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CONFORMING MIXED TRIANGULAR PRISM ELEMENTS FOR THE LINEAR ELASTICITY PROBLEM

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This paper is dedicated to Professor Ben-yu Guo

Abstract. We propose a family of conforming mixed triangular prism finite elements for solving the classical Hellinger-Reissner mixed problem of the linear elasticity equations in three dimensions. These elements are constructed by product of elements on triangular meshes and elements in one dimension. The well-posedness is established for all elements with $k \geq 1$, which are of $k+1$ order convergence for both the stress and displacement. Besides, a family of reduced stress spaces is proposed by dropping the degrees of polynomial functions associated with faces. As a result, the lowest order conforming mixed triangular prism element has 93 plus 33 degrees of freedom on each element.

Key words. Mixed finite element, triangular prism element, linear elasticity.

1. Introduction

In the Hellinger-Reissner mixed formulation of the linear elasticity equations, it is a challenge to design stable mixed finite element spaces mainly due to the symmetric constraint of the stress tensor, see some earlier work for composite elements and weakly symmetric methods in [2, 6, 7, 30, 34, 35, 36]. In [9], Arnold and Winther designed the first family of mixed finite element methods in two dimensions, based on polynomial shape function spaces. The analogue of the results on tetrahedral meshes can be found in [1, 4], and rectangular and cuboid meshes in [3, 11, 18]. Since the conforming symmetric stress elements have too many degrees of freedom, there are some other methods to overcome this drawback. We refer interested readers to nonconforming mixed elements, see [5, 10, 15, 21, 37] on simplicial meshes, and [26, 31, 39, 40] on rectangular and cuboid meshes. For the weakly symmetric mixed finite element methods for linear elasticity, we also refer to some recent work in [8, 12, 19, 32].

Recently, Hu [23] proposed a family of conforming mixed elements on simplical meshes for any dimension, see [27] and [28] the elements in two and three dimensions, respectively. This new class of elements has fewer degrees of freedom than those in the earlier literature. For $k \geq n$, the stress tensor is discretized by P_{k+1} finite element subspace of $H(\mathrm{div})$ and the displacement by piecewise P_k polynomials. Moreover, a new idea was proposed to analyze the discrete inf-sup condition and the basis functions therein are easy to obtain. For the case that $1 \leq k \leq n-1$, the symmetric tensor spaces are enriched by proper high order $H(\mathrm{div})$ bubble functions to stabilize the discretization in [29]. Another method by stabilization technique to deal with this case can be found in [16]. We also refer to [20] for interior penalty mixed finite element methods by using nonconforming symmetric stress spaces, where the stability is established by introducing the conforming $H(\mathrm{div})$ bubble

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spaces from [23] and nonconforming face-bubble spaces. Corresponding mixed elements on both rectangular and cuboid meshes were constructed in [22], also see [17, 24] for the lowest order mixed elements, while the simplest nonconforming mixed element on n-rectangular meshes can be found in [25].

In this paper, we propose a family of conforming mixed triangular prism elements for the linear elasticity problem. Triangular prism meshes can deal with some columnar regions, and in this case, the triangular prism partition is more easily achieved than the tetrahedral partition. The key idea here of constructing triangular prism elements is using a product structure that each prism can be treated as the product of a triangle and an interval. By dividing the stress variable into three parts, we construct the stress space through a combination of the mixed elasticity element [23, 27] and the Brezzi-Douglas-Marini element [14] on triangular meshes, and some other basic elements in one and two dimensions. In this way, we obtain conforming mixed triangular prism elements for any integer $k \geq 1$. The stability analysis is established by the theory developed in [22, 23, 27, 28, 29]. A family of reduced stress spaces is also proposed by dropping the degree of polynomials associated with faces. The reduced elements still preserve the same order of convergence. The lowest order case has 93 plus 33 degrees of freedom on each element. In addition, by using the lowest order nonconforming mixed element in [10, 21] on triangular meshes, we obtain a nonconforming mixed triangular prism element of first order convergence, of which degrees of freedom are 81 plus 33.

The rest of the paper is organized as follows. In Section 2, we define the conforming mixed triangular prism finite element methods and present the basis functions. In Section 3, we prove the well-posedness of these elements, i.e. the K-ellipticity and the discrete inf-sup condition. By which, the optimal order convergence of the new elements follows. In Section 4, we propose a family of reduced triangular prism elements. In the end, we provide some numerical results.

2. The family of conforming mixed triangular prism elements

Based on the Hellinger-Reissner principle, the linear elasticity problem within a stress-displacement (σ-u) form reads: Find $(\sigma, u) \in \Sigma \times V := H(\mathrm{div}, \Omega; \mathbb{S} :=$ symmetric $\mathbb{R}^{3\times 3}) \times L^2(\Omega; \mathbb{R}^3)$, such that

$$(1) \qquad \begin{cases} (A\sigma, \tau) + (\mathrm{div}\tau, u) = 0 & \text{for all } \ \tau \in \Sigma, \\ (\mathrm{div}\sigma, v) = (f, v) & \text{for all } \ v \in V. \end{cases}$$

Here the symmetric tensor space for the stress Σ and the space for the vector displacement V are, respectively,

$$(2) \qquad H(\mathrm{div}, \Omega; \mathbb{S}) := \left\{ \tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \in H(\mathrm{div}, \Omega; \mathbb{R}^{3\times 3}), \ \tau^{\mathrm{T}} = \tau \right\},$$

$$(3) \qquad L^2(\Omega; \mathbb{R}^3) := \{ v = (v_1 \ \ v_2 \ \ v_3)^{\mathrm{T}} | \ v_i \in L^2(\Omega; \mathbb{R}), i = 1, 2, 3 \}.$$

This paper denotes by $H^k(\omega; X)$ the Sobolev space consisting of functions with domain ω, taking values in the finite-dimensional vector space X, and with all derivatives of order at most k square-integrable. For our purposes, the range space X will be either $\mathbb{S}$, $\mathbb{R}^3$, $\mathbb{R}^2$, or $\mathbb{R}$, and in some cases, X will be $\mathbb{S}_2 :=$ symmetric $\mathbb{R}^{2\times 2}$ as well. Let $\| \cdot \|_{k,\omega}$ be the norm of $H^k(\omega)$ and $H(\mathrm{div}, \omega; \mathbb{S})$ consist of square-integrable symmetric matrix fields with square-integrable divergence. The $H(\mathrm{div})$ norm is defined by

$$\|\tau\|^2_{H(\mathrm{div},\omega)} := \|\tau\|^2_{0,\omega} + \|\mathrm{div}\tau\|^2_{0,\omega}.$$

Here, the compliance tensor $A = A(\boldsymbol{x}) : \mathbb{S} \to \mathbb{S}$, characterizing the properties of the material, is bounded and symmetric positive definite uniformly for $\boldsymbol{x} \in \Omega$, namely, there exists $C > 0$ such that $(A\tau, \tau) \geq C\|\tau\|_{0,\Omega}^2$ for any $\tau \in \Sigma$. While for the nearly incompressible materials, it holds only for functions τ which satisfy $\mathrm{div}\tau \equiv 0$, $\int_\Omega \mathrm{tr}\tau \, dx = 0$, see remarks in [9].

This paper deals with a pure displacement problem (1) with the homogeneous boundary condition that $u \equiv 0$ on $\partial\Omega$. But the method and the analysis work for mixed boundary value problems and the pure traction boundary problem as well.

2.1. The discrete stress and displacement spaces. To obtain triangular prism partitions, we suppose that the domain $\Omega = \Omega_{xy} \times \Omega_z$, where Ω_{xy} is a polygon on the (x, y)-plane and Ω_z is an interval on the z-axis. The domain Ω is subdivided into the union of non-overlapping shape-regular triangular prism elements such that the non-empty intersection of any distinct pair of elements is a single common vertex, edge or face. Let $\mathcal{T}_h$ be the set consisting of all these elements (with the mesh size h). In fact, we also obtain partitions of Ω_{xy} and Ω_z, which are denoted by $\mathcal{X}_h$ and $\mathcal{Z}_h$, respectively. Given triangle $\Delta_{xy} \in \mathcal{X}_h$ and interval $\Delta_z \in \mathcal{Z}_h$, $K = \Delta_{xy} \times \Delta_z$ is thus a triangular prism element in $\mathcal{T}_h$. Then, each element K in $\mathcal{T}_h$ is equipped with a product structure. Given element, face or edge ω, let $|\omega|$ denote the measure of ω. Let div_{xy}, ∇_{xy} and curl_{xy} denote the divergence, gradient and curl operators with respect to the variables x and y, respectively. Given any nonnegative integer k, let $P_k(\omega; X)$ denote the space of polynomials over ω of total degrees not greater than k, taking values in the finite-dimensional vector space X. Let $P_k(z)$ be the space of polynomials of degree not greater than k with respect to the variable z, and let $P_k(x, y)$ be the space of polynomials of degree not greater than k with respect to the variables x and y. Given face F of K satisfying $F = e \times \Delta_z$, where $e \subset \partial\Delta_{xy}$, let $Q_{k_1,k_2}(F) = P_{k_1}(e; \mathbb{R}) \times P_{k_2}(\Delta_z; \mathbb{R})$ for any nonnegative integers k_1 and k_2.

We define the following spaces associated with partition $\mathcal{Z}_h$ for $s = 0, 1$ and $k \geq s$

$$\mathcal{L}_k^s(\mathcal{Z}_h) := \{ \, v \in H^s(\Omega_z; \mathbb{R}) \mid v|_{\Delta_z} \in P_k(z) \text{ for any } \Delta_z \in \mathcal{Z}_h\},$$

and the space associated with partition $\mathcal{X}_h$ for $k \geq 0$

$$\mathcal{L}_k^0(\mathcal{X}_h) := \{v \in L^2(\Omega_{xy}; \mathbb{R}) \mid v|_{\Delta_{xy}} \in P_k(x, y) \text{ for any } \Delta_{xy} \in \mathcal{X}_h\}.$$

Before defining the space for the stress, we introduce the mixed elasticity finite element in two dimensions of [23, 27] and the Brezzi-Douglas-Marini (BDM hereafter) space of [14] for the mixed Poisson problem. We recall some notations in [23, 27]. Let $\lambda_i (1 \leq i \leq 3)$ denote the barycentric coordinates with respect to the vertices $\boldsymbol{x}_i$ of triangle Δ_{xy}. For any edge $\boldsymbol{x}_i\boldsymbol{x}_j (1 \leq i < j \leq 3)$ of Δ_{xy}, let $\boldsymbol{t}_{i,j} = \boldsymbol{x}_j - \boldsymbol{x}_i$ denote associated tangent vectors, which allow for us to introduce the following linearly independent symmetric matrices of rank one

$$T_{i,j} = \boldsymbol{t}_{i,j}\boldsymbol{t}_{i,j}^T, \; 1 \leq i < j \leq 3.$$

With these symmetric matrices $T_{i,j}$ of rank one, we define a $H(\mathrm{div}_{xy}, \Delta_{xy}; \mathbb{S}_2)$ polynomial bubble function space

$$H_{\Delta_{xy},k,b} := \sum_{1 \leq i < j \leq 3} \lambda_i \lambda_j P_{k-2}(\Delta_{xy}; \mathbb{R}) T_{i,j},$$

which satisfies

$$H_{\Delta_{xy},k,b} = \{\tau \in P_k(\Delta_{xy}; \mathbb{S}_2) \mid \tau\nu_{xy}|_{\partial\Delta_{xy}} = 0\}.$$

Here and throughout the paper, let $H(\mathrm{div}_{xy}, \omega; X)$ consist of square-integrable functions over ω with values in X and square-integrable divergence with respect to x and y. Here X will be either $\mathbb{S}_2$ or $\mathbb{R}^2$. The finite element space of order k ($k \geq 3$) for the stress approximation in two dimensions is

$$H_{k,h} := \Big\{ \tau \in H(\mathrm{div}_{xy}, \Omega_{xy}; \mathbb{S}_2) \;\Big|\; \tau = \tau_c + \tau_b,\ \tau_c \in H^1(\Omega_{xy}; \mathbb{S}_2), \tau_c|_{\Delta_{xy}} \in P_k(\Delta_{xy}; \mathbb{S}_2)\,,\ \tau_b|_{\Delta_{xy}} \in H_{\Delta_{xy},k,b} \text{ for any } \Delta_{xy} \in \mathcal{X}_h \Big\}. \tag{4}$$

A matrix field $\tau \in P_k(\Delta_{xy}; \mathbb{S}_2)$ can be uniquely determined by the following degrees of freedom [23]

(1) the values of τ at three vertices of Δ_{xy},
(2) $\int_e \tau\nu_{xy} \cdot p\, ds$ for any $p \in P_{k-2}(e; \mathbb{R}^2)$ and $e \subset \partial\Delta_{xy}$,
(3) $\int_{\Delta_{xy}} \tau : p\, dxdy$ for any $p \in H_{\Delta_{xy},k,b}$.

Hereafter ν_{xy} is the normal vector of $\partial\Delta_{xy}$.

The spaces of the BDM element are defined as follows for $k \geq 1$

$$\mathrm{BDM}_k := \{\tau \in H(\mathrm{div}_{xy}, \Omega_{xy}; \mathbb{R}^2) \mid \tau|_{\Delta_{xy}} \in P_k(\Delta_{xy}; \mathbb{R}^2) \text{ for any } \Delta_{xy} \in \mathcal{X}_h\}.$$

The vector-valued function $\tau \in P_k(\Delta_{xy}; \mathbb{R}^2)$ can be determined by the following conditions (see e.g. [13]):

(1) $\int_e \tau \cdot \nu_{xy} p\, ds$ for any $p \in P_k(e; \mathbb{R})$ and $e \subset \partial\Delta_{xy}$,
(2) $\int_{\Delta_{xy}} \tau \cdot \nabla_{xy} p\, dxdy$ for any $p \in P_{k-1}(x, y)$,
(3) $\int_{\Delta_{xy}} \tau \cdot p\, dxdy$ for any $p \in \Psi_k(\Delta_{xy})$

with

$$\Psi_k(\Delta_{xy}) := \{w | w = \mathrm{curl}_{xy}(b_{xy}v), v \in P_{k-2}(x, y)\}, \tag{5}$$

where $b_{xy} := \lambda_1\lambda_2\lambda_3$ denotes the cubic bubble on Δ_{xy}. We also introduce the bubble function space

$$\mathrm{BDM}_{\Delta_{xy},k,b} := \{\tau \in P_k(\Delta_{xy}; \mathbb{R}^2) \mid \tau \cdot \nu_{xy}|_{\partial\Delta_{xy}} = 0\}.$$

This space can be uniquely determined by the conditions in (2) and (3) above.

Based on the above finite element spaces, we use a product structure to define the stress space of the conforming mixed triangular prism elements for $k \geq 1$:

$$\begin{aligned}\Sigma_{k,h} :=& \{\tau = \begin{pmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\ \tau_{21} & \tau_{22} & \tau_{23}\\ \tau_{31} & \tau_{32} & \tau_{33}\end{pmatrix} \in L^2(\Omega; \mathbb{S}) \;\Big|\; \begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{pmatrix} \in H_{k+2,h} \times \mathcal{L}_k^0(\mathcal{Z}_h),\\ & (\tau_{13}, \tau_{23})^T \in \mathrm{BDM}_{k+1} \times \mathcal{L}_{k+1}^1(\mathcal{Z}_h), \tau_{33} \in \mathcal{L}_k^0(\mathcal{X}_h) \times \mathcal{L}_{k+2}^1(\mathcal{Z}_h)\}.\end{aligned} \tag{6}$$

It is straightforward to show that $\Sigma_{k,h} \subset \Sigma$ and the shape function space of the element is

$$\begin{aligned}\Sigma_k(K) :=& \{\tau = \begin{pmatrix}\tau_1 & \tau_2\\ \tau_2^T & \tau_3\end{pmatrix} \in H^1(K; \mathbb{S}) \;\Big|\; \tau_1 := \begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{pmatrix} \in P_{k+2}(\Delta_{xy}; \mathbb{S}_2) \times P_k(z),\\ & \tau_2 := (\tau_{13}, \tau_{23})^T \in P_{k+1}(\Delta_{xy}; \mathbb{R}^2) \times P_{k+1}(z), \tau_3 := \tau_{33} \in P_k(x, y) \times P_{k+2}(z)\}.\end{aligned} \tag{7}$$

Note that ν_{xy} is the normal vector of $\partial\Delta_{xy}$ and thus defined on $\partial\Delta_{xy}$, then it is also well defined on each face F of K that parallels the z-axis and each edge e of K that parallels the (x, y)-plane. We present the degrees of freedom in the following lemma.

Lemma 2.1. *A matrix field $\tau \in \Sigma_k(K)$ can be uniquely determined by the following conditions:*

(1) *the values of* τ_1 *at* $k+1$ *distinct points on edge* e *of* K *that parallels the* z*-axis,*
(2) $\int_F \tau_1\nu_{xy}\cdot p\,ds$ *for any* $p\in (Q_{k,k}(F))^2$ *and face* F *of* K *that parallels the* z*-axis,*
(3) $\int_K \tau_1 : p\,dxdydz$ *for any* $p\in H_{\Delta_{xy},k+2,b}\times P_k(z)$*;*
(4) $\int_e \tau_2\cdot\nu_{xy}p\,ds$ *for any* $p\in P_{k+1}(e;\mathbb{R})$ *and edge* e *of* K *that parallels the* (x,y)*-plane,*
(5) $\int_F \tau_2\cdot\nu_{xy}p\,ds$ *for any* $p\in Q_{k+1,k-1}(F)$ *and face* F *of* K *that parallels the* z*-axis,*
(6) $\int_F \tau_2\cdot\nabla_{xy}p\,dxdy$ *for any* $p\in P_k(x,y)$ *and face* F *that parallels the* (x,y)*-plane,*
(7) $\int_F \tau_2\cdot p\,dxdy$ *for any* $p\in \Psi_{k+1}(\Delta_{xy})$ *and face* F *that parallels the* (x,y)*-plane,*
(8) $\int_K \tau_2\cdot\nabla_{xy}p\,dxdydz$ *for any* $p\in P_k(x,y)\times P_{k-1}(z)$*,*
(9) $\int_K \tau_2\cdot p\,dxdydz$ *for any* $p\in \Psi_{k+1}(\Delta_{xy})\times P_{k-1}(z)$*;*
(10) $\int_F \tau_3 p\,dxdy$ *for any* $p\in P_k(x,y)$ *and face* F *that parallels the* (x,y)*-plane,*
(11) $\int_K \tau_3 p\,dxdydz$ *for any* $p\in P_k(x,y)\times P_k(z)$*.*

Here τ_1,τ_2 *and* τ_3 *are defined in* (7)*.*

Proof. Since the dimensions of the space $\Sigma_k(K)$ are equal to the number of these conditions, it suffices to prove that $\tau\equiv 0$ if these conditions vanish. The first and second conditions show that $\tau_1\nu_{xy}=0$ on side faces of triangular prism K. Moreover it follows from (3) that $\tau_1=0$. Note that (4) plus (5) and (4), (6) plus (7) yield that $\tau_2\cdot\nu_{xy}=0$ on side faces and $\tau_2=0$ on top and bottom faces, respectively. Thus it follows from (8) and (9) that $\tau_2=0$. It remains to prove $\tau_3=0$. Actually, condition (10) implies that

$$\tau_3=b_z g,$$

where b_z is the quadratic bubble function on interval Δ_z and $g\in P_k(x,y)\times P_k(z)$. Using condition (11), we immediately obtain $\tau_3=0$. □

On each element K, the space for the displacement is taken as

$$V_k(K):=\{v=(v_1,v_2,v_3)^T\in H^1(K;\mathbb{R}^3)\mid v_i\in P_{k+1}(x,y)\times P_k(z), i=1,2,\\ v_3\in P_k(x,y)\times P_{k+1}(z)\}. \tag{8}$$

Then the global space for displacement reads

$$V_{k,h}:=\{v\in V\mid v|_K\in V_k(K)\text{ for any }K\in\mathcal{T}_h\}. \tag{9}$$

The mixed finite element approximation of Problem (1) reads: Find $(\sigma_h,u_h)\in\Sigma_{k,h}\times V_{k,h}$, such that

$$\begin{cases}(A\sigma_h,\tau)+(\mathrm{div}\tau,u_h)=0 & \text{for all }\ \tau\in\Sigma_{k,h},\\ (\mathrm{div}\sigma_h,v)=(f,v) & \text{for all }\ v\in V_{k,h}.\end{cases} \tag{10}$$

2.2. Basis functions of the stress space. For convenience, we provide the basis of the stress space $\Sigma_{k,h}$ on element K. In fact, we only need to give the basis of $H_{k,h}(k\ge 3)$ and $\mathrm{BDM}_k(k\ge 1)$. Thus we immediately obtain the basis of $\Sigma_{k,h}$ by the product structure. For any edge $\boldsymbol{x}_i\boldsymbol{x}_j(1\le i<j\le 3)$ of Δ_{xy}, $\boldsymbol{x}_m$ being the opposite vertex, let $\boldsymbol{\nu}_{i,j}$ denote its associated normal vector and h_m denote the height of the triangle from $\boldsymbol{x}_m$ to the opposite edge $\boldsymbol{x}_i\boldsymbol{x}_j$.

The canonical basis of $\mathbb{S}_2$ reads

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, the basis functions of $H_{k,h}(k \geq 3)$ on triangle Δ_{xy} are as follows [27, 29]:

(1) Given vertex $\boldsymbol{x}_i$, the corresponding basis functions are

$$\lambda_i T_j,\ j = 1, 2, 3;$$

(2) Given edge $\boldsymbol{x}_i\boldsymbol{x}_j$, its associated basis functions with nonzero fluxes read

$$\lambda_i\lambda_j \widetilde{P}_{k-2}(\lambda_i, \lambda_j)\boldsymbol{\nu}_{i,j}\boldsymbol{\nu}_{i,j}^T,\ \lambda_i\lambda_j \widetilde{P}_{k-2}(\lambda_i, \lambda_j)\frac{\boldsymbol{t}_{i,j}\boldsymbol{\nu}_{i,j}^T + \boldsymbol{\nu}_{i,j}\boldsymbol{t}_{i,j}^T}{2};$$

(3) The basis functions of $H_{\Delta_{xy},k,b}$ are

$$\lambda_i\lambda_j P_{k-2}(\Delta_{xy}; \mathbb{R})\boldsymbol{t}_{i,j}\boldsymbol{t}_{i,j}^T,\ 1 \leq i < j \leq 3.$$

Here

$$\widetilde{P}_k(\lambda_i, \lambda_j) := \mathrm{span}\{\lambda_i^{m_1}\lambda_j^{m_2}, m_1 + m_2 = k\}. \tag{11}$$

For BDM_k, the hierarchical basis functions can be found in [38]. We give another basis functions following [13]:

(1) Given edge $\boldsymbol{x}_i\boldsymbol{x}_j$, the corresponding basis functions are

$$\frac{1}{h_m}\lambda_i \boldsymbol{t}_{m,i},\ \frac{1}{h_m}\lambda_j \boldsymbol{t}_{m,j},\ \frac{1}{2h_m}\lambda_i\lambda_j \widetilde{P}_{k-2}(\lambda_i, \lambda_j)(\boldsymbol{t}_{m,i} + \boldsymbol{t}_{m,j});$$

(2) The basis functions of $\mathrm{BDM}_{\Delta_{xy},k,b}$ are

$$\lambda_i\lambda_j \widetilde{P}_{k-2}(\lambda_i, \lambda_j)\boldsymbol{t}_{i,j},\ 1 \leq i < j \leq 3,$$
$$\lambda_1\lambda_2\lambda_3 P_{k-3}(\Delta_{xy}; \mathbb{R}^2).$$

Using the above two families of basis functions, we can easily construct the basis functions of $\Sigma_{k,h}(k \geq 1)$ on element $K = \Delta_{xy} \times \Delta_z$ by the product technique. We shall make explicit the lowest order case, of which the stress space is as follows

$$\begin{aligned}\Sigma_{1,h} =& \{\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2^T & \tau_3 \end{pmatrix} \in L^2(\Omega; \mathbb{S}) \bigm| \tau_1 \in H_{3,h} \times \mathcal{L}_1^0(\mathcal{Z}_h), \tau_2 \in \mathrm{BDM}_2 \times \mathcal{L}_2^1(\mathcal{Z}_h),\\ & \tau_3 \in \mathcal{L}_1^0(\mathcal{X}_h) \times \mathcal{L}_3^1(\mathcal{Z}_h)\}.\end{aligned}$$

Let $\{\phi_i\}_{i=1}^{30}$ and $\{\psi_i\}_{i=1}^{12}$ be the collection of basis functions of $H_{3,h}$ and BDM_2 on triangle Δ_{xy}, respectively. Suppose that $\Delta_z = [z_0, z_0 + h_0]$, we introduce the affine invertible transformation

$$F_{\Delta_z} : [0, 1] \to [z_0, z_0 + h_0], z = h_0\xi + z_0, \xi \in [0, 1].$$

Thus we select τ such that $\tau_2 = \tau_3 = 0$ and

$$\tau_1 \in \{\phi_i\xi, \phi_i(1 - \xi)\}_{i=1}^{30},$$

$\tau_1 = \tau_3 = 0$ and

$$\tau_2 \in \{\psi_i\xi(2\xi - 1), \psi_i(1 - \xi)(1 - 2\xi), \psi_i\xi(1 - \xi)\}_{i=1}^{12},$$

and $\tau_1 = \tau_2 = 0$

$$\begin{aligned}\tau_3 \in& \{\lambda_i\xi(3\xi - 1)(3\xi - 2), \lambda_i\xi(1 - \xi)(2 - 3\xi), \lambda_i\xi(1 - \xi)(3\xi - 1),\\ & \lambda_i(1 - \xi)(3\xi - 1)(3\xi - 2)\}_{i=1}^3.\end{aligned}$$

In this way, we obtain the basis functions of $\Sigma_{1,h}$ on K. Thus, the degrees of freedom on each element of the lowest order element are 108 plus 33.

3. The stability analysis for the mixed triangular prism elements

In this section, we consider the well-posedness of the discrete problem (10). By the standard theory, we only need to prove the following two conditions, based on their counterparts at the continuous level.

- K-ellipticity. There exists a constant $C > 0$, independent of the meshsize h such that
$$(A\tau_h, \tau_h) \geq C\|\tau_h\|^2_{H(\mathrm{div},\Omega)} \text{ for any } \tau_h \in W_h,$$
where W_h is the divergence-free space defined as follows
$$W_h := \{\tau_h \in \Sigma_{k,h} \mid (\mathrm{div}\,\tau_h, v) = 0 \text{ for all } v \in V_{k,h}\}.$$
- Discrete inf-sup condition. There exists a positive constant $C > 0$ independent of the meshsize h, such that
$$\sup_{0\neq\tau_h\in\Sigma_{k,h}} \frac{(\mathrm{div}\tau_h, v_h)}{\|\tau_h\|_{H(\mathrm{div},\Omega)}} \geq C\|v_h\|_{0,\Omega} \quad \text{for any } v_h \in V_{k,h}.$$

It can be easily checked that $\mathrm{div}\Sigma_{k,h} \subset V_{k,h}$. Hence $\mathrm{div}\tau_h = 0$ for any $\tau_h \in W_h$ and this implies the above K-ellipticity condition. It remains to show the discrete inf-sup condition. We first introduce the following lemma in [23, 27], which is a key ingredient to prove the discrete inf-sup condition for mixed triangular elasticity elements.

Let $R_{xy}(\Delta_{xy})$ be the rigid motion space in two dimensions, which reads
$$R_{xy}(\Delta_{xy}) := \mathrm{span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}y\\-x\end{pmatrix} \right\}.$$
Define the orthogonal complement space of $R_{xy}(\Delta_{xy})$ with respect to $P_{k+1}(\Delta_{xy};\mathbb{R}^2)$ by
$$R^{\perp}_{xy}(\Delta_{xy}) := \{v \in P_{k+1}(\Delta_{xy};\mathbb{R}^2) \mid (v,w)_{\Delta_{xy}} = 0 \text{ for any } w \in R_{xy}(\Delta_{xy})\},$$
where the inner product $(v,w)_{\Delta_{xy}}$ over Δ_{xy} reads $(v,w)_{\Delta_{xy}} = \int_{\Delta_{xy}} v \cdot w \, dxdy$.

Lemma 3.1. *It holds that*
$$\mathrm{div}_{xy} H_{\Delta_{xy},k+2,b} = R^{\perp}_{xy}(\Delta_{xy}).$$

Next we follow the arguments in [22, 23, 27, 28, 29] to analyze the discrete inf-sup condition. To this end, we define the bubble function space
$$\Sigma_{K,k,b} := \{\tau \in \Sigma_k(K), \tau\nu = 0 \text{ on } \partial K\}.$$
Here ν denotes the normal vector of ∂K. Let $RM(K)$ be the rigid motion space in three dimensions, which reads
$$RM(K) := \mathrm{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-y\\x\\0\end{pmatrix}, \begin{pmatrix}-z\\0\\x\end{pmatrix}, \begin{pmatrix}0\\-z\\y\end{pmatrix} \right\}.$$
Define the orthogonal complement space of the rigid motion space $RM(K)$ with respect to $V_k(K)$ by
$$RM^{\perp}(K) := \{v \in V_k(K) \mid (v,w)_K = 0 \text{ for any } w \in RM(K)\},$$
where the inner product $(v,w)_K$ over K reads $(v,w)_K = \int_K v \cdot w \, dxdydz$.

Lemma 3.2. *For any $k \geq 1$, it holds that*
$$\mathrm{div}\Sigma_{K,k,b} = RM^{\perp}(K).$$

Proof. Since it is straightforward to see that $\mathrm{div}\Sigma_{K,k,b} \subset RM^{\perp}(K)$, we only need to prove the converse. If $\mathrm{div}\Sigma_{K,k,b} \neq RM^{\perp}(K)$, there is a nonzero $v = (v_1, v_2, v_3)^T \in RM^{\perp}(K)$ such that

$$\int_K \mathrm{div}\tau \cdot v \, dxdydz = 0 \text{ for any } \tau \in \Sigma_{K,k,b}.$$

First, we choose $\tau = \begin{pmatrix} \tau_1 & 0 \\ 0^T & 0 \end{pmatrix} \in \Sigma_{K,k,b}$ such that $\tau_1 \in H_{\Delta_{xy},k+2,b} \times P_k(z)$. It follows that

$$0 = \int_K \mathrm{div}\, \tau \cdot v \, dxdydz = \int_K \mathrm{div}_{xy}\, \tau_1 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dxdydz.$$

From (8), we have $(v_1, v_2)^T \in P_{k+1}(\Delta_{xy}; \mathbb{R}^2) \times P_k(z)$. This, together with Lemma 3.1 shows that

$$(v_1, v_2)^T \in R_{xy}(\Delta_{xy}) \times P_k(z). \tag{12}$$

Second, we take τ such that $\tau_{11} = \tau_{12} = \tau_{22} = \tau_{13} = \tau_{23} = 0$ and

$$\tau_{33} \in b_z \times P_k(x, y) \times P_k(z),$$

where the bubble function b_z is defined in Lemma 2.1. An integration by parts yields

$$0 = \int_K \mathrm{div}\, \tau \cdot v \, dxdydz = -\int_K \tau_{33} \frac{\partial v_3}{\partial z} \, dxdydz.$$

Since $\frac{\partial v_3}{\partial z} \in P_k(x, y) \times P_k(z)$, it holds that

$$v_3 \in P_k(x, y). \tag{13}$$

Third, we use degrees of freedom of τ_{13} and τ_{23} to deal with the remaining part of $(v_1, v_2, v_3)^T$ in (12) and (13). Given $\tau \in \Sigma_{K,k,b}$ such that $\tau_{11} = \tau_{12} = \tau_{22} = \tau_{33} = 0$ and $(\tau_{13}, \tau_{23})^T \in b_z \times \mathrm{BDM}_{\Delta_{xy},k+1,b} \times P_{k-1}(z)$, we have

$$0 = \int_K \left((\frac{\partial \tau_{13}}{\partial z} v_1 + \frac{\partial \tau_{23}}{\partial z} v_2) + v_3 \, \mathrm{div}_{xy}(\tau_{13}, \tau_{23})^T \right) dxdydz.$$

Recall that $K = \Delta_{xy} \times \Delta_z$. An integration by parts gives rises to

$$\int_{\Delta_z} \left(\int_{\Delta_{xy}} (\tau_{13}, \tau_{23})^T \cdot (\frac{\partial}{\partial z}(v_1, v_2)^T + \nabla_{xy} v_3) \, dxdy \right) dz = 0. \tag{14}$$

It follows from (12) that there exist two constants c_1 and c_2, two polynomials $p_1, p_2 \in P_{k-2}(z)$, and a polynomial $p_3 \in P_{k-1}(z)$ such that

$$\frac{\partial}{\partial z}(v_1, v_2)^T + \nabla_{xy} v_3 = \nabla_{xy}(c_1 x + c_2 y + v_3 + xzp_1 + yzp_2) + p_3 \begin{pmatrix} y \\ -x \end{pmatrix}.$$

Then, the choice $(\tau_{13}, \tau_{23})^T = b_z p_3 \mathrm{curl}_{xy} b_{xy}$ in (14) implies that

$$-\frac{|\Delta_{xy}|}{30} \int_{\Delta_z} b_z p_3^2 \, dz = 0, \tag{15}$$

where b_{xy} is defined in (5). Indeed, a simple computation shows that

$$\mathrm{div}_{xy} \mathrm{curl}_{xy} b_{xy} = 0 \text{ and } \mathrm{curl}_{xy} b_{xy} \in \mathrm{BDM}_{\Delta_{xy},k+1,b},$$

and

$$\int_{\Delta_{xy}} \mathrm{curl}_{xy} b_{xy} \cdot (y, -x)^T \, dxdy = -\frac{|\Delta_{xy}|}{30} \neq 0. \tag{16}$$

Further, using (15), we obtain $p_3 = 0$. Next we show that $\nabla_{xy}(c_1x + c_2y + v_3) = 0$. If otherwise, it follows from the second degrees of freedom for the BDM space in Section 2 that there exists $w \in \mathrm{BDM}_{\Delta_{xy},k+1,b}$ such that

$$\int_{\Delta_{xy}} w \cdot \nabla_{xy}(c_1x + c_2y + v_3)\, dxdy = 1.$$

Besides, there exists $q_1 \in P_{k-1}(z)$ satisfying

$$\int_{\Delta_z} b_z q_1\, dz = 1 \text{ and } \int_{\Delta_z} b_z q_1 z p_i\, dz = 0 \text{ for } i = 1, 2,$$

Actually, it is straightforward for $k \geq 3$. While for $k = 1$, since $p_1 = p_2 = 0$, we can just take $q_1 = 1/(\int_{\Delta_z} b_z\, dz)$. And for $k = 2$, since $p_1, p_2 \in P_0(z)$ and then p_1 and p_2 are dependent, there exits $q_1 \in P_1(z)$ satisfying the above conditions. However, selecting $(\tau_{13}, \tau_{23})^T = b_z q_1 w$ in (14) leads to a contradiction that $1 = 0$. Therefore, we obtain $\nabla_{xy}(c_1x + c_2y + v_3) = 0$. On the other hand, we select $w \in \mathrm{BDM}_{\Delta_{xy},k+1,b}$ such that $\int_{\Delta_{xy}} w \cdot \nabla_{xy}x\, dxdy = 1$ and $\int_{\Delta_{xy}} w \cdot \nabla_{xy}y\, dxdy = 0$, and $q_1 = zp_1$. This gives $p_1 = 0$. Similar choice yields $p_2 = 0$. Hence, a collection of the above arguments yields

$$\frac{\partial}{\partial z}(v_1, v_2)^T + \nabla_{xy} v_3 = 0. \tag{17}$$

Consequently, we conclude, by (12), (13) and (17),

$$v = (v_1, v_2, v_3)^T \in RM(K),$$

which completes the proof. □

Before giving the following lemma, we present the H^1 conforming triangular prism element ($k \geq 1$)

$$U_{k,h} = \{v \in H^1(\Omega; \mathbb{S}) \mid v|_K \in P_k(\Delta_{xy}; \mathbb{S}) \times P_k(z) \text{ for any } K \in \mathcal{T}_h\}.$$

Let $\widetilde{I}_h : H^1(\Omega; \mathbb{S}) \to U_{k,h}$ denote the Scott-Zhang interpolation operator in [33] that satisfies

$$\|\tau - \widetilde{I}_h\tau\|_{0,\Omega} + h\|\nabla \widetilde{I}_h\tau\|_{0,\Omega} \leq Ch\|\nabla\tau\|_{0,\Omega}. \tag{18}$$

Lemma 3.3. *Given any integer $k \geq 1$, there exists an interpolation operator I_h : $H^1(\Omega; \mathbb{S}) \to \Sigma_{k,h}$ satisfying for any $\tau \in H^1(\Omega; \mathbb{S})$,*

$$\int_K \operatorname{div}(\tau - I_h\tau) \cdot w\, dxdydz = 0 \textit{ for any } w \in RM(K) \textit{ and any } K \in \mathcal{T}_h \tag{19}$$

and

$$\|I_h\tau\|_{H(\operatorname{div},\Omega)} \leq C\|\tau\|_{1,\Omega}. \tag{20}$$

Proof. We use notations τ_1, τ_2, τ_3 to denote the corresponding parts of τ as in (7), and $\widetilde{\tau}_{1,h}, \widetilde{\tau}_{2,h}, \widetilde{\tau}_{3,h}$ are similar defined for $\widetilde{I}_h\tau$ such that $\widetilde{I}_h\tau = \begin{pmatrix} \widetilde{\tau}_{1,h} & \widetilde{\tau}_{2,h} \\ \widetilde{\tau}_{2,h}^T & \widetilde{\tau}_{3,h} \end{pmatrix}$. It follows from degrees of freedom in Lemma 2.1 that there exists $\tau_{1,h} \in H_{k+2,h} \times \mathcal{L}_k^0(\mathcal{Z}_h)$, $\tau_{2,h} \in \mathrm{BDM}_{k+1} \times \mathcal{L}_{k+1}^1(\mathcal{Z}_h)$ and $\tau_{3,h} \in \mathcal{L}_k^0(\mathcal{X}_h) \times \mathcal{L}_{k+2}^1(\mathcal{Z}_h)$ such that for face F that parallels the z-axis,

$$\int_F \tau_{1,h}\nu_{xy} \cdot p\, ds = \int_F (\tau_1 - \widetilde{\tau}_{1,h})\nu_{xy} \cdot p\, ds \text{ for any } p \in (Q_{1,1}(F))^2,$$

$$\int_F \tau_{2,h} \cdot \nu_{xy} p\, ds = \int_F (\tau_2 - \widetilde{\tau}_{2,h}) \cdot \nu_{xy} p\, ds \text{ for any } p \in Q_{1,0}(F),$$

and for face F that parallels the (x, y)-plane,

$$\int_F \tau_{2,h} \cdot p \, dxdy = \int_F (\tau_2 - \widetilde{\tau}_{2,h}) \cdot p \, dxdy \text{ for any } p \in R_{xy}(\Delta_{xy}), \tag{21}$$

$$\int_F \tau_{3,h} p \, dxdy = \int_F (\tau_3 - \widetilde{\tau}_{3,h}) p \, dxdy \text{ for any } p \in P_1(x, y).$$

Note that (21) is a combination of (6) and a slight modification of (7) in Lemma 2.1, replacing $p = \mathrm{curl} b_{xy} \in \Psi_{k+1}(\Delta_{xy})$ with $p = (y, -x)^T$ there. This is valid because of the result (16). In addition, the remaining degrees of freedom vanish for $\tau_{1,h}, \tau_{2,h}$ and $\tau_{3,h}$.

Since $U_{k,h} \subset \Sigma_{k,h}$, we define $I_h \tau = \widetilde{I}_h \tau + \begin{pmatrix} \tau_{1,h} & \tau_{2,h} \\ \tau_{2,h}^T & \tau_{3,h} \end{pmatrix}$. An integration by parts immediately yields that (19) holds true. The stability estimate follows from (18) and the definition of the correction $\begin{pmatrix} \tau_{1,h} & \tau_{2,h} \\ \tau_{2,h}^T & \tau_{3,h} \end{pmatrix}$. □

Theorem 3.4. *For $k \geq 1$, there exists a positive constant C independent of the meshsize h with*

$$\sup_{0 \neq \tau_h \in \Sigma_{k,h}} \frac{(\mathrm{div}\, \tau_h, v_h)}{\|\tau_h\|_{H(\mathrm{div},\Omega)}} \geq C \|v_h\|_{0,\Omega} \quad \text{for any } v_h \in V_{k,h}.$$

Proof. By the stability of the continuous formulation, see [9, 23], there exists a $\tau \in H^1(\Omega; \mathbb{S})$ such that

$$\mathrm{div}\, \tau = v_h \text{ and } \|\tau\|_{1,\Omega} \leq C \|v_h\|_{0,\Omega}.$$

This plus Lemma 3.3 implies that

$$\int_K (\mathrm{div}\, I_h \tau - v_h) \cdot w dx = 0 \text{ for any } w \in RM(K) \text{ and any element } K \tag{22}$$

and

$$\|I_h \tau\|_{H(\mathrm{div},\Omega)} \leq C \|v_h\|_{0,\Omega}. \tag{23}$$

By Lemma 3.2, there exists a $\delta_h \in \Sigma_{k,h}$ such that

$$\mathrm{div}\, \delta_h = v_h - \mathrm{div}\, I_h \tau \text{ and } \|\delta_h\|_{H(\mathrm{div},\Omega)} \leq C \|v_h - \mathrm{div}\, I_h \tau\|_{0,\Omega}. \tag{24}$$

Let $\tau_h = I_h \tau + \delta_h$. Then we have $\mathrm{div}\, \tau_h = v_h$ and $\|\tau_h\|_{H(\mathrm{div},\Omega)} \leq C \|v_h\|_{0,\Omega}$. □

Remark 3.5. *Similarly as mentioned in* [23], *it follows from Lemma 3.2 and Lemma 3.3 that there exists an interpolation operator $\Pi_h : H^1(\Omega; \mathbb{S}) \to \Sigma_{k,h}$ such that*

$$(\mathrm{div}(\tau - \Pi_h \tau), v_h)_K = 0 \text{ for any } K \text{ and } v_h \in V_{k,h}$$

for any $\tau \in H^1(\Omega; \mathbb{S})$. Further, if $\tau \in H^{k+1}(\Omega; \mathbb{S})$, it holds that

$$\|\tau - \Pi_h \tau\|_{0,\Omega} \leq C h^{k+1} \|\tau\|_{k+1,\Omega}.$$

Theorem 3.6. *Let $(\sigma, u) \in \Sigma \times V$ be the exact solution of problem* (1) *and $(\sigma_h, u_h) \in \Sigma_{k,h} \times V_{k,h}$ the finite element solution of* (10). *Then, for $k \geq 1$,*

$$\|\sigma - \sigma_h\|_{H(\mathrm{div},\Omega)} + \|u - u_h\|_{0,\Omega} \leq C h^{k+1} (\|\sigma\|_{k+2,\Omega} + \|u\|_{k+1,\Omega}). \tag{25}$$

Proof. We follow the standard error estimate of mixed finite element methods in [13]

$$\|\sigma-\sigma_h\|_{H(\mathrm{div},\Omega)}+\|u-u_h\|_{0,\Omega}\le C\inf_{\tau_h\in\Sigma_{k,h},v_h\in V_{k,h}}(\|\sigma-\tau_h\|_{H(\mathrm{div},\Omega)}+\|u-v_h\|_{0,\Omega}).$$

Let P_h denote the local L^2 projection operator, from V to $V_{k,h}$, satisfying the error estimate

$$\|v-P_hv\|_{0,\Omega}\le Ch^{k+1}\|v\|_{k+1,\Omega}\text{ for any }v\in H^{k+1}(\Omega;\mathbb{R}^3).$$

Choosing $\tau_h=\Pi_h\sigma$ where Π_h is defined in Remark 3.5, and note that $\operatorname{div}\Pi_h\sigma=P_h\operatorname{div}\sigma$, we have

$$\|\sigma-\Pi_h\sigma\|_{H(\mathrm{div},\Omega)}\le Ch^{k+1}\|\sigma\|_{k+2,\Omega}.$$

Consequently, a choice of $v_h=P_hu$ and $\tau_h=\Pi_h\sigma$ completes the proof. □

4. Reduced mixed triangular prism elements

In this section, we provide a family of reduced spaces of $\Sigma_{k,h}$ in (6). According to Lemma 3.2, we only need the degrees of freedom of bubble function space $\Sigma_{K,k,b}$ to deal with the space $RM^{\perp}(K)$. From the proof of Lemma 3.3, we only need degrees of freedom on faces of the lowest order element to deal with the rigid motion space $RM(K)$ on each element K. Hence the stress finite elements can be reduced by replacing $H_{k+2,h}$ and BDM_{k+1} in (6) by $\widetilde{H}_{k+2,h}$ and $\widetilde{\mathrm{BDM}}_{k+1}$ as follows

$$\widetilde{H}_{k+2,h}:=\Big\{\tau\in H(\mathrm{div}_{xy},\Omega_{xy};\mathbb{S}_2)\;\Big|\;\tau=\tau_\ell+\tau_b,\ \tau_\ell\in H_{k,h},\\ \tau_b|_{\Delta_{xy}}\in H_{\Delta_{xy},k+2,b}\text{ for any }\Delta_{xy}\in\mathcal{X}_h\Big\},$$

$$\widetilde{\mathrm{BDM}}_{k+1}:=\Big\{\tau\in H(\mathrm{div}_{xy},\Omega_{xy};\mathbb{R}^2)\;\Big|\;\tau=\tau_\ell+\tau_b,\ \tau_\ell\in \mathrm{BDM}_k,\\ \tau_b|_{\Delta_{xy}}\in \mathrm{BDM}_{\Delta_{xy},k+1,b}\text{ for any }\Delta_{xy}\in\mathcal{X}_h\Big\}.$$

Remark 4.1. *We know that $H_{k,h}$ is defined for $k\ge 3$ in* (4). *When $k=1,2$, we refer interested readers to* [29] *for those two cases and omit the specific definitions herein. Thus, the degrees of freedom on each element of our lowest order case, which is of second order convergence, are 93 plus 33.*

We use $\widetilde{\Sigma}_{k,h}$ to denote the new stress spaces. The reduced elements preserve the same convergence order.

Theorem 4.2. *Let $(\sigma,u)\in\Sigma\times V$ be the exact solution of problem* (1) *and $(\sigma_h,u_h)\in\widetilde{\Sigma}_{k,h}\times V_{k,h}$ the discrete solution by the reduced triangular prism elements. Then, for $k\ge 1$,*

$$\|\sigma-\sigma_h\|_{H(\mathrm{div},\Omega)}+\|u-u_h\|_{0,\Omega}\le Ch^{k+1}(\|\sigma\|_{k+2,\Omega}+\|u\|_{k+1,\Omega}).$$

Remark 4.3. *For $k=1$, if we utilize the first order nonconforming stress space of* [10, 21] *in two dimensions instead of the first order conforming element of* [29], *we obtain a nonconforming mixed triangular prism element of first order convergence, with 81 plus 33 degrees of freedom. The following error estimate holds*

$$\|\sigma-\sigma_h\|_{0,\Omega}+\|u-u_h\|_{0,\Omega}\le Ch\|u\|_{2,\Omega}.$$

5. Numerical results

5.1. Pure displacement problem. It is a pure displacement problem on the unit cube $\Omega = (0,1)^3$ with a homogeneous boundary condition that $u \equiv 0$ on $\partial\Omega$. In the computation, let

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + 3\lambda}\operatorname{tr}(\sigma)\delta\right),$$

where $\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\mu = 1/2$ and $\lambda = 1$ are the Lamé constants.

Let the exact solution on the unit square $[0,1]^3$ be

$$u = \begin{pmatrix} 2^4 \\ 2^5 \\ 2^6 \end{pmatrix} x(1-x)y(1-y)z(1-z). \tag{26}$$

Then, the true stress function σ and the load function f are defined by the equations in (1), for the given solution u.

We use the triangular prism element of $k = 1$ in Section 2. In the computation, each mesh is refined into a half-sized mesh uniformly, see the initial mesh in Figure 1. In Table 1, the errors and the convergence order in various norms are listed for the true solution (26), by the mixed finite element in (6) and (9), with $k = 1$ there. The optimal order of convergence is achieved in Table 1, coinciding with Theorem 3.6.

Table 1. The error and the order of convergence by the triangular prism element, $k = 1$ in (6) and (9), for (26).

	$\|\sigma - \sigma_h\|_{0,\Omega}$	h^n	$\|u - u_h\|_{0,\Omega}$	h^n	$\|\operatorname{div}(\sigma - \sigma_h)\|_{0,\Omega}$	h^n
1	1.61682569	0.0	0.21093411	0.0	6.10467990	0.0
2	0.48388087	1.74	0.06461602	1.71	1.74304423	1.81
3	0.12795918	1.92	0.01699145	1.92	0.45537323	1.94
4	0.03244990	1.98	0.00429655	1.98	0.11514501	1.98
5	0.00814520	1.99	0.00108161	1.99	0.02886873	1.99

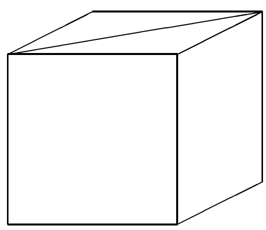

Figure 1. The initial grid for the triangular prism partitions.

5.2. Cantilever beam problem. We consider a straight beam of dimension $10 \times 1 \times 1$ from [41]. The Young's modulus is $E = 2.1 \times 10^6$. The top is loaded with normal stress $q = 100$. See Figure 2 for the boundary conditions and the initial triangular prism grid. We use the mixed triangular prism element of $k = 1$ in Section 2 to compute the vertical displacements at points A and B, which are denoted by u_3^A and u_3^B respectively. For comparison, the first (P1) and second (P2) order Lagrangian elements are also used to produce the vertical displacements at points A and B. The initial tetrahedral grid is obtained by dividing each triangular prism into three tetrahedrons. In this example, we also test the behavior of elements under the incompressible limit condition (Poisson's ratio ν is 0.3 and 0.499). The Lamé constants λ and μ satisfy

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \ \mu = \frac{E}{2(1+\nu)}.$$

The results are listed in Table 2. From Table 2, we see that the first order mixed triangular prism element is locking free.

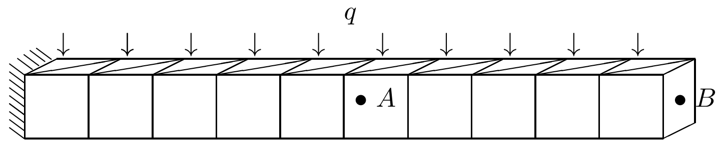

FIGURE 2. The initial triangular prism grid for the cantilever beam.

TABLE 2. Cantilever beam.

	Poisson's ratio $\nu = 0.3$					
	Prism ($k=1$)		P1		P2	
mesh	u_3^A	u_3^B	u_3^A	u_3^B	u_3^A	u_3^B
1	-0.2571	-0.7201	-0.0588	-0.1596	-0.2472	-0.6998
2	-0.2566	-0.7183	-0.1305	-0.3636	-0.2534	-0.7119
3	-0.2560	-0.7171	-0.2023	-0.5671	-0.2549	-0.7148
4	-0.2557	-0.7165	-0.2388	-0.6698	-0.2553	-0.7156
	Poisson's ratio $\nu = 0.499$					
1	-0.2561	-0.7177	-0.0074	-0.0110	-0.2010	-0.6006
2	-0.2540	-0.7128	-0.0095	-0.0171	-0.2339	-0.6716
3	-0.2520	-0.7086	-0.0174	-0.0398	-0.2449	-0.6943
4	-0.2510	-0.7066	-0.0427	-0.1133	-0.2484	-0.7014

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