INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 15, Number 1-2, Pages 243–259

FULLY DIAGONALIZED CHEBYSHEV SPECTRAL METHODS FOR SECOND AND FOURTH ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

JING-MIN LI, ZHONG-QING WANG*, AND HUI-YUAN LI

Abstract. Fully diagonalized Chebyshev spectral methods for solving second and fourth order elliptic boundary value problems are proposed. They are based on appropriate base functions for the Galerkin formulations which are complete and biorthogonal with respect to certain Sobolev inner product. The suggested base functions lead to diagonalization of discrete systems. Accordingly, both the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier series. Numerical results demonstrate the effectiveness and the spectral accuracy.

Key words. Spectral method, biorthogonal Chebyshev polynomials, elliptic boundary value problems, numerical results.

1. Introduction

Chebyshev spectral methods for solving ordinary/partial differential equations on bounded domains have gained a rapid development during the last few decades, due to the Fast Fourier Transforms (FFT) for Chebyshev polynomials, see [1, 2, 3, 5, 7, 8, 9, 10, 11, 14, 17, 18]. The approximations for the general second and fourth order equations with constant coefficients (see for instance (19) and (31) below) also achieve the optimal convergence rates. However, as pointed out in [16], it is very important to choose an appropriate basis such that the resulting linear system is as simple as possible.

For the second order equation (19), one usually chose the basis in the early years as (cf. [6])

$$V_N = \text{span}\{\phi_2(x), \phi_3(x), \cdots, \phi_N(x)\},$$

where

$$\phi_k(x) = \begin{cases} T_k(x) - T_0(x), & k \text{ even}, \\ T_k(x) - T_1(x), & k \text{ odd}, \end{cases}$$

with $T_k(x)$ being the kth degree Chebyshev polynomial. Unfortunately this basis leads to a linear system with full matrix and hence its usage is virtually prohibited in practice (see [16]). To this end, Shen [16] presented a new basis by choosing $\phi_k(x) = T_k(x) - T_{k+2}(x)$. Note that

$$-(\phi_j'', \phi_k)_\omega = \begin{cases} 2\pi(k+1)(k+2), & j = k, \\ 4\pi(k+1), & j = k+2, k+4, k+6, \cdots, \\ 0, & j > k \text{ or } j+k \text{ odd}, \end{cases}$$

where $\omega(x)$ is the Chebyshev weight function. Hence the matrices of the resulting linear systems are sparse and possess special structures. For the fourth order

Received by the editors January 21, 2017 and, in revised form, March 24, 2017.
2000 *Mathematics Subject Classification.* 76M22, 33C45, 35J40.

*Corresponding author.

equation (31), Shen [16] also proposed a new basis

$$\psi_k(x) = T_k(x) - \frac{2(k+2)}{k+3}T_{k+2}(x) + \frac{k+1}{k+3}T_{k+4}(x), \quad 0 \le k \le N-4.$$

The matrix with the term $(\psi_j'', (\psi_k\omega)'')$ in the resulting linear system is not sparse, but still possesses special structures. Benefiting from these special matrix structures, Shen [16] further derive some efficient algorithms. However, in many cases, people still want to obtain a set of Fourier-like basis functions (see [4, 15]), which are orthogonal to each other with respect to certain Sobolev inner product involving derivatives, and thus the corresponding algebraic system is diagonal (see [19]).

Recently, Liu, Li and Wang [12, 13] constructed the Fourier-like Sobolev orthogonal basis functions based on generalized Laguerre functions, and applied them to the Dirichlet and Robin boundary value problems of second and fourth order elliptic equations on the half line. The numerical experiments indicate the suggested algorithms in [12, 13] are simple, fast and stable, and possess high accuracy.

Motivated by [12, 13, 19], the main purpose of this paper is to construct the Fourier-like basis functions for Chebyshev-Galerkin spectral methods of elliptic boundary value problems on bounded domain. Since the Chebyshev weight function will destroy the symmetry in the weak form of differential equations, we cannot design the basis functions which are mutually orthogonal with respect to the Sobolev inner product. Alternatively, we shall construct two kinds of basis functions which are biorthogonal with respect to the Sobolev inner product originated from the coercive bilinear form of the elliptic equation. For this purpose, we first design four kinds of special polynomials composed of Chebyshev polynomials, from which we further derive the basis functions for fully diagonalized Chebyshev-Galerkin spectral methods, which are biorthogonal with respect to the Sobolev inner product. Then stable and efficient algorithms are proposed for second and fourth order Dirichlet boundary value problems. Particularly, both the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier series, respectively.

The remainder of the paper is organized as follows. In Section 2, we first make conventions on the frequently used notations, and then design four kinds of special polynomials and introduce their basic properties. In Section 3, we construct the biorthogonal basis functions with respect to the Sobolev inner product associated with the second order Dirichlet boundary value problems, and present some numerical results. Section 4 is then devoted to the implementation of the fully diagonalized Chebyshev-Galerkin spectral methods for the fourth order Dirichlet boundary value problems. The final section is for some concluding remarks.

2. Chebyshev polynomials

2.1. Notations and preliminaries. Let $I = (-1, 1)$ and $\chi(x)$ be a weight function. Define

$$L^2_\chi(I) = \{v \mid v \text{ is measurable on } I \text{ and } \|v\|_\chi < \infty\},$$

with the following inner product and norm,

$$(u, v)_\chi = \int_I u(x)v(x)\chi(x)dx, \quad \|v\|_\chi = (v, v)_\chi^{\frac{1}{2}}, \quad \forall u, v \in L^2_\chi(I).$$

For simplicity, we denote $\frac{d^k v}{dx^k} = v^{(k)}$, $\frac{d^2 v}{dx^2} = v''$ and $\frac{dv}{dx} = v'$. For any integer $m \ge 0$, we define

$$H^m_\chi(I) = \{v \mid v^{(k)} \in L^2_\chi(I),\ 0 \le k \le m\},$$

with the following semi-norm and norm,

$$|v|_{m,\chi} = \|v^{(m)}\|_\chi, \qquad \|v\|_{m,\chi} = \Big(\sum_{k=0}^{m} |v|_{k,\chi}^2\Big)^{\frac{1}{2}}.$$

In cases where no confusion arises, χ may be dropped from the notations whenever $\chi(x) \equiv 1$. Specifically, we shall use the Chebyshev weight function $\omega(x) = \frac{1}{\sqrt{1-x^2}}$ in the subsequent sections. We also denote by $\mathbb{P}_k$ the space of polynomials of degree $\leq k$.

2.2. Some basic properties. Let $T_n(x)$, $x \in (-1,1)$ be the standard Chebyshev polynomial of degree n. We recall that $T_n(x)$ is the eigenfunction of the singular Sturm-Liouville problem:

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0, \quad n \geq 0. \tag{1}$$

The Chebyshev polynomials satisfy the following recurrence relations (cf. [20]),

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \tag{2}$$

$$2T_n(x) = \frac{1}{n+1}T_{n+1}'(x) - \frac{1}{n-1}T_{n-1}'(x), \quad n \geq 2, \tag{3}$$

$$(1-x^2)T_n'(x) = \frac{n}{2}T_{n-1}(x) - \frac{n}{2}T_{n+1}(x), \tag{4}$$

with $T_0(x) = 1$ and $T_1(x) = x$.

The Chebyshev polynomials are orthogonal with respect to the weight function $\omega(x)$, namely,

$$(T_n, T_m)_\omega = \frac{\tilde{c}_n \pi}{2}\delta_{m,n}, \tag{5}$$

where $\delta_{m,n}$ is the Kronecker symbol, $\tilde{c}_0 = 2$ and $\tilde{c}_n = 1$ for $n \geq 1$.

Next, let $T_j(x) \equiv 0$ for any $j < 0$. We consider the following four kinds of polynomials which will be used for constructing new biorthogonal basis functions in the fully diagonalized Chebyshev spectral methods.

$$\phi_n(x) = \frac{T_n(x) - T_{n-2}(x)}{2(n-1)}, \qquad n \geq 2, \tag{6}$$

$$\psi_n(x) = \frac{(1+\delta_{n,2})T_n(x) - (2-\delta_{n,3})T_{n-2}(x) + T_{n-4}(x)}{2n}, \qquad n \geq 2, \tag{7}$$

$$\mathcal{R}_n(x) = \frac{(n-3)T_n(x) - 2(n-2)T_{n-2}(x) + (n-1)T_{n-4}(x)}{2(n-1)(n-3)}, \qquad n \geq 4, \tag{8}$$

$$\begin{aligned} \mathcal{S}_n(x) = \ & \frac{1}{2n}\Big[T_n(x) - (4-\delta_{n,5}-\delta_{n,6}-\delta_{n,7})T_{n-2}(x) \\ & +(6-3\delta_{n,4}-4\delta_{n,5}-3\delta_{n,6}-3\delta_{n,7})T_{n-4}(x) \\ & -(4-3\delta_{n,6}-3\delta_{n,7})T_{n-6}(x) + T_{n-8}(x)\Big], \qquad n \geq 4. \end{aligned} \tag{9}$$

Lemma 2.1. *For any $n \geq 2$, we have*

$$\Big(\phi_n(x)\omega(x)\Big)' = T_{n-1}(x)\omega(x). \tag{10}$$

Proof. Clearly,

$$
\begin{aligned}
\Big(\frac{\phi_n(x)}{\sqrt{1-x^2}}\Big)' &= \frac{\sqrt{1-x^2}\phi_n'(x)-(\sqrt{1-x^2})'\phi_n(x)}{1-x^2} \\
&= \frac{T_n'(x)-T_{n-2}'(x)}{2(n-1)\sqrt{1-x^2}}+\frac{x(T_n(x)-T_{n-2}(x))}{2(n-1)(1-x^2)^{\frac{3}{2}}} \\
&\overset{(4)}{=} \frac{T_n'(x)-T_{n-2}'(x)}{2(n-1)\sqrt{1-x^2}}-\frac{xT_{n-1}'(x)}{(n-1)^2\sqrt{1-x^2}} \\
&= \frac{(n-1)(T_n'(x)-T_{n-2}'(x))-2xT_{n-1}'(x)}{2(n-1)^2\sqrt{1-x^2}} \\
&\overset{(2)}{=} \frac{(n-1)(T_n'(x)-T_{n-2}'(x))-(T_n'(x)+T_{n-2}'(x)-2T_{n-1}(x))}{2(n-1)^2\sqrt{1-x^2}} \\
&= \frac{(n-2)T_n'(x)-nT_{n-2}'(x)+2T_{n-1}(x)}{2(n-1)^2\sqrt{1-x^2}} \\
&\overset{(3)}{=} \frac{T_{n-1}(x)}{\sqrt{1-x^2}}.
\end{aligned}
$$

This ends the proof. $\square$

Lemma 2.2. *For any $n \geq 2$, we have*

$$\psi_n'(x) = (1+\delta_{n,2})T_{n-1}(x) - \frac{n-4}{n}T_{n-3}(x). \tag{11}$$

Moreover, for $m, n \geq 2$, the following results hold:

$$((\omega\phi_m)', \psi_n') = \frac{\pi}{2} \times \begin{cases} 1+\delta_{n,2}, & n=m, \\ \frac{4-n}{n}, & n=m+2, \\ 0, & \text{otherwise}; \end{cases} \tag{12}$$

$$(\phi_m, \psi_n')_\omega = \frac{\pi}{4(m-1)} \times \begin{cases} \frac{4-n}{n}, & n=m+3, \\ 1+\frac{n-4}{n}(1+\delta_{n,3}), & n=m+1, \\ -1-\delta_{n,2}, & n=m-1, \\ 0, & \text{otherwise}; \end{cases} \tag{13}$$

$$(\phi_m, \psi_n)_\omega = \frac{\pi}{8n(m-1)} \times \begin{cases} 1, & n=m+4, \\ -3-\delta_{n,4}, & n=m+2, \\ (1+\delta_{n,2})(3-\delta_{n,3}), & n=m, \\ -1-\delta_{n,2}, & n=m-2, \\ 0, & \text{otherwise}. \end{cases} \tag{14}$$

Proof. We first verify the result (11). Clearly, by (3) we know that

$$
\begin{aligned}
T'_n(x) &= 2nT_{n-1}(x) + \frac{n}{n-2}T'_{n-2}(x), \quad n > 2, \\
T'_{n-2}(x) &= 2(n-2)T_{n-3}(x) + \frac{n-2}{n-4}T'_{n-4}(x), \quad n > 4.
\end{aligned} \tag{15}
$$

Hence, a direct computation gives that for $n > 4$,

$$\psi'_n(x) = T_{n-1}(x) - \frac{n-4}{n}T_{n-3}(x).$$

Moreover, it is obvious that the results (11) hold for $n = 2, 3, 4$. This ends the proof of (11). Further, by using (10), (11) and (5), we can derive readily the results (12)-(14). □

Lemma 2.3. *For any $n \geq 4$, we have*

$$
\begin{aligned}
&(\mathcal{R}_n(x)\omega(x))' = (T_{n-1}(x) - T_{n-3}(x))\omega(x), \\
&(\mathcal{R}_n(x)\omega(x))'' = 2(n-2)T_{n-2}(x)\omega(x).
\end{aligned} \tag{16}
$$

Proof. By (10), we deduce that for any $n \geq 4$,

$$
\begin{aligned}
(\mathcal{R}_n(x)\omega(x))' &= \Big(\frac{(n-3)T_n(x) - 2(n-2)T_{n-2}(x) + (n-1)T_{n-4}(x)}{2(n-1)(n-3)}\omega(x)\Big)' \\
&= \Big(\big(\frac{T_n(x) - T_{n-2}(x)}{2(n-1)} - \frac{T_{n-2}(x) - T_{n-4}(x)}{2(n-3)}\big)\omega(x)\Big)' \\
&= \big(T_{n-1}(x) - T_{n-3}(x)\big)\omega(x).
\end{aligned}
$$

This, along with (10), gives that

$$(\mathcal{R}_n(x)\omega(x))'' = \Big(\big(T_{n-1}(x) - T_{n-3}(x)\big)\omega(x)\Big)' = 2(n-2)T_{n-2}(x)\omega(x).$$

Thus, we obtain the desired results. □

Lemma 2.4. *For any $n \geq 4$, we have*

$$
\begin{aligned}
\mathcal{S}'_n(x) = \quad & T_{n-1}(x) - \frac{3n - 8 - 3\delta_{n,5} - 4\delta_{n,6} - 5\delta_{n,7}}{n}T_{n-3}(x) \\
&+ \frac{(3n-16)(1-\delta_{n,6})}{n(1+4\delta_{n,7})}T_{n-5}(x) - \frac{(n-8)(1-\delta_{n,6})(1-\delta_{n,7})}{n}T_{n-7}(x), \\
\mathcal{S}''_n(x) = \quad & 2(n-1)T_{n-2}(x) - \frac{4(n-2-\delta_{n,6})(n-6-2\delta_{n,6})(1+\delta_{n,5})}{n(1+\delta_{n,4})(1-2\delta_{n,7})}T_{n-4}(x) \\
&+ \frac{2(n-7+3\delta_{n,6}+3\delta_{n,7})(n-8+5\delta_{n,6}+5\delta_{n,7})}{n}T_{n-6}(x).
\end{aligned} \tag{17}
$$

Proof. By (15) and (3), we get that for any $n \geq 8$,

(18)

$$\begin{aligned}
\mathcal{S}_n'(x) = &\ \frac{1}{2n}\Big(T_n(x) - 4T_{n-2}(x) + 6T_{n-4}(x) - 4T_{n-6}(x) + T_{n-8}(x)\Big)' \\
= &\ \frac{1}{2n}\Big(2nT_{n-1}(x) + \frac{n}{n-2}T_{n-2}'(x) - 4T_{n-2}'(x) + 6T_{n-4}'(x) - 4T_{n-6}'(x) \\
&\ -2(n-8)T_{n-7}(x) + \frac{n-8}{n-6}T_{n-6}'(x)\Big) \\
= &\ \frac{1}{2n}\Big(2nT_{n-1}(x) - \frac{3n-8}{n-2}T_{n-2}'(x) + 6T_{n-4}'(x) \\
&\ -\frac{3n-16}{n-6}T_{n-6}'(x) - 2(n-8)T_{n-7}(x)\Big) \\
= &\ \frac{1}{2n}\Big(2nT_{n-1}(x) - \frac{3n-8}{n-4}T_{n-4}'(x) - 2(3n-8)T_{n-3}(x) + 6T_{n-4}'(x) \\
&\ -\frac{3n-16}{n-4}T_{n-4}'(x) + 2(3n-16)T_{n-5}(x) - 2(n-8)T_{n-7}(x)\Big) \\
= &\ T_{n-1}(x) - \frac{3n-8}{n}T_{n-3}(x) + \frac{3n-16}{n}T_{n-5}(x) - \frac{n-8}{n}T_{n-7}(x).
\end{aligned}$$

This leads to the first result of (17) for $n \geq 8$. It remains to verify the second result of (17). In fact, by (18) we deduce that for $n \geq 8$,

$$\begin{aligned}
\mathcal{S}_n''(x) &= T_{n-1}'(x) - \frac{3n-8}{n}T_{n-3}'(x) + \frac{3n-16}{n}T_{n-5}'(x) - \frac{n-8}{n}T_{n-7}'(x) \\
&\overset{(15)}{=} 2(n-1)T_{n-2}(x) + \frac{n-1}{n-3}T_{n-3}'(x) - \frac{3n-8}{n}T_{n-3}'(x) + \frac{3n-16}{n}T_{n-5}'(x) \\
&\quad -\frac{(n-7)(n-8)}{n(n-5)}T_{n-5}'(x) + \frac{2(n-7)(n-8)}{n}T_{n-6}(x) \\
&= 2(n-1)T_{n-2}(x) + \frac{2(n-7)(n-8)}{n}T_{n-6}(x) \\
&\quad -\frac{2(n-2)(n-6)}{n}\Big(\frac{T_{n-3}'(x)}{n-3} - \frac{T_{n-5}'(x)}{n-5}\Big) \\
&= 2(n-1)T_{n-2}(x) - \frac{4(n-2)(n-6)}{n}T_{n-4}(x) + \frac{2(n-7)(n-8)}{n}T_{n-6}(x).
\end{aligned}$$

This yields the second result of (17) for $n \geq 8$. Moreover, it is easy to verify the results of (17) for any $4 \leq n \leq 7$. This ends the proof. $\square$

3. A fully diagonalized Chebyshev spectral method for second-order problem

In this section, we propose a fully diagonalized Chebyshev spectral method for solving second-order elliptic boundary value problem. The main idea is to find biorthogonal polynomials with respect to the coercive bilinear form arising from differential equations, such that both the exact solution and the approximate solution can be explicitly expressed as a Fourier series.

3.1. Second-order elliptic boundary value problem. Consider the second-order homogeneous elliptic boundary value problem:

$$\begin{cases} -\epsilon u''(x) + \lambda u(x) = f(x), & x \in I, \\ u(\pm 1) = 0, \end{cases} \tag{19}$$

where $\epsilon > 0$ and $\lambda \geq 0$ are given constants.

Let $H^1_{0,\omega}(I) = \{u \in H^1_\omega(I) : u(\pm 1) = 0\}$. Then, a weak formulation of (19) is to find $u \in H^1_{0,\omega}(I)$ such that

$$\langle u, v\rangle_{1,I} := \epsilon(u', (v\omega)') + \lambda(u, v)_\omega = (f, v)_\omega, \qquad \forall v \in H^1_{0,\omega}(I). \tag{20}$$

Clearly, we have (cf. [7])

$$\langle u, u\rangle_{1,I} \geq c\|u\|^2_{1,\omega}, \qquad |\langle u, v\rangle_{1,I}| \leq c\|u\|_{1,\omega}\|v\|_{1,\omega}.$$

Hence, by Lax-Milgram lemma, (20) admits a unique solution if $f \in (H^1_{0,\omega}(I))'$.

Next, denote

$$\mathbb{P}^0_N := \{u \in \mathbb{P}_N : u(\pm 1) = 0\}.$$

The Chebyshev spectral scheme for (19) is to find $u_N \in \mathbb{P}^0_N$, such that for any $v_N \in \mathbb{P}^0_N$,

$$\langle u_N, v_N\rangle_{1,I} = (f, v_N)_\omega. \tag{21}$$

For an efficient approximation scheme, one usually chooses the linear combination of Chebyshev polynomials $\{T_n(x) - T_{n-2}(x)\}_{n\geq 2}$ as the basis functions for problem (21) (cf. [16]). However, this formulation will only lead to a sparse linear system. Here, we are eager for an ideal approximation scheme whose total stiff matrix, in analogue to the Fourier spectral method for periodic problem, is diagonal.

3.2. The diagonalized Chebyshev spectral method.

The diagonalized Chebyshev spectral method is to construct new basis functions $\{r_n(x)\}_{n\geq 2}$ and $\{s_n(x)\}_{n\geq 2}$, which are biorthogonal with respect to the Sobolev inner product $\langle u, v\rangle_{1,I}$.

Lemma 3.1. *Assume that for any $n \leq 1$, $r_n(x) = s_n(x) \equiv 0$, and*

$$r_2(x) := \psi_2(x) = \tfrac{T_2(x)-T_0(x)}{2} \in \mathbb{P}^0_2, \qquad s_2(x) := \phi_2(x) = \tfrac{T_2(x)-T_0(x)}{2} \in \mathbb{P}^0_2,$$
$$r_3(x) := \psi_3(x) = \tfrac{T_3(x)-T_1(x)}{6} \in \mathbb{P}^0_3, \qquad s_3(x) := \phi_3(x) = \tfrac{T_3(x)-T_1(x)}{4} \in \mathbb{P}^0_3.$$

Let $r_n(x) \in \mathbb{P}^0_n$ and $s_n(x) \in \mathbb{P}^0_n$, whose leading coefficients are respectively the same as the polynomials $\psi_n(x)$ and $\phi_n(x)$, satisfying the biorthogonality with respect to the Sobolev inner product $\langle\cdot,\cdot\rangle_{1,I}$,

$$\langle r_n, s_m\rangle_{1,I} = \eta_m\delta_{m,n}, \quad m, n \geq 2. \tag{22}$$

Then the following recurrence relations hold:

$$\psi_n(x) = r_n(x) + a_{n-2}r_{n-2}(x) + b_{n-4}r_{n-4}(x), \qquad \forall n \geq 2, \tag{23}$$

$$\phi_n(x) = s_n(x) + c_{n-2}s_{n-2}(x), \qquad \forall n \geq 2, \tag{24}$$

where $\phi_n(x)$ *and* $\psi_n(x)$ *are defined in* (6) *and* (7), *and*

$$\begin{aligned}\eta_n = \quad & \frac{(1+\delta_{n,2})\pi\epsilon}{2} + \frac{(1+\delta_{n,2})(3-\delta_{n,3})\pi\lambda}{8n(n-1)} \\ & -\frac{(1+\delta_{n,4})\pi\lambda}{8(n-1)(n-2)}\Big(\frac{(n-4)\pi\epsilon}{2n\eta_{n-2}} + \frac{(3+\delta_{n,4})\pi\lambda}{8n(n-3)\eta_{n-2}}\Big) \\ & +\frac{(1+\delta_{n,4})\pi\lambda}{8(n-1)(n-2)}\Big(\frac{(1+\delta_{n,6})\pi^2\lambda^2}{64n(n-3)(n-4)(n-5)\eta_{n-4}\eta_{n-2}}\Big), \qquad n \geq 6,\end{aligned}$$

$$b_{n-4} = \frac{\pi\lambda}{8n(n-5)\eta_{n-4}}, \qquad n \geq 6,$$

$$c_{n-2} = -\frac{(1+\delta_{n,4})\pi\lambda}{8(n-1)(n-2)\eta_{n-2}}, \qquad n \geq 4,$$

$$\begin{aligned}a_{n-2} = \quad & -\frac{(n-4)\pi\epsilon}{2n\eta_{n-2}} - \frac{(3+\delta_{n,4})\pi\lambda}{8n(n-3)\eta_{n-2}} \\ & +\frac{(1+\delta_{n,6})\pi^2\lambda^2}{64n(n-3)(n-4)(n-5)\eta_{n-2}\eta_{n-4}}, \qquad n \geq 6.\end{aligned}$$

Particularly,

$$a_2 = -\frac{\lambda}{8\epsilon+3}, \qquad a_3 = -\frac{24\epsilon+9\lambda}{120\epsilon+10\lambda}, \qquad \eta_2 = \pi\epsilon + \frac{3\pi\lambda}{8}, \qquad \eta_3 = \frac{\pi\epsilon}{2} + \frac{\pi\lambda}{24},$$
$$\eta_4 = \frac{\pi\epsilon}{2} + \frac{\pi\lambda}{32} - \frac{\pi\lambda^2}{192\epsilon+72\lambda}, \qquad \eta_5 = \frac{\pi\epsilon}{2} + \frac{3\pi\lambda}{160} - \frac{8\pi\epsilon\lambda+3\pi\lambda^2}{3840\epsilon+320\lambda}.$$

Proof. Let

$$\psi_n(x) = r_n(x) + \sum_{k=2}^{n-1} a_{n,k} r_k(x), \qquad \phi_n(x) = s_n(x) + \sum_{k=2}^{n-1} c_{n,k} s_k(x), \qquad n \geq 4. \tag{25}$$

We first use mathematical induction to verify (23) and (24). According to the definitions,

$$\begin{aligned}\psi_4 &= \tfrac{T_4(x)-2T_2(x)+T_0(x)}{8} = r_4(x) + a_{4,3}r_3(x) + a_{4,2}r_2(x), \\ \phi_4 &= \tfrac{T_4(x)-T_2(x)}{6} = s_4(x) + c_{4,3}s_3(x) + c_{4,2}s_2(x).\end{aligned}$$

Then, by (11), (10), (6), (7) and (5) we know that

$$\begin{aligned}\langle \psi_4, \phi_3 \rangle_{1,I} &= \epsilon(\psi_4', (\phi_3\omega)') + \lambda(\psi_4, \phi_3)_\omega \\ &= \epsilon(T_3, T_2)_\omega + \lambda(\tfrac{T_4-2T_2+T_0}{8}, \tfrac{T_3-T_1}{4})_\omega = 0.\end{aligned}$$

On the other hand, by (22) we get

$$\langle \psi_4, \phi_3 \rangle_{1,I} = \langle r_4 + a_{4,3}r_3 + a_{4,2}r_2, s_3 \rangle_{1,I} = a_{4,3}\eta_3.$$

Hence, we have $a_{4,3} = 0$, which means $\psi_4(x) = r_4(x) + a_{4,2}r_2(x)$. Similarly, we have

$$\begin{aligned}\langle \psi_3, \phi_4 \rangle_{1,I} &= \epsilon(\psi_3', (\phi_4\omega)') + \lambda(\psi_3, \phi_4)_\omega \\ &= \epsilon(T_2 + \tfrac{T_0}{3}, T_3)_\omega + \lambda(\tfrac{T_3-T_1}{6}, \tfrac{T_4-T_2}{6})_\omega = 0,\end{aligned}$$

and

$$\langle \psi_3, \phi_4 \rangle_{1,I} = \langle r_3, s_4 + c_{4,3}s_3 + c_{4,2}s_2 \rangle_{1,I} = c_{4,3}\eta_3. \tag{26}$$

Thereby, we have $c_{4,3} = 0$, which means $\phi_4(x) = s_4(x) + c_{4,2}s_2(x)$, In the same manner, we can verify the results of (23) and (24) for $n = 5, 6$.

Next, assume that for any $2 \le k \le n-1$ and $n \ge 7$,

$$\psi_k(x) = r_k(x) + a_{k,k-2}r_{k-2}(x) + a_{k,k-4}r_{k-4}(x), \quad \phi_k(x) = s_k(x) + c_{k,k-2}s_{k-2}(x).$$

We shall prove that for $n \ge 7$,

$$\psi_n(x) = r_n(x) + a_{n,n-2}r_{n-2}(x) + a_{n,n-4}r_{n-4}(x), \quad \phi_n(x) = s_n(x) + c_{n,n-2}s_{n-2}(x).$$

Clearly, by (7) and (25) we have for $n \ge 7$,

$$\langle \psi_n, \phi_j \rangle_{1,I} = \langle r_n + \sum_{k=2}^{n-1} a_{n,k}r_k, \phi_j \rangle_{1,I} = \Big\langle \frac{T_n - 2T_{n-2} + T_{n-4}}{2n}, \phi_j \Big\rangle_{1,I}.$$

Taking $j = 2, 3, \cdots, n-5$, successively and using the induction assumption, we derive readily that $a_{n,j} = 0$ for any $2 \le j \le n-5$. Hence, we have

$$\psi_n(x) = r_n(x) + a_{n,n-1}r_{n-1}(x) + a_{n,n-2}r_{n-2}(x) + a_{n,n-3}r_{n-3}(x) + a_{n.n-4}r_{n-4}(x).$$

Moreover, by (11), (10), (7), (6), (5), (22) and the induction assumption, we know that for $n \ge 7$,

$$\begin{aligned} \langle \psi_n, \phi_{n-3} \rangle_{1,I} &= \epsilon(\psi_n', (\phi_{n-3}\omega)') + \lambda(\psi_n, \phi_{n-3})_\omega \\ &= \epsilon(T_{n-1} - \tfrac{n-4}{n}T_{n-3}, T_{n-4})_\omega + \lambda(\tfrac{T_n - 2T_{n-2} + T_{n-4}}{2n}, \tfrac{T_{n-3} - T_{n-5}}{2(n-4)})_\omega = 0, \end{aligned}$$

and

$$\begin{aligned} \langle \psi_n, \phi_{n-3} \rangle_{1,I} &= \langle r_n + a_{n,n-1}r_{n-1} + a_{n,n-2}r_{n-2} + a_{n,n-3}r_{n-3} \\ &\quad + a_{n,n-4}r_{n-4}, s_{n-3} + c_{n-3,n-5}s_{n-5} \rangle_{1,I} \\ &= a_{n,n-3}\eta_{n-3}. \end{aligned}$$

Hence, we get $a_{n,n-3} = 0$. Similarly, we have $a_{n,n-1} = 0$. This means

$$\psi_n(x) = r_n(x) + a_{n,n-2}r_{n-2}(x) + a_{n,n-4}r_{n-4}(x).$$

In the same manner, we derive

$$\phi_n(x) = s_n(x) + c_{n,n-2}s_{n-2}(x).$$

For simplicity of notations, we take $a_{n-2} := a_{n,n-2}$, $b_{n-4} := a_{n,n-4}$ and $c_{n-2} := c_{n,n-2}$, then we obtain the results (23) and (24).

It remains to confirm the coefficients a_{n-2}, b_{n-4}, c_{n-2} and η_n. By using (12), (14) and (22), we know that for $n \ge 4$,

$$\langle \psi_{n-2}, \phi_n \rangle_{1,I} = \epsilon(\psi_{n-2}', (\phi_n\omega)') + \lambda(\psi_{n-2}, \phi_n)_\omega = -\frac{(1+\delta_{n,4})\pi\lambda}{8(n-1)(n-2)}, \tag{27}$$

and

$$\langle \psi_{n-2}, \phi_n \rangle_{1,I} = \langle r_{n-2} + a_{n-4}r_{n-4} + b_{n-6}r_{n-6}, s_n + c_{n-2}s_{n-2} \rangle_{1,I} = c_{n-2}\eta_{n-2}. \tag{28}$$

Hence

$$c_{n-2}\eta_{n-2} = -\frac{(1+\delta_{n,4})\pi\lambda}{8(n-1)(n-2)}, \qquad n \ge 4. \tag{29}$$

Similarly, by using

$$\langle \psi_j, \phi_n \rangle_{1,I} = \epsilon(\psi_j', (\phi_n\omega)') + \lambda(\psi_j, \phi_n)_\omega = \langle r_j + a_{j-2}r_{j-2} + b_{j-4}r_{j-4}, s_n + c_{n-2}s_{n-2} \rangle_{1,I}$$

and taking $j = n, n+2, n+4$, respectively, we get that for $n \geq 2$,

$$
\begin{aligned}
&\eta_n + a_{n-2}c_{n-2}\eta_{n-2} = \frac{(1+\delta_{n,2})\pi\epsilon}{2} + \frac{(1+\delta_{n,2})(3-\delta_{n,3})\pi\lambda}{8n(n-1)},\\
&a_n\eta_n + b_{n-2}c_{n-2}\eta_{n-2} = -\frac{(n-2)\pi\epsilon}{2(n+2)} - \frac{(3+\delta_{n,2})\pi\lambda}{8(n-1)(n+2)},\\
&b_n\eta_n = \frac{\pi\lambda}{8(n-1)(n+4)}.
\end{aligned}
\tag{30}
$$

A combination of (29) and (30) gives that

$$
\begin{aligned}
\eta_n = \;& \frac{(1+\delta_{n,2})\pi\epsilon}{2} + \frac{(1+\delta_{n,2})(3-\delta_{n,3})\pi\lambda}{8n(n-1)}\\
&-\frac{(1+\delta_{n,4})\pi\lambda}{8(n-1)(n-2)}\Big(\frac{(n-4)\pi\epsilon}{2n\eta_{n-2}} + \frac{(3+\delta_{n,4})\pi\lambda}{8n(n-3)\eta_{n-2}}\Big)\\
&+\frac{(1+\delta_{n,4})\pi\lambda}{8(n-1)(n-2)}\Big(\frac{(1+\delta_{n,6})\pi^2\lambda^2}{64n(n-3)(n-4)(n-5)\eta_{n-4}\eta_{n-2}}\Big), \qquad n \geq 6,
\end{aligned}
$$

$$
b_n = \frac{\pi\lambda}{8(n-1)(n+4)\eta_n}, \qquad n \geq 2,
$$

$$
\begin{aligned}
c_{n-2} = & -\frac{(1+\delta_{n,4})\pi\lambda}{8(n-1)(n-2)\eta_{n-2}}, \qquad n \geq 4,\\
a_n = \;& -\frac{(n-2)\pi\epsilon}{2(n+2)\eta_n} - \frac{(3+\delta_{n,2})\pi\lambda}{8(n-1)(n+2)\eta_n}\\
&+\frac{(1+\delta_{n,4})\pi^2\lambda^2}{64(n-1)(n-2)(n-3)(n+2)\eta_{n-2}\eta_n}, \qquad n \geq 4.
\end{aligned}
$$

Moreover, by (11), (10), (7), (6), (5) and (22), we derive

$$
\eta_2 = \langle r_2, s_2\rangle_{1,I} = \langle \psi_2, \phi_2\rangle_{1,I} = \pi\epsilon + \frac{3\pi\lambda}{8}, \quad \eta_3 = \langle r_3, s_3\rangle_{1,I} = \langle \psi_3, \phi_3\rangle_{1,I} = \frac{\pi\epsilon}{2} + \frac{\pi\lambda}{24}.
$$

Similarly

$$
\begin{aligned}
a_2\eta_2 &= \langle r_4 + a_2 r_2, s_2\rangle_{1,I} = \langle \psi_4, \phi_2\rangle_{1,I} = \epsilon(\psi_4', (\phi_2\omega)') + \lambda(\psi_4, \phi_2)_\omega\\
&= \epsilon(T_3, T_1)_\omega + \lambda(\tfrac{T_4-2T_2+T_0}{8}, \tfrac{T_2-T_0}{2})_\omega = -\frac{\pi\lambda}{8},\\
a_3\eta_3 &= \langle r_5 + a_3 r_3, s_3\rangle_{1,I} = \langle \psi_5, \phi_3\rangle_{1,I} = \epsilon(\psi_5', (\phi_3\omega)') + \lambda(\psi_5, \phi_3)_\omega\\
&= \epsilon(T_4 - \tfrac{1}{5}T_2, T_2)_\omega + \lambda(\tfrac{T_5-2T_3+T_1}{10}, \tfrac{T_3-T_1}{4})_\omega = -\frac{\pi\epsilon}{10} - \frac{3\pi\lambda}{80}.
\end{aligned}
$$

Therefore

$$
a_2 = -\frac{\lambda}{8\epsilon + 3\lambda}, \qquad a_3 = -\frac{24\epsilon + 9\lambda}{120\epsilon + 10\lambda}.
$$

In the same manner, we obtain

$$
\eta_4 = \frac{\pi\epsilon}{2} + \frac{\pi\lambda}{32} - \frac{\pi\lambda^2}{192\epsilon + 72\lambda}, \qquad \eta_5 = \frac{\pi\epsilon}{2} + \frac{3\pi\lambda}{160} - \frac{8\pi\epsilon\lambda + 3\pi\lambda^2}{3840\epsilon + 320\lambda}.
$$

This ends the proof. $\square$

Theorem 3.1. *Let u and u_N be the solutions of* (19) *and* (21), *respectively. Then both u and u_N have the explicit representations in* $\{r_n(x)\}$,

$$u(x) = \sum_{k=2}^{\infty} \hat{u}_k r_k(x), \qquad u_N(x) = \sum_{k=2}^{N} \hat{u}_k r_k(x),$$
$$\hat{u}_k = \frac{1}{\eta_k}\langle u, s_k\rangle_{1,I} = \frac{1}{\eta_k}(f, s_k)_\omega.$$

3.3. Numerical results. In this subsection, we examine the effectiveness and the accuracy of the fully diagonalized Chebyshev spectral method for solving second-order elliptic equations on $I = (-1, 1)$. The righthand term $\{(f, s_k)_\omega\}_{k=2}^N$, as well as the discrete errors, is evaluated through the Chebyshev-Gauss quadrature with $2N + 1$ nodes.

We take $\epsilon = 1$ and $\lambda = 1$ in (19) and consider the following two cases:

- $u(x) = e^{-x}(x^2 - 1)$, where the solution is smooth. In Figure 1, we plot the $\log_{10}$ of the discrete L^2-errors.
- $u(x) = \sin(10x)(x^2 - 1)$, where the solution is oscillating. In Figure 2, we plot the $\log_{10}$ of the discrete L^2-errors.

The two near straight lines indicate that the L^2-errors decay like e^{-cN}.

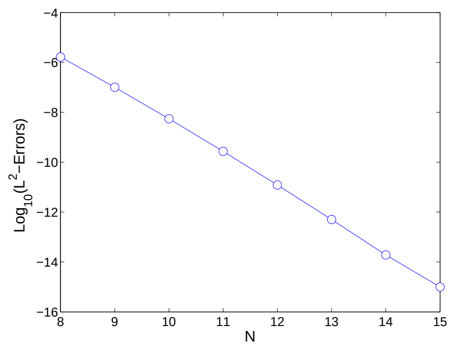

FIGURE 1. L^2-errors.

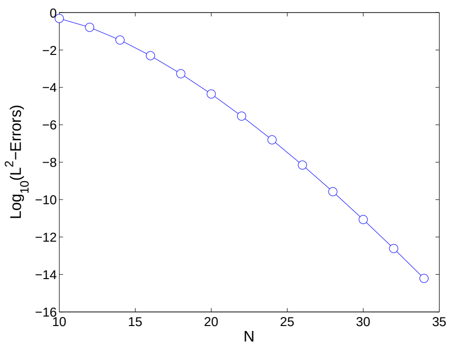

FIGURE 2. L^2-errors.

4. A fully diagonalized Chebyshev spectral method for fourth-order problem

In this section, we propose a fully diagonalized Chebyshev spectral method for solving fourth-order elliptic boundary value problem. We shall also find biorthogonal polynomials with respect to the coercive bilinear form, such that both the exact solution and the approximate solution can be explicitly expressed as a Fourier series.

4.1. Fourth-order elliptic boundary value problem. Consider the fourth order homogeneous elliptic boundary value problem:

$$\begin{cases} u^{(4)}(x) - r_1 u''(x) + r_2 u(x) = f(x), & x \in I, \\ u(\pm 1) = u'(\pm 1) = 0, \end{cases} \tag{31}$$

where $r_1 \geq 0$ and $r_2 \geq 0$ are given constants.

Let $H^2_{0,\omega}(I) = \{u \in H^2_\omega(I) : u(\pm 1) = u'(\pm 1) = 0\}$. Then, a weak formulation of (31) is to find $u \in H^2_{0,\omega}(I)$ such that

$$\langle u, v\rangle_{2,I} := (u'', (v\omega)'') + r_1(u', (v\omega)') + r_2(u, v)_\omega = (f, v)_\omega, \quad \forall v \in H^2_{0,\omega}(I). \tag{32}$$

Next, denote

$$\mathbb{X}_N^0 := \{u \in \mathbb{P}_N : u(\pm 1) = u'(\pm 1) = 0\}.$$

The Chebyshev spectral scheme for (32) is to find $u_N \in \mathbb{X}_N^0$, such that for any $v_N \in \mathbb{X}_N^0$,

$$\langle u_N, v_N \rangle_{2,I} = (f, v_N)_\omega. \tag{33}$$

4.2. The diagonalized Chebyshev spectral method. The diagonalized Chebyshev spectral method is to construct new basis functions $\{p_n(x)\}_{n\geq 4}$ and $\{q_n(x)\}_{n\geq 4}$, which are biorthogonal with respect to the Sobolev inner product $\langle u, v\rangle_{2,I}$.

Lemma 4.1. *Assume that for any $n \leq 3$, $p_n(x) = q_n(x) \equiv 0$, and*

$$\begin{aligned}
p_4(x) &:= \mathcal{S}_4(x) = \tfrac{T_4(x) - 4T_2(x) + 3T_0(x)}{8} \in \mathbb{X}_4^0,\\
q_4(x) &:= \mathcal{R}_4(x) = \tfrac{T_4(x) - 4T_2(x) + 3T_0(x)}{6} \in \mathbb{X}_4^0,\\
p_5(x) &:= \mathcal{S}_5(x) = \tfrac{T_5(x) - 3T_3(x) + 2T_1(x)}{10} \in \mathbb{X}_5^0,\\
q_5(x) &:= \mathcal{R}_5(x) = \tfrac{T_5(x) - 3T_3(x) + 2T_1(x)}{8} \in \mathbb{X}_5^0.
\end{aligned}$$

Let $p_n(x) \in \mathbb{X}_n^0$ and $q_n(x) \in \mathbb{X}_n^0$, whose leading coefficients are respectively the same as the polynomials $\mathcal{S}_n(x)$ and $\mathcal{R}_n(x)$, satisfying the biorthogonality with respect to the Sobolev inner product $\langle \cdot, \cdot \rangle_{2,I}$,

$$\langle p_n, q_m \rangle_{2,I} = \rho_m \delta_{m,n}, \quad m, n \geq 4. \tag{34}$$

Then the following recurrence relations hold:

$$\begin{aligned}
\mathcal{S}_n(x) =& p_n(x) + a_{n-2}p_{n-2}(x) + b_{n-4}p_{n-4}(x)\\
&+ c_{n-6}p_{n-6}(x) + d_{n-8}p_{n-8}(x), \qquad \forall n \geq 4,
\end{aligned} \tag{35}$$

$$\mathcal{R}_n(x) = q_n(x) + e_{n-2}q_{n-2}(x) + h_{n-4}q_{n-4}(x), \qquad \forall n \geq 4, \tag{36}$$

where $\mathcal{R}_n(x)$ *and* $\mathcal{S}_n(x)$ *are defined in* (8) *and* (9)*,* $\rho_n \equiv 0$ *for any* $n \le 3$ *and*

(i). $\rho_n + a_{n-2}e_{n-2}\rho_{n-2} + b_{n-4}h_{n-4}\rho_{n-4}$

$$= 2\pi(n-1)(n-2) + \frac{(4n-8-3\delta_{n,5}-4\delta_{n,6}-5\delta_{n,7})\pi r_1}{2n}$$

$$+ \frac{\pi r_2}{8n(n-1)} + \frac{(n-2)(4-\delta_{n,5}-\delta_{n,6}-\delta_{n,7})\pi r_2}{4n(n-1)(n-3)}$$

$$+ \frac{(6-3\delta_{n,4}-4\delta_{n,5}-3\delta_{n,6}-3\delta_{n,7})(1+\delta_{n,4})\pi r_2}{8n(n-3)}, \quad n \ge 4,$$

(ii). $a_n\rho_n + b_{n-2}e_{n-2}\rho_{n-2} + c_{n-4}h_{n-4}\rho_{n-4}$

$$= -\frac{4\pi(n-\delta_{n,4})(n-4-2\delta_{n,4})(n-2)}{(n+2)(1-2\delta_{n,5})} - \frac{(3n-10)(1-\delta_{n,4})\pi r_1}{2(n+2)(1+4\delta_{n,5})}$$

$$- \frac{(3n-2-4\delta_{n,4}-5\delta_{n,5})\pi r_1}{2(n+2)} - \frac{(4-\delta_{n,4}-\delta_{n,5})\pi r_2}{8(n-1)(n+2)}$$

$$- \frac{(6-3\delta_{n,4}-3\delta_{n,5})(n-2)\pi r_2}{4(n-1)(n+2)(n-3)} - \frac{(4-3\delta_{n,4}-3\delta_{n,5})(1+\delta_{n,4})\pi r_2}{8(n+2)(n-3)}, \; n \ge 4,$$

(iii). $e_n\rho_n + a_{n-2}h_{n-2}\rho_{n-2}$

$$= -\frac{\pi r_1}{2} - \frac{\pi r_2}{4(n-1)(n+1)} - \frac{(4-\delta_{n,5}-\delta_{n,6}-\delta_{n,7})\pi r_2}{8n(n-1)}, \quad n \ge 4,$$

(iv). $h_n\rho_n = \dfrac{\pi r_2}{8n(n+1)}, \quad n \ge 4,$

(v). $b_n\rho_n + c_{n-2}e_{n-2}\rho_{n-2} + d_{n-4}h_{n-4}\rho_{n-4}$

$$= \frac{2(n-2)(n-3)(n-4)\pi}{n+4} + \frac{(2n-4)\pi r_1}{n+4}$$

$$+ \frac{(7n-17)\pi r_2}{4(n-1)(n-3)(n+4)} + \frac{(1+\delta_{n,4})\pi r_2}{8(n-3)(n+4)}, \quad n \ge 4,$$

(vi). $c_n\rho_n + d_{n-2}e_{n-2}\rho_{n-2} = -\dfrac{(n-2)\pi r_1}{2(n+6)} - \dfrac{(3n-8)\pi r_2}{4(n-1)(n-3)(n+6)}, \quad n \ge 4,$

(vii). $d_n\rho_n = \dfrac{\pi r_2}{8(n-1)(n+8)}, \quad n \ge 4.$

Proof. Let

$$\text{(37)} \quad \mathcal{S}_n(x) = p_n(x) + \sum_{k=4}^{n-1} a_{n,k}p_k(x), \qquad \mathcal{R}_n(x) = q_n(x) + \sum_{k=4}^{n-1} c_{n,k}q_k(x), \qquad n \ge 6.$$

We first use mathematical induction to verify (35) and (36). According to the definitions of (8) and (9),

$$\mathcal{S}_6(x) = \frac{T_6(x) - 3T_4(x) + 3T_2(x) - T_0(x)}{12} = p_6(x) + a_{6,5}p_5(x) + a_{6,4}p_4(x),$$
$$\mathcal{R}_6(x) = \frac{3T_6(x) - 8T_4(x) + 5T_2(x)}{30} = q_6(x) + c_{6,5}q_5(x) + c_{6,4}q_4(x).$$

Then, by (16), (17), (8), (9) and (5) we know that

$$\begin{aligned}\langle \mathcal{S}_6, \mathcal{R}_5\rangle_{2,I} &= (\mathcal{S}_6'', (\mathcal{R}_5\omega)'') + r_1(\mathcal{S}_6', (\mathcal{R}_5\omega)') + r_2(\mathcal{S}_6, \mathcal{R}_5)_\omega \\ &= (10T_4 + 4T_2 + 2T_0(x), 6T_3)_\omega + r_1(T_5 - T_3, T_4 - T_2)_\omega \\ &\quad + r_2\Big(\frac{T_6(x) - 3T_4(x) + 3T_2(x) - T_0}{12}, \frac{T_5 - 3T_3 + 2T_1}{8}\Big)_\omega \\ &= 0.\end{aligned}$$

On the other hand, by (34) we get

$$\langle \mathcal{S}_6, \mathcal{R}_5\rangle_{2,I} = \langle p_6 + a_{6,5}p_5 + a_{6,4}p_4, q_5\rangle_{2,I} = a_{6,5}\rho_5.$$

Hence, we have $a_{6,5} = 0$, which means $\mathcal{S}_6(x) = p_6(x) + a_{6,4}p_4(x)$. Similarly, we have

$$\begin{aligned}\langle \mathcal{S}_5, \mathcal{R}_6\rangle_{2,I} &= (\mathcal{S}_5'', (\mathcal{R}_6\omega)'') + r_1(\mathcal{S}_5', (\mathcal{R}_6\omega)') + r_2(\mathcal{S}_5, \mathcal{R}_6)_\omega \\ &= (8T_3 + \tfrac{24}{5}T_1, 8T_4)_\omega + r_1(T_4 - \tfrac{4}{5}T_2 - \tfrac{1}{5}T_0, T_5 - T_3)_\omega \\ &\quad + r_2(\tfrac{1}{10}T_5 - \tfrac{3}{10}T_3 + \tfrac{1}{5}T_1, \tfrac{1}{10}T_6 - \tfrac{4}{15}T_4 + \tfrac{1}{6}T_2)_\omega \\ &= 0,\end{aligned}$$

and

$$\langle \mathcal{S}_5, \mathcal{R}_6\rangle_{2,I} = \langle p_5, q_6 + c_{6,5}q_5 + c_{6,4}q_4\rangle_{2,I} = c_{6,5}\rho_5. \tag{38}$$

Thereby, we have $c_{6,5} = 0$, which means $\mathcal{R}_6(x) = q_6(x) + c_{6,4}q_4(x)$. In the same manner, we can verify the results of (35) and (36) for $7 \le n \le 12$.

Next, assume that for any $4 \le k \le n-1$ and $n \ge 13$,

$$\mathcal{S}_k(x) = p_k(x) + a_{k,k-2}p_{k-2}(x) + a_{k,k-4}p_{k-4}(x) + a_{k,k-6}p_{k-6}(x) + a_{k,k-8}p_{k-8}(x),$$
$$\mathcal{R}_k(x) = q_k(x) + c_{k,k-2}q_{k-2}(x) + c_{k,k-4}q_{k-4}(x).$$

We shall prove that for $n \ge 13$,

$$\mathcal{S}_n(x) = p_n(x) + a_{n,n-2}p_{n-2}(x) + a_{n,n-4}p_{n-4}(x) + a_{n,n-6}p_{n-6}(x) + a_{n,n-8}p_{n-8}(x),$$
$$\mathcal{R}_n(x) = q_n(x) + c_{n,n-2}q_{n-2}(x) + c_{n,n-4}q_{n-4}(x).$$

Clearly, by (9), (37), (16) and (17), we have that for $n \ge 13$,

$$\begin{aligned}\langle \mathcal{S}_n, \mathcal{R}_j\rangle_{2,I} &= \langle p_n + \sum_{k=4}^{n-1} a_{n,k}p_k, \mathcal{R}_j\rangle_{2,I} \\ &= \Big\langle \frac{T_n - 4T_{n-2} + 6T_{n-4} - 4T_{n-6} + T_{n-8}}{2n}, \mathcal{R}_j\Big\rangle_{2,I}.\end{aligned}$$

Taking $j = 4, 5, \cdots, n-9$, successively and using the induction assumption, we derive readily that $a_{n,j} = 0$ for any $4 \le j \le n-9$. Hence, we have

$$\begin{aligned}\mathcal{S}_n(x) &= p_n(x) + a_{n,n-1}p_{n-1}(x) + a_{n,n-2}p_{n-2}(x) \\ &\quad + a_{n,n-3}p_{n-3}(x) + a_{n,n-4}p_{n-4}(x) + a_{n,n-5}p_{n-5}(x) + a_{n,n-6}p_{n-6}(x) \\ &\quad + a_{n,n-7}p_{n-7}(x) + a_{n,n-8}p_{n-8}(x).\end{aligned}$$

Moreover, by (17), (16), (9), (8), (5) and (34) and the induction assumption, we know that for $n \geq 13$,

$$\begin{aligned}
\langle \mathcal{S}_n, \mathcal{R}_{n-7}\rangle_{2,I} &= (\mathcal{S}_n'', (\mathcal{R}_{n-7}\omega)'') + r_1(\mathcal{S}_n', (\mathcal{R}_{n-7}\omega)') + r_2(\mathcal{S}_n, \mathcal{R}_{n-7})_\omega \\
&= (2(n-1)T_{n-2} - \tfrac{4(n-2)(n-6)}{n}T_{n-4} + \tfrac{2(n-7)(n-8)}{n}T_{n-6}, 2(n-9)T_{n-9})_\omega \\
&\quad + r_1(T_{n-1} - \tfrac{3n-8}{n}T_{n-3} + \tfrac{3n-16}{n}T_{n-5} - \tfrac{n-8}{n}T_{n-7}, T_{n-8} - T_{n-10})_\omega \\
&\quad + r_2(\tfrac{T_n - 4T_{n-2} + 6T_{n-4} - 4T_{n-6} + T_{n-8}}{2n}, \tfrac{(n-10)T_{n-7} - 2(n-9)T_{n-9} + (n-8)T_{n-11}}{2(n-8)(n-10)})_\omega \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathcal{S}_n, \mathcal{R}_{n-7}\rangle_{2,I} &= \langle p_n + a_{n,n-1}p_{n-1} + a_{n,n-2}p_{n-2} + a_{n,n-3}p_{n-3} + a_{n,n-4}p_{n-4} \\
&\quad + a_{n,n-5}p_{n-5} + a_{n,n-6}p_{n-6} + a_{n,n-7}p_{n-7} + a_{n,n-8}p_{n-8}, \\
&\quad\ q_{n-7} + c_{n-7,n-9}q_{n-9} + c_{n-7,n-11}q_{n-11}\rangle_{2,I} \\
&= a_{n,n-7}\rho_{n-7}.
\end{aligned}$$

Hence, we get $a_{n,n-7} = 0$. Similarly, we have $a_{n,n-5} = a_{n,n-3} = a_{n,n-1} = 0$. This means

$$\mathcal{S}_n(x) = p_n(x) + a_{n,n-2}p_{n-2}(x) + a_{n,n-4}p_{n-4}(x) + a_{n,n-6}p_{n-6}(x) + a_{n,n-8}p_{n-8}(x).$$

In the same manner, we derive that

$$\mathcal{R}_n(x) = q_n(x) + c_{n,n-2}q_{n-2}(x) + c_{n,n-4}q_{n-4}(x).$$

For simplicity of notations, we take $a_{n-2} := a_{n,n-2}$, $b_{n-4} := a_{n,n-4}$, $c_{n-6} := a_{n,n-6}$, $d_{n-8} := a_{n,n-8}$ and $e_{n-2} := c_{n,n-2}$, $h_{n-4} := c_{n,n-4}$, then we obtain the results of (35) and (36).

It remains to confirm the coefficients a_{n-2}, b_{n-4}, c_{n-6}, d_{n-8}, e_{n-2}, h_{n-4} and ρ_n. By using (17), (16), (9), (8), (5) and (34), we know that for $n \geq 8$,

(39)

$$\begin{aligned}
\langle \mathcal{S}_{n-4}, \mathcal{R}_n\rangle_{2,I} &= (\mathcal{S}_{n-4}'', (\mathcal{R}_n\omega)'') + r_1(\mathcal{S}_{n-4}', (\mathcal{R}_n\omega)') + r_2(\mathcal{S}_{n-4}, \mathcal{R}_n)_\omega \\
&= (2(n-5)T_{n-6} - \tfrac{4(n-6)(n-10)(1+\delta_{n,9})}{(n-4)(1+\delta_{n,8})}T_{n-8} + \tfrac{2(n-11)(n-12)}{(n-4)(1+\delta_{n,10})}T_{n-10}, 2(n-2)T_{n-2})_\omega \\
&\quad + r_1(T_{n-5} - \tfrac{3n-20-3\delta_{n,9}}{n-4}T_{n-7} + \tfrac{3n-28}{n-4}T_{n-9} - \tfrac{n-12}{(n-4)(1+\delta_{n,11})}T_{n-11}, T_{n-1} - T_{n-3})_\omega \\
&\quad + r_2(\tfrac{1}{2(n-4)}T_{n-4} - \tfrac{4-\delta_{n,9}}{2(n-4)}T_{n-6} + \tfrac{6-3\delta_{n,8}-4\delta_{n,9}}{2(n-4)}T_{n-8} - \tfrac{2}{n-4}T_{n-10} + \tfrac{1}{2(n-4)}T_{n-12}, \\
&\quad\quad \tfrac{1}{2(n-1)}T_n - \tfrac{n-2}{(n-1)(n-3)}T_{n-2} + \tfrac{1}{2(n-3)}T_{n-4})_\omega \\
&= \tfrac{\pi r_2}{8(n-3)(n-4)},
\end{aligned}$$

and

(40)

$$\begin{aligned}
\langle \mathcal{S}_{n-4}, \mathcal{R}_n\rangle_{2,I} &= \langle p_{n-4} + a_{n-6}p_{n-6} + b_{n-8}p_{n-8} + c_{n-10}p_{n-10} + d_{n-12}p_{n-12}, \\
&\quad\ q_n + e_{n-2}q_{n-2} + h_{n-4}q_{n-4}\rangle_{2,I} = h_{n-4}\rho_{n-4}.
\end{aligned}$$

Hence

$$h_{n-4}\rho_{n-4} = \frac{\pi r_2}{8(n-3)(n-4)}, \qquad n \geq 8. \tag{41}$$

Similarly, by using

$$\begin{aligned}\langle \mathcal{S}_j, \mathcal{R}_n\rangle_{2,I} &= (\mathcal{S}_j'', (\mathcal{R}_n\omega)'') + r_1(\mathcal{S}_j', (\mathcal{R}_n\omega)') + r_2(\mathcal{S}_j, \mathcal{R}_n)_\omega \\ &= \langle p_j + a_{j-2}p_{j-2} + b_{j-4}p_{j-4} + c_{j-6}p_{j-6} + d_{j-8}p_{j-8}, \\ &\quad q_n + e_{n-2}q_{n-2} + h_{n-4}q_{n-4}\rangle_{2,I}\end{aligned}$$

and taking $j = n-2,\ n,\ n+2,\ n+4,\ n+6,\ n+8$, respectively, we derive the results of (i)-(vii) in Lemma 4.1. This ends the proof. □

Theorem 4.1. *Let u and u_N be the solutions of* (31) *and* (33)*, respectively. Then both u and u_N have the explicit representations in $\{p_n(x)\}$,*

$$u(x) = \sum_{k=4}^{\infty} \hat{u}_k p_k(x), \qquad u_N(x) = \sum_{k=4}^{N} \hat{u}_k p_k(x),$$

$$\hat{u}_k = \frac{1}{\rho_k}\langle u, q_k\rangle_{2,I} = \frac{1}{\rho_k}(f, q_k)_\omega.$$

4.3. Numerical results. In this subsection, we examine the effectiveness and the accuracy of the fully diagonalized Chebyshev spectral method for solving fourth-order elliptic equations on $I = (-1,1)$. The righthand terms $\{(f, q_k)_\omega\}_{k=4}^N$, as well as the discrete errors, is also evaluated through the Chebyshev-Gauss quadrature with $2N+1$ nodes.

We take $r_1 = 1$ and $r_2 = 1$ in (31) and consider the following two cases:

- $u(x) = e^{-x}(x^2-1)^2$, where the solution is smooth. In Figure 3, we plot the $\log_{10}$ of the discrete L^2-errors.
- $u(x) = \sin(10x)(x^2-1)^2$, where the solution is oscillating. In Figure 4, we plot the $\log_{10}$ of the discrete L^2-errors.

The two near straight lines indicate that the L^2-errors decay like e^{-cN}.

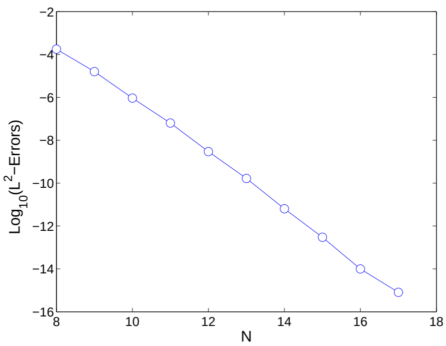

FIGURE 3. L^2-errors.

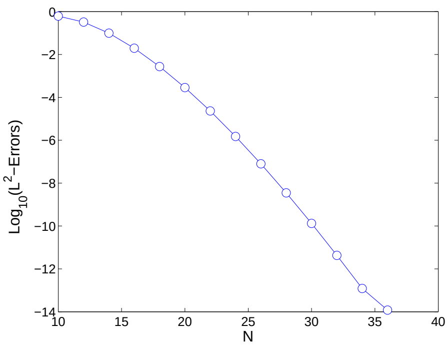

FIGURE 4. L^2-errors.

5. Concluding Remarks

In this paper, we construct two kinds of basis functions which are biorthogonal with respect to the Sobolev inner product originated from the coercive bilinear form of the elliptic equation. We also propose stable and efficient algorithms for second and fourth order Dirichlet boundary value problems. Particularly, both the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier series, respectively. Numerical experiments demonstrate that the suggested methods possess high-order accuracy. Although we only consider two model problems in this paper, the main idea and technology are also suitable for some other problems, such as the Neumann and Robin boundary value problems.

Acknowledgments

The second author was supported by National Natural Science Foundation of China (No. 11571238); The third author was supported by National Natural Science Foundation of China (Nos. 91130014, 11471312 and 91430216).

References

[1] C. Bernardi and Y. Maday, Spectral methods, in Handbook of Numerical Analysis, Vol. 5, edited by P. G. Ciarlet and J. L. Lions, North-Holland, 1997.
[2] J. P. Boyd, Chebyshev and Fourier spectral methods, Second edition, Dover, New York, 2001.
[3] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, Spectral Methods: Fundamentals in single domains, Springer-Verlag, Berlin, 2006.
[4] L. Fernandez, F. Marcellan, T. Pérez, M. Piñar and Xu Yuan, Sobolev orthogonal polynomials on product domains. t J. Comp. Anal. Appl., 284(2015), 202-215.
[5] D. Funaro, Polynomial Approximations of Differential Equations, Springer-Verlag, 1992.
[6] D. Gottlieb and S. A. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, SIAM-CBMS, Philadelphia, 1977.
[7] B.Y. Guo, Spectral Methods and Their Applications, World Scietific, Singapore, 1998.
[8] B.Y. Guo, Some progress in spectral methods, Sci. China Math., 56(2013), 2411-2438.
[9] B.Y. Guo, S.N. He and H.P. Ma, Chebyshev spectral-finite element method for two-dimensional unsteady Navier-Stokes equation, J. Comput. Math., 20(2002), 65-78.
[10] B.Y. Guo and J. Li, Fourier-Chebyshev spectral method for the two-dimensional Navier-Stokes equations, SIAM J. Numer. Anal., 33(1996), 1169-1187.
[11] B.Y. Guo, H.P. Ma, W.M. Cao and H. Huang, The Fourier-Chebyshev spectral method for solving two-dimensional unsteady vorticity equations, J. Comput. Phys., 101(1992), 207-217.
[12] F.J. Liu, Z.Q. Wang and H.Y. Li, A fully diagonalized spectral method using generalized Laguerre functions on the half line, Adv. Comput. Math., DOI:10.1007/s10444-017-9522-3.
[13] F.J. Liu, H.Y. Li and Z.Q. Wang, Spectral methods using generalized Laguerre functions for second and fourth order problems, Numer. Algorithms, 75(2017), 1005-1040.
[14] H.P. Ma and B.Y. Guo, The Chebyshev spectral method for Burgers-like equations, J. Comput. Math., 6(1988), 48-53.
[15] F. Marcellán and Y. Xu, On Sobolev orthogonal polynomials, Expo. Math., 33(2015), 308-352.
[16] J. Shen, Efficient spectral-Galerkin method II. Direct solvers of second- and fourth-order equations using Chebyshev polynomials, SIAM J. Sci. Comput., 16(1995), 74-87.
[17] J. Shen and T. Tang, Spectral and High-order Methods with Applications, Science Press, Beijing, 2006.
[18] J. Shen, T. Tang and L.L. Wang, Spectral Methods: Algorithms, Analysis and Applications, Springer Series in Computational Mathematics, Vol.41, Springer, 2011.
[19] J. Shen and L.L. Wang, Fourierization of the Legendre-Galerkin method and a new space-time spectral method, Appl. Numer. Math., 57(2007), 710-720.
[20] G. Szegö, Orthogonal Polynomials (fourth edition), Vol. 23, AMS Coll. Publ., 1975.

School of Science, University of Shanghai for Science and Technology, Shanghai, 200093, China
E-mail: ljm_destiny@163.com and zqwang@usst.edu.cn

State Key Laboratory of Computer Science/Laboratory of Parallel Computing, Institute of Software, Chinese Academy of Sciences, Beijing 100190, China
E-mail: huiyuan@iscas.ac.cn