev imbedding $H^1 \hookrightarrow L^6$. □

Now we define the Hilbert spaces $\mathbb{Y}$ and $\mathbb{V}$ by

$$\mathbb{Y} = \mathcal{H} \times H^1(\mathcal{M}), \quad \mathbb{V} = \mathcal{U} \times D(A_\gamma), \tag{43}$$

endowed with the scalar products whose associated norms are

$$\begin{aligned} |(u,\phi)|^2_{\mathbb{Y}} &= \mathcal{K}^{-1}|u|^2_{\mathcal{H}} + \nu_2(|\nabla\phi|^2_{L^2} + \gamma|\phi|^2_{L^2}) =: \mathcal{K}^{-1}|u|^2_{\mathcal{H}} + \nu_2\|\phi\|^2_\gamma, \\ \|(u,\phi)\|^2_{\mathbb{V}} &= \|u\|^2_{\mathcal{U}} + |A_\gamma\phi|^2_{L^2}. \end{aligned} \tag{44}$$

We also set

$$f_\gamma(r) = f(r) - \alpha_0^{-1}\nu_2\gamma r \tag{45}$$

and observe that f_γ still satisfies (8) with γ in place of 2γ since $\nu_2 \le \alpha_0$. Also its primitive $F_\gamma(r) = \int_0^r f_\gamma(\zeta)d\zeta$ is bounded from below.

Throughout this article, we will denote by c a generic positive constant depending on the domain $\mathcal{M}$.

Using the notations above, we rewrite (1)–(4) as (see [23] for details)

$$\begin{cases} \dfrac{d\tilde{u}}{dt} + \nu_1 Au + B_0(u,u) - \mathcal{K}R_0(\nu_2 A_\gamma\phi, \phi) = g, \quad \text{a.e., in } \mathcal{M}\times(0,+\infty), \\ \tilde{u} = u + \alpha^2 A_0 u, \quad \text{a.e., in } \mathcal{M}\times(0,+\infty), \\ \mu = \nu_2 A_\gamma\phi + \alpha_0 f_\gamma(\phi), \quad \text{a.e., in } \mathcal{M}\times(0,+\infty), \\ \dfrac{d\phi}{dt} + \mu + B_1(u,\phi) = 0, \quad \text{a.e., in } \mathcal{M}\times(0,+\infty). \end{cases} \tag{46}$$

Remark 2.2. In the weak formulation (46), the term $\mu\nabla\phi$ is replaced by $\nu_2 A_\gamma\phi\nabla\phi$. This is justified since $f'_\gamma(\phi)\nabla\phi$ is the gradient $F_\gamma(\phi)$ and can be incorporated into the pressure gradient, see [23] for details.

Definition 2.1. Suppose that $(u_0,\phi_0) \in \mathbb{Y}$, $g \in L^2(0,T;\mathcal{U}^*)$ and $T > 0$. A pair (u,ϕ) is called a weak solution to (46), (5) on $[0,T]$ if it satisfies (46), (5) in a weak sense on $[0,T]$ and

$$(u,\phi) \in \mathcal{C}([0,T];\mathbb{Y}) \cap L^2([0,T];\mathbb{V}), \ \frac{du}{dt} \in L^2([0,T];\mathcal{U}^*), \ \frac{d\phi}{dt}, \ \mu \in L^2([0,T];L^2(\mathcal{M})). \tag{47}$$

If $(u_0,\phi_0) \in \mathbb{V}$, a weak solution (u,ϕ) is called a strong solution on the time interval $[0,T]$ if in addition to (47), it satisfies

$$u \in \mathcal{C}([0,T];\mathcal{U}) \cap L^2(0,T;D(A)), \ \phi \in \mathcal{C}([0,T];D(A_\gamma)) \cap L^2(0,T;D(A_\gamma^{3/2})). \tag{48}$$

Remark 2.3. To justify the first part of (46), we note that if (u,ϕ) is a weak solution to (46) satisfying (47), then we have $B_0(u,u) \in L^2([0,\infty);\mathcal{U}^*)$, $Au \in L^2([0,\infty);\mathcal{U}^*)$. Moreover, we have

$$\begin{aligned} |(R_0(A_\gamma\phi,\phi),v)_{L^2}| &= |b_1(v,\phi,A_\gamma\phi)| \\ &\le c|v|_{L^\infty}\|\phi\||A_\gamma\phi|_{L^2} \le c\|v\|_{\mathcal{U}}\|\phi\||A_\gamma\phi|_{L^2}, \end{aligned} \tag{49}$$

therefore $R_0(A\phi,\phi) \in L^2([0,\infty);\mathcal{U}^*)$.

Remark 2.4. The existence and uniqueness of weak and strong solutions to (46) is proved in [34], (see also [21], where the convergence of the solutions as α goes to zero is studied). In particular, the author in [34] proved that the system (46) has a unique solution $(u,\phi) \in C(0,T;\mathbb{Y}) \cap L^2(0,T;\mathbb{V})$.

We define the continuous (nonlinear) semigroup from $\mathbb{Y}$ to $\mathbb{Y}$ by:

$$S(t)(u_0,\phi_0) = (u(t),\phi(t)), \tag{50}$$

where (u,ϕ) is the solution to (46) with the initial condition (u_0,ϕ_0). The following result is proved in [34].

Proposition 2. The semi-group $S(t)$ from $\mathbb{Y}$ to $\mathbb{Y}$ associated with (46) is such that
i) There exists bounded absorbing sets in $\mathbb{Y}$ and in $\mathbb{V}$.
ii) There exists a maximal attractor $\mathcal{A}$ which is compact and connected in $\mathbb{Y}$ and bounded in $\mathbb{V}$. Its basin of attraction is the whole space $\mathbb{Y}$.

3. Implicit Euler scheme for (46)

In this article we discretize (46) using the fully implicit Euler scheme,

$$\begin{cases} \frac{\tilde{u}^n-\tilde{u}^{n-1}}{k} + \nu_1 A u^n + B_0(u^n,u^n) - \mathcal{K}R_0(\nu_2 A_\gamma \phi^n,\phi^n) = g^n, \\ \tilde{u}^n = u^n + \alpha^2 A_0 u^n, \\ \mu^n = \nu_2 A_\gamma \phi^n + \alpha_0 f_\gamma(\phi^n), \\ \frac{\phi^n-\phi^{n-1}}{k} + \mu^n + B_1(u^n,\phi^n) = 0, \\ u^0 = u_0, \phi^0 = \phi_0, \end{cases} \tag{51}$$

and prove that the attractors generated by the above system converge to the attractor generated by the continuous system (46) as the time-step converges to zero. To prove the existence of the discrete attractors we need to use the theory of the multi-valued attractors, that we discuss in Subsection 4.1. The need of employing the theory of the multi-valued attractors comes from the fact that the uniqueness of the solution of the system (51) can be proved under the restriction $k \le \kappa(\|(u_0,\phi_0)\|_{\mathbb{V}})$, for some $\kappa(\|(u_0,\phi_0)\|_{\mathbb{V}})$ depending on the initial data. To see that, we let (u^n,ϕ^n) and (v^n,ψ^n) be two solutions corresponding to the same initial data $(u_0,\phi_0) \in \mathbb{V}$. Letting $w^n = u^n - v^n$ and $\theta^n = \phi^n - \psi^n$, we note that (w^n,θ^n) is a solution of the system

$$\begin{cases} \tilde{w}^n + \nu_1 k A \tilde{w}^n + k\left[B_0(u^n,u^n) - B_0(v^n,v^n)\right] \\ \qquad\qquad -\mathcal{K}k\left[R_0(\nu_2 A_\gamma\phi^n,\phi^n) - R_0(\nu_2 A_\gamma \psi^n,\psi^n)\right] = 0, \\ \tilde{w}^n = w^n + \alpha^2 A_0 w^n, \\ \theta^n + \nu_2 k A_\gamma \theta^n + \alpha_0 k f_\gamma(\phi^n) - \alpha_0 k f_\gamma(\psi^n) + k\left[B_1(u^n,\phi^n) - B_1(v^n,\psi^n)\right] = 0, \end{cases} \tag{52}$$

which can be rewritten as

$$\begin{cases} \tilde{w}^n + \nu_1 k A \tilde{w}^n + k\left[B_0(w^n,u^n) + B_0(v^n,w^n)\right] \\ \qquad\qquad -\mathcal{K}k\left[R_0(\nu_2 A_\gamma\theta^n,\phi^n) + R_0(\nu_2 A_\gamma \psi^n,\theta^n)\right] = 0, \\ \tilde{w}^n = w^n + \alpha^2 A_0 w^n, \\ \theta^n + \nu_2 k A_\gamma \theta^n + \alpha_0 k f_\gamma(\phi^n) - \alpha_0 k f_\gamma(\psi^n) + k\left[B_1(w^n,\phi^n) + B_1(v^n,\theta^n)\right] = 0. \end{cases} \tag{53}$$

Taking the scalar product of the first equation with w^n, of the third equation with $\nu_2 A_\gamma \theta^n$ in $L^2(\Omega)$, integrating over Ω and adding the resulting equations, we obtain (using (31)):

$$\begin{aligned} \left(|w^n|^2_{\mathcal{H}} + \nu_2\|\theta^n\|^2_\gamma\right) &+ k\left(\nu_1\|w^n\|^2_{\mathcal{U}} + \nu_2^2|A_\gamma\theta^n|^2_{L^2}\right) + kb_0(v^n,w^n,w^n) \\ &- \mathcal{K}k\left[b_1(w^n,\phi^n,\nu_2 A_\gamma\theta^n) + b_1(w^n,\theta^n,\nu_2 A_\gamma\psi^n)\right] \\ &+ k\left[b_1(w^n,\phi^n,\nu_2 A_\gamma\theta^n) + b_1(v^n,\theta^n,\nu_2 A_\gamma\theta^n)\right] \\ &+ \alpha_0\nu_2 k\left(f_\gamma(\phi^n) - f_\gamma(\psi^n), A_\gamma\theta^n\right)_{L^2} = 0, \end{aligned} \tag{54}$$

Using (33) and (37), we bound the nonlinear terms are bounded as follows:

$$\begin{aligned} k|b_0(v^n,w^n,w^n)| &\le c_b k\|v^n\|_{\mathcal{U}}\|w^n\|_{\mathcal{U}}|w^n|_{\mathcal{H}} \\ &\le \frac{\nu_1}{6}k\|w^n\|_{\mathcal{U}}^2 + ck|w^n|_{\mathcal{H}}^2\|v^n\|_{\mathcal{U}}^2, \end{aligned} \tag{55}$$

$$\begin{aligned} & k\left[b_1(w^n,\phi^n,\nu_2 A_\gamma\theta^n) + b_1(w^n,\theta^n,\nu_2 A_\gamma\psi^n)\right] \\ &\le \tilde{c}_b\nu_2 k|w^n|_{\mathcal{H}}^{1/2}\|w^n\|_{\mathcal{U}}^{1/2}\|\phi^n\|^{1/2}|A_\gamma\phi^n|_{L^2}^{1/2}|A_\gamma\theta^n|_{L^2} \\ &\quad + \tilde{c}_b\nu_2 k|w^n|_{\mathcal{H}}^{1/2}\|w^n\|_{\mathcal{U}}^{1/2}\|\theta^n\|^{1/2}|A_\gamma\theta^n|_{L^2}^{1/2}|A_\gamma\psi^n|_{L^2} \\ &\le \frac{\nu_1}{6}k\|w^n\|_{\mathcal{U}}^2 + \frac{\nu_2^2}{6}k|A_\gamma\theta^n|_{L^2}^2 + ck|w^n|_{\mathcal{H}}^2\|\phi^n\|^2|A_\gamma\phi^n|_{L^2}^2 \\ &\quad + ck|w^n|_{\mathcal{H}}\|\theta^n\||A_\gamma\psi^n|_{L^2}^2 \\ &\le \frac{\nu_1}{6}k\|w^n\|_{\mathcal{U}}^2 + \frac{\nu_2^2}{6}k|A_\gamma\theta^n|_{L^2}^2 + ck|w^n|_{\mathcal{H}}^2\|\phi^n\|^2|A_\gamma\phi^n|_{L^2}^2 \\ &\quad + ck|w^n|_{\mathcal{H}}^2|A_\gamma\psi^n|_{L^2}^2 + ck\|\theta^n\|^2|A_\gamma\psi^n|_{L^2}^2, \end{aligned} \tag{56}$$

$$\begin{aligned} & k\left[b_1(w^n,\phi^n,\nu_2 A_\gamma\theta^n) + b_1(v^n,\theta^n,\nu_2 A_\gamma\theta^n)\right] \\ &\le \tilde{c}_b\nu_2 k|w^n|_{\mathcal{H}}^{1/2}\|w^n\|_{\mathcal{U}}^{1/2}\|\phi^n\|^{1/2}|A_\gamma\phi^n|_{L^2}^{1/2}|A_\gamma\theta^n|_{L^2} \\ &\quad + \tilde{c}_b\nu_2 k|v^n|_{\mathcal{H}}^{1/2}\|v^n\|_{\mathcal{U}}^{1/2}\|\theta^n\|^{1/2}|A_\gamma\theta^n|_{L^2}^{3/2} \\ &\le \frac{\nu_1}{6}k\|w^n\|_{\mathcal{U}}^2 + \frac{\nu_2^2}{6}k|A_\gamma\theta^n|_{L^2}^2 + ck|w^n|_{\mathcal{H}}^2\|\phi^n\|^2|A_\gamma\phi^n|_{L^2}^2 \\ &\quad + ck|v^n|_{\mathcal{H}}^2\|v^n\|_{\mathcal{U}}^2\|\theta^n\|^2. \end{aligned} \tag{57}$$

The last term in (54) is bounded as

$$\begin{aligned} k|\left(f_\gamma(\phi^n)-f_\gamma(\psi^n),A_\gamma\theta^n\right)_{L^2}| &\le k|f_\gamma(\phi^n)-f_\gamma(\psi^n)|_{L^2}|A_\gamma\theta^n|_{L^2} \\ &\le \frac{\nu_2^2}{6}k|A_\gamma\theta^n|_{L^2}^2 + ck|f_\gamma(\phi^n)-f_\gamma(\psi^n)|_{L^2}^2. \end{aligned} \tag{58}$$

Recalling (45) and (6) we have

$$\begin{aligned} & |f_\gamma(\phi^n)-f_\gamma(\psi^n)|_{L^2}^2 \\ &= \int_\Omega |f(\phi^n(x)) - f(\psi^n(x)) - \alpha_0^{-1}\nu_2\gamma(\phi^n(x)-\psi^n(x))|^2\,dx \\ &= \int_\Omega |f'(\zeta^n(x)) - \alpha_0^{-1}\nu_2\gamma|^2(\phi^n(x)-\psi^n(x))^2\,dx \\ &\quad \text{(for some } \zeta^n(x)\in(\phi^n(x),\psi^n(x)) \text{ or } \zeta^n(x)\in(\psi^n(x),\phi^n(x))) \\ &\le 2\int_\Omega \left(2c_f^2(1+|\zeta^n(x)|^4) + \alpha_0^{-2}{\nu_2}^2\gamma^2\right)(\phi^n(x)-\psi^n(x))^2\,dx \\ &\le 2\left(2c_f^2 + \alpha_0^{-2}{\nu_2}^2\gamma^2\right)|\phi^n-\psi^n|_{L^2}^2 + 4c_f^2\int_\Omega |\gamma^n(x)|^4(\phi^n(x)-\psi^n(x))^2\,dx \\ &\quad \text{(where } \gamma^n(x)=\phi^n(x) \text{ or } \psi^n(x)) \\ &\le 2\left(2c_f^2 + \alpha_0^{-2}{\nu_2}^2\gamma^2\right)|\phi^n-\psi^n|_{L^2}^2 + 4c_f^2|\gamma^n|_{L^6}^4|\phi^n-\psi^n|_{L^6}^2 \\ &\le 2\left(2c_f^2 + \alpha_0^{-2}{\nu_2}^2\gamma^2\right)|\phi^n-\psi^n|_{L^2}^2 + c\|\gamma^n\|^4\|\phi^n-\psi^n\|^2 \\ &\quad \text{(by the Sobolev imbedding } H^1(\Omega)\hookrightarrow L^6(\Omega)) \\ &\le 2\left(2c_f^2 + \alpha_0^{-2}{\nu_2}^2\gamma^2\right)|\phi^n-\psi^n|_{L^2}^2 + cK_1^4\|\phi^n-\psi^n\|^2 \quad \text{(by (64))} \\ &\le K_7^2\|\theta^n\|_\gamma^2, \end{aligned} \tag{59}$$

for some $K_7 = K_7(\|(u_0,\phi_0)\|_{\mathbb{Y}})$ independent on n, and thus (58) gives

$$k|\left(f_\gamma(\phi^n) - f_\gamma(\psi^n), A_\gamma\theta^n\right)_{L^2}| \le \frac{\nu_2^2}{6}k|A_\gamma\theta^n|^2_{L^2} + kK_7^2\|\theta^n\|^2_\gamma. \tag{60}$$

Gathering relations (54)–(57) and (60), and using (64) and (142), we obtain

$$\left(1 - ckK_1^2 - ckK_1^2K_3^2 - ckK_3^2\right)|w^n|^2_{\mathcal{H}} + \left(\nu_2 - ckK_3^2 - ckK_1^4 - kK_7^2\right)\|\theta^n\|^2_\gamma \le 0. \tag{61}$$

For k such that

$$k \le \min\left\{1, \frac{1}{\kappa}, \frac{1}{2c\,(K_1^2 + K_1^2K_3^2 + K_3^2)}, \frac{\nu_2}{2\,(cK_3^2 + cK_1^4 + K_7^2)}\right\} =: \kappa_0(\|(u_o,\phi_0)\|_{\mathbb{V}}), \tag{62}$$

relation (61) implies $w^n = \theta^n = 0$. Hence, the system (51) possesses a unique solution, provided that the time-step satisfies constraint (62). This dependence of the time step on the initial value prevents us from defining a single-valued attractor in the classical sense. This is why we need the theory of the multi-valued attractors, that we discuss in Subsection 4.1.

Throughout the article, we assume that $g \in L^\infty(0,T;L^2)$ and we let $\|g\|_\infty := \|g\|_{L^\infty(0,T;L^2)}$.

3.1. $\mathbb{Y}$-Uniform Boundedness. We begin with the first main result, which proves the uniform boundedness of the approximate solution (u^n,ϕ^n) in $\mathbb{Y}$.

Theorem 1. Let (u^n,ϕ^n) be a solution of (51). Then there exists $\kappa > 0$ such that for every $k > 0$, we have

$$\|(u^n,\phi^n)\|^2_{\mathbb{Y}} \le (1+\kappa k)^{-n}Q^2(\|(u_0,\phi_0)\|_{\mathbb{Y}}) + \rho_0^2\left[1 - (1+\kappa k)^{-n}\right],\ \forall\, n \ge 0, \tag{63}$$

where the monotonically increasing function Q, given in (102) below, is independent of n, and ρ_0, given in (103) below, is independent of the initial data.

Moreover, there exists $K_1 = K_1(\|(u_0,\phi_0)\|_{\mathbb{Y}}, \|g\|_\infty)$ such that for every $k > 0$, we have

$$\|(u^n,\phi^n)\|_{\mathbb{Y}} \le K_1,\ \forall\, n \ge 0, \tag{64}$$

and for every $i = 1,\cdots,n$ there exist $M_1 = M_1(\|(u^{i-1},\phi^{i-1})\|_{\mathbb{Y}}, \|g\|_\infty, (n-i+1)k)$ and $M_2 = M_2(\|(u^{i-1},\phi^{i-1})\|_{\mathbb{Y}}, \|g\|_\infty, (n-i+1)k)$ such that

$$k\sum_{j=i}^{n}\left(\frac{\nu_1}{2\mathcal{K}}\|u^n\|^2_{\mathcal{U}} + 2|\mu^n|^2_{L^2}\right) \le M_1, \tag{65}$$

$$k\sum_{j=i}^{n}|A_\gamma(\phi^j)|^2_{L^2} \le M_2. \tag{66}$$

Proof. Taking the scalar product of the first equation of (51) with $2ku^n$ and using (13), the skew property (31), as well as the relation

$$2(\varphi - \psi, \varphi)_H = |\varphi|^2_H - |\psi|^2_H + |\varphi - \psi|^2_H, \tag{67}$$

where H is any Hilbert space, we obtain

$$|u^n|^2_{\mathcal{H}} - |u^{n-1}|^2_{\mathcal{H}} + |u^n - u^{n-1}|^2_{\mathcal{H}} + 2\nu_1 k\|u^n\|^2_{\mathcal{U}} - 2\mathcal{K}kb_1(u^n,\phi^n,\nu_2A_\gamma\phi^n) = 2k(g^n,u^n)_{L^2}. \tag{68}$$

Taking the scalar product of the fourth equation of (51) in L^2 by $2k\mu^n$, we obtain

$$2(\phi^n - \phi^{n-1},\mu^n)_{L^2} + 2k|\mu^n|^2_{L^2} + 2kb_1(u^n,\phi^n,\mu^n) = 0. \tag{69}$$

Dividing (68) by $\mathcal{K}$ and adding the resulting equation to (69), we obtain (recalling the third equation of (51) and (35))

$$\begin{aligned}\frac{1}{\mathcal{K}}\left[|u^n|_{\mathcal{H}}^2-|u^{n-1}|_{\mathcal{H}}^2+|u^n-u^{n-1}|_{\mathcal{H}}^2\right]+\frac{2\nu_1}{\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2+2(\phi^n-\phi^{n-1},\mu^n)_{L^2}\\ +2k|\mu^n|_{L^2}^2=\frac{2}{\mathcal{K}}k(g^n,u^n)_{L^2}.\end{aligned}\tag{70}$$

Using the third equation of (51) and (20), we obtain

$$\begin{aligned}&2(\phi^n-\phi^{n-1},\mu^n)_{L^2}\\ &=2(\phi^n-\phi^{n-1},\nu_2A_\gamma\phi^n+\alpha_0f_\gamma(\phi^n))_{L^2}\\ &=\nu_2\left(\|\phi^n\|_\gamma^2-\|\phi^{n-1}\|_\gamma^2+\|\phi^n-\phi^{n-1}\|_\gamma^2\right)+2\alpha_0(\phi^n-\phi^{n-1},f_\gamma(\phi^n))_{L^2}.\end{aligned}\tag{71}$$

To evaluate the second term on the right-hand side of Eq. (71) we fix $x\in\Omega$ and we write

(72)

$$\begin{aligned}F_\gamma(\phi^n(x))-F_\gamma(\phi^{n-1}(x))&=\int_0^1\frac{d}{dt}[F_\gamma\left(\phi^{n-1}(x)+t(\phi^n(x)-\phi^{n-1}(x))\right)]dt\\ =\int_0^1&\left[f_\gamma\left(\phi^{n-1}(x)+t(\phi^n(x)-\phi^{n-1}(x))\right)-f_\gamma(\phi^n(x))\right](\phi^n(x)-\phi^{n-1}(x))\,dt\\ &+(\phi^n(x)-\phi^{n-1}(x))f_\gamma(\phi^n(x)).\end{aligned}$$

Thus

$$2\alpha_0(\phi^n-\phi^{n-1},f_\gamma(\phi^n))_{L^2}=2\alpha_0\mathcal{F}_\gamma(\phi^n)-2\alpha\mathcal{F}_\gamma(\phi^{n-1})+2\alpha_0\mathbb{R}_\gamma^n,\tag{73}$$

where

$$\mathcal{F}_\gamma(\phi^n)=\int_\Omega F_\gamma(\phi^n(x))dx,\tag{74}$$

and

$$\begin{aligned}\mathbb{R}_\gamma^n=&-\int_\Omega\int_0^1\left[f_\gamma\left(\phi^{n-1}(x)+t(\phi^n(x)-\phi^{n-1}(x))\right)-f_\gamma(\phi^n(x))\right]\\ &\times(\phi^n(x)-\phi^{n-1}(x))\,dt\,dx\\ =&-\int_\Omega\int_0^1\left[f\left(\phi^{n-1}(x)+t(\phi^n(x)-\phi^{n-1}(x))\right)-f(\phi^n(x))\right]\\ &\times(\phi^n(x)-\phi^{n-1}(x))\,dt\,dx\\ &+\frac{\nu_2\gamma}{\alpha_0}\int_\Omega\int_0^1(t-1)(\phi^n(x)-\phi^{n-1}(x))^2\,dt\,dx\quad\text{(by (45))}.\end{aligned}\tag{75}$$

Relations (71) and (73) give

$$\begin{aligned}2(\phi^n-\phi^{n-1},\mu^n)_{L^2}=\nu_2\left(\|\phi^n\|_\gamma^2-\|\phi^{n-1}\|_\gamma^2+\|\phi^n-\phi^{n-1}\|_\gamma^2\right)\\ +2\alpha_0\mathcal{F}_\gamma(\phi^n)-2\alpha_0\mathcal{F}_\gamma(\phi^{n-1})+2\alpha_0\mathbb{R}_\gamma^n,\end{aligned}\tag{76}$$

and recalling (70) we obtain

(77)

$$\begin{aligned}&\frac{1}{\mathcal{K}}\left[|u^n|_{\mathcal{H}}^2-|u^{n-1}|_{\mathcal{H}}^2+|u^n-u^{n-1}|_{\mathcal{H}}^2\right]+\nu_2\left(\|\phi^n\|_\gamma^2-\|\phi^{n-1}\|_\gamma^2+\|\phi^n-\phi^{n-1}\|_\gamma^2\right)\\ &+\frac{2\nu_1}{\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2+2\alpha_0\mathcal{F}_\gamma(\phi^n)-2\alpha_0\mathcal{F}_\gamma(\phi^{n-1})+2\alpha_0\mathbb{R}_\gamma^n+2k|\mu^n|_{L^2}^2=\frac{2}{\mathcal{K}}k(g^n,u^n)_{L^2}.\end{aligned}$$

Multiplying the fourth equation of (51) by $2k\phi^n$ and integrating we obtain (recalling the third equation of (51), (20), and (35))

$$|\phi^n|_{L^2}^2 - |\phi^{n-1}|_{L^2}^2 + |\phi^n - \phi^{n-1}|_{L^2}^2 + 2k\nu_2\|\phi^n\|_\gamma^2 + 2\alpha_0 k(f_\gamma(\phi^n), \phi^n)_{L^2} = 0. \tag{78}$$

Adding (77) and (78) we find

(79)
$$\begin{aligned}
&\frac{1}{\mathcal{K}}\left[|u^n|_{\mathcal{H}}^2 - |u^{n-1}|_{\mathcal{H}}^2 + |u^n - u^{n-1}|_{\mathcal{H}}^2\right] + \nu_2\left(\|\phi^n\|_\gamma^2 - \|\phi^{n-1}\|_\gamma^2 + \|\phi^n - \phi^{n-1}\|_\gamma^2\right)\\
&+ |\phi^n|_{L^2}^2 - |\phi^{n-1}|_{L^2}^2 + |\phi^n - \phi^{n-1}|_{L^2}^2 + 2\alpha_0\mathcal{F}_\gamma(\phi^n) - 2\alpha_0\mathcal{F}_\gamma(\phi^{n-1})\\
&+ \frac{2\nu_1}{\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2 + 2k\nu_2\|\phi^n\|_\gamma^2 + 2k|\mu^n|_{L^2}^2 + 2\alpha_0 k(f_\gamma(\phi^n), \phi^n)_{L^2} + 2\alpha_0\mathbb{R}_\gamma^n\\
=&\frac{2}{\mathcal{K}}k(g^n, u^n)_{L^2}.
\end{aligned}$$

Now, for any $n \geq 1$, we set

$$x_n = \frac{1}{\mathcal{K}}|u^n|_{\mathcal{H}}^2 + \nu_2\|\phi^n\|_\gamma^2 + 2\alpha_0\mathcal{F}_\gamma(\phi^n) + |\phi^n|_{L^2}^2 + 2\alpha_0 C_{F_\gamma}|\Omega|, \tag{80}$$

where C_{F_γ} is taken large enough to ensure that $x_n \geq 0$ (recall that F_γ is bounded from below by a constant independent of ν_1 and α_0). We rewrite (86) in the form

$$x_n - x_{n-1} + \kappa k x_n = y_n, \tag{81}$$

where $\kappa \in (0,1)$ is to be determined, and

(82)
$$\begin{aligned}
y_n = &\frac{\kappa}{\mathcal{K}}k|u^n|_{\mathcal{H}}^2 - (2-\kappa)\nu_2 k\|\phi^n\|_\gamma^2 + 2\alpha_0 k\kappa\mathcal{F}_\gamma(\phi^n) + \kappa k|\phi^n|_{L^2}^2 + 2\alpha_0 k\kappa C_{F_\gamma}|\Omega|\\
&+ \frac{2}{\mathcal{K}}k(g^n, u^n)_{L^2} - \frac{1}{\mathcal{K}}|u^n - u^{n-1}|_{\mathcal{H}}^2 - \nu_2\|\phi^n - \phi^{n-1}\|_\gamma^2 - |\phi^n - \phi^{n-1}|_{L^2}^2\\
&- \frac{2\nu_1}{\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2 - 2k|\mu^n|_{L^2}^2 - 2\alpha_0 k(f_\gamma(\phi^n), \phi^n)_{L^2} - 2\alpha_0\mathbb{R}_\gamma^n.
\end{aligned}$$

Using the Cauchy–Schwarz inequality and the Poincaré inequality (14), we have

$$\begin{aligned}
\frac{2}{\mathcal{K}}k(g^n, u^n)_{L^2} \leq &\frac{2}{\mathcal{K}}k|g^n|_{L^2}|u^n|_{L^2}\\
\leq &\frac{2}{\mathcal{K}\sqrt{\lambda_1}}k|g^n|_{L^2}\|u^n\|_{\mathcal{U}} \leq \frac{\nu_1}{\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2 + \frac{1}{\nu_1\lambda_1\mathcal{K}}k|g^n|_{L^2}^2.
\end{aligned} \tag{83}$$

Hereafter, we assume that the potential function f satisfies the following additional condition:

$$f'(r) \geq -\frac{1}{2\alpha}, \ \forall r \in \mathbb{R}. \tag{84}$$

Then using the mean value theorem and (75), we obtain

$$\begin{aligned}
2\alpha_0\mathbb{R}_\gamma^n \geq &\int_\Omega\int_0^1 (1-t)(\phi^n(x) - \phi^{n-1}(x))^2\,dt\,dx - \nu_2\gamma|\phi^n - \phi^{n-1}|_{L^2}^2\\
= &-\frac{1}{2}|\phi^n - \phi^{n-1}|_{L^2}^2 - \nu_2\gamma|\phi^n - \phi^{n-1}|_{L^2}^2.
\end{aligned} \tag{85}$$

Combining (82), (83) and (85), we obtain

$$\begin{aligned}
y_n \leq k\Big[&-\frac{\nu_1}{\mathcal{K}}\|u^n\|_{\mathcal{U}}^2 + \frac{\kappa}{\mathcal{K}}|u^n|_{\mathcal{H}}^2 - (2-\kappa)\nu_2\|\phi^n\|_\gamma^2\\
&+\kappa|\phi^n|_{L^2}^2 + 2\alpha_0\kappa C_{F_\gamma}|\Omega| + \frac{1}{\nu_1\lambda_1\mathcal{K}}|g^n|_{L^2}^2 - 2|\mu^n|_{L^2}^2\\
&+2\alpha_0\left[\kappa\left(F_\gamma(\phi^n) - f_\gamma(\phi^n)\phi^n, 1\right)_{L^2} - (1-\kappa)\left(f_\gamma(\phi^n)\phi^n, 1\right)_{L^2}\right]\Big].
\end{aligned} \tag{86}$$

Note that due to (6), we have (for any $r \in \mathbb{R}$)

$$f_\gamma(r)r \geq \frac{c_\star}{2}|f_\gamma(r)|(1+|r|) - \frac{c_f}{2}(1+\alpha_0^{-1}\nu_2), \tag{87}$$

$$F_\gamma(r) - f_\gamma(r)r \leq c_f'(1+\alpha_0^{-1}\nu_2)|r|^2 + c_f'', \tag{88}$$

$$|F_\gamma(r)| \leq |f_\gamma(r)|(1+|r|) + c_1, \tag{89}$$

where $c_f, c_\star, c_f', c_f'', c_1$ are positive, sufficiently large constants that depend on f only.

Thus

$$\begin{aligned}(F_\gamma(\phi^n) - f_\gamma(\phi^n)\phi^n, 1)_{L^2} &= \int_\Omega \left(F_\gamma(\phi^n(x)) - f_\gamma(\phi^n(x))\phi^n(x)\right) dx \\ &\leq c_f'(1+\alpha_0^{-1}\nu_2)|\phi^n|_{L^2}^2 + c_f''|\Omega|,\end{aligned} \tag{90}$$

$$\begin{aligned}2\alpha_0 (f_\gamma(\phi^n)\phi^n, 1) &= 2\alpha_0 \int_\Omega f_\gamma(\phi^n(x))\phi^n(x)\, dx \\ &\geq c_\star\alpha_0(|f_\gamma(\phi^n)|, 1+|\phi^n|)_{L^2} - c_f\alpha_0(1+\alpha_0^{-1}\nu_2)|\Omega| \\ &\geq c_\star\alpha_0(|F_\gamma(\phi^n)|, 1)_{L^2} - (c_1c_\star\alpha_0 + c_f\alpha_0 + c_f\nu_2)|\Omega|.\end{aligned} \tag{91}$$

Using the above inequalities and the Poincaré inequality (14), we bound y_n as

$$\begin{aligned}y_n \leq k \Bigg[&-\frac{1}{\mathcal{K}}\left(\nu_1 - \frac{\kappa}{\lambda_1}\right)\|u^n\|_{\mathcal{U}}^2 - (2-\kappa)\nu_2\|\phi^n\|_\gamma^2 + \kappa|\phi^n|_{L^2}^2 + 2\alpha_0\kappa C_{F_\gamma}|\Omega| \\ &-2|\mu^n|_{L^2}^2 + \frac{1}{\nu_1\mathcal{K}\lambda_1^2}|g^n|_{L^2}^2 + 2\alpha_0\kappa\left[c_f'(1+\alpha_0^{-1}\nu_2)|\phi^n|_{L^2}^2 + c_f''|\Omega|\right] \\ &-(1-\kappa)\left[c_\star\alpha_0(|F_\gamma(\phi^n)|, 1)_{L^2} - (c_1c_\star\alpha_0 + c_f\alpha_0 + c_f\nu_2)|\Omega|\right]\Bigg]\end{aligned} \tag{92}$$

where

$$c_2 = \left[(1-\kappa)(c_1c_\star\alpha_0 + c_f\alpha_0 + c_f\nu_2) + 2\alpha_0\kappa C_{F_\gamma} + 2\alpha_0\kappa c_f''\right]|\Omega|. \tag{93}$$

Now we choose $\kappa \in (0,1)$ to be

$$\kappa = \min\left\{\frac{\nu_1\lambda_1}{2}, \frac{\nu_2\gamma}{1+\nu_2\gamma+2c_f'(\alpha_0+\nu_2)}\right\}. \tag{94}$$

Then relation (92) gives

$$\begin{aligned}y_n \leq k \Bigg(&-\frac{\nu_1}{2\mathcal{K}}\|u^n\|_{\mathcal{U}}^2 - \nu_2\|\phi^n\|_\gamma^2 - 2|\mu^n|_{L^2}^2 + \frac{1}{\nu_1\mathcal{K}\lambda_1^2}|g^n|_{L^2}^2 \\ &-(1-\kappa)c_\star\alpha_0(|F_\gamma(\phi^n)|, 1)_{L^2} + c_2\Bigg),\end{aligned} \tag{95}$$

and recalling (81), we find

$$\begin{aligned}x_n - x_{n-1} + \kappa k x_n &+ \frac{1}{2}k\left(\frac{\nu_1}{\mathcal{K}}\|u^n\|_{\mathcal{U}}^2 + \nu_2\|\phi^n\|_\gamma^2\right) + 2k|\mu^n|_{L^2}^2 \\ &+ c_3k|F_\gamma(\phi^n)|_{L^1} \leq \frac{1}{\nu_1\mathcal{K}\lambda_1^2}k|g^n|_{L^2}^2 + c_2k.\end{aligned} \tag{96}$$

Neglecting some positive terms, we obtain

$$x_n \leq \frac{1}{\beta}x_{n-1} + \frac{1}{\beta}k\left(\frac{1}{\nu_1\mathcal{K}\lambda_1^2}|g^n|_{L^2}^2 + c_2\right), \tag{97}$$

where

$$\beta = 1 + \kappa k. \tag{98}$$

Using recursively (97), we find

$$x_n \le \frac{1}{\beta^n} x_0 + k \sum_{i=1}^{n} \frac{1}{\beta^i} \left(\frac{1}{\nu_1 \mathcal{K} \lambda_1^2} |g^{n+1-i}|_{L^2}^2 + c_2 \right) \tag{99}$$
$$\le (1+\kappa k)^{-n} x_0 + \frac{1}{\kappa} \left(\frac{1}{\nu_1 \mathcal{K} \lambda_1^2} \|g\|_\infty^2 + c_2 \right) \left[1 - (1+\kappa k)^{-n} \right].$$

Now observe that, due to (6), we can find $C_f > 0$ such that

$$x_n \le C_f \left(1 + \|(u^n, \phi^n)\|_{\mathbb{Y}}^2 + |\phi^n|_{L^4}^4 \right), \tag{100}$$

and thus, relation (99) yields

$$x_n \le (1+\kappa k)^{-n} Q^2(\|(u_0, \phi_0)\|_{\mathbb{Y}}) + \rho_0^2 \left[1 - (1+\kappa k)^{-n} \right], \tag{101}$$

where

$$Q^2(\|(u^n, \phi^n)\|_{\mathbb{Y}}) = C_f \left(1 + \|(u^n, \phi^n)\|_{\mathbb{Y}}^2 + \frac{c}{\nu_2^2} \|(u^n, \phi^n)\|_{\mathbb{Y}}^4 \right), \tag{102}$$

for some constant $c > 0$, and

$$\rho_0^2 = \frac{1}{\kappa} \left(\frac{1}{\nu_1 \mathcal{K} \lambda_1^2} \|g\|_\infty^2 + c_2 \right). \tag{103}$$

Taking

$$K_1^2(\|(u_0, \phi_0)\|_{\mathbb{Y}}, \|g\|_\infty) = Q^2(\|(u_0, \phi_0)\|_{\mathbb{Y}}) + \rho_0^2, \tag{104}$$

we obtain

$$x_n \le K_1^2, \ \forall n \ge 0. \tag{105}$$

Since $\|(u^n, \phi^n)\|_{\mathbb{Y}}^2 \le x_n$, relations (101) and (105) give (63) and (64), respectively.

Now adding inequalities (96) with n from i to N and dropping some positive terms, we find

$$k \sum_{n=i}^{N} \left[\frac{1}{2} \left(\frac{\nu_1}{\mathcal{K}} \|u^n\|_{\mathcal{U}}^2 + \nu_2 \|\phi^n\|_\gamma^2 \right) + 2|\mu^n|_{L^2}^2 + c_3 |F_\gamma(\phi^n)|_{L^1} \right] \tag{106}$$
$$\le x_{i-1} + \left(\frac{1}{\nu_1 \mathcal{K} \lambda_1^2} \|g\|_\infty^2 + c_2 \right) (N - i + 1) k.$$

Recalling (100), the above inequality gives conclusion (65) of the theorem, with

$$M_1(\|(u^{i-1}, \phi^{i-1})\|_{\mathbb{Y}}, \|g\|_\infty, (n-i+1)k) \tag{107}$$
$$= Q^2(\|(u^{i-1}, \phi^{i-1})\|_{\mathbb{Y}}) + \rho_0^2 + \left(\frac{1}{\nu_1 \mathcal{K} \lambda_1^2} \|g\|_\infty^2 + c_2 \right) (n - i + 1) k.$$

Recalling (51), (45), (7) and the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$|A_\gamma \phi^n|_{L^2}^2 \le \frac{3}{\nu_2^2} |\mu^n|_{L^2}^2 + \frac{3\alpha_0^2}{\nu_2^2} |f(\phi^n)|_{L^2}^2 + 3\gamma^2 |\phi^n|_{L^2}^2 \tag{108}$$
$$\le \frac{3}{\nu_2^2} |\mu^n|_{L^2}^2 + \frac{6\alpha_0^2}{\nu_2^2} c_f^2 \left(|\Omega| + |\phi^n|_{L^6}^6 \right) + 3\gamma^2 |\phi^n|_{L^2}^2$$
$$\le \frac{3}{\nu_2^2} |\mu^n|_{L^2}^2 + \frac{6\alpha_0^2}{\nu_2^2} c_f^2 |\Omega| + \frac{6\alpha_0^2}{\nu_2^2} c_f''(\|\phi^n\|_\gamma^2)^3 + 3\gamma^2 |\phi^n|_{L^2}^2,$$

for some positive constant c_f'' depending on c_f. Summing with n from i to N and using (65) and (105), we obtain conclusion (66) of the theorem, with

$$\begin{aligned} &M_2(\|(u^{i-1},\phi^{i-1})\|_{\mathbb{Y}},\|g\|_\infty,(n-i+1)k)\\ =&\frac{3}{2\nu_2^2}M_1(\|(u^{i-1},\phi^{i-1})\|_{\mathbb{Y}})+\frac{6\alpha_0^2}{\nu_2^2}c_f^2|\Omega|(n-i+1)k\\ &+\frac{6\alpha_0^2}{\nu_2^5}c_f''K_1^6(\|(u^{i-1},\phi^{i-1})\|_{\mathbb{Y}})(n-i+1)k\\ &+\frac{3\gamma}{\nu_2}K_1^2(\|(u^{i-1},\phi^{i-1})\|_{\mathbb{Y}})(n-i+1)k. \end{aligned} \tag{109}$$

Thus, the theorem has been proved. □

Corollary 3.1. If

$$0<k\le\frac{1}{\kappa}, \tag{110}$$

then then $B_{\mathbb{Y}}(0,\sqrt{2}\rho_0)$, the ball in $\mathbb{Y}$ centered at 0 and radius $\sqrt{2}\rho_0$, is an absorbing ball for (u^n,ϕ^n) in $\mathbb{Y}$.

Proof. Let $\mathcal{B}$ be any bounded set in $\mathbb{Y}$ and assume that it is included in a ball $B(0,R)$ of $\mathbb{Y}$. For any initial data $(u^0,\phi^0)\in\mathcal{B}$, the bound (63) on $\|(u^n,\phi^n)\|_{\mathbb{Y}}^2$ gives

$$\|(u^n,\phi^n)\|_{\mathbb{Y}}^2\le\big(1+\kappa k\big)^{-n}Q^2(R)+\rho_0^2.$$

Using assumption (110) on k and the fact that $1+x\ge\exp(x/2)$ if $x\in(0,1)$ we obtain

$$\|(u^n,\phi^n)\|_{\mathbb{Y}}^2\le\exp\Big(-nk\frac{\kappa}{2}\Big)Q^2(R)+\rho_0^2,$$

and thus $\|(u^n,\phi^n)\|_{\mathbb{Y}}^2\le 2\rho_0^2$, $\forall n\ge N_0(R,k)$, where

$$N_0(R,k):=\frac{4}{\kappa k}\ln\left(\frac{Q(R)}{\rho_0}\right). \tag{111}$$

This completes the proof of the corollary. □

3.2. $\mathbb{V}$-Uniform Boundedness. We are now going to derive uniform bounds for (u^n,ϕ^n) in $\mathbb{V}$, similar to those we have already obtained in $\mathbb{Y}$ (see (63) above). In order to do this, we will first use the discrete Gronwall lemma to derive an upper bound on $\|(u^n,\phi^n)\|_{\mathbb{V}}$, $n\le N$, for some $N>0$, and then we will employ the discrete uniform Gronwall lemma to obtain an upper bound on $\|(u^n,\phi^n)\|_{\mathbb{V}}$, $n\ge N$.

We begin with some preliminary results.

Lemma 2. For every $k>0$, we have

$$\|(u^n,\phi^n)\|_{\mathbb{V}}^2\le K_2\|(u^{n-1},\phi^{n-1})\|_{\mathbb{V}}^2+c_3(|f_\gamma(\phi^n)|_{L^2}^2+|g^n|_{L^2}^2),\ \forall n\ge1, \tag{112}$$

where $K_2=K_2(\|(u_0,\phi_0)\|_{\mathbb{Y}},\|g\|_\infty)$ and $c_3>0$ are given below, in (122) and (123), respectively.

Proof. Taking the scalar product of the first equation of (51) with $2k(u^n-u^{n-1})$ we obtain

$$\begin{aligned} 2|u^n-u^{n-1}|_{\mathcal{H}}^2&+\nu_1k\|u^n\|_{\mathcal{U}}^2-\nu_1k\|u^{n-1}\|_{\mathcal{U}}^2\\ &+\nu_1k\|u^n-u^{n-1}\|_{\mathcal{U}}^2+2k\,b_0(u^n,u^n,u^n-u^{n-1})\\ &-2\mathcal{K}kb_1(u^n-u^{n-1},\phi^n,\nu_2A_\gamma\phi^n)=2k(g^n,u^n-u^{n-1})_{L^2}. \end{aligned} \tag{113}$$

Multiplying the fourth equation of (51) by $2\nu_2 k A_\gamma(\phi^n - \phi^{n-1})$ and integrating we obtain (recalling the third equation of (51))

$$\begin{aligned}(114)\quad &2\nu_2\|\phi^n-\phi^{n-1}\|_\gamma^2 + \nu_2^2 k|A_\gamma\phi^n|_{L^2}^2 - \nu_2^2 k|A_\gamma\phi^{n-1}|_{L^2}^2 + \nu_2^2 k|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}^2\\ &+ 2\nu_2 k b_1(u^n,\phi^n,A_\gamma(\phi^n-\phi^{n-1})) + 2\nu_2\alpha_0 k(f_\gamma(\phi^n),A_\gamma(\phi^n-\phi^{n-1}))_{L^2} = 0.\end{aligned}$$

Dividing (113) by $\mathcal{K}$ and adding the resulting equation to (114), we obtain

$$\begin{aligned}(115)\quad &\frac{2}{\mathcal{K}}|u^n-u^{n-1}|_{\mathcal{H}}^2 + 2\nu_2\|\phi^n-\phi^{n-1}\|_\gamma^2 + \frac{\nu_1}{\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2 + \nu_2^2 k|A_\gamma\phi^n|_{L^2}^2\\ &- \frac{\nu_1}{\mathcal{K}}k\|u^{n-1}\|_{\mathcal{U}}^2 - \nu_2^2 k|A_\gamma\phi^{n-1}|_{L^2}^2 + \frac{\nu_1}{\mathcal{K}}k\|u^n-u^{n-1}\|_{\mathcal{U}}^2 + \nu_2^2 k|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}^2\\ &+ \frac{2}{\mathcal{K}}k\, b_0(u^n,u^n,u^n-u^{n-1}) - 2kb_1(u^n-u^{n-1},\phi^n,\nu_2 A_\gamma\phi^n)\\ &+ 2\nu_2 k b_1(u^n,\phi^n,A_\gamma(\phi^n-\phi^{n-1}))\\ &= \frac{2}{\mathcal{K}}k(g^n,u^n-u^{n-1})_{L^2} - 2\nu_2\alpha_0 k(f_\gamma(\phi^n),A_\gamma(\phi^n-\phi^{n-1}))_{L^2}.\end{aligned}$$

To bound the right-hand side of (115) we use the Cauchy–Schwarz inequality and we obtain

$$\begin{aligned}(116)\quad \frac{2}{\mathcal{K}}k(g^n,u^n-u^{n-1})_{L^2} &\le \frac{2}{\mathcal{K}}k|g^n|_{L^2}|u^n-u^{n-1}|_{L^2}\\ &\le \frac{2}{\mathcal{K}\sqrt{\lambda_1}}\,k|g^n|_{L^2}\|u^n-u^{n-1}\|_{\mathcal{U}} \quad \text{(by (14))}\\ &\le \frac{\nu_1}{4\mathcal{K}}k\|u^n-u^{n-1}\|_{\mathcal{U}}^2 + \frac{4}{\nu_1\lambda_1\mathcal{K}}k|g^n|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}(117)\quad &2\nu_2\alpha_0 k|(f_\gamma(\phi^n),A_\gamma(\phi^n-\phi^{n-1}))_{L^2}|\\ &\le 2\nu_2\alpha_0 k|f_\gamma(\phi^n)|_{L^2}|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}\\ &\le \frac{\nu_2^2}{4}k|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}^2 + 4\alpha_0^2 k|f_\gamma(\phi^n)|_{L^2}^2.\end{aligned}$$

We bound the nonlinear terms as:

$$\begin{aligned}(118)\quad \frac{2}{\mathcal{K}}kb_0(u^n,u^n,u^n-u^{n-1}) &= -\frac{2}{\mathcal{K}}kb_0(u^n,u^n,u^{n-1}) \quad \text{(by (31))}\\ &\le \frac{2}{\mathcal{K}}c_b k|u^n|_{\mathcal{H}}\|u^n\|_{\mathcal{U}}\|u^{n-1}\|_{\mathcal{U}} \quad \text{(by (32))}\\ &\le \frac{\nu_1}{2\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2 + \frac{2c_b^2}{\nu_1\mathcal{K}}k|u^n|_{\mathcal{H}}^2\|u^{n-1}\|_{\mathcal{U}}^2\\ &\le \frac{\nu_1}{2\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2 + \frac{2c_b^2}{\nu_1}K_1^2 k\|u^{n-1}\|_{\mathcal{U}}^2 \quad \text{(by (64))},\end{aligned}$$

(119)

$$\begin{aligned}
&-2kb_1(u^n-u^{n-1},\phi^n,\nu_2A_\gamma\phi^n)+2\nu_2kb_1(u^n,\phi^n,A_\gamma(\phi^n-\phi^{n-1}))\\
&\quad=-2\nu_2kb_1(u^n-u^{n-1},\phi^n,A_\gamma\phi^{n-1})+2\nu_2kb_1(u^{n-1},\phi^n,A_\gamma(\phi^n-\phi^{n-1}))\\
&\quad\le 2\nu_2\tilde{c}_bk|u^n-u^{n-1}|_{\mathcal{H}}^{1/2}\|u^n-u^{n-1}\|_{\mathcal{U}}^{1/2}\|\phi^n\|^{1/2}|A_\gamma\phi^n|_{L^2}^{1/2}|A_\gamma\phi^{n-1}|_{L^2}\\
&\qquad+2\nu_2\tilde{c}_bk|u^{n-1}|_{\mathcal{H}}^{1/2}\|u^{n-1}\|_{\mathcal{U}}^{1/2}\|\phi^n\|^{1/2}|A_\gamma\phi^n|_{L^2}^{1/2}|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}\quad\text{(by (37))}\\
&\quad\le\frac{\nu_1}{4\mathcal{K}}k\|u^n-u^{n-1}\|_{\mathcal{U}}^2+\frac{\nu_2^2}{2}k|A_\gamma\phi^n|_{L^2}^2+ck|u^n-u^{n-1}|_{\mathcal{H}}\|\phi^n\|\|A_\gamma\phi^{n-1}|_{L^2}^2\\
&\qquad+\frac{\nu_2^2}{4}k|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}^2+ck|u^{n-1}|_{\mathcal{H}}^2\|u^{n-1}\|_{\mathcal{U}}^2\|\phi^n\|^2\\
&\quad\le\frac{\nu_1}{4\mathcal{K}}k\|u^n-u^{n-1}\|_{\mathcal{U}}^2+\frac{\nu_2^2}{2}k|A_\gamma\phi^n|_{L^2}^2+cK_1^2k|A_\gamma\phi^{n-1}|_{L^2}^2\\
&\qquad+\frac{\nu_2^2}{4}k|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}^2+cK_1^4k\|u^{n-1}\|_{\mathcal{U}}^2\quad\text{(by (64))}.
\end{aligned}$$

Relations (115)–(119) give

(120)

$$\begin{aligned}
&\frac{2}{\mathcal{K}}|u^n-u^{n-1}|_{\mathcal{H}}^2+2\nu_2\|\phi^n-\phi^{n-1}\|_\gamma^2+\frac{\nu_1}{2\mathcal{K}}k\|u^n\|_{\mathcal{U}}^2+\frac{\nu_2^2}{2}k|A_\gamma\phi^n|_{L^2}^2\\
&-\left(\frac{\nu_1}{\mathcal{K}}+\frac{2c_b^2}{\nu_1}K_1^2+cK_1^4\right)k\|u^{n-1}\|_{\mathcal{U}}^2-(\nu_2^2+cK_1^2)k|A_\gamma\phi^{n-1}|_{L^2}^2+\frac{\nu_1}{2\mathcal{K}}k\|u^n-u^{n-1}\|_{\mathcal{U}}^2\\
&+\frac{\nu_2^2}{2}k|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}^2\le\frac{4}{\nu_1\lambda_1\mathcal{K}}k|g^n|_{L^2}^2+4\alpha_0^2k|f_\gamma(\phi^n)|_{L^2}^2,
\end{aligned}$$

and after neglecting some positive terms we find

$$\begin{aligned}
\frac{\nu_1}{2\mathcal{K}}\|u^n\|_{\mathcal{U}}^2+\frac{\nu_2^2}{2}|A_\gamma\phi^n|_{L^2}^2&\le\left(\frac{\nu_1}{\mathcal{K}}+\frac{2c_b^2}{\nu_1}K_1^2+cK_1^4\right)\|u^{n-1}\|_{\mathcal{U}}^2\\
&+(\nu_2^2+cK_1^2)|A_\gamma\phi^{n-1}|_{L^2}^2+\frac{4}{\nu_1\lambda_1\mathcal{K}}|g^n|_{L^2}^2+4\alpha_0^2|f_\gamma(\phi^n)|_{L^2}^2.
\end{aligned}\tag{121}$$

Taking

$$K_2=2\left(\frac{\mathcal{K}}{\nu_1}+\frac{1}{\nu_2^2}\right)\left(\frac{\nu_1}{\mathcal{K}}+\frac{2c_b^2}{\nu_1}K_1^2+cK_1^4+\nu_2^2+cK_1^2\right),\tag{122}$$

and

$$c_3=2\left(\frac{\mathcal{K}}{\nu_1}+\frac{1}{\nu_2^2}\right)\left(\frac{4}{\nu_1\lambda_1\mathcal{K}}+4\alpha_0^2\right),\tag{123}$$

we obtain conclusion (112) of the lemma. □

Lemma 3. For every $k>0$ and $n\ge1$, we have

$$\begin{aligned}
&c_4(1+K_1^2)k\|(u^n,\phi^n)\|_{\mathbb{V}}^4-\|(u^n,\phi^n)\|_{\mathbb{V}}^2+\|(u^{n-1},\phi^{n-1})\\
&\|_{\mathbb{V}}^2+c_5k\left(|f_\gamma{}'(\phi^n)\nabla\phi^n|_{L^2}^2+|g^n|_{L^2}^2\right)\ge0,
\end{aligned}\tag{124}$$

for some positive constants c_4 and c_5.

Proof. Taking the scalar product of the first equation of (51) by $2kA_0u^n$ and integrating, we obtain

$$\begin{aligned}
\|u^n\|_{\mathcal{U}}^2-\|u^{n-1}\|_{\mathcal{U}}^2+\|u^n-u^{n-1}\|_{\mathcal{U}}^2+2\nu_1k|A_0u^n|_{\mathcal{H}}^2+2kb_0(u^n,u^n,A_0u^n)\\
-2\mathcal{K}kb_1(A_0u^n,\phi^n,\nu_2A_\gamma\phi^n)=2k(g^n,A_0u^n)_{L^2}.
\end{aligned}\tag{125}$$

Multiplying the fourth equation of (51) by $2kA_\gamma^2\phi^n$ and integrating, we obtain

$$\begin{aligned}|A_\gamma\phi^n|_{L^2}^2 - |A_\gamma\phi^{n-1}|_{L^2}^2 + |A_\gamma(\phi^n - \phi^{n-1})|_{L^2}^2 + 2k(\mu^n, A_\gamma^2\phi^n)_{L^2} \\ +2kb_1(u^n, \phi^n, A_\gamma^2\phi^n) = 0.\end{aligned} \tag{126}$$

Recalling the third equation of (51) we find

$$2k(\mu^n, A_\gamma^2\phi^n)_{L^2} = 2k\nu_2|A_\gamma{}^{3/2}\phi^n|_{L^2}^2 + 2\alpha_0 k(f_\gamma(\phi^n), A_\gamma^2\phi^n)_{L^2}, \tag{127}$$

and then (126) gives

$$\begin{aligned}|A_\gamma\phi^n|_{L^2}^2 - |A_\gamma\phi^{n-1}|_{L^2}^2 + |A_\gamma(\phi^n - \phi^{n-1})|_{L^2}^2 + 2k\nu_2|A_\gamma{}^{3/2}\phi^n|_{L^2}^2 \\ + 2\alpha_0 k(f_\gamma(\phi^n), A_\gamma^2\phi^n)_{L^2} + 2kb_1(u^n, \phi^n, A_\gamma^2\phi^n) = 0.\end{aligned} \tag{128}$$

Adding (125) and (128) we obtain

$$\begin{aligned}&\|u^n\|_{\mathcal{U}}^2 + |A_\gamma\phi^n|_{L^2}^2 - (\|u^{n-1}\|_{\mathcal{U}}^2 + |A_\gamma\phi^{n-1}|_{L^2}^2) + \|u^n - u^{n-1}\|_{\mathcal{U}}^2 \\ &\quad + |A_\gamma(\phi^n - \phi^{n-1})|_{L^2}^2 + 2\nu_1 k|A_0u^n|_{\mathcal{H}}^2 + 2k\nu_2|A_\gamma{}^{3/2}\phi^n|_{L^2}^2 \\ &\quad + 2kb_0(u^n, u^n, A_0u^n) - 2\mathcal{K}kb_1(A_0u^n, \phi^n, \nu_2 A_\gamma\phi^n) \\ &\quad + 2kb_1(u^n, \phi^n, A_\gamma^2\phi^n) = 2k(g^n, A_0u^n)_{L^2} - 2\alpha_0 k(f_\gamma(\phi^n), A_\gamma^2\phi^n)_{L^2}.\end{aligned} \tag{129}$$

To bound the right-hand side of the above equality we use the Cauchy-Schwarz inequality and obtain

$$2k(g^n, A_0u^n)_{L^2} \le 2k|g^n|_{L^2}|A_0u^n|_{L^2} \le \frac{\nu_1}{4}k|A_0u^n|_{\mathcal{H}}^2 + \frac{4}{\nu_1}k|g^n|_{L^2}^2, \tag{130}$$

$$\begin{aligned}2\alpha_0 k|(f_\gamma(\phi^n), A_\gamma^2\phi^n)_{L^2}| &= 2\alpha_0 k|(A_\gamma^{1/2}f_\gamma(\phi^n), A_\gamma^{3/2}\phi^n)_{L^2}| \\ &\le 2\alpha_0 k|A_\gamma^{1/2}f_\gamma(\phi^n)|_{L^2}|A_\gamma^{3/2}\phi^n|_{L^2} \\ &\le c|\nabla f_\gamma(\phi^n)|_{L^2}|A_\gamma^{3/2}\phi^n|_{L^2} \\ &\le \frac{\nu_2}{3}k|A_\gamma^{3/2}\phi^n|_{L^2}^2 + ck|f_\gamma{}'(\phi^n)\nabla\phi^n|_{L^2}^2.\end{aligned} \tag{131}$$

We bound the nonlinear terms as follows:

$$\begin{aligned}&2k|b_0(u^n, u^n, A_0u^n)| \\ &\le 2c_b k\|u^n\|_{\mathcal{U}}^2|A_0u^n|_{\mathcal{H}} \quad \text{(by (33))} \\ &\le \frac{\nu_1}{4}k|A_0u^n|_{\mathcal{H}}^2 + ck\|u^n\|_{\mathcal{U}}^4,\end{aligned} \tag{132}$$

$$\begin{aligned}&2\mathcal{K}k|b_1(A_0u^n, \phi^n, \nu_2A_\gamma\phi^n)| \\ &\le 2\tilde{c_b}\nu_2\mathcal{K}|A_0u^n|_{\mathcal{H}}\|\phi^n\|^{1/2}|A_\gamma^{3/2}\phi^n|_{L^2}^{1/2}|A_\gamma\phi^n|_{L^2} \quad \text{(by (38))} \\ &\le \frac{\nu_2}{3}k|A_\gamma^{3/2}\phi^n|_{L^2}^2 + \frac{\nu_1}{4}k|A_0u^n|_{\mathcal{H}}^2 + ck\|\phi^n\|^2|A_\gamma\phi^n|_{L^2}^4,\end{aligned} \tag{133}$$

(134)
$$\begin{aligned}2k|b_1(u^n,\phi^n,A_\gamma^2\phi^n)| &= 2k|(A_\gamma^{1/2}B_1(u^n,\phi^n),A_\gamma^{3/2}\phi^n)_{L^2}|\\ &\le 2k|A_\gamma^{1/2}B_1(u^n,\phi^n)|_{L^2}|A_\gamma^{3/2}\phi^n|_{L^2}\\ &\le 2\tilde{c_b}k|\nabla u^n|_{\mathcal{H}}^{1/2}\|\nabla u^n\|_{\mathcal{U}}^{1/2}\|\phi^n\|^{1/2}|A_\gamma\phi^n|_{L^2}^{1/2}|A_\gamma^{3/2}\phi^n|_{L^2}\\ &\quad + 2\tilde{c_b}k|u^n|_{\mathcal{H}}^{1/2}\|u^n\|_{\mathcal{U}}^{1/2}\|\nabla\phi^n\|^{1/2}|A_\gamma^{3/2}\phi^n|_{L^2}^{3/2} \quad \text{(by (38))}\\ &\le ck\|u^n\|_{\mathcal{U}}^{1/2}|A_0u^n|_{\mathcal{H}}^{1/2}\|\phi^n\|^{1/2}|A_\gamma\phi^n|_{L^2}^{1/2}|A_\gamma^{3/2}\phi^n|_{L^2}\\ &\quad + ck|u^n|_{\mathcal{H}}^{1/2}\|u^n\|_{\mathcal{U}}^{1/2}|A_\gamma\phi^n|_{L^2}^{1/2}|A_\gamma^{3/2}\phi^n|_{L^2}^{3/2}\\ &\le \frac{\nu_2}{3}k|A_\gamma^{3/2}\phi^n|_{L^2}^2 + \frac{\nu_1}{4}k|A_0u^n|_{\mathcal{H}}^2 + ck\|u^n\|_{\mathcal{U}}^2\|\phi^n\|^2|A_\gamma\phi^n|_{L^2}^2\\ &\quad + ck|u^n|_{\mathcal{H}}^2\|u^n\|_{\mathcal{U}}^2|A_\gamma\phi^n|_{L^2}^2.\end{aligned}$$

Recalling (64), relations (129)–(134) give

$$\begin{aligned}&\|u^n\|_{\mathcal{U}}^2 + |A_\gamma\phi^n|_{L^2}^2 - (\|u^{n-1}\|_{\mathcal{U}}^2 + |A_\gamma\phi^{n-1}|_{L^2}^2)\\ &\quad + \|u^n-u^{n-1}\|_{\mathcal{U}}^2 + |A_\gamma(\phi^n-\phi^{n-1})|_{L^2}^2\\ &\quad + \nu_1k|A_0u^n|_{\mathcal{H}}^2 + \nu_2k|A_\gamma{}^{3/2}\phi^n|_{L^2}^2 \le ck\|u^n\|_{\mathcal{U}}^4 + cK_1^2k|A_\gamma\phi^n|_{L^2}^4\\ &\quad + cK_1^2k\|u^n\|_{\mathcal{U}}^2|A_\gamma\phi^n|_{L^2}^2 + ck|f_\gamma{}'(\phi^n)\nabla\phi^n|_{L^2}^2 + \frac{4}{\nu_1}k|g^n|_{L^2}^2,\end{aligned} \tag{135}$$

and the conclusion of the Lemma follows right away. □

We now recall the following two lemmas, whose proofs can be found in [39]:

Lemma 4. Given $k>0$ and positive sequences ξ_n, η_n and ζ_n such that

$$\xi_n \le \xi_{n-1}(1+k\eta_{n-1}) + k\zeta_n, \qquad \text{for } n\ge 1, \tag{136}$$

we have, for any $n\ge 2$,

$$\xi_n \le \left(\xi_0 + \sum_{i=1}^n k\zeta_i\right)\exp\left(\sum_{i=0}^{n-1}k\eta_i\right). \tag{137}$$

Lemma 5. Given $k>0$, a positive integer n_0, positive sequences ξ_n, η_n and ζ_n such that

$$\xi_n \le \xi_{n-1}(1+k\eta_{n-1}) + k\zeta_n, \qquad \text{for } n\ge n_0, \tag{138}$$

and given the bounds

$$\begin{aligned}&\sum_{n=k_0}^{N+k_0} k\eta_n \le a_1, \qquad \sum_{n=k_0}^{N+k_0} k\zeta_n \le a_2,\\ &\sum_{n=k_0}^{N+k_0} k\xi_n \le a_3,\end{aligned} \tag{139}$$

for any $k_0\ge n_0$, we have,

$$\xi_n \le \left(\frac{a_3}{Nk} + a_2\right)e^{a_1}, \qquad \forall n\ge N+n_0. \tag{140}$$

We are now able to prove the $\mathbb{V}$-uniform stability of the scheme:

Proposition 3. Let $(u_0,\phi_0)\in\mathbb{V}$ and (u^n,ϕ^n) be the solution of the numerical scheme (51). Also, let k be such that

$$k\le \min\left\{\frac{1}{\kappa},1\right\} =: \kappa_1, \tag{141}$$

where κ is given in (94). Then there exists $K_3\big(\|(u_0,\phi_0)\|_{\mathbb{V}},\|g\|_\infty\big)$, such that

$$\|(u^n,\phi^n)\|_{\mathbb{V}} \le K_3\big(\|(u_0,\phi_0)\|_{\mathbb{V}},\|g\|_\infty\big),\ \forall\, n\ge 0, \tag{142}$$

and for all $i=1,\cdots,m$, we have

$$\begin{aligned}
&\sum_{n=i}^{m}\left(\|u^n-u^{n-1}\|_{\mathcal{U}}^2+|A_\gamma(\phi^n-\phi^{n-1})|_{L^2}^2\right)\\
\le & K_3^2+c(1+K_1^2)K_3^4(m-i+1)k+\frac{4}{\nu_1\lambda_1}\|g\|_\infty^2(m-i+1)k\\
&+c\left(c_f^2K_1^4K_3^2+2(2c_f^2+\alpha_0^{-2}\nu_2^2\gamma^2)K_1^2\right)(m-i+1)k.
\end{aligned} \tag{143}$$

Moreover, there exists $K_4=K_4(\|g\|_\infty)$, independent of the initial data, such that

$$\|(u^n,\phi^n)\|_{\mathbb{V}} \le K_4(\|g\|_\infty),\ \forall n\ge 2N_0+1, \tag{144}$$

where N_0 is given in (111).

Proof. Using (112), we infer from (124)

$$\begin{aligned}
\|(u^n,\phi^n)\|_{\mathbb{V}}^2 \le\ & c_4(1+K_1^2)k\left[K_2\|(u^{n-1},\phi^{n-1})\|_{\mathbb{V}}^2+c_3(|f_\gamma(\phi^n)|_{L^2}^2+|g^n|_{L^2}^2)\right]^2\\
&+\|(u^{n-1},\phi^{n-1})\|_{\mathbb{V}}^2+c_5k\left(|f_\gamma{}'(\phi^n)\nabla\phi^n|_{L^2}^2+|g^n|_{L^2}^2\right)\\
\le\ & \|(u^{n-1},\phi^{n-1})\|_{\mathbb{V}}^2\left[1+2c_4(1+K_1^2)K_2^2k\|(u^{n-1},\phi^{n-1})\|_{\mathbb{V}}^2\right]\\
&+2c_3^2c_4(1+K_1^2)k(|f_\gamma(\phi^n)|_{L^2}^2+|g^n|_{L^2}^2)^2\\
&+c_5k\left(|f_\gamma{}'(\phi^n)\nabla\phi^n|_{L^2}^2+|g^n|_{L^2}^2\right).
\end{aligned} \tag{145}$$

We rewrite (145) in the form

$$\xi_n\le\xi_{n-1}(1+k\eta_{n-1})+k\zeta_n, \tag{146}$$

with

(147)

$$\begin{aligned}
\xi_n=&\|(u^n,\phi^n)\|_{\mathbb{V}}^2, \qquad \eta_n=2c_4(1+K_1^2)K_2^2\|(u^n,\phi^n)\|_{\mathbb{V}}^2,\\
\zeta_n=&\ 2c_3^2c_4(1+K_1^2)(|f_\gamma(\phi^n)|_{L^2}^2+|g^n|_{L^2}^2)^2+c_5\left(|f_\gamma{}'(\phi^n)\nabla\phi^n|_{L^2}^2+|g^n|_{L^2}^2\right),
\end{aligned}$$

and recalling (65) and (66), we compute:

$$\begin{aligned}
\sum_{i=0}^{n-1}k\eta_i=&\ 2c_4(1+K_1^2)K_2^2k\sum_{i=0}^{n-1}\|(u^n,\phi^n)\|_{\mathbb{V}}^2\\
\le&\ \frac{4c_4\mathcal{K}}{\nu_1}(1+K_1^2)K_2^2M_1(\|(u_0,\phi_0)\|_{\mathbb{Y}},\|g\|_\infty,(n-1)k)\\
&+2c_4(1+K_1^2)K_2^2M_2(\|(u_0,\phi_0)\|_{\mathbb{Y}},\|g\|_\infty,(n-1)k)\\
&+2c_4(1+K_1^2)K_2^2k\|(u_0,\phi_0)\|_{\mathbb{V}}^2.
\end{aligned} \tag{148}$$

Now using (45), (6), (7) and recalling (64) and the Sobolev imbedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$, we obtain

$$\begin{aligned}
|f_\gamma(\phi^n)|_{L^2}^2=&\int_\Omega|f_\gamma(\phi^n(x))|^2\,dx=\int_\Omega|f(\phi^n(x))-\alpha_0^{-1}\nu_2\gamma\phi^n(x)|^2\,dx\\
\le&\ 4c_f^2\left(|\Omega|+|\phi^n|_{L^6}^6\right)+2\alpha_0^{-2}\nu_2^2\gamma^2|\phi^n|_{L^2}^2\\
\le&\ cc_f^2\left(|\Omega|+\|\phi^n\|^6\right)+2\alpha_0^{-2}\nu_2^2\gamma^2|\phi^n|_{L^2}^2\\
\le&\ cc_f^2\left(|\Omega|+K_1^6\right)+2\alpha_0^{-2}\nu_2^2\gamma^2K_1^2,
\end{aligned} \tag{149}$$

(150)
$$\begin{aligned}|f_\gamma{}'(\phi^n)\nabla\phi^n|_{L^2}^2 &= \int_\Omega |f_\gamma{}'(\phi^n(x))\nabla\phi^n(x)|^2\,dx\\ &= \int_\Omega |f'(\phi^n(x)) - \alpha_0^{-1}\nu_2\gamma|^2|\nabla\phi^n(x)|^2\,dx\\ &\le 4c_f^2\int_\Omega |\phi^n(x)|^4|\nabla\phi^n(x)|^2\,dx + 2(2c_f^2+\alpha_0^{-2}\nu_2^2\gamma^2)|\nabla\phi^n(x)|_{L^2}^2\\ &\le 4c_f^2|\phi^n|_{L^6}^4|\nabla\phi^n|_{L^6}^2 + 2(2c_f^2+\alpha_0^{-2}\nu_2^2\gamma^2)|\nabla\phi^n|_{L^2}^2\\ &\le cc_f^2\|\phi^n\|^4|A_\gamma\phi^n|_{L^2}^2 + 2(2c_f^2+\alpha_0^{-2}\nu_2^2\gamma^2)|\nabla\phi^n|_{L^2}^2\\ &\le cc_f^2K_1^4|A_\gamma\phi^n|_{L^2}^2 + 2(2c_f^2+\alpha_0^{-2}\nu_2^2\gamma^2)K_1^2.\end{aligned}$$

Recalling (65) and (66), relations (149), (150) give

$$\sum_{i=1}^{n} k\zeta_i \le K_5\big(\|(u_0,\phi_0)\|_{\mathbb{V}}, \|g\|_\infty, nk\big). \tag{151}$$

Then conclusion (137) of Lemma 4 together with (148) and (151) give

$$\xi_n = \|(u^n,\phi^n)\|_{\mathbb{V}}^2 \le K_6^2\big(\|(u_0,\phi_0)\|_{\mathbb{V}}, \|g\|_\infty, 2N_0k\big), \quad \forall n = 1,\cdots,2N_0, \tag{152}$$

for some continuous functions $K_5(\cdot,\cdot,\cdot)$, $K_6(\cdot,\cdot,\cdot)$, increasing in all their arguments.

Now, to derive an upper bound on $\|(u^n,\phi^n)\|_{\mathbb{V}}$, $n \ge 2N_0$, we apply Lemma 5 to (146). For that, we recall that $\|(u^n,\phi^n)\|_{\mathbb{Y}} < \rho_0$, for $n \ge N_0$, and for $k_0 \ge N_0+1$ we compute:

(153)
$$\begin{aligned}\sum_{n=k_0}^{N_0+k_0} k\xi_n &= k\sum_{n=k_0}^{N_0+k_0} \|(u^n,\phi^n)\|_{\mathbb{V}}^2\\ &\le \frac{2\mathcal{K}}{\nu_1}M_1(\rho_0,T_0+1) + M_2(\rho_0,T_0+1) =: a_3, \quad \text{(by (65) and (66))}\end{aligned}$$

(154)
$$\sum_{n=k_0}^{N_0+k_0} k\eta_n \le 2c_4(1+\rho_0^2)K_2^2(\rho_0)\left(\frac{2\mathcal{K}}{\nu_1}M_1(\rho_0,T_0+1) + M_2(\rho_0,T_0+1)\right) =: a_1,$$

Recalling (149), (150), (65) and (66), we find

$$\sum_{n=k_0}^{N_0+k_0} k\zeta_n \le a_2, \tag{155}$$

for some a_2 independent of the initial data. Then Lemma 5 gives

$$\xi_n = \|(u^n,\phi^n)\|_{\mathbb{V}}^2 \le \left(\frac{a_3}{T_0} + a_2\right)\mathrm{e}^{a_1} =: K_4(\|g\|_\infty), \qquad \forall n \ge 2N_0+1, \tag{156}$$

which is exactly conclusion (144). Combining (156) with (151), we obtain conclusion (142) of the proposition.

Taking the sum of (135) with n from i to m and using (142) and (150) gives conclusion (143) and thus the proof of Proposition 3 is complete. □

4. Convergence of Attractors

In this section we address the issue of the convergence of the attractors generated by the discrete system (51) to the attractor generated by the continuous system (46). Whereas for the continuous system (46) one can prove both the existence and uniqueness of the solution (see [34]) and, therefore, define a global attractor for the discrete system (51) one can prove (using Proposition 3) the uniqueness of the solution provided that $k \le \kappa(\|(u_0, \phi_0)\|_{\mathbb{V}})$, for some $\kappa(\|(u_0, \phi_0)\|_{\mathbb{V}}) > 0$. Since the time restriction depends on the initial data, one cannot define a single-valued attractor in the classical sense, and this is why we need to use the attractor theory for the so-called multi-valued mappings. Multi-valued dynamical systems have been investigated by many authors (see, e.g., [1], [2], [5], [35], [37], [38]), but in this article we use the tools developed in [16] (see also, [17]) to study the convergence of the discrete (multi-valued) attractors to the continuous (single-valued) attractor. For convenience, we recall those results in Subsection 4.1, and then we apply them to the two-phase flow model in Subsection 4.2.

4.1. Attractors for multi-valued mappings. Throughout this subsection, we consider $(H, |\cdot|)$ to be a Hilbert space and $\mathbb{T}$ to be either $\mathbb{R}^+ = [0, \infty)$ or $\mathbb{N}$.

Definition 4.1. A one-parameter family of set-valued maps $S(t) : 2^H \to 2^H$ is a **multi-valued semigroup** (m-semigroup) if it satisfies the following properties:

(S.1) $S(0) = I_{2^H}$ (identity in 2^H);
(S.2) $S(t + s) = S(t)S(s)$, for all $t, s \in \mathbb{T}$.

Moreover, the m-semigroup is said to be **closed** if $S(t)$ is a closed map for every $t \in \mathbb{T}$, meaning that if $x_n \to x$ in H and $y_n \in S(t)x_n$ is such that $y_n \to y$ in H, then $y \in S(t)x$. (To simplify the notation, hereafter we have written $S(t)x$ in place of $S(t)\{x\}$.)

Definition 4.2. The **positive orbit** of $\mathcal{B}$, starting at $t \in \mathbb{T}$, is the set

$$\gamma_t(\mathcal{B}) = \bigcup_{\tau \ge t} S(\tau)\mathcal{B},$$

where

$$S(t)\mathcal{B} = \bigcup_{x \in \mathcal{B}} S(t)x.$$

Definition 4.3. For any $\mathcal{B} \in 2^H$, the set

$$\omega(\mathcal{B}) = \bigcap_{t \in \mathbb{T}} \overline{\gamma_t(\mathcal{B})}$$

is called the **ω-limit set** of $\mathcal{B}$.

Definition 4.4. A nonempty set $\mathcal{B} \in 2^H$ is **invariant** for $S(t)$ if

$$S(t)\mathcal{B} = \mathcal{B}, \qquad \forall t \in \mathbb{T}.$$

Definition 4.5. A set $\mathcal{B}_0 \in 2^H$ is an **absorbing set** for the m-semigroup $S(t)$ if for every bounded set $\mathcal{B} \in 2^H$ there exists $t_{\mathcal{B}} \in \mathbb{T}$ such that

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \qquad \forall t \ge t_{\mathcal{B}}.$$

Definition 4.6. A nonempty set $\mathcal{C} \in 2^H$ is **attracting** if for every bounded set $\mathcal{B}$ we have

$$\lim_{t \to \infty} \operatorname{dist}(S(t)\mathcal{B}, \mathcal{C}) = 0,$$

where $\mathrm{dist}(\cdot,\cdot)$ is the **Hausdorff semidistance**, defined as

$$\mathrm{dist}(\mathcal{B},\mathcal{C}) = \sup_{b\in\mathcal{B}} \inf_{c\in\mathcal{C}} |b-c|, \forall\, \mathcal{B},\mathcal{C} \subset H. \tag{157}$$

Definition 4.7. A nonempty compact set $\mathcal{A} \in 2^X$ is said to be the **global attractor** of $S(t)$ if $\mathcal{A}$ is an invariant attracting set.

Remark 4.1. The global attractor, if it exists, is necessarily unique. Moreover, it enjoys the following maximality and minimality properties:

(i) if $\tilde{\mathcal{A}}$ is a bounded invariant set, then $\mathcal{A} \supset \tilde{\mathcal{A}}$;
(ii) if $\tilde{\mathcal{A}}$ is a closed attracting set, then $\mathcal{A} \subset \tilde{\mathcal{A}}$.

Definition 4.8. Given a bounded set $\mathcal{B} \in 2^H$, the **Kuratowski measure of noncompactness** $\alpha(\mathcal{B})$ of $\mathcal{B}$ is defined as

$$\alpha(\mathcal{B}) = \inf\big\{\delta\,:\,\mathcal{B} \text{ has a finite cover by balls of } X \text{ of diameter less than } \delta\big\}.$$

We note that $\alpha(\mathcal{B}) = 0$ if and only if $\overline{\mathcal{B}}$ is compact.

The following theorem, whose proof can be found in [16], gives conditions under which a global attractor exists.

Theorem 2. Suppose that the closed m-semigroup $S(t)$ possesses a bounded absorbing set $\mathcal{B}_0 \in 2^H$ and

$$\lim_{t\to\infty} \alpha(S(t)\mathcal{B}_0) = 0. \tag{158}$$

Then $\omega(\mathcal{B}_0)$ is the global attractor of $S(t)$.

For the purpose of this article, we need to introduce the notion of *discrete m-semigroups*. More precisely, we have the following:

Definition 4.9. Given a set-valued map $S : 2^H \to 2^H$, we define a **discrete m-semigroup** by

$$S(n) = S^n, \qquad \forall n \in \mathbb{N},$$

and we will denote it by $\{S\}_{n\in\mathbb{N}}$ (instead of $\{S^n\}_{n\in\mathbb{N}}$).

Remark 4.2. Given two nonempty sets $\mathcal{B},\mathcal{C} \in 2^H$, we write

$$\mathcal{B}-\mathcal{C} = \{b-c :\, b\in\mathcal{B},\, c\in\mathcal{C}\} \qquad \text{and} \qquad |\mathcal{B}| = \sup_{b\in\mathcal{B}} |b|.$$

In order to prove the convergence of the attractors generated by the discrete system (51) to the attractor generated by the continuous system (46) we will use the following result, whose proof can be found in [16] (see also [42], [41], [17]).

Theorem 3. Let $S(t)$ be a closed m-semigroup, possessing the global attractor $\mathcal{A}$, and for $\kappa_0 > 0$, let $\{S_k,\, 0 < k \le \kappa_0\}_{n\in\mathbb{N}}$ be a family of discrete closed m-semigroups, with global attractor $\mathcal{A}_k$. Assume the following:

(H1) [Uniform boundedness]: there exists $\kappa_1 \in (0,\kappa_0]$ such that the set

$$\mathcal{K} = \bigcup_{k\in(0,\kappa_1]} \mathcal{A}_k$$

is bounded in H;

(H2) [Finite time uniform convergence]: there exists $t_0 \ge 0$ such that for any $T^\star > t_0$,

$$\lim_{k\to 0} \sup_{x\in\mathcal{A}_k,\, nk\in[t_0,T^\star]} |S_k^n x - S(nk)x| = 0.$$

Then

$$\lim_{k\to 0} \mathrm{dist}(\mathcal{A}_k, \mathcal{A}) = 0,$$

where dist denotes the Hausdorff semidistance defined in (157).

4.2. Application: The 3D NS-α model with phase transition. As proven in [34], the system (46) possesses a unique solution and generates a continuous single-valued dynamical system $S(t) : \mathbb{Y} \to \mathbb{Y}$, with global attractor $\mathcal{A}$, bounded in $\mathbb{V}$. Using Proposition 3, one can prove that the discrete system (51) has a unique solution provided that $k \le \kappa_0(\|(u_0, \phi_0)\|_{\mathbb{V}})$, for $\kappa_0(\|(u_0, \phi_0)\|_{\mathbb{V}}) > 0$ given in (62). The dependence of the time step k on the initial data prevents us from defining a single-valued attractor in the classical sense, but this difficulty can be overcome by the theory of the multi-valued attractors recalled above. More precisely, in this part we will prove that there exists $\kappa_1 > 0$ such that if $0 < k \le \kappa_1$, the system (51) generates a closed discrete m-semigroup $\{S_k\}_{n\in\mathbb{N}}$, with global attractors $\mathcal{A}_k$, that will converge to $\mathcal{A}$ in the sense of Theorem 3. For that, we define, for $k > 0$, the multi-valued map $S_k : 2^{\mathbb{Y}} \to 2^{\mathbb{Y}}$ as follows: for every $(v, \psi) \in \mathbb{Y}$,

$$S_k(v, \psi) = \{(u, \phi) \in \mathbb{V} : (u, \phi) \text{ solves (159) below with time-step } k\} :$$

$$\begin{cases} \tilde{u} + \nu_1 k A u + k B_0(u, u) - \mathcal{K} k R_0(\nu_2 A_\gamma \phi, \phi) = \tilde{v} + kg, \\ \tilde{u} = u + \alpha^2 A_0 u, \\ \tilde{v} = v + \alpha^2 A_0 v, \\ \mu = \nu_2 A_\gamma \phi + \alpha_0 f_\gamma(\phi), \\ \phi + k\mu + k B_1(u, \phi) = \psi. \end{cases} \tag{159}$$

Using the same ideas as in [16] (see also [17]), one can prove the following:

Theorem 4. The multi-valued map S_k associated with the implicit Euler scheme (51) generates a closed discrete m-semigroup $\{S_k\}_{n\in\mathbb{N}}$.

Proposition 4. Let $k \le \kappa_1$, where κ_1 is given in Proposition 3. Then there exists a constant $R_1 > 0$ such that for every $R \ge 0$ and $\|(u_0, \phi_0)\|_{\mathbb{Y}} \le R$, there exists $N_1 = N_1(R, k) \ge 0$ such that

$$\|S_k^n(u_0, \phi_0)\|_{\mathbb{V}} \le R_1, \qquad \forall n \ge N_1. \tag{160}$$

Thus, the set

$$\mathcal{B}_1 = \{(u, \phi) \in \mathbb{V} : \|(u, \phi)\|_{\mathbb{V}} \le R_1\}$$

is a $\mathbb{V}$-bounded absorbing set for $\{S_k\}_{n\in\mathbb{N}}$, for $k \in (0, \kappa_1]$.

Proof. Let k be as in the hypothesis. Also, let $R \ge 0$ and $\|(u_0, \phi_0)\|_{\mathbb{Y}} \le R$. By Corollary 3.1, there exists $N_0 = N_0(R, k) \ge 0$ such that

$$\|(u^n, \phi^n)\|_{\mathbb{Y}} \le \sqrt{2}\rho_0, \qquad \forall n \ge N_0. \tag{161}$$

Let $m := N_0 + \left\lfloor \frac{1}{k} \right\rfloor$. By (65) and (66), we have

$$k \sum_{j=N_0+1}^{m} \left(\frac{\nu_1}{2\mathcal{K}} \|u^j\|_{\mathcal{U}}^2 + |A_\gamma \phi^j|^2 \right) \le M_1(\sqrt{2}\rho_0, (m - N_0)k) + M_2(\sqrt{2}\rho_0, (m - N_0)k). \tag{162}$$

Arguing by contradiction, we obtain that there exists $l \in \{N_0 + 1, \cdots, m\}$ such that

$$\frac{\nu_1}{2\mathcal{K}} \|u^l\|_{\mathcal{U}}^2 + |A_\gamma \phi^l|^2 \le \frac{1}{(m - N_0)k} \Big(M_1(\sqrt{2}\rho_0, (m - N_0)k) + M_2(\sqrt{2}\rho_0, (m - N_0)k) \Big). \tag{163}$$

Since $(m - N_0)k = k\lfloor 1/k \rfloor \in [1/2, 1]$ and recalling that M_1 and M_2 are increasing functions of their arguments, the above relation yields

$$\|u^l\|_{\mathcal{U}}^2 + |A_\gamma \phi^l|^2 \leq 2\left(\frac{2\mathcal{K}}{\nu_1} + 1\right)\Big(M_1(\sqrt{2}\rho_0, 1) + M_2(\sqrt{2}\rho_0, 1)\Big) =: R_*^2. \tag{164}$$

Applying Proposition 3 with initial data (u^l, ϕ^l) we obtain that there exists $K_3\big(\|(u^l, \phi^l)\|_{\mathbb{V}}, \|g\|_\infty\big)$, such that

$$\|(u^n, \phi^n)\|_{\mathbb{V}} \leq K_3\big(\|(u^l, \phi^l)\|_{\mathbb{V}}, \|g\|_\infty\big), \forall\, n \geq l. \tag{165}$$

Recalling (164) and the fact that $K_3(\cdot,\cdot)$ is an increasing function of its arguments, relation (165) gives

$$\|(u^n, \phi^n)\|_{\mathbb{V}} \leq K_3\big(R_*, \|g\|_\infty\big) =: R_1, \forall\, n \geq N_1 = N_1(R, k) := N_0 + \left\lfloor \frac{1}{k} \right\rfloor. \tag{166}$$

This completes the proof of Proposition 4. □

We can now prove the existence of the discrete global attractors. More precisely, we have:

Proposition 5. For every $k \in (0, \kappa_1]$, there exists the global attractor $\mathcal{A}_k$ of the m-semigroup $\{S_k\}_{n\in\mathbb{N}}$.

Proof. Let $\mathcal{B}_0 = B_{\mathbb{Y}}(0, \sqrt{2}\rho_0)$ be the bounded absorbing set given in Corollary 3.1. Then Proposition 4 implies that $S_k^n \mathcal{B}_0$ is bounded in $\mathbb{V}$, for all $n \geq N_1(\sqrt{2}\rho_0, k)$. Since $\mathbb{V}$ is compactly embedded in $\mathbb{Y}$, we obtain that $S_k^n \mathcal{B}_0$ is relatively compact in $\mathbb{Y}$ and, thus, $\alpha(S_k^n \mathcal{B}_0) = 0$, for all $n \geq N_1(\sqrt{2}\rho_0, k)$. Condition (158) of Theorem 2 is therefore satisfied and then the existence of the discrete global attractor $\mathcal{A}_k$ follows right away. □

Remark 4.3. Since the global attractor $\mathcal{A}_k$ is the smallest closed attracting set of $\mathbb{Y}$, Proposition 4 implies

$$\mathcal{A}_k \subset \mathcal{B}_1, \forall k \in (0, \kappa_1], \tag{167}$$

and thus

$$\bigcup_{k\in(0,\kappa_1]} \mathcal{A}_k \subset \mathcal{B}_1. \tag{168}$$

Relation (168) shows that condition $(H1)$ of Theorem 3 is satisfied. In order to prove the convergence of the discrete global attractors $\mathcal{A}_k$ to the continuous global attractor $\mathcal{A}$ we also need to prove that condition $(H2)$ is satisfied. To do so, we define, for any function ψ and for any $k > 0$, the following:

$$\psi_k(t) = \psi^n, \quad t \in [(n-1)k, nk), \tag{169}$$

$$\hat{\psi}_k(t) = \psi^n + \frac{t - nk}{k}(\psi^n - \psi^{n-1}), \quad t \in [(n-1)k, nk), \tag{170}$$

and we note that, for any $t \in [(n-1)k, nk)$, we have

$$\hat{\psi}_k(t) - \psi_k(t) = \frac{t - nk}{k}(\psi^n - \psi^{n-1}). \tag{171}$$

With the above notations, the system (51) can be rewritten in the following form (for $t \in [(n-1)k, nk)$):

(172)
$$\begin{cases} \dfrac{d\hat{\tilde{u}}_k(t)}{dt} + \nu_1 A\hat{u}_k(t) + B_0(\hat{u}_k(t), \hat{u}_k(t)) - \mathcal{K}R_0(\nu_2 A_\gamma \hat{\phi}_k(t), \hat{\phi}_k(t)) = g + g_k(t), \\ \mu_k = \nu_2 A_\gamma \phi_k + \alpha f_\gamma(\phi_k), \\ \dfrac{d\hat{\phi}_k(t)}{dt} + \mu_k + B_1(\hat{u}_k(t), \hat{\phi}_k(t)) = B_1(\hat{u}_k(t), \hat{\phi}_k(t)) - B_1(u_k(t), \phi_k(t)), \end{cases}$$

where

$$\begin{aligned} g_k(t) =& \nu_1 A(\hat{u}_k(t) - u_k(t)) + B_0(\hat{u}_k(t), \hat{u}_k(t)) - B_0(u_k(t), u_k(t)) \\ & - \mathcal{K}\left(R_0(\nu_2 A_\gamma \hat{\phi}_k(t), \hat{\phi}_k(t)) - R_0(\nu_2 A_\gamma \phi_k(t), \phi_k(t))\right). \end{aligned} \tag{173}$$

Subtracting (172) from (46) we obtain

$$\begin{cases} \dfrac{d\tilde{\xi}_k(t)}{dt} + \nu_1 A\xi_k(t) + B_0(u(t), u(t)) - B_0(\hat{u}_k(t), \hat{u}_k(t)) \\ \qquad -\mathcal{K}\left(R_0(\nu_2 A_\gamma \phi(t), \phi(t)) - R_0(\nu_2 A_\gamma \hat{\phi}_k(t), \hat{\phi}_k(t))\right) = -g_k(t), \\ \tilde{\xi}_k = \xi_k + \alpha^2 A_0 \xi_k, \\ \mu - \mu_k = \nu_2 A_\gamma(\phi(t) - \phi_k(t)) + \alpha f_\gamma(\phi(t)) - \alpha f_\gamma(\phi_k(t)), \\ \dfrac{d\eta_k(t)}{dt} + \mu - \mu_k + B_1(u(t), \phi(t)) - B_1(\hat{u}_k(t), \hat{\phi}_k(t)) \\ \qquad = -B_1(\hat{u}_k(t), \hat{\phi}_k(t)) + B_1(u_k(t), \phi_k(t)), \end{cases} \tag{174}$$

where

$$\xi_k(t) = u(t) - \hat{u}_k(t), \qquad \eta_k(t) = \phi(t) - \hat{\phi}_k(t). \tag{175}$$

Using the linearity of the operators B_0, B_1 and R_0, the above system can be rewritten as

$$\begin{cases} \dfrac{d\tilde{\xi}_k(t)}{dt} + \nu_1 A\xi_k(t) + B_0(\xi_k(t), u(t)) + B_0(\hat{u}_k(t), \xi_k(t)) \\ \qquad -\mathcal{K}\left(R_0(\nu_2 A_\gamma \eta_k(t), \phi(t)) + R_0(\nu_2 A_\gamma \hat{\phi}_k(t), \eta_k(t))\right) = -g_k(t), \\ \dfrac{d\eta_k(t)}{dt} + \nu_2 A_\gamma \eta_k(t) + B_1(\xi_k(t), \phi(t)) + B_1(\hat{u}_k(t), \eta_k(t)) \\ \qquad = -\alpha\left(f_\gamma(\phi(t)) - f_\gamma(\hat{\phi}_k(t))\right) - h_k(t), \end{cases} \tag{176}$$

where $g_k(t)$ is given in (173) and

$$\begin{aligned} h_k(t) =& B_1(\hat{u}_k(t) - u_k(t), \hat{\phi}_k(t)) + B_1(u_k(t), \hat{\phi}_k(t) - \phi_k(t)) \\ & + \nu_2 A_\gamma(\hat{\phi}_k(t) - \phi_k(t)) + \alpha\left(f_\gamma(\hat{\phi}_k(t)) - f_\gamma(\phi_k(t))\right). \end{aligned} \tag{177}$$

Lemma 6. Let $T^* > 0$ be arbitrarily fixed and let $k < \kappa_1$, where κ_1 is given in Proposition 3. Assume that $(u_0, \phi_0) \in \mathcal{A}_k$ and let (u^n, ϕ^n) be the solution of the numerical scheme (51). Then there exist $K_8(T^*)$ and $K_9(T^*)$ such that

$$\|g_k\|^2_{L^2(0,T^*;\mathcal{U}')} \le kK_8(T^*), \tag{178}$$

and

$$\|h_k\|^2_{L^2(0,T^*;D(A_\gamma)')} \le kK_9(T^*). \tag{179}$$

Proof. Note that since $(u_0, \phi_0) \in \mathcal{A}_k$, relation (167) implies $\|(u_0, \phi_0)\|_{\mathbb{V}} \le R_1$ and then Proposition 3 gives

$$\|(u^n, \phi^n)\|_{\mathbb{V}} \le K_3(R_1),\ \forall n \ge 0. \tag{180}$$

Now let $v \in \mathcal{U}$ be such that $\|v\|_{\mathcal{U}} \leq 1$, and let $t \in [(n-1)k, nk)$ be fixed. Using properties (32) and (37) of the trilinear forms b_0 and b_1, respectively, and recalling (171) and (142), we have the following bounds

$$\begin{aligned}
&|b_0(\hat{u}_k(t), \hat{u}_k(t), v) - b_0(u_k(t), u_k(t), v)| \\
&= |b_0(\hat{u}_k(t) - u_k(t), \hat{u}_k(t), v) + b_0(u_k(t), \hat{u}_k(t) - u_k(t), v)| \\
&\leq c(\|\hat{u}_k(t)\|_{\mathcal{U}} + \|u_k(t)\|_{\mathcal{U}})\|\hat{u}_k(t) - u_k(t)\|_{\mathcal{U}} \\
&\leq cK_3\|u^n - u^{n-1}\|_{\mathcal{U}},
\end{aligned} \tag{181}$$

$$\begin{aligned}
&\mathcal{K}\nu_2|b_1(v, \hat{\phi}_k(t), A_\gamma\hat{\phi}_k(t)) - b_1(v, \phi_k(t), \nu_2 A_\gamma\phi_k(t))| \\
&= \mathcal{K}\nu_2|b_1(v, \hat{\phi}_k(t), A_\gamma(\hat{\phi}_k(t) - \phi_k(t)) + b_1(v, \hat{\phi}_k(t) - \phi_k(t), A_\gamma\phi_k(t))) \\
&\leq c\mathcal{K}\nu_2\left(|A_\gamma\hat{\phi}_k(t)|_{L^2} + |A_\gamma\phi_k(t)|_{L^2}\right)|A_\gamma(\hat{\phi}_k(t) - \phi_k(t))|_{L^2} \\
&\leq cK_3|A_\gamma(\phi^n - \phi^{n-1})|_{L^2}.
\end{aligned} \tag{182}$$

We also have

$$\begin{aligned}
\nu_1|\langle A(\hat{u}_k(t) - u_k(t)), v\rangle| =& ((\hat{u}_k(t) - u_k(t), v))_{\mathcal{U}} \\
\leq& \nu_1\|\hat{u}_k(t) - u_k(t)\|_{\mathcal{U}}\|v\|_{\mathcal{U}} \leq \nu_1\|u^n - u^{n-1}\|_{\mathcal{U}}.
\end{aligned} \tag{183}$$

Relations (181)–(183) imply

$$\|g_k(t)\|_{\mathcal{U}'} \leq cK_3(\|u^n - u^{n-1}\|_{\mathcal{U}} + |A_\gamma(\phi^n - \phi^{n-1})|_{L^2}), \tag{184}$$

and thus, setting $N^* = \lfloor T^\star/k \rfloor$ and recalling that $\|(u_0, \phi_0)\|_{\mathbb{V}} \leq R_1$, we obtain

$$\begin{aligned}
\|g_k\|^2_{L^2(0,T^*;\mathcal{U}')} &= \int_0^{T^*} \|g_k(t)\|^2_{\mathcal{U}'}dt = \sum_{n=1}^{N^*+1} \int_{(n-1)k}^{nk} \|g_k(t)\|^2_{\mathcal{U}'}dt \\
&\leq kK_8(T^*) \quad \text{(by (184) and (143))},
\end{aligned} \tag{185}$$

which proves (178).

Now let $\phi \in D(A_\gamma)$ be such that $|A_\gamma\phi|_{L^2} \leq 1$, and let $t \in [(n-1)k, nk)$ be fixed. Using (36), (37) and (64), we obtain

$$\begin{aligned}
|b_1(\hat{u}_k(t) - u_k(t), \hat{\phi}_k(t), \phi)| &\leq c\|\hat{u}_k(t) - u_k(t)\|_{\mathcal{U}}|A_\gamma\phi|_{L^2}|\hat{\phi}_k(t)|_{L^2} \\
&\leq cK_1\|u^n - u^{n-1}\|_{\mathcal{U}},
\end{aligned} \tag{186}$$

$$\begin{aligned}
|b_1(u_k(t), \hat{\phi}_k(t) - \phi_k(t), \phi)| &\leq c\|u_k(t)\|_{\mathcal{U}}|A_\gamma\phi|_{L^2}|\hat{\phi}_k(t) - \phi_k(t)|_{L^2} \\
&\leq cK_3|\phi^n - \phi^{n-1}|_{L^2}.
\end{aligned} \tag{187}$$

We also have

$$|(A_\gamma(\hat{\phi}_k(t) - \phi_k(t)), \phi)_{L^2}| \leq |\hat{\phi}_k(t) - \phi_k(t)|_{L^2}|A_\gamma\phi|_{L^2} \leq |\phi^n - \phi^{n-1}|_{L^2}. \tag{188}$$

$$\begin{aligned}
|(f_\gamma(\hat{\phi}_k(t)) - f_\gamma(\phi_k(t)), \phi)_{L^2}| &\leq |f_\gamma(\hat{\phi}_k(t)) - f_\gamma(\phi_k(t))|_{L^2}|\phi|_{L^2} \\
&\leq K_7\|\hat{\phi}_k(t) - \phi_k(t)\| \quad \text{(by (59))} \\
&\leq K_7\|\phi^n - \phi^{n-1}\|.
\end{aligned} \tag{189}$$

Gathering relations (186)–(189) we obtain

$$\|h_k(t)\|_{D(A_\gamma)'} \leq c(K_1 + K_3 + K_7)(\|u^n - u^{n-1}\|_{\mathcal{U}} + |A_\gamma(\phi^n - \phi^{n-1})|_{L^2}), \tag{190}$$

and thus setting $N^* = \lfloor T^\star/k \rfloor$ and recalling that $\|(u_0, \phi_0)\|_{\mathbb{V}} \le R_1$, we obtain

$$\|h_k\|^2_{L^2(0,T^*;D(A_\gamma)')} = \int_0^{T^*} \|h_k(t)\|^2_{D(A_\gamma)'} dt = \sum_{n=1}^{N^*+1} \int_{(n-1)\,k}^{nk} \|h_k(t)\|^2_{D(A_\gamma)'}\, dt \le kK_9(T^*) \quad \text{(by (190) and (143))}, \tag{191}$$

which proves (179) and the proof of the lemma is complete. □

We can now prove that condition (H2) of Theorem 3 is satisfied. More precisely, we have

Proposition 6 (Finite time uniform convergence). For any $T^* > 0$ we have

$$\lim_{k\to 0} \sup_{(u_0,\phi_0)\in\mathcal{A}_k,\, nk\in[0,T^*]} \|S^n_k(u_0, \phi_0) - S(nk)(u_0, \phi_0)\|_{\mathbb{Y}} = 0. \tag{192}$$

Proof. Taking the scalar product of the first equation of (176) with $\xi(t)$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\xi_k(t)|^2_{\mathcal{H}} + \nu_1\|\xi_k(t)\|^2_{\mathcal{U}} + b_0(\hat{u}_k(t), \xi_k(t), \xi_k(t))\\ &- \mathcal{K}\Big(b_1(\xi_k(t), \phi(t), \nu_2 A_\gamma \eta_k(t)) + b_1(\xi_k(t), \eta_k(t), \nu_2 A_\gamma \hat{\phi}_k(t))\Big) = -(g_k(t), \xi_k(t))_{L^2}.\end{aligned} \tag{193}$$

Using (32) and (37) we bound the nonlinear terms as follows:

$$\begin{aligned}|b_0(\xi_k(t), u(t), \xi_k(t))| &\le c_b|\xi_k(t)|_{\mathcal{H}}\|u(t)\|_{\mathcal{U}}\|\xi_k(t)\|_{\mathcal{U}}\\ &\le \frac{\nu_1}{8}\|\xi_k(t)\|^2_{\mathcal{U}} + c|\xi_k(t)|^2_{\mathcal{H}}\|u(t)\|^2_{\mathcal{U}},\end{aligned} \tag{194}$$

$$\begin{aligned}&\mathcal{K}|b_1(\xi_k(t), \phi(t), \nu_2 A_\gamma \eta_k(t))|\\ &\le \tilde{c}_b\mathcal{K}\nu_2|\xi_k(t)|^{1/2}_{\mathcal{H}}\|\xi_k(t)\|^{1/2}_{\mathcal{U}}\|\phi(t)\|^{1/2}|A_\gamma\phi(t)|^{1/2}_{L^2}|A_\gamma\eta_k(t)|_{L^2}\\ &\le \frac{\nu_2^2}{4}\mathcal{K}|A_\gamma\eta_k(t)|^2_{L^2} + \frac{\nu_1}{8}\|\xi_k(t)\|^2_{\mathcal{U}} + c|\xi_k(t)|^2_{\mathcal{H}}\|\phi(t)\|^2|A_\gamma\phi(t)|^2_{L^2},\end{aligned} \tag{195}$$

$$\begin{aligned}&\mathcal{K}|b_1(\xi_k(t), \eta_k(t), \nu_2 A_\gamma\hat{\phi}_k(t))\\ &\le \tilde{c}_b\mathcal{K}\nu_2|\xi_k(t)|^{1/2}_{\mathcal{H}}\|\xi_k(t)\|^{1/2}_{\mathcal{U}}\|\eta_k(t)\|^{1/2}|A_\gamma\eta_k(t)|^{1/2}_{L^2}|A_\gamma\hat{\phi}_k(t)|_{L^2}\\ &\le \frac{\nu_2^2}{4}\mathcal{K}|A_\gamma\eta_k(t)|^2_{L^2} + \frac{\nu_1}{8}\|\xi_k(t)\|^2_{\mathcal{U}} + c|\xi_k(t)|_{\mathcal{H}}\|\eta_k(t)\||A_\gamma\hat{\phi}_k(t)|^2_{L^2}.\end{aligned} \tag{196}$$

Using the Cauchy-Schwarz inequality, we bound the right-hand side of (193) as

$$|(g_k(t), \xi_k(t))_{L^2}| \le \|g_k(t)\|_{\mathcal{U}'}\|\xi_k(t)\|_{\mathcal{U}} \le \frac{\nu_1}{8}\|\xi_k(t)\|^2_{\mathcal{U}} + c\|g_k(t)\|^2_{\mathcal{U}'}. \tag{197}$$

Relations (193)–(197) imply

$$\begin{aligned}&\frac{d}{dt}|\xi_k(t)|^2_{\mathcal{H}} + \nu_1\|\xi_k(t)\|^2_{\mathcal{U}}\\ \le &c|\xi_k(t)|^2_{\mathcal{H}}\|u(t)\|^2_{\mathcal{U}} + \frac{\nu_2^2}{2}\mathcal{K}|A_\gamma\eta_k(t)|^2_{L^2} + c|\xi_k(t)|^2_{\mathcal{H}}\|\phi(t)\|^2|A_\gamma\phi(t)|^2_{L^2}\\ &+ c\left(|\xi_k(t)|^2_{\mathcal{H}} + \|\eta_k(t)\|^2_{\mathcal{U}}\right)|A_\gamma\hat{\phi}_k(t)|^2_{L^2} + c\|g_k(t)\|^2_{\mathcal{U}'}.\end{aligned} \tag{198}$$

Now multiplying the second equation of (176) by $\nu_2 A_\gamma\eta_k(t)$ and integrating we obtain

$$\begin{aligned}&\frac{\nu_2}{2}\frac{d}{dt}\|\eta_k(t)\|^2_\gamma + \nu_2^2|A_\gamma\eta_k(t)|^2_{L^2}\\ &+ b_1(\xi_k(t), \phi(t), \nu_2 A_\gamma\eta_k(t)) + b_1(\hat{u}_k(t), \eta_k(t), \nu_2 A_\gamma\eta_k(t))\\ =& -\alpha(f_\gamma(\phi(t)) - f_\gamma(\hat{\phi}_k(t)), \nu_2 A_\gamma\eta_k(t))_{L^2} - (h_k(t), \nu_2 A_\gamma\eta_k(t))_{L^2}.\end{aligned} \tag{199}$$

Using (37) we bound the nonlinear terms as follows:

$$\begin{aligned}
&|b_1(\xi_k(t),\phi(t),\nu_2A_\gamma\eta_k(t))|\\
&\le \tilde{c}_b\nu_2|\xi_k(t)|_{\mathcal{H}}^{1/2}\|\xi_k(t)\|_{\mathcal{U}}^{1/2}\|\phi(t)\|^{1/2}|A_\gamma\phi(t)|_{L^2}^{1/2}|A_\gamma\eta_k(t)|_{L^2}\\
&\le \frac{\nu_2^2}{8}|A_\gamma\eta_k(t)|_{L^2}^2+\frac{\nu_1}{4\mathcal{K}}\|\xi_k(t)\|_{\mathcal{U}}^2+c|\xi_k(t)|_{\mathcal{H}}^2\|\phi(t)\|^2|A_\gamma\phi(t)|_{L^2}^2,
\end{aligned} \tag{200}$$

$$\begin{aligned}
&|b_1(\hat{u}_k(t),\eta_k(t),\nu_2A_\gamma\eta_k(t))|\\
&\le c_b\nu_2|\hat{u}_k(t)|_{\mathcal{H}}^{1/2}\|\hat{u}_k(t)\|_{\mathcal{U}}^{1/2}\|\eta_k(t)\|^{1/2}|A_\gamma\eta_k(t)|_{L^2}^{3/2}\\
&\le \frac{\nu_2^2}{8}|A_\gamma\eta_k(t)|_{L^2}^2+c|\hat{u}_k(t)|_{\mathcal{H}}^2\|\hat{u}_k(t)\|_{\mathcal{U}}^2\|\eta_k(t)\|^2.
\end{aligned} \tag{201}$$

Using the Cauchy-Schwarz inequality, we bound the right-hand side of (199) as

$$\begin{aligned}
&\alpha|(f_\gamma(\phi(t))-f_\gamma(\hat{\phi}_k(t)),\nu_2A_\gamma\eta_k(t))_{L^2}|\\
&\le \nu_2|f_\gamma(\phi(t))-f_\gamma(\hat{\phi}_k(t))|_{L^2}|A_\gamma\eta_k(t)|_{L^2}\\
&\le \frac{\nu_2^2}{8}|A_\gamma\eta_k(t)|_{L^2}^2+c|f_\gamma(\phi(t))-f_\gamma(\hat{\phi}_k(t))|_{L^2}^2\\
&\le \frac{\nu_2^2}{8}|A_\gamma\eta_k(t)|_{L^2}^2+cK_7^2\|\eta_k(t)\|^2 \quad \text{(by (59))},
\end{aligned} \tag{202}$$

$$\begin{aligned}
|(h_k(t),\nu_2A_\gamma\eta_k(t))_{L^2}| &\le \nu_2\|h_k(t)\|_{D(A_\gamma)'}|A_\gamma\eta_k(t))|_{L^2}\\
&\le \frac{\nu_2^2}{8}|A_\gamma\eta_k(t)|_{L^2}^2+c\|h_k(t)\|_{D(A_\gamma)'}^2.
\end{aligned} \tag{203}$$

Relations (199)–(203) yield

$$\begin{aligned}
\nu_2\frac{d}{dt}\|\eta_k(t)\|_\gamma^2+\nu_2^2|A_\gamma\eta_k(t)|_{L^2}^2 &\le \frac{\nu_1}{2\mathcal{K}}\|\xi_k(t)\|_{\mathcal{U}}^2+c|\xi_k(t)|_{\mathcal{H}}^2\|\phi(t)\|^2|A_\gamma\phi(t)|_{L^2}^2\\
&+c|\hat{u}_k(t)|_{\mathcal{H}}^2\|\hat{u}_k(t)\|_{\mathcal{U}}^2\|\eta_k(t)\|^2+cK_7^2\|\eta_k(t)\|^2+c\|h_k(t)\|_{D(A_\gamma)'}^2.
\end{aligned} \tag{204}$$

Dividing (198) by $\mathcal{K}$ and adding the resulting equation to (204) we obtain

$$\begin{aligned}
&\frac{d}{dt}\|(\xi_k(t),\eta_k(t)\|_{\mathbb{Y}}^2+\frac{\nu_1}{2\mathcal{K}}\|\xi_k(t)\|_{\mathcal{U}}^2+\frac{\nu_2^2}{2}|A_\gamma\eta_k(t)|_{L^2}^2\\
\le& c|\xi_k(t)|_{\mathcal{H}}^2\|u(t)\|_{\mathcal{U}}^2+c|\xi_k(t)|_{\mathcal{H}}^2\|\phi(t)\|^2|A_\gamma\phi(t)|_{L^2}^2\\
&+c\left(|\xi_k(t)|_{\mathcal{H}}^2+\|\eta_k(t)\|^2\right)|A_\gamma\hat{\phi}_k(t)|_{L^2}^2\\
&+c\|g_k(t)\|_{V'}^2+c|\hat{u}_k(t)|_{\mathcal{H}}^2\|\hat{u}_k(t)\|_{\mathcal{U}}^2\|\eta_k(t)\|^2\\
&+cK_7^2\|\eta_k(t)\|^2+c\|h_k(t)\|_{D(A_\gamma)'}^2.
\end{aligned} \tag{205}$$

Neglecting some positive terms, the above relation implies

$$\frac{d}{dt}\|(\xi_k(t),\eta_k(t)\|_{\mathbb{Y}}^2 \le \mathcal{G}(t)\|(\xi_k(t),\eta_k(t)\|_{\mathbb{Y}}^2+c\|g_k(t)\|_{V'}^2+c\|h_k(t)\|_{D(A_\gamma)'}^2, \tag{206}$$

where

(207)

$$\mathcal{G}(t)=c\left(\|u(t)\|_{\mathcal{U}}^2+\|\phi(t)\|^2|A_\gamma\phi(t)|_{L^2}^2+|A_\gamma\hat{\phi}_k(t)|_{L^2}^2+|\hat{u}_k(t)|_{\mathcal{H}}^2\|\hat{u}_k(t)\|_{\mathcal{U}}^2+K_7^2\right).$$

By the Gronwall Lemma and using the fact that $\xi_k(0)=\eta_k(0)=0$, we obtain

$$\|(\xi_k(t),\eta_k(t)\|_{\mathbb{Y}}^2 \le c\int_0^t \exp\left(\int_\tau^t \mathcal{G}(s)\,ds\right)\left(\|g_k(\tau)\|_{V'}^2+\|h_k(\tau)\|_{D(A_\gamma)'}^2\right)d\tau. \tag{208}$$

Since the solution (u,ϕ) of the continuous problem is uniformly bounded in $\mathbb{V}$ for all $t \geq 0$ (cf. [23]), we have

$$\sup_{t\geq 0}\sup_{(u_0,\phi_0)\in\mathcal{B}_1}\|S(t)(u_0,\phi_0)\|_{\mathbb{V}} \leq c, \tag{209}$$

and thus

$$\int_\tau^t \mathcal{G}(s)\,ds \leq c\int_\tau^t\left(c+|A_\gamma\hat{\phi}_k(s)|^2_{L^2}+|\hat{u}_k(s)|^2_{\mathcal{H}}\|\hat{u}_k(s)\|^2_{\mathcal{U}}\right)ds. \tag{210}$$

Recalling (170) and (142) we obtain the following bound

$$\begin{aligned}\int_\tau^t |A_\gamma\hat{\phi}_k(s)|^2_{L^2}\,ds &\leq \int_0^{T^\star}|A_\gamma\hat{\phi}_k(s)|^2_{L^2}\,ds\\ &\leq c\sum_{n=1}^{\lfloor T^\star/k\rfloor+1}\int_{(n-1)k}^{nk}\left(|A_\gamma\phi^n|^2_{L^2}+|A_\gamma(\phi^n-\phi^{n-1})|^2_{L^2}\right)ds \leq c_6,\end{aligned} \tag{211}$$

for some constant $c_6 = c_6(T^*) > 0$. Similarly, we have

$$\int_\tau^t |\hat{u}_k(s)|^2_{\mathcal{H}}\|\hat{u}_k(s)\|^2_{\mathcal{U}}\,ds \leq c_7, \tag{212}$$

and thus

$$\int_\tau^t \mathcal{G}(s)\,ds \leq c_8, \tag{213}$$

for some constants $c_7 = c_7(T^*) > 0$ and $c_8 = c_8(T^*) > 0$.

Relations (208), (213), (178) and (179) give

$$\|(\xi_k(t),\eta_k(t)\|^2_{\mathbb{Y}} \leq kc_9, \tag{214}$$

and thus

$$\begin{aligned}&\lim_{k\to 0}\sup_{(u_0,\phi_0)\in\mathcal{A}_k,\,nk\in[0,T^*]}\|S^n_k(u_0,\phi_0)-S(nk)(u_0,\phi_0)\|_{\mathbb{Y}}\\ &=\lim_{k\to 0}\sup_{(u_0,\phi_0)\in\mathcal{A}_k,\,nk\in[0,T^*]}\sup_{(u^n,\phi^n)\in S^n_k(u_0,\phi_0)}\|(u^n,\phi^n)-(u(nk),\phi(nk))\|_{\mathbb{Y}}\\ &=\lim_{k\to 0}\sup_{(u_0,\phi_0)\in\mathcal{A}_k,\,nk\in[0,T^*]}\sup_{(u^n,\phi^n)\in S^n_k(u_0,\phi_0)}\|(\hat{u}_k(nk),\hat{\phi}_k(nk))-(u(nk),\phi(nk))\|_{\mathbb{Y}}\\ &=\lim_{k\to 0}\sup_{(u_0,\phi_0)\in\mathcal{A}_k,\,nk\in[0,T^*]}\sup_{(u^n,\phi^n)\in S^n_k(u_0,\phi_0)}\|(\xi_k(nk),\eta_k(nk))\|_{\mathbb{Y}}=0,\end{aligned} \tag{215}$$

which concludes the proof of the lemma. □

Since conditions (H1) and (H2) of Theorem 3 are satisfied we obtain that the discrete attractors converge to the continuous attractor as the time-step approaches zero. More precisely, we have the following:

Theorem 5. The family of attractors $\{\mathcal{A}_k\}_{k\in(0,\kappa_1]}$ converges, as $k \to 0$, to $\mathcal{A}$, in the following sense:

$$\lim_{k\to 0}\operatorname{dist}(\mathcal{A}_k,\mathcal{A}) = 0,$$

where dist denotes the Hausdorff semidistance in $\mathbb{Y}$, namely

$$\operatorname{dist}(\mathcal{A}_k,\mathcal{A}) = \sup_{x_k\in\mathcal{A}_k}\inf_{x\in\mathcal{A}}\|x_k - x\|_{\mathbb{Y}}.$$

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