INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 15, Number 1-2, Pages 102–110

THE SINGULARITY-SEPARATED METHOD FOR THE SINGULAR PERTURBATION PROBLEMS IN 1-D

CHUANMIAO CHEN[1] AND JING YANG[1,2]

Abstract. The singularity-separated method(SSM) for the singular perturbation problem $-\epsilon u''+bu'+cu=f(x), u(0)=u(1)=0$, is proposed for the first time. The solution is expressed as $u=w-v$, where w is the solution of corresponding third boundary value problem and v is an exact singular function. We have proved a global regularity, $||w||_2 \le C$, where the constant C is independent of ϵ, and discussed three kinds of finite element (FE) methods with SSM. Numerical results show that these FE-solutions have the high accuracy when only one element in boundary layer is taken.

Key words. Singular perturbation problem, singularity-separated method, third boundary value, second order regularity, finite elements.

1. Introduction

L.Prandtl in 1904 found that the speed of fluid moved in a pipeline would decrease acutely along the wall, which was called boundary layer phenomenon. This phenomenon is common in many practical fields. Mathematically, if the coefficient of the highest order derivative term in a differential equation is a small parameter, its solution often has a boundary layer. It is very difficult to simulate numerically the singular perturbation solution. In this paper we will propose a new idea to deal with this difficulty.

We consider one-dimensional singular perturbation problem

$$\begin{cases} Au=-\epsilon u''+bu'+cu=f(x) \quad in\ J=[0,1] \\ u(0)=0, \quad u(1)=0\ or\ u'(1)=0 \end{cases} \tag{1}$$

where ϵ is a small parameter, e.g., $\epsilon=10^{-3}\sim 10^{-10}$, constants $b>0, c>0$. The solution has the singularity $e^{b(x-1)/\epsilon}$ in the boundary layer near $x=1$. Denote by $\tau=p_0\epsilon|\ln\epsilon|/b$ the width of boundary layer. Early the finite difference method(FDM) was used and its error in boundary layer vibrates strongly [11]. For this situation G.Shishkin[10, 12] proposed a famous Shishkin-mesh, i.e., J is divided into two subintervals $J_0=(0,1-\tau)$ and $J_1=(1-\tau,1)$, in which the smooth subinterval J_0 is subdivided into N-uniform meshes with step-size $h=(1-\tau)/N\approx 1/N$, and the boundary layer J_1 is subdivided into N-uniform meshes with the much smaller step-size $h'=\tau/N<<h$. The convergence of FDM under the Shishkin mesh was studied [1, 10, 12, 13]. Besides, the finite element method (FEM) was also discussed [4, 8]. The Shishkin-mesh is successful to simulate the singular perturbation problems up to now.

We are interesting in (local) discontinuous Galerkin finite element methods(LDG) much more. It is well-known that LDG can simulate the acute change of singular solution very well [18]. By introducing a new variable $q=u'$, the original equation

Received by the editors January 31, 2017.
2000 *Mathematics Subject Classification.* 34E10, 34E15, 65N30.
Corresponding author: Jing Yang.

(1) becomes a first order elliptic system

$$\begin{cases} -\epsilon q' + bq + cu = f(x), \\ u' - q = 0, \\ u(0) = 0, \quad u(1) = 0 \ or \ q(1) = 0, \end{cases} \tag{2}$$

Celiker-Cockburn [2] discussed the highest order superconvergence for LDG solution when $\epsilon = 1$. At the same time Xie-Zhang [15, 16, 17] independently studied LDG method on Shishkin meshes for the singular perturbation problem, and the vibration of error was weakened greatly. They got the uniform convergence and superconvergence independent of ϵ.

The accuracy of numerical approximation depends not only on the number of the grid nodes, but also on the parameter ϵ. The smaller ϵ is, the thinner the boundary layer is, and the more acutely the solution varies. We feel that Shishkin meshes have some defects for high dimensional problems. For example, more and more nodes concentrate in the boundary layer of domain so that the geometrical aspect ratios of the meshes become very large, even the meshes near corner are extremely small, thus many troubles will appear in FE-approximation, such as the accuracy, efficiency, stability and so on. In numerical simulation, we gradually realize that it is necessary to improve the mathematical expression of singular perturbation solution. Numerical algorithms strongly depend on the regularity estimates of the solution, and constructing various singular functions is a fundamental method to study the regularity. They are systematically summarized in monograph [11], and the singular functions in 1-D are also discussed in it. Generally speaking, these singular functions are applied to study the regularity, but (maybe due to their inexactness) they have not been taken as a correct function to construct a new high-performance algorithm.

So we propose a new idea, called singularity-separated method(SSM). We decompose the original problem into two sub-problems as follows. Firstly introduce an auxiliary third boundary problem, whose solution has weaker singularity and the free term f is eliminated. Secondly construct a singular correct function exactly, which is a special solution with homogenous free term $f = 0$, and is a part of numerical solution we need. It should be emphasized that these solutions of two sub-problems have the weaker singularity, which are directly and simply applied in FE-computation. In particular only one or two elements in boundary layer are needed, thus the local refinement of Shishkin mesh is not necessary. This is the main aim of us. Although only one-dimensional case is discussed in this paper, the theoretical and computational framework proposed are valid for multi-dimensional singularly perturbed elliptic and convection-diffusion problems, which need some new analysis techniques yet.

We also have another motivation. The solutions u of fluid dynamics problems with viscosity, such as Navier-Stokes equation, have classical energy estimate $\|u\|_1 \le \|f\|/\epsilon^2$. For small ϵ, the bound becomes awfully large, unless f is also small. Thus the essential difficulty will occur in studying the solvability by the fixed point principle and numerical simulations. We want to know that whether SSM can solve the trouble f. This is an expectation in future.

An outline of this article is as follows. In Section 2, we present the singularity separated method and state two results which provide the theoretical basis of SSM. The proofs of those two theorems are carried out in details in Section 3. In Section 4, we present three kinds of FEMs with SSM and the numerical experiments provided show the robustness of SSM.

We use the Sobolev space $W^{n,p}(J)$ and $H^n(J)$ with norms $||u||_{n,p,J}$ and $||u||_{n,J}$, respectively. The index J is often omitted. The constant C is independent of u and ϵ.

2. The singularity-separated method

Consider the problem (1). The homogenous equation $Au = 0$ has two eigenvalues

$$\lambda_1 = \frac{-2c}{b+\sqrt{b^2+4c\epsilon}} \approx -\frac{c}{b}, \quad \lambda_2 = \frac{b+\sqrt{b^2+4c\epsilon}}{2\epsilon} \approx \frac{b}{\epsilon}, \quad \lambda_1+\lambda_2 = \frac{b}{\epsilon}.$$

Let u_0 be a special solution of $Au = f$, its general solution is

$$u(x) = u_0(x) - C_1\phi_1(x) - C_2\phi_2(x), \quad \phi_1(x) = e^{\lambda_1 x}, \quad \phi_2(x) = e^{\lambda_2(x-1)}. \tag{3}$$

For $b = c = 1, f = e^x$, $u_0(x) = e^x/(2-\epsilon)$. By $u(0) = u(1) = 0$, the solution of (1) is

$$u(x) = \frac{1}{2-\epsilon}\{\frac{e^{1-\lambda_2}-1}{1-e^{\lambda_1-\lambda_2}}\phi_1(x) + \frac{e^{\lambda_1}-e}{1-e^{\lambda_1-\lambda_2}}\phi_2(x) + e^x\}, \tag{4}$$

where $\phi_2(1) = 1$, $\phi_2'(1) = O(\epsilon^{-1})$. Meanwhile $\phi_2(1-\tau) = e^{-\tau\lambda_2} = e^{-p_0|\ln\epsilon|} = \epsilon^{p_0}$ is very small, and we choose $p_0 \geq m+1$ where m is the degree of FE-solution.

Now we shall rewrite the expression (4). Note that if $\epsilon = 10^{-k}$ and $k \geq 3$, $\phi_2(0) = e^{-\lambda_2} \approx e^{-1/\epsilon} = 10^{-0.43*10^k} \leq 10^{-430} = 0$ can be neglected. By (3) we have $u(0) = u_0(0) - C_1 - C_2e^{-\lambda_2} = 0$, $u(1) = u_0(1) - C_1e^{\lambda_1} - C_2 = 0$, and then define $C_1 = u_0(0)$ and $C_2 = u_0(1) - u_0(0)e^{\lambda_1}$. So the solution $u(x)$ is composed of two parts, $u = w - v$, in which $w(x) = u_0(x) - C_1\phi_1(x)$ is the regular term and $v = C_2\phi_2(x)$ is the singular term. This make us realize the separation of singularity. For any $f(x)$, how to construct the regular solution $w(x)$ is key. So we propose the singularity-separated method as follows.

Step 1. Introduce an auxiliary third boundary value problem

$$\begin{cases} Aw = -\epsilon w'' + bw' + cw = f(x), \\ w(0) = 0, \quad A_1 w = bw'(1) + cw(1) = f(1) \end{cases} \tag{5}$$

then $\epsilon w''(1) = 0$ and the singularity of solution is weakened. We shall prove:

Theorem 1. For $f(x) \in W^{n+1,\infty}(J)$, the solution of (5) can be decomposed into

$$w = F(x) - v(x), \quad F(x) = \sum_{j=0}^{n} \epsilon^j w_j(x) + \epsilon^{n+1}R_n, \quad v = C_2\phi_2(x), \tag{6}$$

where $|C_2| \leq C\epsilon^2$, and there are global regularities

$$\begin{cases} ||w_j||_{n+2-j,\infty,J0} \leq CM, & ||w_j||_{n+2-j,\infty,J1} \leq CM\tau^{0.5}, & n \leq p_0+2, \\ ||v||_{m,J_0} \leq CM\epsilon^{2-m}, & ||v||_{m,J_1} \leq CM\epsilon^{2.5-m}, & 0 \leq m \leq 2, \\ \max|R_n(x)| \leq CM, & ||R_n(x)||_{1,J} \leq CM, & 0 \leq x \leq 1, \end{cases} \tag{7}$$

where $M = ||f||_{n+1,\infty,J}$ and the constant C is independent of ϵ.

Remark 1. In Theorem 1, $|C_2| \leq C\epsilon^2$ is key to the most important global regularity $||w||_2 \leq ||F||_2 + ||v||_2 \leq CM$. Because each term $\epsilon^j w_j$ in (6) includes a factor ϵ^j, $F(x)$ is of the higher regularity.

Step 2. Construct an explicit singular function

$$g(x) = w(1)\phi_2(x), \quad \phi_2(x) = e^{\lambda_2(x-1)}, \tag{8}$$

which satisfies $Ag = 0$ and then $g(1) = w(1)$ and $g(0) = w(1)e^{-\lambda_2} \approx 0$.

Step 3. The difference $z = u - (w - g)$ satisfies

$$Az = 0, \quad z(0) = g(0) = w(1)e^{-\lambda_2} \approx 0, \quad z(1) = 0. \tag{9}$$

By the following Theorem 2, we can think $z = 0$. Thus we have the decomposition

$$u(x) = w(x) - g(x) + O(e^{-b/\epsilon}), \quad u'(x) = w'(x) - g'(x) + O(e^{-b/\epsilon})/\epsilon. \tag{10}$$

Theorem 2(Estimates of z). The solution $z = u - (w - g)$ of (9) has

$$|z(x)| \le |g(0)|, \quad |z'(x)| \le C|g(0)|/\epsilon, \quad g(0) = O(e^{-b/\epsilon}). \tag{11}$$

These two theorems above provide theoretical basis for our new high-performance algorithm.

3. The proofs of Theorem 1 and Theorem 2

Theorem 3. The solution for the problem

$$\begin{cases} Az = -\epsilon z'' + bz' + cz = f, \\ z(0) = \alpha, z(1) = \beta \end{cases} \tag{12}$$

has the estimates

$$|z(x)| \le M = \max(|\alpha|, |\beta|) + C\max|f(x)|, \quad |z'(x)| \le CM/\epsilon. \tag{13}$$

Proof. The first estimate is derived by maximum principle. Actually setting $f_m = max|f(x)|$, we can construct $v = z - f_m/c$ satisfying

$$\begin{cases} Av = -\epsilon v'' + bv' + cv = Az - f_m = f - f_m \le 0, \\ v(0) = \alpha - f_m/c, v(1) = \beta - f_m/c \end{cases} \tag{14}$$

If v have a positive maximum value $v(x_0) > 0$ at $x_0 \in (0, 1)$, there are $v'(x_0) = 0, v''(x_0) < 0$, and then $Av(x_0) > 0$ which contradicts (14). So the positive maximum will occur at the endpoints, and we have $v(x) \le \max\{\max\{v(0), v(1)\}, 0\} \le \max\{|\alpha|, |\beta|\}$. Similarly, we can construct $v = z + f_m/c$ and prove $v(x) \ge \min\{\min\{v(0), v(1)\}, 0\} \ge -\max\{|\alpha|, |\beta|\}$. Thus the first estimate in (13) follows by $z = v \pm f_m/c$.

Next, multiplying $Az = f$ by x and integrating by parts in $(0, x)$, we have

$$-\epsilon xz' + bxz + \epsilon z(x) - \epsilon z(0) + \int_0^x (-bz + cxz)dx = \int_0^x xf dx.$$

Thus $\epsilon|z'|x \le CM$, and $\epsilon|z'(x)| \le CM$ for $x \ge 1/2$. Similarly, multiplying $Az = f$ by $x - 1$ and integrating by parts in $(x, 1)$, we have

$$\begin{aligned} &\epsilon(x-1)z'(x) - b(x-1)z(x) + \epsilon z(1) - \epsilon z(x) \\ &\quad + c\textstyle\int_x^1 (x-1)zdx - \int_x^1 bzdx = \int_x^1 (x-1)fdx. \end{aligned}$$

If $x \le 1/2$, $\epsilon|z'(x)| \le CM$ is also obtained. Theorem 3 is proved.

In particular, if $f = 0, \alpha = g(0), \beta = 0$, Theorem 2 is a corollary of Theorem 3.

Theorem 4. The energy estimates for the solution $w(x)$ of (5) are

$$\begin{cases} \epsilon||w'||^2 + |w(1)|^2 + ||w||^2 & \le C||f||^2 + C\epsilon^2|f(1)|^2 + Cw^2(0), \\ \epsilon|w'(0)|^2 + ||w'||^2 + \max|w(x)|^2 & \le C||f||^2 + C\epsilon|f(1)|^2 + Cw^2(0). \end{cases} \tag{15}$$

The second estimate is invalid under the first boundary condition $w(1) = \beta$.

Proof. Set $w(0) = \gamma$. Integrating $2wAw = 2wf$ over J, and integrating by parts, under the third boundary condition in (5), we have

$$\begin{aligned} &(b + 2cb^{-1}\epsilon)w^2(1) + 2\epsilon||w'||^2 + 2c||w||^2 \\ &\quad = 2(f, w) + 2\epsilon f(1)w(1)/b - 2\epsilon w'(0)\alpha + b\gamma^2. \end{aligned}$$

Using Young's inequality to eliminate w and $w(1)$ on the right hand side, we get

$$w^2(1) + \epsilon||w'||^2 + ||w||^2 \le C||f||^2 + C\epsilon^2 f^2(1) + C\epsilon^2|w'(0)|^2 + C\gamma^2. \tag{16}$$

Similarly, integrating $2w'Aw = 2w'f$ over J and integrating by parts, we have

$$\begin{aligned}\epsilon|w'(0)|^2 + 2b||w'||^2 + (c - \epsilon(c/b)^2)w^2(1)\\ = 2(f, w') + \epsilon b^{-2}(f^2(1) - 2cf(1)w(1)) + c\gamma^2.\end{aligned}$$

Using Young's inequality and $|w(x)| = |w(0) + \int_0^x w'(x)dx| \le |\gamma| + ||w'||$, we get

$$\epsilon|w'(0)|^2 + ||w'||^2 + \max|w(x)|^2 \le C||f||^2 + C\epsilon|f(1)|^2 + C\gamma^2, \tag{17}$$

which can also be employed to delete $|w'(0)|^2$ in (16). Thus (15) is proved.

Now we turn to

The proof of Theorem 1. For the first order equation

$$A_1 w_0 = bw_0' + cw_0 = f, \quad w_0(0) = 0, \tag{18}$$

firstly we construct an explicit solution

$$w_0(x) = L(f) = b^{-1}e^{-cx/b}\int_0^x e^{ct/b}f(t)dt,$$

where $L(f)$ is an Volterra's integral operator: $L^\infty \to W^{1,\infty}$. Assume that $f \in W^{n+1,\infty}(J)$ for some $n \ge 1$, and let $M = ||f||_{n+1,\infty,J}$. By differentiating $w_0(x)$, obviously

$$||w_0||_{n+2,\infty,J} \le CM. \tag{19}$$

To construct the solution $F(x)$ in (6) by w_0, substituting F into (5) we have

$$AF - f \quad = \textstyle\sum_{j=1}^n \epsilon^j(A_1 w_j - w''_{j-1}) + \epsilon^{n+1}(AR_n - w''_n) = 0,$$

and define one-by-one

$$\begin{cases} A_1 w_j & = bw_j' + cw_j = w''_{j-1}, \\ w_j(0) & = 0, \quad j = 1, 2, ..., n, \\ AR_n & = -\epsilon R''_n + bR'_n + cR_n = w''_n, \\ R_n(0) & = 0, \ bR'_n(1) + cR_n(1) = w''_n(1), \end{cases} \tag{20}$$

thus $w_j = L(w''_{j-1})$. In view of (19), there are

$$||w_1||_{n+2-1,\infty} \le C||w''_0||_n \le CM, \ ||w_2||_{n+2-2,\infty} \le C||w''_1||_{n-1} \le CM$$

and so on. In general,

$$||w_j||_{n+2-j,\infty,J} \le CM, \quad j = 0, 1, ..., n. \tag{21}$$

Besides, by Theorem 4 we have

$$\max|R_n(x)| + ||R_n||_1 \le C||w''_n|| + C\epsilon^{0.5}|w''_n(1)| \le CM. \tag{22}$$

Note that the $F(x)$ satisfies $AF = f$, $F(0) = 0$ and

$$\begin{aligned}A_1F(1) - f(1) \quad &= A_1w_0(1) + \textstyle\sum_{j=1}^n \epsilon^j A_1 w_j(1) + \epsilon^{n+1}A_1R_n(1) - f(1)\\ &= \epsilon(\textstyle\sum_{j=1}^n \epsilon^{j-1}w''_{j-1}(1) + \epsilon^n A_1R_n(1)) \ne 0,\end{aligned}$$

thus we have to correct $F(x)$ at $x = 1$. Setting $\epsilon\nu = A_1F(1) - f(1)$, and by (21) and (22), we have

$$|\nu| = |\sum_{j=1}^n \epsilon^{j-1}w''_{j-1}(1) + \epsilon^n A_1R_n(1)| \le CM(\sum_{j=1}^n \epsilon^{j-1} + \epsilon^n) \le CM.$$

Finally we can express the solution w of (5) in the form

$$w(x) = F(x) - v(x), \quad v(x) = C_2\phi_2(x), \quad \epsilon\lambda_i^2 + b\lambda_i + c = 0, \ i = 1, 2, \tag{23}$$

which satisfies the following boundary conditions

$$(24)\qquad \begin{cases} w(0) & = F(0) - C_2 e^{-\lambda_2} = -C_2 e^{-\lambda_2} \approx 0, \\ (A_1 w - f)(1) & = (A_1 F - f)(1) - C_2(b\lambda_2 + c) = \epsilon(\nu - C_2\lambda_2^2) = 0. \end{cases}$$

Thus we can define the constant $C_2 = \nu\lambda_2^{-2} = O(\epsilon^2)$, which is the most important fact. Based on the estimates $||\phi_2||_{J_0} \le \epsilon^{p_0+0.5}$ and $||\phi_2||_{J_1} \le ||\phi_2||_J \le \epsilon^{0.5}$, the regularities (7) and Theorem 1 are proved.

We should point out that the proof of the regularity of w seems to be a little complicated, actually its FE-computation is simple and direct.

4. Singularity-separated FEM

We compute the first order system corresponding to (5),

$$(25)\qquad \begin{cases} -\epsilon p' + bw' + cw = f, \\ w' - p = 0, \\ w(0) = 0, \quad (bp' + cw)(1) = f(1). \end{cases}$$

Let $\tau = p_0\epsilon|\ln\epsilon|/b,\ p_0 \ge m+1$. Take N_0 uniform meshes in $J_0 = (0, 1-\tau)$ with step-size $h = (1-\tau)/N_0$ and nodes $x_j = jh, j = 0, 1, ..., N_0$, and take N_1 uniform meshes in J_1 with small step-size $h' = \tau/N_1 \le h$ and nodes $x_j = x_{N_0} + (j - N_0)h'$, $j = N_0+1, ..., N_0+N_1$, which are called N_0+N_1 meshes. Total number of elements is $N = N_0 + N_1$. For small ϵ, we can even take $N_1 = 1$.

To give a unified formulation for the continuous and discontinuous Galerkin finite element methods, we denote the m-degree piecewise polynomial space by

$$S_m^h = \{v \in P_m(I_j),\ I_j = (x_j, x_{j+1})\}.$$

Denote the left and right limits $v_j^{\pm} = v(x_j \pm 0)$ at node x_j, and the averaging flux (or trace) $\hat{v}_j = (v_j^+ + v_j^-)/2$. If v is continuous in J, then $v_j^+ = v_j^-$ and $\hat{v}_j = v_j$. Define the inner product

$$G_j(v', \xi) = -(v, \xi')_{I_j} + (\hat{v}\xi^-)_{j+1} - (\hat{v}\xi^+)_j,\ G(v', \xi) = \sum_{j=0}^{N-1} G_j(v', \xi).$$

We should point out that only two boundary value conditions in (25) are given, but four freedoms occur, thus we need to supplement two constraints in order to guarantee the solvability of linear systems of equation. These constraints are different for different FEMs.

We are interested in three kinds of FEM as follows.

1. Continuous Galerkin finite element method (CG).

The CG-solution $\{W, P\} \in S_m^h$ satisfies

$$(26)\qquad \begin{cases} -\epsilon G(P, \xi) + b(P, \xi) + c(W, \xi) = (f, \xi), & \xi \in P_{m-1}, \\ (W' - P, \eta) = 0, & \eta \in P_{m-1}, \end{cases}$$

and the trial functions and test functions take the boundary conditions as follows

$$\begin{array}{r|ll|ll} at\ x = 0, & W = 0, & \eta = 0, & P = free, & \xi = free, \\ at\ x = 1, & W = free, & \eta = free, & P = (f - cW)/b, & \xi = 0, \end{array}$$

where $\{P, \xi\}$ at $x = 0$ and $\{W, \eta\}$ at $x = 1$ are free, thus their total freedoms are matched, and the linear system (26) can be solved simultaneously.

2. Local discontinuous Galerkin method(LDG).

In each cell I_j we find $\{W, P\} \in S_m^h$ satisfting

$$(27)\qquad \begin{cases} -\epsilon G_j(P', \xi) + bG_j(W', \xi) + c(W, \xi)_{I_j} & = (f, \xi)_{I_j}, \quad \xi \in P_m, \\ -G_j(W', \eta) + (P, \eta)_{I_j} & = 0, \quad \eta \in P_m, \end{cases}$$

and take the numerical fluxes as follows,

$$\hat{W}_j = \begin{cases} W_j^- = 0, & j = 0, \\ W_j^-, & j = 1, 2 \cdots N; \end{cases}$$

$$\hat{P}_j = \begin{cases} P_j^+, & j = 0, 1 \cdots, N-1 \\ (-c\hat{W}_N + f(1))/b, & j = N, \end{cases}$$

where $W_N^+ = W_N^-$ and $P_0^- = P_0^+$ are postulated. The total freedoms are matched, so the linear systems of equations in (27) can be solved simultaneously.

3. Average discontinuous Galerkin method (ADG).

In each cell I_j we search for $\{W, P\} \in S_m^h$ such that

$$\begin{cases} -\epsilon G_j(P', \xi) + bG_j(W', \xi) + c(W, \xi)_{I_j} & = (f, \xi)_{I_j}, \quad \xi \in P_m, \\ -G_j(W', \eta) + (P, \eta)_{I_j} & = 0, \quad \eta \in P_m, \end{cases} \tag{28}$$

and take the numerical fluxes (or traces) as follows,

$$\hat{W}_j = \begin{cases} W_0^+ = W_0^- = 0, & j = 0, \\ (W_j^- + W_j^+)/2, & j = 1, 2 \cdots, N-1, \\ W_N^- = W_N^+, & j = N; \end{cases}$$

$$\hat{P}_j = \begin{cases} P_0^- = P_0^+, & j = 0, \\ (P_j^- + P_j^+)/2, & j = 1, 2 \cdots, N-1, \\ (-c\hat{W}_N + f(1))/b, & j = N. \end{cases}$$

where $W_N^- = W_N^+$ and $P_0^- = P_0^+$ are postulated, thus the linear systems of equation in (28) can be solved simultaneously. Note that the ADG is a symmetrical scheme, and of several elegant properties [5, 9, 14].

Now we propose the singularity-separated FEM as follows.

1). Compute m-degree(continuous or discontinuous) FE-solutions $\{W, P\}$ as before.

2). Construct $g_h(x) = W(1)\phi_2(x)$, which is not a polynomial.

3). Define FE-solutions $\{U, Q\}$ in (2) by the following expressions

$$U(x) = W(x) - g_h(x), \quad Q(x) = P(x) - g_h'(x). \tag{29}$$

Thus their errors can be expressed by $e_w = w - W, e_p = w' - P$ as follows.

Corollary 1. FE-solutions $\{U, Q\}$ have the error expressions

$$\begin{cases} e_u(x) & \equiv u(x) - U(x) = e_w(x) - \hat{e}_w(1)\phi_2(x) + O(e^{-1/\epsilon}), \\ e_q(x) & \equiv p(x) - Q(x) = e_p(x) - \hat{e}_w(1)\lambda_2\phi_2(x) + O(e^{-1/\epsilon}/\epsilon), \end{cases} \tag{30}$$

where $\hat{W}(1)$ is the flux of FE-solution $W(1)$ and $\hat{e}_w(1) = w(1) - \hat{W}(1)$ is the key error.

We summarize superconvergence results for three classes of FEM as follows.

Lemma 1. If $\epsilon = 1$, for the m-degree FEM, then

1). CG has errors at nodes,$\{e_w, e_p\} = O(h^{2m}), m \geq 1$, see [3, 6, 7];

2). LDG [2, 18] has errors at nodes, $\{e_w^-, e_p^+\} = O(h^{2m+1}), m \geq 0$;

3). ADG [5, 9, 14] has errors at nodes,

$$\begin{cases} \{\hat{e}_w, \hat{e}_p\} & = O(h^{2m}), \quad \textit{for odd } m, \\ \{\hat{e}_w, \hat{e}_p\} & = O(h^{2m+2}), \quad \textit{for even } m \geq 0, \end{cases}$$

(The latter is the highest superconvergence known till now in various DG-methods).

We think these superconvergence results are valid for singular problems on the whole.

Numerical comparison of three FEMs. We use the quadratic FEM based on SSM to solve (2) and (25) with $b = c = 1, f = e^x$. For different ϵ we take the same mesh numbers $N_0 = 10, N_1 = 1$. The errors for three kinds of FEMs at $x = 1$ are listed in table 1. We see that $\hat{e}_w(1)$ is of high accuracy independent of ϵ, whereas the error $\hat{e}_q(1) \approx \hat{e}_w(1)/\epsilon$ increases greatly, which verifies an important fact that $-\hat{e}_w(1)\lambda_2\phi_2(x)$ is the main error source of $e_q(x)$ as mentioned in Corollary 1. However, LDG and ADG solutions still have high accuracy thanks to the high accuracy of $\hat{e}_w(1)$. Under so coarse 10+1 meshgrids, the numerial results are satisfactory. These properties are available to solve multi-dimensional problems.

Table 1. The errors of quadratic FEMs at $x = 1$ on $10 + 1$ meshes.

	CG-method		LDG-method		ADG-method	
ϵ	$-e_q(1)$	$e_w(1)$	$-e_q^+(1)$	$e_w^-(1)$	$-\hat{e}_q(1)$	$\hat{e}_w(1)$
E-3	2.7949E-2	2.7433E-5	8.5292E-4	-1.5933E-9	8.5534E-4	3.4358E-9
E-4	4.9672E-1	4.9713E-5	9.9934E-5	7.5376E-10	9.7497E-5	4.4155E-10
E-5	5.3163E+0	5.3168E-5	1.1288E-4	1.0319E-9	1.9478E-5	9.0529E-11
E-6	5.3530E+1	5.3530E-5	1.0617E-3	1.0607E-9	5.6341E-5	5.4718E-11
E-7	5.3567E+2	5.3567E-5	1.0637E-2	1.0637E-9	5.1196E-4	5.1133E-11
E-8	5.3570E+3	5.3570E-5	1.0640E-1	1.0640E-9	5.0779E-3	5.0774E-11
Rate	$O(h^4/\epsilon)$	$O(h^4)$	$O(h^5/\epsilon)$	$O(h^5)$	$O(h^6/\epsilon)$	$O(h^6)$

Acknowledgments

This research was supported by The NSF of China (No. 10771063) and Key Laboratory of HPCSIP(Ministry of Education).

References

[1] G.Beckett, J.A.Mackenzie. Convergence analysis of finite difference approximations on equidistributed grids to a singularly perturbed boundary value problem. Appl. Numer. Math.,**35:2** (2000): 87C109.

[2] F.Celiker, B.Cockburn. Superconvergence of the numerical trace of discontinuous Galerkin and hybridized methods for convection-diffusion problems in one space dimension. Math. Comp.,**76**(2007): 67-96.

[3] C.M.Chen. Structure theory of superconvergence for finite element method(in Chinese). Changsha: Hunan Science and Technique Press. 2001, pp. 1-464.

[4] L.Chen, J.Xu. Stability and accuracy of adapted finite element methods for singularly perturbed problems. Numer. Math.,**109:2**(2008): 167C191.

[5] M.Delfour, W.Hager, F. Trochu. Discontinuous Galerkin methods for ordinary differential equations. Math. Comp., **36**(1981): 453-473.

[6] J.Douglas, T.Dupont. Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary value problems, Topics in Numerical Analysis, Academic Press, 1973,pp.89-92.

[7] J.Douglas, T.Dupont. Galerkin approximations for the two-point boundary problem using continuous, piecewise polynomial spaces, Numer. Math., **22**(1974):99-109.

[8] W.Lenferink. Pointwise convergence of approximations to a convection-diffusion equation on a Shishkin mesh. Appl. Numer. Math.,**32:1**(2000): 69-86.

[9] C.H.Li, C.M.Chen. Ultraconvergence for averaging discontinuous finite elements and its applications in Hamiltonian system. Appl. Math. Mech.,**32:7**(2011):343-356.

[10] J.J.Miller, E.O'Riordan, G.I.Shishkin. Fitted Numerical Methods for Singular Perturbation Problems. Singapore: World Scientific, 1996.

[11] H.Roos, M.Stynes, L.Tobiska. Robust numerical methods for singularly perturbed differential equations. Berlin: Springer,2008.

[12] G.Shishkin. Methods of constructing grid approximations for singularly perturbed boundary-value problems. Condensing grid methods. Russ. J. Numer. Anal. Math. Modeling. **7**(1992): 537-562.

[13] M.Stynes, L.Tobiska. A finite difference analysis of a streamline diffusion method on a Shishkin mesh. Numer. Algorithms, **18:3**(1998):337-360.

[14] J.X.Wang, C.M.Chen, Z.Q.Xie. The highest superconvergence of the averaging discontinuous Galerkin(ADG) method for two-point boundary problems. J. Sci. Comp., **70:1**(2017):175-191.

[15] Z.Q.Xie, Z.M.Zhang. Superconvergence of DG method for one-dimensional singular perturbed problems. J.Comp.Math., **25:2**(2007):185-200.
[16] Z.Q.Xie, Z. Z.Zhang, and Z.Zhang. A numerical study of uniform superconvergence of LDG method for solving perturbed problems. J. Comp. Math.,**27:2**(2009):280-298.
[17] Z.Q.Xie, Z.M.Zhang. Uniform superconvergence analysis of the discontinuous Galerkin method for a singularly perturbed problem in 1-D. Math. Comp., **79:269**(2010):35-45
[18] Y.Xu, C.W.Shu. Local discontinuous Galerkin methods for high-order time-dependent partial differential equations(Review Article). Commun. Comput. Phys.**7:1**(2010): 1-47.

[1] College of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, PRC

[2] College of Science and Agricultural mathematical modeling and data processing center, Hunan Agricultural University, Changsha 410128, PRC

E-mail: cmchen@hunnu.edu.cn and jingyang@hunau.edu.cn