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ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS

R. WONG

In memory of Professor Benyu Guo

Abstract. In this survey article, we present some recent results on the asymptotic behavior of four systems of orthogonal polynomials. These are Stieltjes-Wigert, Hahn, Racah and pseudo-Jacobi polynomials. In each case, the variable z is allowed to be in any part of the complex plane. In some cases, asymptotic formulas are also given for their zeros.

Key words. Asymptotics, orthogonal polynomials, Riemann-Hilbert method, difference-equation techniques.

1. Introduction

Most of the asymptotic results of classical orthogonal polynomials (namely, Hermite $H_n(z)$, Laguerre $L_n^{(\alpha)}(z)$, and Jacobi $P_n^{(\alpha,\beta)}(z)$) can be found in the books of Szegö [45] and Erdélyi et al. [15]. Some of the asymptotic results of a few familiar discrete orthogonal polynomials (e.g., Charlier $C_n(z;a)$, Meixner $M_n(z;\beta,c)$, and Krawtchouk $K_n(z;p,N)$) have been summarized in a survey article of Wong [56]. With the new developments in asymptotic methods based on the Riemann-Hilbert approach [10, 11] and difference equations [48, 50, 51], asymptotic problems of some orthogonal polynomials, which have been considered to be more difficult to tackle, have also been resolved; for instance, discrete Chebyshev [33], Conrad-Flajolet [8], polynomials orthogonal with respect to Freud weights [27, 57], and Tricomi-Carlitz [30, 58]. The reasons for the asymptotic problems of the last few mentioned orthogonal polynomials being more difficult are : (i) they do not satisfy second-order linear ordinary differential equations, and (ii) they do not have integral representations to which one can apply the classical methods of steepest descent or stationary phase.

The purpose of this paper is to give a summary of asymptotic results obtained recently for four systems of orthogonal polynomials; they are Stieltjes-Wigert $S_n(z;q)$, Hahn $Q_n(z;\alpha,\beta,N)$, Racah $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, and pseudo-Jacobi $P_n(z;a,b)$. The problem of finding asymptotic formulas for Stieltjes-Wigert polynomials has been around for some time. However, serious work began only at the beginning of this century. These polynomials do not have integral representations, and neither do they satisfy any second-order linear differential equation. Although they satisfy a three-term recurrence relation [6, p.174], the coefficients of the recurrence relation contain exponentially large terms of the form $q^{-n}, 0 < q < 1$. As a consequence, none of the existing methods for second-order difference equations can be applied. The Hahn polynomial was first introduced by Chebyshev in 1858. Despite its long history, there seems to be no literature on asymptotic results of this polynomial. A difficulty in dealing with asymptotic problems of this polynomial is that it has three free real parameters, whereas the other classical discrete orthogonal polynomials involve only two or less free parameters. This remark also applies to the

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Racah polynomial, which has even one more free parameter. We pick Pseudo-Jacobi polynomials as our last example for presentation, since it also has a long history (over hundred years) and it is not well-known even to the researchers in the field of orthogonal polynomials until just recently. Another reason is that this example illustrates the asymptotic method developed for differential equations with a large parameter [38, Chapter 11], which can be applied to many applied disciplines, and deserves a larger audience.

2. Stieltjes-Wigert polynomials

Let $k > 0$ be a fixed number and

$$q = \exp\{-(2k^2)^{-1}\}. \tag{1}$$

Note that $0 < q < 1$. The q-shifted factorial is given by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{j=0}^{n-1}(1 - aq^j), \qquad n = 1, 2, \cdots .$$

The Stieltjes-Wigert polynomials

$$S_n(z;q) := \sum_{j=0}^{n} \frac{q^{j^2}}{(q;q)_j (q;q)_{n-j}} (-z)^j, \quad n = 0, 1, 2, \cdots , \tag{2}$$

are orthogonal with respect to the weight function

$$w(x) = k\pi^{-\frac{1}{2}} \exp\{-k^2 \log^2 x\} \tag{3}$$

for $0 < x < \infty$; see [26, (3.27.1)] and [39, (18.27.18)]. It is known that these polynomials belong to the indeterminate moment class and the weight function in (3) is not unique; see [7]. By changing the index j to $n-j$ in the explicit expression given in (2), one can easily verify the symmetry relation

$$S_n(z;q) = (-zq^n)^n S_n\left(\frac{1}{zq^{2n}}; q\right). \tag{4}$$

In some literatures, the variable z in (2) is replaced by $q^{\frac{1}{2}}z$; see, for instance, [6], [45] and [52]. The notation for the Stieltjes-Wigert polynomials used in these literatures is

$$p_n(z) = (-1)^n q^{n/2+1/4} \sqrt{(q;q)_n}\, S_n(q^{\frac{1}{2}}z; q). \tag{5}$$

These polynomials arise in random walks and random matrix formulation of Chern-Simons theory on Seifert manifolds; see [4, 12].

The asymptotics of the Stieltjes-Wigert polynomials, as the degree tends to infinity, has been studied by several authors. First, Wigert [53] in 1923 proved that the polynomials have the limiting behavior

$$\lim_{n\to\infty} (-1)^n q^{-n/2} p_n(z) = \frac{q^{1/4}}{\sqrt{(q;q)_\infty}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k^2+k/2}}{(q;q)_k} z^k, \tag{6}$$

which can be put in terms of the q-Airy function (also known as the Ramanujan function)

$$A_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} (-z)^k. \tag{7}$$

In terms of this function, Wigert's result can be stated as

$$\lim_{n\to\infty} S_n(z;q) = \frac{1}{(q;q)_\infty} A_q(z). \tag{8}$$

It has been shown in [52] that for large values of n, all zeros of $S_n(z;q)$ lie in the interval $(\frac{1}{4}, 4q^{-2n})$. Thus, it is natural to introduce a new scale

$$z = uq^{-tn} \tag{9}$$

with $u \in \mathbb{C}\backslash\{0\}$ and $t \in \mathbb{R}$. The values of $t = 0$ and $t = 2$ can be regarded as the turning points of $S_n(q^{-nt}u;q)$. In view of the symmetry relation in (4), one may restrict oneself to the case $t \geqslant 1$; see [47, (1.4)]. (However, we shall not make this restriction in our discussion here.) The case $t = 2$ has been studied by Ismail [20], and he proved that for $t = 2$

$$\lim_{n\to\infty} q^{n^2(t-1)}(-u)^{-n} S_n(uq^{-nt};q) = \frac{1}{(q;q)_\infty} A_q\left(\frac{q^{n(t-2)}}{u}\right) \tag{10}$$

uniformly for u in compact subsets of $\mathbb{C} \setminus \{0\}$; see [20, Theorem 2.5]. This result can in fact be derived directly from Wigert's formula in (8) by using the symmetry relation mentioned in (4). In [21], Ismail and Zhang extended the validity of this result to $t \geqslant 2$. For $1 \leqslant t < 2$, Ismail and Zhang [21] gave asymptotic formulas for these polynomials in terms of the theta-type function (cf. also [17])

$$\Theta_q(z) = \sum_{k=-\infty}^{\infty} q^{k^2} z^k, \tag{11}$$

but in a very complicated manner. Their result was then simplified by Wang and Wong in [46]. For instance, when $1 \leqslant t < 2$, Wang and Wong proved that

$$\begin{aligned} S_n(uq^{-nt};q) = {} & \frac{(-u)^{n-m} q^{n^2(1-t)-m[n(2-t)-m]}}{(q;q)_n (q;q)_\infty} \\ & \times \left\{ \Theta_q\left(\frac{q^{2m-n(2-t)}}{-u}\right) + O(q^{n(l-\delta)}) \right\}, \end{aligned} \tag{12}$$

where $l = \frac{1}{2}(2-t), m = \lfloor nl \rfloor$ and $\delta > 0$ is any small number; see [46, Corollary 2]. However, none of these results is valid in a neighborhood of $t = 2$, one of the turning points. To address this issue, Wang and Wong in a second paper [47] presented a uniform asymptotic formula: for $z := uq^{-nt}$ with $t > 2(1-\delta), \delta > 0$, they showed that

$$S_n(z;q) = \frac{(-z)^n q^{n^2}}{(q;q)_n} [A_{q,n}(q^{-2n}/z) + r_n(z)], \tag{13}$$

where $r_n(z)$ is the remainder and $A_{q,n}(z)$ is the q-Airy polynomial obtained by truncating the infinite series in (7) at $k = n$, i.e.,

$$A_{q,n}(z) := \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k} (-z)^k.$$

This result, however, is not totally satisfactory, since the q-Airy polynomial is not a known special function and has not been well studied. (The situation is like that with the exponential polynomial $\sum_{k=0}^{n} z^k/k!$, which is not as well-known as the exponential function e^z itself.)

A more satisfactory solution was provided by Li and Wong [32] three years later. Their result is stated below.

THEOREM 2.1 *Let* $z := uq^{-nt}$ *with* $-\infty < t < 2, u \in \mathbb{C}$ *and* $|u| \leqslant R$, *where* $R > 0$ *is any fixed positive number. We have*

$$S_n(z;q) = \frac{1}{(q;q)_n}[A_q(z) + r_n(z)], \tag{14}$$

where the remainder satisfies

$$|r_n(z)| \leqslant \left[\frac{q^{n(1-\sigma)}}{1-q} + \frac{2}{1-q}\left(\frac{1}{2}\right)^{\lfloor n\sigma \rfloor}\right] A_q(-|z|) \tag{15}$$

with $\sigma = \max\{\frac{1}{2}, \frac{1}{2} + \frac{t}{4}\}$.

Let $z := uq^{-nt}$ *with* $0 < t < \infty, u \in \mathbb{C}$ *and* $|u| \geqslant 1/R$, *where* $R > 0$ *is any fixed number. We have*

$$S_n(z;q) = \frac{(-z)^n q^{n^2}}{(q;q)_n}\left[A_q(q^{-2n}/z) + r_n(z)\right], \tag{16}$$

where the remainder satisfies

$$|r_n(z)| \leqslant \left[\frac{q^{n(\delta-1)}}{1-q} + \frac{2}{1-q}\left(\frac{1}{2}\right)^{\lfloor n(2-\delta) \rfloor}\right] A_q(-q^{-2n}/|z|) \tag{17}$$

with $\delta = \min\{\frac{3}{2}, 1 + \frac{t}{4}\}$.

Since the quantities inside the square brackets in (15) and (17) are exponentially small, the two asymptotic formulas in (14) and (16) are quite satisfactory.

The above mentioned investigations all began with the explicit expression of $S_n(z;q)$ given in (2). In [52], Wang and Wong used a different method, namely, the Riemann-Hilbert approach, to construct a uniform asymptotic expression of the Stieltjes-Wigert polynomials $p_n(z)$ in terms of Airy functions. Unfortunately, the main result in [52] is not quite correct; it requires an additional condition. The additional condition is that the parameter k in (1) should depend on the degree n and tend to infinity as $n \to \infty$. More precisely, we need to impose the condition that there are two constants $c > 0$ and $\sigma > 0$ such that

$$\lim_{n\to\infty} \frac{k}{n^\sigma} = c; \tag{18}$$

see also Baik and Suidan [4]. For completeness, we state the amended result below.

First, we introduce some notations. Let γ_n denote the leading coefficient of the polynomial $p_n(z)$ given in (5), and put $\pi_n(z) := p_n(z)/\gamma_n$. The Mhaskar-Rakhmanov-Saff (MRS) numbers α_n and β_n for our present problem are given by

$$\begin{aligned} \alpha_n = {} & 2e^{(n+1/2)/k^2} - e^{(n+1/2)/2k^2} \\ & - 2e^{(n+1/2)/2k^2}\sqrt{e^{(n+1/2)/k^2} - e^{(n+1/2)/2k^2}}, \end{aligned} \tag{19}$$

$$\begin{aligned} \beta_n = {} & 2e^{(n+1/2)/k^2} - e^{(n+1/2)/2k^2} \\ & + 2e^{(n+1/2)/2k^2}\sqrt{e^{(n+1/2)/k^2} - e^{(n+1/2)/2k^2}}. \end{aligned} \tag{20}$$

The probability density function supported on $[\alpha_n, \beta_n]$ is given by

$$\mu_n(x) = \frac{2k^2}{N\pi x}\arctan\frac{\sqrt{(x-\alpha_n)(\beta_n - x)}}{x + \sqrt{\alpha_n\beta_n}},$$

where $N = n + \frac{1}{2}$. Note that $\mu_n(x) \geqslant 0$ and

$$\int_{\alpha_n}^{\beta_n} \mu_n(x)dx = 1. \tag{21}$$

The logarithmic potential of $\mu_n(x)$, i.e., the g-function is given by

$$g(z) = \int_{\alpha_n}^{\beta_n} \log(z-s)\mu_n(s)ds. \tag{22}$$

A closely related function is

$$\begin{aligned}\phi_n(z) = \frac{k^2}{N}\int_{\beta_n}^{z} \frac{1}{\zeta}\Big\{2\log\Big[\zeta + \sqrt{\alpha_n\beta_n} + \sqrt{(\zeta-\alpha_n)(\zeta-\beta_n)}\Big] \\ - \log\Big[(\sqrt{\alpha_n}+\sqrt{\beta_n})^2\zeta\Big]\Big\}d\zeta\end{aligned} \tag{23}$$

for $z \in \mathbb{C} \setminus (-\infty, \beta_n]$; see [52, (3.30)-(3.31)]. The function

$$\zeta_n(z) := \left[\frac{3}{2}\phi_n(z)\right]^{2/3} \tag{24}$$

plays an important role in the theory of uniform asymptotic expansions. It arises in the Liouville transformation for differential equations [38, p. 398], as well as in the cubic transformation for integrals [55, p. 367]. Since $(\zeta_n)_+(x) = (\zeta_n)_-(x)$ for $x \in (\alpha_n, \beta_n)$, $\zeta_n(z)$ can be analytically continued to $\mathbb{C} \setminus (-\infty, \alpha_n]$. Finally, we introduce the Lagrange constant

$$l_n = 2\left(n+\frac{1}{2}\right)g(\beta_n) + \log w(\beta_n) - \log\frac{\beta_n-\alpha_n}{4}. \tag{25}$$

Note that the g-function in (22) has been explicitly evaluated in [52]; see (23) above and (3.7) and (3.37) in that paper.

THEOREM 2.2 *With $\pi_n(z), \zeta_n(z)$ and l_n defined as above, we assume that the condition in (18) holds. Then, we have*

$$\begin{aligned}\pi_n(z) = \frac{\sqrt{\pi}e^{l_n/2}}{\sqrt{w(z)}}\Bigg\{\left(n+\frac{1}{2}\right)^{1/6}\mathrm{Ai}\left(\left(n+\frac{1}{2}\right)^{2/3}\zeta_n\right)\boldsymbol{A}(z,n) \\ - \left(n+\frac{1}{2}\right)^{-1/6}\mathrm{Ai}'\left(\left(n+\frac{1}{2}\right)^{2/3}\zeta_n\right)\boldsymbol{B}(z,n)\Bigg\},\end{aligned} \tag{26}$$

where $\boldsymbol{A}(z,n)$ and $\boldsymbol{B}(z,n)$ are analytic functions of z in $\mathbb{C} \setminus S_\delta$, $S_\delta = \{z : \frac{2\pi}{3} \leqslant \arg(z-(\alpha_n+\delta)) \leqslant \frac{4\pi}{3}\}$ and δ is any small positive number. Further, the asymptotic expansions

$$\boldsymbol{A}(z,n) \sim \frac{\zeta_n^{1/4}(\beta_n-\alpha_n)^{1/2}}{[(z-\alpha_n)(z-\beta_n)]^{1/4}}\left[1+\sum_{k=1}^{\infty}\frac{A_k(z)}{n^k}\right], \tag{27}$$

$$\boldsymbol{B}(z,n) \sim \frac{[(z-\alpha_n)(z-\beta_n)]^{1/4}}{\zeta_n^{1/4}(\beta_n-\alpha_n)^{1/2}}\sum_{k=1}^{\infty}\frac{B_k(z)}{n^k} \tag{28}$$

hold uniformly, and the coefficient functions $A_k(z)$ and $B_k(z)$ are analytic functions in $\mathbb{C} \setminus S_\delta$.

3. Hahn polynomials

The Hahn polynomials are explicitly given by

$$\begin{aligned}Q_n(x;\alpha,\beta,N) &:= {}_3F_2(-n,-x,n+\alpha+\beta+1;-N,\alpha+1;1) \\ &= \sum_{k=0}^{n}\frac{(-n)_k(-x)_k(n+\alpha+\beta+1)_k}{(-N)_k(\alpha+1)_k k!},\end{aligned} \tag{29}$$

where $\alpha, \beta > -1$. These polynomials are orthogonal on the discrete set $\{0, 1, \cdots, N\}$ with respect to the weight function

$$\rho(x; \alpha, \beta, N) = \frac{(\alpha+1)_x}{x!} \frac{(\beta+1)_{N-x}}{(N-x)!}, \qquad x = 0, 1, \cdots, N; \tag{30}$$

that is,

$$\sum_{k=0}^{N} Q_n(k; \alpha, \beta, N) Q_m(k; \alpha, \beta, N) \rho(k; \alpha, \beta, N) = h_{N,n} \delta_{n,m}, \tag{31}$$

$n, m = 0, 1, \cdots, N$, where

$$h_{N,n} = \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n (-N)_n N!}. \tag{32}$$

Formula (31) indicates that these polynomials are orthogonal on an interval which is unbounded as $n \to \infty$. To make the interval of orthogonality bounded, we introduce a rescaling. Let X_N denote the set defined by

$$X_N := \left\{ x_{N,k} \right\}_{k=0}^{N-1}, \qquad \text{where } x_{N,k} := \frac{k+1/2}{N}. \tag{33}$$

The points $x_{N,k}$ are called *nodes*, and they all lie in the interval $(0, 1)$. Furthermore, we let

$$P_{N,n}(z) := Q_n(Nz - \frac{1}{2}; \alpha, \beta, N-1) \tag{34}$$

and

$$w(z) := N^{-\alpha-\beta} \alpha! \beta! \rho(Nz - 1/2; \alpha, \beta, N-1). \tag{35}$$

It is readily verified that the polynomials $P_{N,n}(z)$ are orthogonal on the nodes $x_{N,n}$ with respect to the weight $w_{N,k} := w(x_{N,k})$; i.e.,

$$\sum_{k=0}^{N-1} P_{N,n}(x_{N,k}) P_{N,m}(x_{N,k}) w_{N,k} = h^*_{N,n} \delta_{n,m}, \tag{36}$$

where $h^*_{N,n} = N^{-\alpha-\beta} \alpha! \beta! h_{N-1,n}$.

In 1989, Sharapudinov [43] considered the asymptotic behavior of $Q_n(x; \alpha, \beta, N)$ when the degree becomes large. His result is an asymptotic formula for $Q_n(x; \alpha, \beta, N)$ with $n = O(N^{1/2})$ as $n \to \infty$, which involves the Jacobi polynomial $P_n^{(\beta,\alpha)}(t)$, where t is related to x via the formula $x = (N-1)(1+t)/2$. More recently, Baik et al [3] used the Riemann-Hilbert method to investigate the asymptotics of discrete orthogonal polynomials with respect to a general weight function. Their results are very general, and it is difficult to use them to write out explicit formulas for specific polynomials. With regard to Hahn polynomials, they only considered the case of varying parameters, namely, $\alpha = NA$ and $\beta = NB$, where A and B are fixed positive numbers. Furthermore, their results are more local in nature; that is, one needs more asymptotic formulas to describe the behavior of these polynomials in the whole complex plane.

By using a modified version of the Riemann-Hilbert method, Lin and Wong [34] constructed globally uniform asymptotic formulas for the Hahn polynomials as $n \to \infty$. They considered the case when the parameters α and β are fixed and the ratio n/N is a constant $c \in (0, 1)$. To present their result, we need to recall some terminologies used in the Riemann-Hilbert method, including equilibrium measure, g-function, Lagrange constants, etc. In the existing literature, equilibrium measures are usually obtained by solving minimization problems of certain quadratic functions (see [3, 9]). In [34], the authors used a method introduced by Kuijlaars and Van Assche [29]. To state their result, we let

$$a := \frac{1}{2} - \frac{1}{2}\sqrt{1-c^2}, \qquad b := \frac{1}{2} + \frac{1}{2}\sqrt{1-c^2}, \tag{37}$$

where $c := n/N$. Define the function

$$\mu(x) = \begin{cases} \dfrac{2}{\pi c} \arcsin\left(\dfrac{c}{2\sqrt{x-x^2}}\right), & x \in (a,b), \\ \\ \dfrac{1}{c}, & x \in [0,a] \cup [b,1]. \end{cases} \tag{38}$$

and note that $\mu(x) \geqslant 0$ and

$$\int_0^1 \mu(x)dx = 1; \tag{39}$$

that is, $\mu(x)$ is a probability density function. Also, note that

$$a + b = 1, \qquad a \cdot b = c^2/4. \tag{40}$$

The equilibrium measure associated with the Hahn polynomials is just the (indefinite) integral of $\mu(x)$. The g-function is the logarithmic potential of $\mu(x)$ defined by

$$g(z) := \int_0^1 \log(z-s)\mu(s)ds, \qquad z \in \mathbb{C} \setminus (-\infty, 1]. \tag{41}$$

An auxiliary function $\phi(z)$ is defined by

$$\phi(z) := l/2 - g(z), \qquad z \in \mathbb{C} \setminus (-\infty, 1], \tag{42}$$

where l is the Lagrange constant given by

$$l := 2\int_0^1 \log|a-s| \cdot \mu(s)ds. \tag{43}$$

Moreover, we need a complex function $\nu(z)$ satisfying the requirement

$$\nu_\pm(x) = \pm\pi i \left(\mu(x) - \frac{1}{c}\right) \tag{44}$$

for $x \in (a,b)$, and the function is

$$\nu(z) := \frac{2}{c}\log\left[c/2 - \sqrt{(z-a)(z-b)}\right] - \frac{1}{c}\log(z-z^2), \tag{45}$$

where $\sqrt{(z-a)(z-b)}$ is analytic in $\mathbb{C} \setminus [a,b]$ and behaves like z as $z \to \infty$, and where $\log(z-z^2)$ is an analytic function with branch cuts along $(-\infty,0] \cup [1,+\infty)$.

Now, we are ready to introduce the auxiliary function

$$\tilde{\phi}(z) := \int_a^z \nu(s)ds, \qquad z \in \mathbb{C} \setminus (-\infty,0] \cup [a,+\infty), \tag{46}$$

where the path of integration from a to z lies entirely in the region $z \in \mathbb{C} \setminus (-\infty, 0] \cup [a,+\infty)$, except for the initial point a. Similarly, we define

$$\phi^*(z) := \int_b^z \nu(s)ds, \qquad z \in \mathbb{C} \setminus (-\infty,b] \cup [1,+\infty), \tag{47}$$

where the path of integration from b to z lies entirely in the region $z \in \mathbb{C} \setminus (-\infty, b] \cup [1,+\infty)$, except for the initial point b. The connection between the $\tilde{\phi}$ - function (ϕ^* - function) and the ϕ - function in (42) is given by the formulas

$$\tilde{\phi}(z) = \phi(z) \pm \pi i \left(1 - \frac{1}{c} z\right) \tag{48}$$

and

$$\phi^*(z) = \phi(z) \pm \pi i \frac{1}{c} (1 - z) \tag{49}$$

for $z \in \mathbb{C}_\pm$.

The two functions $\tilde{\phi}(z)$ and $\phi^*(z)$ defined above play an important role in the construction of globally uniform asymptotic approximations given by Lin and Wong [34], since the arguments of the leading terms in our two asymptotics expansions are

$$\tilde{f}(z) = \left(-\frac{3}{2}\tilde{\phi}(z)\right)^{2/3}, \tag{50}$$

which is analytic in $\mathbb{C} \setminus (-\infty, 0] \cup [b, +\infty)$, and

$$f^*(z) = \left(-\frac{3}{2}\phi^*(z)\right)^{2/3}, \tag{51}$$

which is analytic in $\mathbb{C} \setminus (-\infty, a] \cup [1, +\infty)$.

Another important function used in our method is the D-function

$$D(z) := \frac{e^{Nz}\Gamma(Nz + 1/2)}{\sqrt{2\pi}(Nz)^{Nz}}, \qquad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{52}$$

Two related functions are

$$D^*(z) := \frac{\sqrt{2\pi}e^{Nz}(-Nz)^{-Nz}}{\Gamma(-Nz + 1/2)}, \qquad z \in \mathbb{C} \setminus [0, +\infty), \tag{53}$$

and

$$\tilde{D}(z) := \begin{cases} D^*(z), & \operatorname{Re} z < x_0, \\ D^*(1 - z) & \operatorname{Re} z > x_0, \end{cases} \tag{54}$$

where x_0 is an arbitrary fixed point in the interval (a, b) and a, b are the numbers defined in (37).

To state the result in [34], we need to first introduce two more functions:

$$\begin{aligned} m(z) :=& \frac{(z-a)^{1/2} + (z-b)^{1/2}}{2(z-a)^{1/4}(z-b)^{1/4}} \left(\frac{z + c/2 + (z-a)^{1/2}(z-b)^{1/2}}{2z}\right)^{\alpha} \\ &\times \left(\frac{1 - z + c/2 - (z-a)^{1/2}(z-b)^{1/2}}{2(1-z)}\right)^{\beta} \end{aligned} \tag{55}$$

and

$$\begin{aligned} m^*(z) :=& \frac{(z-b)^{1/2} - (z-a)^{1/2}}{2(z-a)^{1/4}(z-b)^{1/4}} \left(\frac{z + c/2 - (z-a)^{1/2}(z-b)^{1/2}}{2z}\right)^{\alpha} \\ &\times \left(\frac{1 - z + c/2 + (z-a)^{1/2}(z-b)^{1/2}}{2(1-z)}\right)^{\beta}. \end{aligned} \tag{56}$$

Note that in view of (40), $m(z)$ is analytical in $\mathbb{C} \setminus [a, b]$. But, the function $m^*(z)$ has additional cuts $(-\infty, 0]$ and $[1, +\infty)$. Next, we divide the complex plane into the three regions I, II and III as shown in Figure 1, where δ is any positive number and x_1 is an arbitrary fixed number in $(0, a)$.

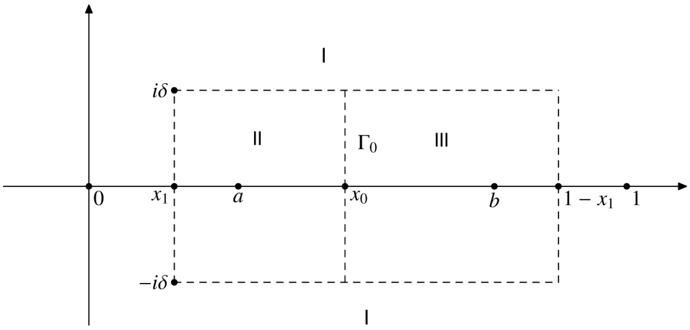

Figure 1: The regions I, II, III.

Recall the polynomial $P_{N,n}(z)$ defined in (34), and note that the leading coefficient of this polynomial is

$$k_{N,n} := \frac{N^n(n+\alpha+\beta+1)_n}{(\alpha+1)_n(-N+1)_n}. \tag{57}$$

Hence, the monic Hahn polynomials $\pi_{N,n}(z)$ are given

$$\pi_{N,n}(z) = \frac{1}{k_{N,n}} P_{N,n}(z). \tag{58}$$

Let $\Omega_+ = \{z \in \mathbb{C} : 0 < \text{Re } z < 1 \text{ and } 0 < \text{Im } z < \delta\}$ and $\Omega_- = \{z : \bar{z} \in \Omega_+\}$.

THEOREM 3.1. *Let* I, II *and* III *be the regions shown in Figure* 1, *and let* l *denote the Lagrange constant given in* (43). *With* $\phi(z)$ *and* $\tilde{D}(z)$ *defined in* (42) *and* (53) – (54), *respectively, the asymptotic formula of the polynomial* $\pi_{N,n}(z)$ *is given by*

$$\pi_{N,n}(z) = e^{nl/2}\left\{\tilde{D}(z)e^{-n\phi(z)}m(z)\Big[1+O\big(\frac{1}{n}\big)\Big] + \delta(n)\right\} \tag{59}$$

for $z \in$ I, *where* $\delta(n) = 0$ *for* $z \in \text{I} \setminus \Omega_\pm$ *and* $\delta(n)$ *is exponentially small in comparison with its leading term for* $z \in \text{I} \cap \Omega_\pm$. *More precisely,* $\delta(n) = O(N^{\max(\alpha,\beta)}e^{n\phi(z)})$, *where* Re $\phi(z)$ *is negative when* Re z *is bounded away from the interval* (a,b).

Let $\tilde{f}(z)$ *be defined as in* (50). *We have*

$$\pi_{N,n}(z) = (-1)^n\sqrt{\pi}e^{nl/2}\left\{\tilde{A}(z,n)\Big[1+O\big(\frac{1}{n}\big)\Big] + \tilde{B}(z,n)\Big[1+O\big(\frac{1}{n}\big)\Big]\right\} \tag{60}$$

for $z \in$ II, *where*

$$\begin{aligned}\tilde{A}(z,n) =& \Big[n^{2/3}\tilde{f}(z)\Big]^{1/4}\Big[m(z)+m^*(z)\Big] \\ &\times \left\{\sin(N\pi z)\text{Ai}\Big(n^{2/3}\tilde{f}(z)\Big) + \cos(N\pi z)\text{Bi}\Big(n^{2/3}\tilde{f}(z)\Big)\right\}\end{aligned} \tag{61}$$

and

$$\begin{aligned}\tilde{B}(z,n) =& \Big[n^{2/3}\tilde{f}(z)\Big]^{-1/4}\Big[m(z)-m^*(z)\Big] \\ &\times \left\{\sin(N\pi z)\text{Ai}'\Big(n^{2/3}\tilde{f}(z)\Big) + \cos(N\pi z)\text{Bi}'\Big(n^{2/3}\tilde{f}(z)\Big)\right\}.\end{aligned} \tag{62}$$

Similarly, with $f^*(z)$ *defined in* (51),

$$\pi_{N,n}(z) = (-1)^N \sqrt{\pi} e^{nl/2} \left\{ A^*(z,n)\left[1 + O\left(\frac{1}{n}\right)\right] + B^*(z,n)\left[1 + O\left(\frac{1}{n}\right)\right] \right\} \tag{63}$$

for $z \in \mathrm{III}$, *where*

$$\begin{aligned} A^*(z,n) = {} & \left[n^{2/3} f^*(z)\right]^{1/4} \left[m(z) - m^*(z)\right] \\ & \times \left\{ \cos(N\pi z)\mathrm{Bi}\left(n^{2/3} f^*(z)\right) - \sin(N\pi z)\mathrm{Ai}\left(n^{2/3} f^*(z)\right) \right\} \end{aligned} \tag{64}$$

and

$$\begin{aligned} B^*(z,n) = {} & \left[n^{2/3} f^*(z)\right]^{-1/4} \left[m(z) + m^*(z)\right] \\ & \times \left\{ \cos(N\pi z)\mathrm{Bi}'\left(n^{2/3} f^*(z)\right) - \sin(N\pi z)\mathrm{Ai}'\left(n^{2/3} f^*(z)\right) \right\}. \end{aligned} \tag{65}$$

From (59), one can deduce asymptotic formulas for the Hahn polynomials $Q_n(x;\alpha, \beta, N-1)$ when x is a fixed number. First, we note that with

$$x = Nz - 1/2, \tag{66}$$

it follows from (34) and (58) that

$$\begin{aligned} Q_n(x;\alpha,\beta,N-1) &= Q_n(Nz - 1/2;\alpha,\beta,N-1) \\ &= \frac{N^n (n+\alpha+\beta+1)_n}{(\alpha+1)_n(-N+1)_n} \pi_{N,n}(z). \end{aligned} \tag{67}$$

The final result is summarized in the following

Corollary 3.2. *For fixed values of* x, *we have*

$$\begin{aligned} Q_n(x;\alpha,\beta,N-1) \sim {} & \frac{\Gamma(\alpha+1)\Gamma(N-n)}{\Gamma(N)\Gamma(-x)e^n n^{2x+2\alpha+2}} \\ & \times (1+c)^{n+N+\alpha+\beta+1/2} N^{x+n+\alpha+1} \end{aligned} \tag{68}$$

as $n \to \infty$.

Note that equation (68) is only an asymptotic equality, and is *not* an equation. So, even when the right-hand side of the equation vanishes when x is a positive integer or zero, the polynomial on the left-hand side may not vanish. An error estimate for the approximation in (68) can be obtained by using the order estimates in (59).

4. Racah polynomials

Within the Askey scheme [39, p. 464] of hypergeometric orthogonal polynomials, Racah polynomials stay on the top of the hierarchy and they generalize all of the discrete hypergeometric orthogonal polynomials. These polynomials are named after Racah, because their orthogonal relation is equivalent to that of Racah coefficients or 6-j symbols; see [2]. In [54], Wilson defined the Racah polynomials in terms of a ${}_4F_3$ hypergeometric function. Let $\lambda(x) := x(x+\gamma+\delta+1)$ and N be a nonnegative integer. Define

(69)

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) := {}_4F_3\left(\begin{matrix} -n,\ n+\alpha+\beta+1, & -x, & x+\gamma+\delta+1 \\ \alpha+1, & \beta+\delta+1, & \gamma+1 \end{matrix} \middle| 1 \right),$$

where $n = 0, \cdots, N$ and one of the following equalities is satisfied : $\alpha + 1 = -N$ or $\beta + \delta + 1 = -N$ or $\gamma + 1 = -N$. Let

$$A_n := -\frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$
$$C_n := -\frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \tag{70}$$

The Racah polynomials (69) satisfy the recurrence relation [26, (9.2.3)]

$$\lambda(x)R_n(\lambda(x)) = -A_nR_{n+1}(\lambda(x)) + (A_n + C_n)R_n(\lambda(x)) - C_nR_{n-1}((\lambda(x)).$$

This recurrence relation can be normalized as

$$\pi_{n+1}(z) = (z - A_n - C_n)\pi_n(z) - A_{n-1}C_n\pi_{n-1}(z),$$
$$\pi_0(z) = 1, \qquad \pi_1(z) = z - A_0, \tag{71}$$

where

$$\pi_n(\lambda(x)) := \frac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(n+\alpha+\beta+1)_n}R_n(\lambda(x)). \tag{72}$$

In addition to the variable x and the degree n, the polynomials in (69) involve four free parameters. This inevitably makes the problem of deriving their asymptotic formulas much more complicated. By approximating the ratio of two shifted factorials in the hypergeometric representation (69), Chen, Ismail and Simeonov [5] obtained several asymptotic formulas in terms of hypergeometric function ${}_3F_2$ or ${}_2F_1$, when the parameters are fixed. Recently, Wang and Wong [49] studied the large-n behavior of $\pi_n(z)$ with varying parameters $\alpha, \beta, \gamma, \delta$. More precisely, we set

$$\alpha + 1 = Na, \quad \beta = Nb, \quad \gamma + 1 = Nc, \quad \delta + 1 = Nd, \tag{73}$$

where either $a = -1$ or $b + d = -1$ or $c = -1$. For simplicity, we assume $A_n > 0$ and $C_n > 0$. From Favard's theorem, these conditions guarantee that the zeros of $\pi_n(z)$ are all real and simple; see [6, Sections 1.4 and 1.5]. Thus, we require some additional conditions:

1. when $a = -1$, we assume $b, c, d > 0$ and $b > c + 1$;
2. when $b + d = -1$, we assume $a, b, c > 0$ and $a + b + 1 < c$;
3. when $c = -1$, we assume $a, b, d > 0$ and $a + 1 < d$.

Let $n/N = p$ be a fixed number in (0,1). To present the results in [49], we need to introduce some notations. From (70) and (73), we have

$$\lim_{N\to\infty}\frac{A_n}{N^2} = A(p); \quad \lim_{N\to\infty}\frac{C_n}{N^2} = C(p),$$

where

$$A(t) := -\frac{(t+a)(t+a+b)(t+b+d)(t+c)}{(2t+a+b)^2}, \qquad 0 \leqslant t \leqslant p,$$
$$C(t) := -\frac{t(t+a+b-c)(t+a-d)(t+b)}{(2t+a+b)^2}, \qquad 0 \leqslant t \leqslant p. \tag{74}$$

To use the asymptotic results from difference equation theory, we also need the concept of transition points (or turning points). These are the points where the characteristic roots of equation (71) coincide; see [50]. For $t \in [0, p]$, we define

$$y_\pm(t) := A(t) + C(t) \pm 2\sqrt{A(t)C(t)}; \tag{75}$$

These are transaction points when $t = p$. From (74), it is readily seen that $C(0) = 0$. Thus, $y_\pm(0) = A(0)$. For simplicity, in [49] Wang and Wong considered only the case when $y_+(x)$ is increasing and $y_-(t)$ is decreasing for $t \in [0,p]$. Following [49], we adopt the notations

$$\mathscr{A}_1(t) := t+a, \quad \mathscr{A}_2(t) := t+a+b, \quad \mathscr{A}_3(t) := t+b+d, \tag{76a}$$

$$\mathscr{A}_4(t) := t+c, \quad \mathscr{C}_1(t) := t, \quad \mathscr{C}_2(t) := t+a+b-c, \tag{76b}$$

$$\mathscr{C}_3(t) := t+a-d, \quad \mathscr{C}_4(t) := t+b, \quad \mathscr{D}(t) := 2t+a+b, \tag{76c}$$

and

$$\mathscr{A}^*(t) := \frac{4\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3\mathscr{A}_4 - (\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 + \mathscr{A}_1\mathscr{A}_2\mathscr{A}_4 + \mathscr{A}_1\mathscr{A}_3\mathscr{A}_4 + \mathscr{A}_2\mathscr{A}_3\mathscr{A}_4)\mathscr{D}}{\mathscr{D}(t)^3} \tag{77a}$$

$$\mathscr{C}^*(t) := \frac{4\mathscr{C}_1\mathscr{C}_2\mathscr{C}_3\mathscr{C}_4 - (\mathscr{C}_1\mathscr{C}_2\mathscr{C}_3 + \mathscr{C}_1\mathscr{C}_2\mathscr{C}_4 + \mathscr{C}_1\mathscr{C}_3\mathscr{C}_4 + \mathscr{C}_2\mathscr{C}_3\mathscr{C}_4)\mathscr{D}}{\mathscr{D}(t)^3} \tag{77b}$$

Furthermore, we put

$$S(y;t) := \sqrt{(y-A(t)-C(t))^2 - 4A(t)C(t)} = \sqrt{[y-y_-(t)][y-y_+(t)]}, \tag{78}$$

$$T(y;t) := y - A(t) - C(t) + S(y;t) = \left[\sqrt{y-y_-(t)}\right] + \left[\sqrt{y-y_+(t)}\right]^2 \Big/ 2, \tag{79}$$

and

$$G(y;t) := T(y;t)\left[A^*(t) + C^*(t)\right] + 2\left[A(t)C^*(t) + A^*(t)C(t)\right]. \tag{80}$$

Despite all these notations, the functions in (76), (77) and (78)−(80) are simple and elementary. To state the results in [49], we also need the two integrals

$$\Omega_0 = \int_0^p \left[\frac{A(t)-C(t)}{\mathscr{D}(t)S(y;t)} - \frac{2A(t)C(t)\mathscr{E}(t)}{S(y;t)T(y;t)}\right] dt \tag{81}$$

and

$$\Omega_1 = \int_0^p \frac{G(y;t)}{2S(y;t)^2} dt. \tag{82}$$

We are now ready to state their main results.

Theorem 4.1. *Let $\pi_n(z)$ be the monic Racah polynomials satisfying the recurrence relation* (71)*. Assume $n/N = p$ is a fixed number in $(0,1)$. Let $A(t), C(t)$ and $y_\pm(t)$ be defined as in* (74) *and* (75)*. Also, let $\Omega_0(y)$ and $\Omega_1(y)$ denote, respectively, the integrals in* (81) *and* (82)*. Assume that $y_+(t)$ is increasing and $y_-(t)$ is decreasing for $t \in [0,p]$. Then, for $y \in \mathbb{C} \setminus [y_-(p), y_+(p)]$, we have*

$$\pi_n(N^2 y) = \left(\frac{N^2}{2}\right)^n e^{Ng(y)+r(y)} \left[1 + O\left(\frac{1}{n}\right)\right], \tag{83}$$

where the main term $g(y)$ is given by

$$g(y) := \int_0^p \ln T(y;t) dt, \tag{84}$$

and the correction term $r(y)$ is given by

$$r(y) := \frac{1}{2} \ln \frac{T(y;p)}{T(y;0)} + \Omega_0(y) + \Omega_1(y). \tag{85}$$

To derive the asymptotic formula of $\pi_n(N^2 y)$ for y in the interval of oscillation, i.e., $y \in [y_-(p), y_+(p)]$, Wang and Wong simply added the limits of the approximant in (83) as y approaches the real axis from above and below. The reason behind their argument is based on the phenomenon that the real part of an asymptotic approximant for orthogonal polynomials in the complex plane is always half of the corresponding asymptotic approximant in the interval of orthogonality on the real line. This fact was apparantly known to Heine [18, p. 175] and Szegö [45, p. 196, Theorem 8.21.9].

Theorem 4.2. *Let $\pi_n(z)$ be the monic Racah polynomials satisfying the recurrence relation* (71). *Assume $n/N = p$ is a fixed number in* $(0, 1)$. *Let $A(t), C(t)$ and $y_\pm(t)$ be defined as in* (74) *and* (75). *We further assume that $y_+(t)$ is increasing and $y_-(t)$ is decreasing for $t \in [0, p]$. Then we have*

$$\pi_n(N^2y) = \left(\frac{N^2}{2}\right)^n \left\{ e^{Ng^+(y)+r^+(t)} \left[1 + O\left(\frac{1}{n}\right)\right] + e^{Ng^-(y)+r^-(y)} \left[1 + O\left(\frac{1}{n}\right)\right] \right\} \tag{86}$$

for $y \in (y_-(p), A(0)) \cup (A(0), y_+(p))$.

As a corollary, Wang and Wong [49] also gave an asymptotic formula for the Racah polynomials when the variable z is fixed. The exact result is stated below.

Corollary 4.1. *Let $\pi_n(z)$ be the monic Racah polynomials satisfying the recurrence relation* (71). *Assume $n/N = p$ is fixed in* $(0, 1)$. *Let $A(t), C(t)$ and $y_\pm(t)$ be defined as in* (74) *and* (75). *We further assume that $A(t) > C(t)$, $y_+(t)$ is increasing and $y_-(t)$ is decreasing for $t \in [0, p]$. Then as $n \to \infty$, we have for any fixed z,*

$$\pi_n(z) = (-N^2)^n \exp\left[N \int_0^p \ln A(t) dt\right] \times \sqrt{\frac{ac(b+d)(2p+a+b)}{(p+a)(p+c)(p+b+d)(p+a+b)}} \left[1 + O\left(\frac{1}{n}\right)\right]. \tag{87}$$

5. Pseudo-Jacobi polynomials

Pseudo-Jacobi (P-J) polynomials $P_n(z; a, b)$ are also known as the Romanovski-Routh polynomials. They are solutions to the second-order differential equation

$$(x^2 + 1)\frac{d^2y}{dx^2} + 2\left[(1 + a)x + b\right]\frac{dy}{dx} - n(n + 2a + 1)y = 0. \tag{88}$$

These polynomials (of degree n) are orthogonal with respect to the weight function

$$w(x) = (x^2 + 1)^a e^{2b \arctan x}, \qquad x \in (-\infty, \infty), \tag{89}$$

only when the moments

$$\mu_n = \int_{-\infty}^{\infty} x^{2n} w(x) dx$$

exist; that is, only when $n < -a - 1/2$. If $a = -N - 1$ with N being a positive integer, then the orthogonal polynomials $P_n(x; a, b)$ are well-defined with respect to $w(x)$ only for $n = 0, \cdots, N$. Therefore, for fixed $a < -1$, we have only a finite number of orthogonal polynomials, in contrast to the classical Hermite, Laguerre and Jacobi polynomials.

The polynomials $P_n(x; a, b)$ were first discovered by Romanovski [41] in 1929 (in French) within the context of probability distribution functions in statistics. They also belong to a general family of Routh polynomials [42] introduced in 1884. For a historical account, see [35, 36]. Since they have not been discussed in standard books on orthogonal polynomials and special functions (e.g., [37, 45]), they are not known to most of the researchers in the field. In view of its close connection with the Jacobi polynomials of purely imaginary argument and complex parameters (see (91) below), in 1996 Lesky [31] (in German) coined the term "Pseudo-Jacobi polynomials". Later these polynomials are also included in the monograph [26]. It was probably the review article [40] that brought attention to the community of people working in orthogonal polynomials. This article discusses various applications of this polynomial ranging from quantum mechanics and quark physics to random matrix theory (see [16] and the references in [40]). For instance, they arise in the solutions of Schrödinger equations with certain non-central electric potential obtained by separation of variables in spherical coordinates; their role is similar to that of the associated Legendre functions; see [1] and [14].

The definition of monic P-J polynomials in terms of hypergeometric functions is given in [26], with parameters N and ν related to the parameters a and b in (89); more precisely, with $a = -N - 1$ and $b = \nu$, the monic P-J polynomials are defined by

$$P_n(x; a, b) = \frac{(-2i)^n (1 + a + bi)_n}{(n + 2a + 1)_n} {}_2F_1\left(\begin{matrix} -n, n + 2a + 1 \\ 1 + a + bi \end{matrix} ; \frac{1 - ix}{2} \right). \tag{90}$$

By comparing the definitions of the P-J polynomials and Jacobi polynomials $P_n^{(\alpha,\beta)}$, we have the relation

$$P_n(x; a, b) = (-i)^n P_n^{(a+bi,\, a-bi)}(ix). \tag{91}$$

This relation has also been used as the definition of P-J polynomials in [19, p. 508] and [23]. In view of the symmetry property of Jacobi polynomials, the P-J polynomials satisfy

$$P_n(z; a, -b) = (-1)^n P_n(-z; a, b); \tag{92}$$

see [23]. Since the coefficients of P-J polynomials are real, it also follows that

$$P_n(\overline{z}; a, b) = \overline{P_n(z; a, b)}, \tag{93}$$

where an overbar means the complex conjugate. For other properties such as orthogonality, recurrence relations and Rodrigues formula, we refer to [26, p. 231-233].

Probably due to their close relation with Jacobi polynomials, recently P-J polynomials have attracted much attention [23, 24]. Properties of the zeros of P-J polynomials, including their asymptotics and bounds, have been studied in [23]. As a special case, the zeros of the pseudo-ultraspherical polynomials have been independently investigated in [13] almost at the same time. In [44], Song and Wong have investigated the asymptotic behavior of the P-J polynomials as the degree n goes to infinity. Note that, for fixed a or $a > -n$, there is no real-line orthogonality anymore; see [23, 28] for their orthogonality. For fixed parameters a and b, in view of (90) the asymptotics of these polynomials can be obtained as a special case of that of the hypergeometric functions investigated in [22, 25]. In [44], Song and Wong consider the case when the parameters depend on the degree n,

$$a_n = -(An + A_0), \quad (A > 1); \qquad b_n = Bn + B_0, \tag{94}$$

where A, A_0, B, B_0 are real constants. Their goal is to derive asymptotic formulas for $P_n(z; a_n, b_n)$ (with varying parameters) for z in the whole complex plane. Their main tool is the well-developed method for differential equations with a large parameter as presented in Olver [38, Chapter 11]. Their other goal is to use this example as an illustration to show that this method is easy to understand and can be used in many problems in physical applications.

They first deal with the asymptotics when the variable z lies outside the interval where the zeros are located. To do this, they let $\{x_{n,j}\}|_{j=1}^{n}$ be the zeros of

$P_n(x; a_n, b_n)$ and define the sequence of discrete measures λ_n by

$$\lambda_n(f) \equiv \frac{1}{n}\sum_{j=1}^{n} f(x_{n,j}) = \int f(x)d\lambda_n(x),$$

where f is any continuous function. They give a brief derivation of the result that there is a unique measure λ, which is the weak-$*$ limit of λ_n as $n \to \infty$, and the measure λ is supported on $[\gamma_-, \gamma_+]$ and given by

$$d\lambda = \frac{(A-1)\sqrt{(x-\gamma_-)(\gamma_+ - x)}}{\pi(x^2+1)}dx, \tag{95}$$

where

$$\gamma_\pm = \frac{AB \pm \sqrt{(2A-1)[(A-1)^2+B^2]}}{(A-1)^2}. \tag{96}$$

This result was actually given earlier in [23]. To state the first main result in [44], we introduce the functions

$$\begin{aligned} \xi^*(z) = {} & 2(A-1)\log(\sqrt{z-\gamma_-} + \sqrt{z-\gamma_+}) - C_1 \\ & - (A+iB)\log(\sqrt{1+i\gamma_-}\sqrt{z-\gamma_+} + \sqrt{1+i\gamma_+}\sqrt{z-\gamma_-}) \\ & - (A-iB)\log(\sqrt{1-i\gamma_-}\sqrt{z-\gamma_+} + \sqrt{1-i\gamma_+}\sqrt{z-\gamma_-}) \end{aligned} \tag{97}$$

and

$$\begin{aligned} \chi^*(z) = {} & 2(A_0-1)\log(\sqrt{z-\gamma_-} + \sqrt{z-\gamma_+}) - C_2 \\ & - (A_0+iB_0)\log(\sqrt{1+i\gamma_-}\sqrt{z-\gamma_+} + \sqrt{1+i\gamma_+}\sqrt{z-\gamma_-}) \\ & - (A_0-iB_0)\log(\sqrt{1-i\gamma_-}\sqrt{z-\gamma_+} + \sqrt{1-i\gamma_+}\sqrt{z-\gamma_-}), \end{aligned} \tag{98}$$

where

$$\begin{aligned} C_1 = 2(A-1)\log 2 - \frac{1}{2}B\pi & - (A+iB)\log(\sqrt{1+i\gamma_-} + \sqrt{1+i\gamma_+}) \\ & - (A-iB)\log(\sqrt{1-i\gamma_-} + \sqrt{1-i\gamma_+}) \end{aligned} \tag{99}$$

and

$$\begin{aligned} C_2 = 2(A_0-1)\log 2 - \frac{1}{2}B_0\pi & - (A_0+iB_0)\log(\sqrt{1+i\gamma_-} + \sqrt{1+i\gamma_+}) \\ & - (A_0-iB_0)\log(\sqrt{1-i\gamma_-} + \sqrt{1-i\gamma_+}). \end{aligned} \tag{100}$$

In the last four equations, A, A_0, B and B_0 are the constants in (94).

We also need the definition of a domain D in three separate cases: (i) $\gamma_- \leqslant 0 \leqslant \gamma_+$, (ii) $0 < \gamma_- < \gamma_+$ and (iii) $\gamma_- < \gamma_+ < 0$. In case (i), D denotes the set

$$D = \{z |\ \mathrm{Re}\ z \geqslant 0,\ \mathrm{Im}\ z \geqslant 0\} \setminus [0, \gamma_+], \qquad \gamma_- < 0 < \gamma_+, \tag{101}$$

where exclusion of the interval $[0, \gamma_+]$ is understood to mean the exclusion of a small neighborhood in the first quadrant containing the interval $[0, \gamma_+]$. In case (ii), we put

$$\xi(z) = \xi^*(z) + \frac{1}{2}(A+iB)\log(1+iz) + \frac{1}{2}(A-iB)\log(1-iz) \tag{102}$$

and let $\xi_0 = \xi(\gamma_+)$. Define a curve L_1 in the first quadrant of the z-plane by

$$\mathrm{Re}\ \xi(z) = \xi_0, \text{ or equivalently } \mathrm{Re}\left(\int_{\gamma_-}^{z} \frac{\sqrt{(t-\gamma_-)(t-\gamma_+)}}{1+t^2}dt\right) = 0, \tag{103}$$

which connects γ_- and a point $z_1 = iy_1\,(0 < y_1 < 1)$ on the positive imaginary axis, and denote the region bounded by the axes and L_1 by Ω_1; i.e.,

$$\begin{aligned}\Omega_1 = \{z|\ &\mathrm{Re}\ z \geq 0,\ \mathrm{Im}\ z \geq 0,\\ &\xi_0 \leqslant \mathrm{Re}\ \xi(z) \leqslant \mathrm{Re}\ \xi(0), \quad \mathrm{Im}\ \xi(z_1) \leqslant \mathrm{Im}\ \xi(z) \leqslant \mathrm{Im}\ \xi(0)\}.\end{aligned} \tag{104}$$

The domain D is now defined by

$$D = \{z|\ \mathrm{Re}\ z \geqslant 0,\ \mathrm{Im}\ z \geqslant 0\} \setminus ([\gamma_-, \gamma_+] \cup \Omega_1), \qquad 0 < \gamma_- < \gamma_+, \tag{105}$$

where small neighborhoods of the interval $[\gamma_-, \gamma_+]$ and the curve L_1 are again excluded. (For details; see Figure 2 in [44].) The final case (i.e., case (iii)) can be handled in a similar manner. We define a curve L_2 in the second quadrant of z-plane by

$$\mathrm{Re}\ \xi(z) = \xi_0, \text{ or equivalently } \mathrm{Re}\left(\int_{\gamma_+}^{z} \frac{\sqrt{(t-\gamma_-)(t-\gamma_+)}}{1+t^2} dt\right) = 0, \tag{106}$$

which connects γ_+ and a point $z_2 = iy_2\,(0 < y_2 < 1)$ on the positive imaginary axis, and denote the region bounded by the axes and L_2 by Ω_2; i.e.,

$$\begin{aligned}\Omega_2 = \{z|\ &\mathrm{Re}\ z \leqslant 0,\ \mathrm{Im}\ z \geqslant 0,\\ &\xi_0 \leqslant \mathrm{Re}\ \xi(z) \leqslant \xi(0), \quad 0 \leqslant \mathrm{Im}\ \xi(z) \leqslant \mathrm{Im}\ \xi(z_2)\}.\end{aligned} \tag{107}$$

The relevant domain D is defined by

$$D = \{z|\ \mathrm{Re}\ z \geqslant 0,\ \mathrm{Im}\ z \geqslant 0\} \cup \Omega_2 \setminus [\gamma_+, \gamma_+ + \delta], \qquad \gamma_- < \gamma_+ < 0, \tag{108}$$

where a small neighborhood of L_2 can be included but a small neighborhood of the point γ_+ is excluded ($\delta > 0$); see Figure 3 in [44].

Note that the excluded region Ω_1 in case (ii) is the mirror image of the region Ω_2 in case (iii) with respect to the imaginary axis. Based on this observation and the symmetry properties (92) and (93) of the P-J polynomials, Song and Wong [44] proved that the domain D defined as in (101), (105) and (108) for each of the three cases considered above can be used to cover the whole complex plane except a neighborhood of the critical interval $[\gamma_-, \gamma_+]$. Their first main result is stated below.

Theorem 5.1. *For $z \in D$ given by* (101), (105) *or* (108), *the* P-J *polynomials $P_n(z; a_n, b_n)$ with parameters in* (94) *have the uniform asymptotic expansion*

$$\begin{aligned}P_n(z; a_n, b_n) \sim\ & e^{b_n \pi/2}[(z-\gamma_-)(z-\gamma_+)]^{-1/4}\\ & \times e^{-n\xi^*(z)-\chi^*(z)} \sum_{s=0}^{\infty} \frac{(-1)^s w_s(z)}{n^s}\end{aligned} \tag{109}$$

as $n \to \infty$, where $\xi^*(z)$ and $\chi^*(z)$ are given in (97) *and* (98), *respectively, and the coefficients $w_s(z)(s \geqslant 1)$ can be determined by a recursive formula with $w_0(z) = 1$.*

The second main result in [44] is an asymptotic expansion for $P_n(z; a_n, b_n)$ which holds uniformly in a region that contains a turning point and a large part of the critical interval $[\gamma_-, \gamma_+]$. By symmetry, we can concentrate on the *turning point* $z = \gamma_+$. In the previous case, the function $\xi(z)$ in (102) is not well-defined near γ_+, which is a branch point. Following the method illustrated in [38, p. 426], Song and Wong modified the function $\xi(z)$ and defined

$$\zeta(z) := \left[\frac{3}{2}(\xi(z) - \xi_0)\right]^{2/3}, \qquad \xi_0 := \xi(\gamma_+). \tag{110}$$

To state their next result, we need some new notations. First, put

$$n := N + \alpha \qquad \text{with } \alpha := -\frac{2A_0 - 1}{2(A-1)}, \tag{111}$$

$$\chi(z) := \chi^*(z) + \frac{1}{2}(A_0 + iB_0)\log(1+iz) + \frac{1}{2}(A_0 - iB_0)\log(1-iz), \tag{112}$$

where $\chi^*(z)$ is given in (98), and

$$\Phi\left(\zeta(z)\right) := \zeta^{-1/2}(z)[\chi(z) - \chi_0], \qquad \chi_0 := \chi(\gamma_+). \tag{113}$$

Define

$$\hat{\Phi}(\zeta) := \left[\Phi(\zeta) + \frac{2}{3}\alpha\zeta\right] \tag{114}$$

and

$$\zeta_N := N^{2/3}\zeta + N^{-1/3}\hat{\Phi}(\zeta). \tag{115}$$

We now restrict z to the domain

$$D^* = \{z|\ \gamma_- + \delta \leqslant \operatorname{Re} z < +\infty, \quad |\operatorname{Im} z| \leqslant \delta_1\}, \tag{116}$$

where δ, δ_1 are some positive numbers. Clearly, D^* is a complex neighborhood of a semi-infinite real interval.

Theorem 5.2. *For $z \in D^*$, the Pseudo-Jacobi polynomials $P_n(z; a_n, b_n)$ with parameters in* (94) *have the uniform asymptotic expansion*

$$\begin{aligned} P_n(z; a_n, b_n) &\sim C_n e^{b_n\pi/2}[w(z)]^{-1/2}\left[\frac{\zeta}{(z-\gamma_-)(z-\gamma_+)}\right]^{1/4} \\ &\times \left\{\operatorname{Ai}(\zeta_N)\sum_{s=0}^{\infty}\frac{X_s(\zeta)}{N^s} + N^{-\frac{4}{3}}\operatorname{Ai}'(\zeta_N)\sum_{s=0}^{\infty}\frac{Y_s(\zeta)}{N^s}\right\} \end{aligned} \tag{117}$$

as $n \to \infty$, where $w(z)$ is given in (89)*, $\gamma_\pm$ are given in* (96) *and $\zeta(z)$ is given by* (110)*. The constant C_n, independent of z, has an asymptotic series representation in powers of N^{-1} with explicit expressions for the coefficients. The coefficient functions $X_s(\zeta)$ and $Y_s(\zeta)$ can be determined by recursive formulas with $X_0(\zeta) \equiv 1$.*

An immediate consequence of the above theorem is the following corollary.

Corollary 5.1. *For $z \in D^*$ defined in* (116)*, the P-J polynomials $P_n(z; a_n, b_n)$ with parameters in* (94) *have the leading asymptotic behavior*

$$\begin{aligned} P_n(z; a_n, b_n) &\sim 2\sqrt{\pi}N^{1/6}e^{-(n\xi_0+\chi_0)+b_n\pi/2}[w(z)]^{-1/2} \\ &\times \left[\frac{\zeta}{(z-\gamma_-)(z-\gamma_+)}\right]^{1/4}\operatorname{Ai}(\zeta_N) \end{aligned} \tag{118}$$

as $n \to \infty$, where $w(z)$ is given in (89)*, $\gamma_\pm$ are given in* (96)*, $\zeta_N = N^{2/3}\zeta + N^{-1/3}\hat{\Phi}$ and $\zeta, \xi_0, \hat{\Phi}, \chi_0, N$ are given by* (110)*,* (114)*,* (113) *and* (111)*.*

As a biproduct, Song and Wong [44] also obtained an asymptotic formula for the zeros of the P-J polynomials. To describe this result, we let the zeros be arranged in decreasing order:

$$\gamma_- < x_{n,n} < x_{n,n-1} < \cdots < x_{n,2} < x_{n,1} < \gamma_+, \tag{119}$$

and introduce the functions

$$f_i(z) = \frac{Q_i(z)}{(1+z^2)^2}, \qquad i = 0, 1, \tag{120}$$

where

$$Q_0(z) = (A-1)^2z^2 - 2ABz + B^2 + 1 - 2A = (A-1)^2(z-\gamma_-)(z-\gamma_+),$$
$$Q_1(z) = (A-1)(2A_0-1)z^2 - 2(A_0B + AB_0)z + 1 - A - 2A_0 + 2BB_0. \tag{121}$$

Corollary 5.2. *Let the parameters a_n and b_n be given as in* (94)*, and let the zeros of $P_n(z;a_n,b_n)$ be arranged as in* (119)*. For each fixed k, we have*

$$x_{n,k} = \gamma_+ + c_0^{-2/3}\widetilde{a}_k N^{-2/3} - f_1(\gamma_+)c_0^{-2}N^{-1} - \frac{f_0''(\gamma_+)}{10c_0^{10/3}}\widetilde{a}_k^2 N^{-4/3} + O(N^{-5/3}) \tag{122}$$

as $n \to \infty$, where N is given in (111)*, $\widetilde{a}_k$ is the kth negative zero of the Airy function $\mathrm{Ai}(x)$ and the O-symbol depends on k. The constant γ_+ is given in* (96)*, and the constant c_0 is given by*

$$c_0 = \frac{(A+1)\sqrt{\gamma_+ - \gamma_-}}{1+\gamma_+^2} > 0. \tag{123}$$

The functions $f_i(z), i = 0, 1,$ are given in (120)*.*

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Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong
E-mail: `rscwong@cityu.edu.hk`