THE \textit{h-p} VERSION OF THE CONTINUOUS PETROV-GALERKIN
METHOD FOR NONLINEAR VOLTERRA FUNCTIONAL
INTEGRO-DIFFERENTIAL EQUATIONS WITH VANISHING
DELAYS

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In memory of Professor Ben-yu Guo

Abstract. We investigate an \textit{h-p} version of the continuous Petrov-Galerkin method for the
nonlinear Volterra functional integro-differential equations with vanishing delays. We derive \textit{h-p}
version a priori error estimates in the $L^2$, $H^1$, and $L^\infty$-norms, which are completely explicit
in the local discretization and regularity parameters. Numerical computations supporting the
theoretical results are also presented.

Key words. \textit{h-p} version, continuous Petrov-Galerkin method, nonlinear Volterra functional
integro-differential equations, vanishing delays.

1. Introduction

We study the numerical solutions for the nonlinear Volterra functional integro-
differential equation (VFIDE) with vanishing delays:

\begin{equation}
\begin{cases}
u'(t) = f(t, u(t), u(\theta(t))) + (Vu)(t) + (V\theta u)(t), & t \in I := [0,T], \\
u(0) = u_0, & \end{cases}
\end{equation}

corresponding to the Volterra integral operators

\begin{align*}
(Vu)(t) &:= \int_0^t K_1(t,s)G_1(s,u(s))ds, \\
(V\theta u)(t) &:= \int_0^{\theta(t)} K_2(t,s)G_2(s,u(s))ds,
\end{align*}

where the delay function $\theta$ is subject to the following conditions:

\begin{enumerate}
\item[(C1)] $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$,
\item[(C2)] $\theta'(t) \geq q_0 > 0$ for all $t \in I$.
\end{enumerate}

We assume that $f$ and $G_i$ with $i = 1, 2$ are given functions. Moreover, the kernels
$K_1(t,s)$ and $K_2(t,s)$ are continuous on $D := \{(t,s) : 0 \leq s \leq t, \quad t \in I\}$ and
$D_\theta := \{(t,s) : 0 \leq s \leq \theta(t), \quad t \in I\}$, respectively.

During the past few decades, many numerical methods have been proposed and
analyzed for the VFIDEs. Among those a large number of methods are based on the
\textit{h}-version approach, which means that the convergence is achieved by decreasing
the size of time steps at a fixed and typically low approximation order. For an
overview of the lower-order methods developed for the VFIDEs, the reader can
refer to monographs \cite{3, 5} and the references therein. In contrast, the higher-
order methods, for example, the \textit{p}- and \textit{h-p} version methods employ (varying)
high order approximation polynomials. Particularly, the \textit{h-p} version method allows
for locally varying time steps and approximation orders, which can significantly
enhance the numerical accuracy. The \textit{h-p} version continuous and discontinuous

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Galerkin methods were introduced for initial-value problems in [9, 17, 19], for delay differential equations in [6], for parabolic problems in [10], and for Volterra integro-differential equations in [3, 8, 18, 20]. Moreover, some other high-order methods, such as the spectral methods were developed for various Volterra integro-differential equations with delays; see, e.g., [1, 13, 14, 15, 16, 21]. However, to the best of our knowledge, there is no work that considers the \( h-p \) version Galerkin method for nonlinear VFIDEs.

The purpose of the current work is to present and analyze an \( h-p \) version of the continuous Petrov-Galerkin (CPG) discretization scheme for the numerical approximation of the VFIDE (1) with vanishing delays. The Petrov-Galerkin method allows the trial and test spaces to be different, and it has become powerful tools for solving many kinds of differential equations (see e.g., [7, 12]). The CPG method presented in this paper is a hybrid of the continuous and discontinuous Galerkin methods with respect to time. More precisely, one uses continuous and piecewise polynomials for the trial spaces, but uses discontinuous and piecewise polynomials for the test spaces. With such choice of the trial and test spaces, we show that the CPG scheme defines a unique approximate solution, provided that a certain condition on the time steps is satisfied (which is completely independent of the approximation orders). We also describe in detail our implementation for the CPG scheme according to certain relationship between the delay function \( \theta(t) \) and nodal points of the time partition. Moreover, we derive \( h-p \) version a priori error estimates that are completely explicit with respect to the local time steps, the local approximation orders, and the local regularity properties of the exact solution.

The remainder of this paper is organized as follows. In Section 2, we introduce the \( h-p \) version of the CPG method for the VFIDE (1) and prove existence and uniqueness of approximate solutions. We also give a detailed description of the computational form of the CPG scheme. In Section 3, we carry out a complete \( h-p \) version error analysis of the CPG method. In Section 4, we present some numerical experiments to verify the theoretical results. We end the paper with a summary and discussion in Section 5.

2. The \( h-p \) version of continuous Petrov-Galerkin method

In this section, we first introduce the \( h-p \) version of the CPG method for the VFIDE (1). We then show the existence and uniqueness of the approximate solutions. Finally, we discuss the numerical implementation of the CPG scheme.

2.1. Continuous Petrov-Galerkin discretization. Let \( T_h \) be a partition of the time interval \( I \) given by the points

\[
0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T.
\]

We set \( I_n = (t_{n-1}, t_n) \) and \( k_n = t_n - t_{n-1} \) for \( 1 \leq n \leq N \). Let \( k = \max_{1 \leq n \leq N} \{k_n\} \). Moreover, we assign to each time interval \( I_n \) an approximation order \( r_n \geq 1 \) and introduce the degree vector \( \mathbf{r} = \{r_n\}_{n=1}^{N} \). Then, the tuple \((T_h, \mathbf{r})\) is called an \( h-p \) discretization of \( I \). Next, we introduce the \( h-p \) version trial and test spaces

\[
S^{r-1}(T_h) = \{u \in H^1(I) : u|_{I_n} \in P_{r_n}(I_n), 1 \leq n \leq N \}
\]

and

\[
S^{r-1,0}(T_h) = \{u \in L^2(I) : u|_{I_n} \in P_{r_n-1}(I_n), 1 \leq n \leq N \},
\]

respectively, where \( P_{r_n}(I_n) \) denotes the space of polynomials of degree at most \( r_n \) on \( I_n \).
The $h$-$p$ version CPG approximation of the VFIDE (1) is now defined as follows: find $U \in S^{r,1}(T_h)$ such that $U(0) = u_0$ and

$$
\sum_{n=1}^{N} \int_{I_n} U'(t)\varphi(t)dt = \sum_{n=1}^{N} \int_{I_n} \left(f(t, U(t), U'(t)) + (VU)(t) + (V_\theta U)(t)\right)\varphi(t)dt
$$

for all $\varphi \in S^{r-1,0}(T_h)$.

**Remark 2.1.** Due to the discontinuous character of the test space $S^{r-1,0}(T_h)$, the CPG method in (2) can be regarded as a time stepping scheme: if $U$ is given on the time intervals $I_m, 1 \leq m \leq n - 1$, we find $U|_{I_n} \in P_{r_n}(I_n)$ on $I_n$ by solving

$$
\int_{I_n} U'(t)\varphi(t)dt = \int_{I_n} \left(f(t, U(t), U'(t)) + (VU)(t) + (V_\theta U)(t)\right)\varphi(t)dt,
$$

$U|_{I_n}(t_{n-1}) = U|_{I_{n-1}}(t_{n-1})$

for all $\varphi \in P_{r_{n-1}}(I_n)$. Here, $U|_{I}(0) = u_0$.

**2.2. Existence and uniqueness of discrete solutions.** We start by showing the following well-known Poincaré-Friedrichs inequality (see, e.g., [2]).

**Lemma 2.1.** Let $u \in H^1(J), J = (a,b) \subset R$. Assume that $u(a) = 0$. Then, there holds

$$
\|u\|_{L^2(J)} \leq h\|u'\|_{L^2(J)},
$$

where $h = b - a$.

We next address the well-posedness of the discrete solutions. For our purpose, let

$$
K_1 := \max_{(t,s) \in D} |K_1(t,s)|, \quad K_2 := \max_{(t,s) \in D} |K_2(t,s)|.
$$

Further, we assume that $f(t, u, v), G_1(t, u)$ and $G_2(t, u)$ fulfill the following Lipschitz conditions:

$$
|f(t, u_1, v) - f(t, u_2, v)| \leq L_1|u_1 - u_2|,
$$

$$
|f(t, u_1, v_1) - f(t, u, v_2)| \leq L_2|v_1 - v_2|,
$$

$$
|G_1(t, u_1) - G_1(t, u_2)| \leq L_3|u_1 - u_2|,
$$

and

$$
|G_2(t, u_1) - G_2(t, u_2)| \leq L_4|u_1 - u_2|
$$

for all $t \in I, |u| < \infty, |v| < \infty$ and $|u_i| < \infty$ ($i = 1, 2$), where $L_1, L_2, L_3$ and $L_4$ are positive constants independent of $t, u$ and $v$.

**Theorem 2.1.** Assume that the partition $T_h$ satisfies

$$
\lambda_n := \left(L_1 + \frac{L_2}{\sqrt{h}} + \frac{K_1 L_3}{\sqrt{2}} k_n + \frac{K_2 L_4}{\sqrt{2}} k_n\right) k_n < 1, \quad 1 \leq n \leq N.
$$

Then the discrete problem (3) has a unique solution $U \in S^{r,1}(T_h)$. 

Proof. Owing to Remark 2.1, it suffices to prove that problem (3) admits a unique solution \( U|_{I_n} \in P_{r_n}(I_n) \), \( 1 \leq n \leq N \). Since the CPG solution is constructed step by step, it is enough to show the existence and uniqueness on the first time interval \( I_1 \), namely, we only have to consider \( n = 1 \) in (3) (for \( n \geq 2 \) the proof is completely analogous).

To this end, we shall show that on \( I_1 \) there is a unique solution \( U \in P_{r_1}(I_1) \) satisfying

\[
\int_{I_1} U'(t) \varphi(t) dt = \int_{I_1} \left( f(t, U(t), U(\theta(t))) + (VU)(t) + (V_\theta U)(t) \right) \varphi(t) dt, \\
U(0) = u_0
\]

for all \( \varphi \in P_{r_1-1}(I_1) \).

Select \( U_0 \in P_{r_1}(I_1) \) with \( U_0(0) = u_0 \). For \( m \geq 1 \), let \( U_m \in P_{r_1}(I_1) \) be the solution of the linear problem

\[
\int_{I_1} U_m'(t) \varphi(t) dt = \int_{I_1} \left( f(t, U_{m-1}(t), U_{m-1}(\theta(t))) + (VU_{m-1})(t) + (V_\theta U_{m-1})(t) \right) \varphi(t) dt, \\
U_m(0) = u_0
\]

for all \( \varphi \in P_{r_1-1}(I_1) \). Then, we have

\[
\int_{I_1} (U_m - U_{m-1})' \varphi(t) dt = \int_{I_1} \left( f(t, U_{m-1}(t), U_{m-1}(\theta(t))) + (VU_{m-1})(t) + (V_\theta U_{m-1})(t) \right) \varphi(t) dt \\
\quad - \int_{I_1} \left( f(t, U_{m-2}(t), U_{m-2}(\theta(t))) + (VU_{m-2})(t) + (V_\theta U_{m-2})(t) \right) \varphi(t) dt
\]

for all \( \varphi \in P_{r_1-1}(I_1) \). Choosing \( \varphi = (U_m - U_{m-1})' \) in the above equation, using (11)-(12) and the Cauchy-Schwarz inequality yields

\[
\begin{align*}
&\| (U_m - U_{m-1})' \|_{L^2(I_1)}^2 \\
\leq &\ L_1 \| U_{m-1} - U_{m-2} \|_{L^2(I_1)} \| (U_m - U_{m-1})' \|_{L^2(I_1)} \\
&+ L_2 \| (U_{m-1} - U_{m-2})(\theta(t)) \|_{L^2(I_1)} \| (U_m - U_{m-1})' \|_{L^2(I_1)} \\
&+ K_1 L_3 \int_0^t \| U_{m-1} - U_{m-2} \|_{L^2(I_1)} \| (U_m - U_{m-1})' \|_{L^2(I_1)} ds \\
&+ K_2 L_4 \int_0^{\theta(t)} \| U_{m-1} - U_{m-2} \|_{L^2(I_1)} ds \\
&\| (U_m - U_{m-1})' \|_{L^2(I_1)}
\end{align*}
\]

which implies

\[
\begin{align*}
&\| (U_m - U_{m-1})' \|_{L^2(I_1)}^2 \\
\leq &\ L_1 \| U_{m-1} - U_{m-2} \|_{L^2(I_1)} + L_2 \| (U_{m-1} - U_{m-2})(\theta(t)) \|_{L^2(I_1)} \\
&+ K_1 L_3 \int_0^t \| U_{m-1} - U_{m-2} \|_{L^2(I_1)} ds \\
&+ K_2 L_4 \int_0^{\theta(t)} \| U_{m-1} - U_{m-2} \|_{L^2(I_1)} ds
\end{align*}
\]

(12)
In view of the conditions (C1) and (C2), we find that
\[
\|(U_{m-1} - U_{m-2})(\theta(t))\|_{L^2(I_t)}^2 \leq \frac{1}{q_0} \int_0^{\theta(t)} \|(U_{m-1} - U_{m-2})(s)\|^2 ds \\
\leq \frac{1}{q_0} \|U_{m-1} - U_{m-2}\|_{L^2(I_t)}^2,
\]
and by the Cauchy-Schwarz inequality we have
\[
\left\| \int_0^t |(U_{m-1} - U_{m-2})(s)| ds \right\|_{L^2(I_t)}^2 \leq \int_0^t \left( \int_0^t |(U_{m-1} - U_{m-2})(s)|^2 ds \right) dt \\
\leq \frac{k_1^2}{2} \|U_{m-1} - U_{m-2}\|_{L^2(I_t)}^2,
\]
\[
\left\| \int_0^t (U_{m-1} - U_{m-2})(s) ds \right\|_{L^2(I_t)}^2 \leq \left\| \int_0^t |(U_{m-1} - U_{m-2})(s)| ds \right\|_{L^2(I_t)}^2 \\
\leq \frac{k_1^2}{2} \|U_{m-1} - U_{m-2}\|_{L^2(I_t)}^2,
\]
which together with (12) gives
\[
\| (U_m - U_{m-1})' \|_{L^2(I_t)}^2 \leq \left( L_1 + \frac{L_2}{\sqrt{q_0}} + \frac{K_1 L_3 k_1}{\sqrt{2}} + \frac{K_2 L_3 k_1}{\sqrt{2}} \right) \|U_{m-1} - U_{m-2}\|_{L^2(I_t)}^2.
\]
Then, by Lemma 2.1 we get
\[
\| (U_m - U_{m-1})' \|_{L^2(I_t)}^2 \leq \left( L_1 + \frac{L_2}{\sqrt{q_0}} + \frac{K_1 L_3 k_1}{\sqrt{2}} + \frac{K_2 L_3 k_1}{\sqrt{2}} \right) k_1 \|U_{m-1} - U_{m-2}'\|_{L^2(I_t)}^2 \\
= \lambda_1 \|U_{m-1} - U_{m-2}'\|_{L^2(I_t)}^2 \leq \lambda_1^{m-1} \|U_1 - U_0\|_{L^2(I_t)}^2
\]
and
\[
\| U_m - U_{m-1} \|_{L^2(I_t)} \leq k_1 \|U_{m-1} - U_{m-2}'\|_{L^2(I_t)} \leq \lambda_1 \|U_{m-1} - U_{m-2}\|_{L^2(I_t)} \leq \lambda_1^{m-1} \|U_1 - U_0\|_{L^2(I_t)}^2
\]
which implies
\[
\| U_m - U_{m-1} \|_{H^\ell(I_t)} \leq \lambda_1^{m-1} \|U_1 - U_0\|_{H^\ell(I_t)}, \quad \ell = 0, 1.
\]
For our purpose, we denote by \([\alpha]\) the smallest integer larger or equal to \(\alpha\). For any \(\varepsilon > 0\), there is an integer \(N = \left\lceil \frac{\varepsilon (1 - \lambda_1)}{\ln \lambda_1} / \ln \lambda_1 \right\rceil\) such that for \(m > n > N\) there holds
\[
\| U_m - U_n \|_{H^\ell(I_t)} \leq \| U_m - U_{m-1} \|_{H^\ell(I_t)} + \cdots + \| U_{n+1} - U_n \|_{H^\ell(I_t)} \\
\leq (\lambda_1^{m-1} + \cdots + \lambda_1^n) \|U_1 - U_0\|_{H^\ell(I_t)} \\
= \lambda_1^n \frac{(1 - \lambda_1^{m-n})}{1 - \lambda_1} \|U_1 - U_0\|_{H^\ell(I_t)} \\
\leq \frac{\lambda_1^N}{1 - \lambda_1} \|U_1 - U_0\|_{H^\ell(I_t)} < \varepsilon,
\]
for all \(m > n > N\).
which implies that \( \{U_n\} \) is a Cauchy sequence in \( H^1(I_1) \). Hence, \( \{U_n\} \) has a limit \( U \in P_{r_1}(I_1) \) such that \( \lim_{n \to \infty} U_n(t) = U(t) \) in \( H^1(I_1) \). Taking the limit on both sides of (11), then \( U(t) \) satisfies the equation (11). Thus the existence is proved. Similarly, using the above arguments can easily lead to the uniqueness. In fact, suppose there are two solutions \( U \) and \( \tilde{U} \) of the problem (11), then we have

\[
\|U - \tilde{U}\|_{L^2(I_1)} \leq \lambda_1 \|U - \tilde{U}\|_{L^2(I_1)}
\]

for \( 0 < \lambda_1 < 1 \), which implies that \( U = \tilde{U} \). This proves the uniqueness.

\[\Box\]

2.3. Computational form of the continuous Petrov-Galerkin method. Let \( L_i(x), x \in [-1, 1] \) be the standard Legendre polynomial of degree \( i \). The shifted Legendre polynomials \( L_{n,i}(t) \) on the interval \( I_n \) are defined by

\[
L_{n,i}(t) = L_i\left(\frac{2t - t_{n-1} - t_n}{k_n}\right), \quad t \in I_n, \quad 0 \leq i \leq n.
\]

Let \( U_n(t) = U|_{I_n} \) be the solution of the discrete problem (3) on the interval \( I_n, 1 \leq n \leq N \). We expand \( U_n(t) \) as

\[
U_n(t) = \sum_{l=0}^{n} \hat{u}_{n,l} L_{n,l}(t).
\]

Inserting the above expression into (3) and choosing \( \varphi = L_{n,j}(t), \quad 0 \leq j \leq r_n - 1 \), we can rewrite (3) as a system of nonlinear algebraic equations for the unknown vector

\[
\hat{U}_n := (\hat{u}_{n,0}, \hat{u}_{n,1}, \ldots, \hat{u}_{n,r_n})^T \in \mathbb{R}^{r_n+1}.
\]

We emphasize that, the structure of the resulted nonlinear system depends strongly on the delay terms in (3) and changes for each value of \( n \) as we pass from Phase I to Phase III (described below).

For our purpose, we introduce the matrices

\[
A_n = (a_{jl})_{0 \leq j \leq r_n, 0 \leq l \leq r_n} \in \mathbb{R}^{(r_n+1) \times (r_n+1)}, \quad 1 \leq n \leq N,
\]

with the entries given by

\[
a_{jl} = \int_{I_n} L_{n,l}'(t)L_{n,j}(t)dt = \int_{-1}^1 L_i'(x)L_j(x)dx, \quad 0 \leq j \leq r_n - 1, \quad 0 \leq l \leq r_n,
\]

and \( a_{rl} = L_{n,l}(t_{n-1}) = L_l(-1), \quad 0 \leq l \leq r_n \). Further, for \( 0 \leq j \leq r_n - 1 \), we set

\[
b_{n,j} := \int_{I_n} (\mathcal{V}U)(t)L_{n,j}(t)dt
\]

\[
= \int_{I_n} \left( \int_{0}^{t} K_1(t,s)G_1(s,U(s))ds \right) L_{n,j}(t)dt
\]

\[
= \sum_{m=1}^{n-1} \int_{I_n} \left( \int_{I_m} K_1(t,s)G_1(s,U_m(s))ds \right) L_{n,j}(t)dt
\]

\[
+ \int_{I_n} \left( \int_{t_{n-1}}^{t} K_1(t,s)G_1(s,U_n(s))ds \right) L_{n,j}(t)dt
\]
\begin{equation}
\begin{split}
\gamma_{n,j}^2 & := \int_{I_n} (V_2 U)(t) L_{n,j}(t) dt \\
& = \int_{I_n} \left( \int_0^{\theta(t)} K_2(t,s) G_2(s,U(s)) ds \right) L_{n,j}(t) dt \\
& = \int_{I_n} \left( \int_0^{\theta(t)} K_2(t,s) G_2(s,U(s)) ds \right) L_{n,j}(t) dt \\
& \quad + \int_{I_n} \left( \int_0^{\theta(t)} K_2(t,s) G_2(s,U(s)) ds \right) L_{n,j}(t) dt \\
& := J_{1,j} + J_{2,j}.
\end{split}
\end{equation}

Obviously, if \( n = 1 \), the summation term in (16) and the term \( J_{1,j} \) in (17) will vanish.

We now introduce the following three distinct phases inspired by [6].

- **Phase I**: \( n = 1 \). In this initial phase we have complete overlap, i.e., for any \( t \in I_1 \) the images \( \theta(t) = 1 \). For \( 0 \leq j \leq r_1 - 1 \), we define

\[ f_{1,j}^I := \int_{I_1} f(t, U(t), U(\theta(t))) L_{1,j}(t) dt = \int_{I_1} f(t, U_1(t), U_1(\theta(t))) L_{1,j}(t) dt. \]

Moreover, we have

\[ b_{1,j}^I = \int_{I_1} \left( \int_0^{\theta(t)} K_2(t,s) G_2(s,U_1(s)) ds \right) L_{1,j}(t) dt. \]

Let \( c_{1,j}^I = f_{1,j}^I + b_{1,j}^I + b_{1,j}^2 \), and

\[ C^I(\tilde{U}_1) := (c_{1,0}^I, c_{1,1}^I, \ldots, c_{1,r_1-1}^I, u_0)^T. \]

Then we can rewrite (13) as the nonlinear system

\begin{equation}
A_1 \tilde{U}_1 = C^I(\tilde{U}_1).
\end{equation}

- **Phase II**: If \( n > 1 \) and \( \theta(t_n) > t_{n-1} \), then we will encounter partial overlap, i.e., for some \( t \in I_n \) the images \( \theta(t) \) are still in \( I_n \), while for some other (smaller) \( t \in I_n \) we have \( \theta(t) \notin I_n \). Clearly, there is an integer \( z \geq 1 \) such that \( \theta(t_{n-1}) \in I_z \).

Let \( t_{0}^* = t_{n-1}, t_{n-z+1}^* = t_n \) and \( t_m^* = \theta^{-1}(t_{z+m-1}) \in I_n \) for \( 1 \leq m \leq n-z \). For \( 0 \leq j \leq r_n - 1 \), we define

\[ f_{n,j}^{II} := \int_{I_n} f(t, U(t), U(\theta(t))) L_{n,j}(t) dt \]

\[ = \sum_{m=1}^{n-z+1} \int_{t_m^*}^{t_{m-1}^*} f(t, U_n(t), U_{z+m-1}(\theta(t))) L_{n,j}(t) dt. \]
In this phase, we have

\[ J_{1,j} = \int_{I_n} \left( \int_0^{t_{z-1}^*} K_2(t,s)G_2(s,U(s))ds \right) L_{n,j}(t)dt 
+ \int_{I_n} \left( \int_{t_z}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U(s))ds \right) L_{n,j}(t)dt 
= \sum_{\alpha=1}^{n-1} \int_{I_{n_{\alpha}}} \left( \int_{t_{z-1}}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U_\alpha(s))ds \right) L_{n,j}(t)dt 
+ \int_{I_n} \left( \int_{t_z}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U_z(s))ds \right) L_{n,j}(t)dt \]

and

\[ J_{2,j} = \sum_{m=1}^{n-2} \int_{t_{m-1}^*}^{t_m^*} \left( \int_{\theta(t_{n-1})}^{\theta(t_m)} K_2(t,s)G_2(s,U(s))ds \right) L_{n,j}(t)dt := \sum_{m=1}^{n-2} J_{2,m}^{II}, \]

with

\[ J_{2,m}^{II} = \sum_{\beta=1}^{m-1} \int_{t_{m-1}^*}^{t_m^*} \left( \int_{\theta(t_{\beta-1})}^{\theta(t_m)} K_2(t,s)G_2(s,U_{z+\beta-1}(s))ds \right) L_{n,j}(t)dt 
+ \int_{t_{m-1}^*}^{t_m^*} \left( \int_{\theta(t_{m-1})}^{\theta(t_m)} K_2(t,s)G_2(s,U_{z+m-1}(s))ds \right) L_{n,j}(t)dt. \]

Let \( c_{n,j}^{II} = f_{n,j}^{II} + b_{n,j}^1 + b_{n,j}^2 \) and

\[ C_{n}^{II}(\hat{U}_n) := (c_{n,0}^{II}, c_{n,1}^{II}, \ldots, c_{n,r_n-1}^{II}, U_n(t_{n-1}))^T. \]

Then we can rewrite \( (3) \) as the nonlinear system

\[ A_n \hat{U}_n = C_{II}(\hat{U}_n). \]

• Phase III: If \( n > 1 \) and \( \theta(t_{n-1}) \leq t_{n-1} \), then we will encounter the pure delay phase, i.e., there is no overlap between \( I_n \) and the images \( \theta(t) \) for any \( t \in I_n \). In this phase, there are two integers \( z_1 \) and \( z_2 \) \((z_1 \leq z_2)\) such that \( \theta(t_{n-1}) \in I_{z_1} \) and \( \theta(t_{n}) \in I_{z_2}. \)

Let \( t_{0}^* = t_{n-1}, \ t_{z_2-z_1+1}^* = t_{n} \) and \( t_{m}^* = \theta^{-1}(t_{z_1+m-1}) \in I_n \) for \( 1 \leq m \leq z_2 - z_1 \) (if \( z_1 < z_2 \)). For \( 0 \leq j \leq r_n - 1 \), we define

\[ f_{n,j}^{III} := \int_{I_n} f(t, U(t), U(\theta(t)))L_{n,j}(t)dt 
= \sum_{m=1}^{z_2-z_1+1} \int_{t_{m-1}^*}^{t_m^*} f(t, U_n(t), U_{z_1+m-1}(\theta(t)))L_{n,j}(t)dt. \]

In this phase, we have

\[ J_{1,j} = \int_{I_n} \left( \int_0^{t_{z-1}^*} K_2(t,s)G_2(s,U(s))ds \right) L_{n,j}(t)dt 
+ \int_{I_n} \left( \int_{t_z}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U(s))ds \right) L_{n,j}(t)dt 
= \sum_{\alpha=1}^{n-1} \int_{I_{n_{\alpha}}} \left( \int_{t_{z-1}}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U_\alpha(s))ds \right) L_{n,j}(t)dt 
+ \int_{I_n} \left( \int_{t_z}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U_z(s))ds \right) L_{n,j}(t)dt \]
and
\[ J_{2,j} = \sum_{m=1}^{z_2-z_1+1} \int_{t_{m-1}}^{t_m} \left( \int_{\theta(t_{m-1})}^{\theta(t)} K_2(t,s)U(s)ds \right) L_{n,j}(t)dt := \sum_{m=1}^{z_2-z_1+1} J_{2,j|m}, \]
with
\[ J_{2,j|m} = \sum_{\beta=1}^{m-1} \int_{t_{m-1}}^{t_m} \left( \int_{\theta(t_{\beta-1})}^{\theta(t_{\beta})} K_2(t,s)G_2(s, U_{z_1+\beta-1}(s))ds \right) L_{n,j}(t)dt \]
(20)
\[ + \int_{t_{m-1}}^{t_m} \left( \int_{\theta(t_{m-1})}^{\theta(t)} K_2(t,s)G_2(s, U_{z_1+m-1}(s))ds \right) L_{n,j}(t)dt. \]
Evidently, for \( m = 1 \) the summation term in the first line of (20) will vanish.

Let \( c_{n,j}^{III} = f_{n,j}^{III} + b_{n,j} + b_{n,j}^\prime \) and
\[ C^{III}(\hat{U}_n) := (c_{n,0}^{III}, c_{n,1}^{III}, \ldots, c_{n,n-1}^{III}, U_{n-1}(t_{n-1}))^T. \]
Then we can rewrite (20) as the nonlinear system
\[ A_n \hat{U}_n = C^{III}(\hat{U}_n). \]
(21)

**Remark 2.2.** In actual computation, the nonlinear systems (18)-(21) can be solved by an iterative process, for example, the Newton-Raphson iteration method or the successive substitution method.

### 3. Error analysis

In this section, we carry out a priori error analysis of the h-p version of the CPG method.

#### 3.1. Preliminaries.

Let \( \Lambda = (-1, 1) \). For a function \( u \in H^1(\Lambda) \), we introduce a projection operator \( \Pi^\Lambda_h : H^1(\Lambda) \rightarrow P_r(\Lambda) \) with \( r \geq 1 \) by
\[ \begin{cases} \int_\Lambda (u - \Pi^\Lambda_h u) \varphi dt = 0, & \forall \varphi \in P_{r-1}(\Lambda), \\ \Pi^\Lambda_h u(-1) = u(-1). \end{cases} \]
(22)

Setting \( \varphi = 1 \) in (22) and using integration by parts gives \( u(1) - \Pi^\Lambda_h u(1) = u(-1) - \Pi^\Lambda_h u(-1) = 0 \), which implies \( \Pi^\Lambda_h u(1) = u(1) \). It is well-known that the projection operator \( \Pi^\Lambda_h \) is well defined (see, e.g., [19]) and there holds
\[ \Pi^\Lambda_h u(x) = \int_\Lambda \left( \sum_{i=0}^{r-1} a_i L_i(\xi) \right) d\xi + u(-1), \]
where \( a_i = \frac{2i+1}{\int_\Lambda u' L_i dx} \) is the Legendre expansion coefficients of \( u' \).

For any interval \( J = (a, b) \) of length \( b - a \), we define \( \Pi^J_h u = [\Pi^\Lambda_h (u \circ M)] \circ M^{-1} \), where \( M : \Lambda \rightarrow J \) is the linear transformation \( x \mapsto t = \frac{a+b+x}{2} \). Then for the exact solution \( u \) of (11) we can define an approximation polynomial \( \mathcal{I}u \in S^{r-1}(T_n) \) as
\[ \mathcal{I}u|_{I_n} = \Pi^J_h u, \quad 1 \leq n \leq N. \]
Thanks to the definition of \( \Pi^J_h \), it is straightforward to show that \( \mathcal{I}u(t_n) = u(t_n) \) for \( 0 \leq n \leq N \), and there holds
\[ \int_{I_n} (u - \mathcal{I}u) \varphi dt = 0, \quad \forall \varphi \in P_{r-1}(I_n). \]
(23)

The polynomial \( \mathcal{I}u \) constructed above has the following approximation properties (cf. [11] (20)).
Lemma 3.2. Let $\mathcal{T}_h$ be any mesh in $I$ and assume that $u \in H^1(I)$ satisfies $u|_{I_n} \in H^{s_0,n+1}(I_n)$ for $s_0,n \geq 0$. Then

\begin{equation}
\|u - Iu\|_{L^2(I)}^2 \leq \sum_{n=1}^{N} \left( \frac{k_n}{2} \right)^{2s_n+2} \frac{\Gamma(r_n + 1 - s_n)}{r_n(r_n + 1)\Gamma(r_n + 1 + s_n)} \|u\|_{H^{s_0,n+1}(I_n)}^2,
\end{equation}

(24)

\begin{equation}
\|u - Iu\|_{H^1(I)}^2 \leq \sum_{n=1}^{N} \left( \frac{k_n}{2} \right)^{2s_n} \frac{\Gamma(r_n + 1 - s_n)}{r_n(r_n + 1 + s_n)} \|u\|_{H^{s_0,n+1}(I_n)}^2
\end{equation}

(25)

for any real $s_n$, $0 \leq s_n \leq \min\{r_n,s_0,n\}$.

Moreover, if $u \in H^1(I)$ satisfies $u|_{I_n} \in W^{s_0,n+1,\infty}(I_n)$ for $s_0,n \geq 0$. Then

\begin{equation}
\|u - Iu\|_{L^\infty(I_n)} \leq C \left( \frac{k_n}{2} \right)^{2s_n+2} \frac{\Gamma(r_n + 1 - s_n)}{r_n(r_n + 1 + s_n)} \|u\|_{W^{s_0,n+1,\infty}(I_n)}^2
\end{equation}

(26)

for any real $s_n$, $0 \leq s_n \leq \min\{r_n,s_0,n\}$.

We note that, the following discrete Gronwall inequality have been proved, for instance, in \cite{[1]}.

Lemma 3.3. Let $\{a_n\}_{n=1}^{N}$ and $\{b_n\}_{n=1}^{N}$ be two sequences of nonnegative real numbers with $b_1 \leq b_2 \leq \cdots \leq b_N$. Assume that for $C \geq 0$ and weights $w_i > 0, 1 \leq i \leq N-1$, there holds

\begin{equation*}
a_1 \leq b_1, \quad a_n \leq b_n + C \sum_{i=1}^{n-1} w_i a_i, \quad 2 \leq n \leq N.
\end{equation*}

Then

\begin{equation*}
a_n \leq b_n \exp(C \sum_{i=1}^{n-1} w_i), \quad 1 \leq n \leq N.
\end{equation*}

3.2. **Abstract error bounds.** Let $u$ be the exact solution of (1) and $U$ be the $h$-$p$ version of the CPG approximation defined by (2). We proceed in a standard way and decompose the error $e = u - U$ into two parts:

\begin{equation}
e = (u - Iu) + (Iu - U) := \eta + \xi.
\end{equation}

Lemma 3.2 can be used to bound $\eta$, and we are left with the task of estimating the term $\xi$.

In view of (1) and (3), there holds

\begin{equation*}
\int_{I_n} e' \varphi dt = \int_{I_n} \left( f(t,u(t),u(\theta(t))) - f(t,U(t),U(\theta(t))) \right) \varphi dt
+ \int_{I_n} (V u - VU) \varphi dt + \int_{I_n} (V_\theta u - V_\theta U) \varphi dt
\end{equation*}

for all $\varphi \in P_{r_{n-1}}(I_n)$. Then, by (26) we have

\begin{equation}
\int_{I_n} \xi' \varphi dt = \int_{I_n} \left( f(t,u(t),u(\theta(t))) - f(t,U(t),U(\theta(t))) \right) \varphi dt
+ \int_{I_n} (V u - VU) \varphi dt + \int_{I_n} (V_\theta u - V_\theta U) \varphi dt
\end{equation}

(28)

for all $\varphi \in P_{r_{n-1}}(I_n)$.

For any $v \in L^2(I_n)$, we define the $L^2$ projection of $v$ onto $P_{r_{n-1}}(I_n)$ by $\Pi_{r_{n-1}} v$, namely,

\begin{equation*}
\int_{I_n} (v - \Pi_{r_{n-1}} v) \varphi dt = 0, \quad \forall \varphi \in P_{r_{n-1}}(I_n).
\end{equation*}
First, we show the following bounds.

Lemma 3.4. Assume that $k$ is sufficiently small, there holds

\begin{equation}
\|\xi\|_{L^2(0,t_n)} \leq C\|\eta\|_{L^2(0,t_n)},
\end{equation}

\begin{equation}
|\xi|_{H^1(0,t_n)} \leq C\|\eta\|_{L^2(0,t_n)},
\end{equation}

and

\begin{equation}
|\xi(t_n)| \leq C\|\eta\|_{L^2(0,t_n)}
\end{equation}

for $1 \leq n \leq N$, where the constant $C > 0$ solely depends on $q_0, L_1, L_2, L_3, K_1, K_2,$ and $t_n$.

Proof. By choosing $\varphi = \Pi^{n-1}\xi$ in (28) and using (14)-(18) we get

\[
\int_{t_n}^{t} \xi \xi dt = \int_{t_n}^{t} (f(t,u(t),u(\theta(t))) - f(t,u(t),U(\theta(t)))) \Pi^{n-1}\xi dt
\]

\[
+ \int_{t_n}^{t} (\mathcal{V}u - \mathcal{V})\Pi^{n-1}\xi dt + \int_{t_n}^{t} (\mathcal{V}_0 u - \mathcal{V}_0 U)\Pi^{n-1}\xi dt
\]

\[
\leq L_1 \int_{t_n}^{t} |e| \cdot |\Pi^{n-1}\xi| dt + L_2 \int_{t_n}^{t} |e(\theta(t))| \cdot |\Pi^{n-1}\xi| dt
\]

\[
+ K_1 K_3 \int_{t_n}^{t} \left( \int_{t_{n-1}}^{t} |e(s)| ds \right) |\Pi^{n-1}\xi| dt
\]

\[
+ K_2 K_4 \int_{t_n}^{t} \int_{0}^{\theta(t)} |e(s)| ds |\Pi^{n-1}\xi| dt,
\]

which together with the Cauchy-Schwarz inequality and the $L^2$-stability of $\Pi^{n-1}$ yields

\[
\frac{1}{2}(|\xi(t_n)|^2 - |\xi(t_{n-1})|^2)
\]

\[
\leq L_1 \|e\|_{L^2(I_{n})}\|\xi\|_{L^2(I_{n})} + L_2 \|e(\theta(t))\|_{L^2(I_{n})}\|\xi\|_{L^2(I_{n})}
\]

\[
+ K_1 K_3 \int_{t_n}^{t} \left( \int_{t_{n-1}}^{t} |e(s)| ds \right) |\Pi^{n-1}\xi| dt
\]

\[
+ K_1 K_3 \int_{t_n}^{t} |\Pi^{n-1}\xi| dt \int_{0}^{\theta(t)} |e(s)| ds
\]

\[
+ K_2 K_4 \int_{0}^{\theta(t)} \left( \int_{0}^{\theta(t)} |e(s)| ds \right) |\Pi^{n-1}\xi| dt
\]

\[
\leq L_1 \|e\|_{L^2(I_{n})}\|\xi\|_{L^2(I_{n})} + L_2 \|e(\theta(t))\|_{L^2(I_{n})}\|\xi\|_{L^2(I_{n})}
\]

\[
+ K_1 K_3 k_n \|e\|_{L^2(I_{n})}\|\xi\|_{L^2(I_{n})}
\]

\[
+ K_1 K_3 k_n^* \|\xi\|_{L^2(I_{n})} \sum_{i=1}^{n-1} k_i^* \|e\|_{L^2(I_{i})}
\]

\[
+ K_2 K_4 \int_{0}^{\theta(t)} \left( \int_{0}^{\theta(t)} |e(s)| ds \right) |\Pi^{n-1}\xi| dt
\]

\[
+ K_2 K_4 \int_{0}^{\theta(t)} \left( \int_{0}^{\theta(t)} |e(s)| ds \right) |\Pi^{n-1}\xi| dt
\]

\[
+ K_2 K_4 \int_{0}^{\theta(t)} \left( \int_{0}^{\theta(t)} |e(s)| ds \right) |\Pi^{n-1}\xi| dt
\]
Here, we have used the fact that

\[
\phi(33)
\]

Consequently,

\[
\left| \xi_{L^n} \right| \leq \frac{L_1}{2} \left( ||e||^2_{L^2(I_n)} + ||\xi||^2_{L^2(I_n)} \right) + \frac{L_2}{2} \left( ||e(\theta(t))||^2_{L^2(I_n)} + ||\xi||^2_{L^2(I_n)} \right)
\]

\[+ \frac{K_1L_3k_n}{2\sqrt{2}} \left( ||e||^2_{L^2(I_n)} + ||\xi||^2_{L^2(I_n)} \right)
\]

\[+ \frac{K_1L_3}{2} \left( ||\xi||^2_{L^2(I_n)} + k_nt_{n-1}||e||^2_{L^2(0,t_{n-1})} \right)
\]

\[+ \frac{K_2L_4}{2} \left( \left\| \int_0^{\theta(t)} |e(s)|ds \right\|^2_{L^2(I_n)} + ||\xi||^2_{L^2(I_n)} \right).
\]

Here, we have used the fact that

\[
\left( \int_{t_{n-1}}^{t} |e(s)|ds \right)_{L^2(I_n)} \leq \left\{ \int_{t_{n-1}}^{t} (t - t_{n-1}) \left( \int_{t_{n-1}}^{t} |e(s)|^2ds \right)dt \right\}^{\frac{1}{2}} \leq \frac{k_n}{\sqrt{2}} ||e||_{L^2(I_n)}.
\]

Consequently,

\[
|\xi(t_n)|^2 \leq |\xi(t_{n-1})|^2 + \left( L_1 + \frac{K_1L_3k_n}{\sqrt{2}} \right) ||e||^2_{L^2(I_n)}
\]

\[+ \left( L_1 + L_2 + \frac{K_1L_3k_n}{\sqrt{2}} + K_1L_3 + K_2L_4 \right) ||\xi||^2_{L^2(I_n)}
\]

\[+ L_2 ||e(\theta(t))||^2_{L^2(I_n)} + K_1L_3k_nt_{n-1}||e||^2_{L^2(0,t_{n-1})}
\]

\[+ \frac{K_2L_4}{2} \left( \left\| \int_0^{\theta(t)} |e(s)|ds \right\|^2_{L^2(I_n)} + ||\xi||^2_{L^2(I_n)} \right)
\]

\[
(33)
\]

Additionally, taking \( \varphi = \Pi^{n-1}(t_{n-1} - t)\xi \) in (28), we find that

\[
\int_{I_n} (t_{n-1} - t)\xi' \xi dt
\]

\[= \int_{I_n} \left( f(t, u(t), u(\theta(t))) - f(t, U(t), U(\theta(t)))) \Pi^{n-1}(t_{n-1} - t)\xi \right) dt
\]

\[+ \int_{I_n} (Vt - VU)\Pi^{n-1}(t_{n-1} - t)\xi dt
\]

\[+ \int_{I_n} (V_{\theta}U - V_{\theta}U)\Pi^{n-1}(t_{n-1} - t)\xi dt,
\]
which together with (41)–(52) gives

\[
\frac{1}{2} \left( -k_n |\xi(t_n)|^2 + \|\xi\|_{L^2(I_n)}^2 \right)
\leq L_1 \|e\|_{L^2(I_n)} \|\Pi^{n-1}((t_{n-1} - t)\xi)\|_{L^2(I_n)}
\quad + L_2 \|e(\theta(t))\|_{L^2(I_n)} \|\Pi^{n-1}((t_{n-1} - t)\xi)\|_{L^2(I_n)}
\]

\[
+ \bar{K}_1 L_3 \int_{I_n} \left( \int_{I_{n-1}}^t |e(s)| ds \right) \|\Pi^{n-1}((t_{n-1} - t)\xi)\| dt
\quad + \bar{K}_1 L_3 \int_{I_n} \left( \int_{I_{n-1}}^t |e(s)| ds \right) \|\Pi^{n-1}((t_{n-1} - t)\xi)\| dt
\]

\[
+ \bar{K}_2 L_4 \left( \int_0^{\theta(t)} |e(s)| ds \right) \|\Pi^{n-1}((t_{n-1} - t)\xi)\|_{L^2(I_n)}.
\]

(34)

We notice that

\[
\|\Pi^{n-1}((t_{n-1} - t)\xi)\|_{L^2(I_n)} \leq \|(t_{n-1} - t)\xi\|_{L^2(I_n)} \leq k_n \|\xi\|_{L^2(I_n)}.
\]

Then, by (51) and (52) we readily find that

\[
\|\xi\|_{L^2(I_n)}^2
\leq k_n |\xi(t_n)|^2 + 2L_1 k_n \|e\|_{L^2(I_n)} \|\xi\|_{L^2(I_n)} + 2L_2 k_n \|e(\theta(t))\|_{L^2(I_n)} \|\xi\|_{L^2(I_n)}
\quad + \frac{2\bar{K}_1 L_3 k_n^2}{\sqrt{2}} \|e\|_{L^2(I_n)} \|\xi\|_{L^2(I_n)} + 2\bar{K}_1 L_3 k_n^2 \|\xi\|_{L^2(I_n)} \left( \sum_{i=1}^{n-1} \|e\|^2_{L^2(I_i)} \right)
\quad + 2\bar{K}_2 L_4 k_n \left( \int_0^{\theta(t)} |e(s)| ds \right) \|\xi\|_{L^2(I_n)}
\]

\[
\leq k_n |\xi(t_n)|^2 + L_1 k_n \left( |\xi|^2_{L^2(I_n)} + \|\xi\|^2_{L^2(I_n)} \right)
\quad + L_2 k_n \left( |e(\theta(t))|^2_{L^2(I_n)} + \|\xi\|^2_{L^2(I_n)} \right)
\quad + \frac{\bar{K}_1 L_3 k_n^2}{\sqrt{2}} \left( |\xi|^2_{L^2(I_n)} + \|\xi\|^2_{L^2(I_n)} \right)
\quad + \bar{K}_1 L_3 k_n \left( k_n t_{n-1} \|e\|_{L^2(0,t_{n-1})} + \|\xi\|^2_{L^2(I_n)} \right)
\quad + \bar{K}_2 L_4 k_n \left( \|\int_0^{\theta(t)} |e(s)| ds \|_{L^2(I_n)} \right)
\]

\[
\leq k_n |\xi(t_n)|^2 + 2 \left( L_1 + \frac{\bar{K}_1 L_3 k_n^2}{\sqrt{2}} \right) k_n |\eta|^2_{L^2(I_n)}
\quad + \left( 3L_1 + L_2 + \frac{3\bar{K}_1 L_3 k_n}{\sqrt{2}} + \bar{K}_1 L_3 + \bar{K}_2 L_4 \right) k_n \|\xi\|^2_{L^2(I_n)}
\quad + L_2 k_n \|e(\theta(t))|^2_{L^2(I_n)} + \bar{K}_1 L_3 k_n^2 t_{n-1} \|e\|^2_{L^2(0,t_{n-1})}
\quad + \bar{K}_2 L_4 k_n \left( \int_0^{\theta(t)} |e(s)| ds \right)^2_{L^2(I_n)}.
\]
For convenience, we define

$$A_n = \left(3L_1 + L_2 + \frac{3K_1L_3k_n}{\sqrt{2}} + K_1L_3 + K_2L_4\right)k_n.$$ 

We observe after elementary manipulation that

$$||\xi||^2_{L^2(I_n)} \leq \frac{k_n}{1 - A_n}||\xi(t_n)||^2 + \frac{2\left(L_1 + \frac{K_1L_3k_n}{\sqrt{2}}\right)k_n}{1 - A_n}||\eta||^2_{L^2(I_n)}$$

$$+ \frac{L_2k_n}{1 - A_n}||e(\theta(t))||^2_{L^2(I_n)} + \frac{K_1L_3k^2_{n-1}A_n}{1 - A_n}||e||^2_{L^2(0,t_{n-1})}$$

$$+ \frac{K_2L_4k_n}{1 - A_n}\left|\int_{0}^{\theta(t_n)}|e(s)|ds\right|^2_{L^2(I_n)},$$

By inserting \(35\) into \(33\), we obtain

$$||\xi(t_n)||^2 \leq ||\xi(t_{n-1})||^2 + 2\left(L_1 + \frac{K_1L_3k_n}{\sqrt{2}}\right)A_n$$

$$+ \frac{A_n}{1 - A_n}||\xi(t_n)||^2 + \frac{2\left(L_1 + \frac{K_1L_3k_n}{\sqrt{2}}\right)A_n}{1 - A_n}||\eta||^2_{L^2(I_n)}$$

$$+ \frac{L_2A_n}{1 - A_n}||e(\theta(t))||^2_{L^2(I_n)} + \frac{K_1L_3k_{n-1}A_n}{1 - A_n}||e||^2_{L^2(0,t_{n-1})}$$

$$+ \frac{K_2L_4A_n}{1 - A_n}\left|\int_{0}^{\theta(t_n)}|e(s)|ds\right|^2_{L^2(I_n)}$$

$$+ K_1L_3k_{n-1}||e||^2_{L^2(0,t_{n-1})} + K_2L_4\left|\int_{0}^{\theta(t_n)}|e(s)|ds\right|^2_{L^2(I_n)},$$

which can be rewritten as

$$||\xi(t_n)||^2 \leq \left(1 + \frac{A_n}{1 - 2A_n}\right)||\xi(t_{n-1})||^2 + \frac{2\left(L_1 + \frac{K_1L_3k_n}{\sqrt{2}}\right)A_n}{1 - 2A_n}||\eta||^2_{L^2(I_n)}$$

$$+ \frac{L_2}{1 - 2A_n}||e(\theta(t))||^2_{L^2(I_n)} + \frac{K_1L_3k_{n-1}A_n}{1 - 2A_n}||e||^2_{L^2(0,t_{n-1})}$$

$$+ \frac{K_2L_4}{1 - 2A_n}\left|\int_{0}^{\theta(t_n)}|e(s)|ds\right|^2_{L^2(I_n)}.$$

Assume that \(k_n\) is sufficiently small, then there exists a positive constant \(\gamma\) such that

$$2A_n \leq \gamma < 1, \quad 1 \leq n \leq N.$$
Summing up (39) over all element $I_i$, $1 \leq i \leq n$, and using the facts that $\xi(0) = 0$ and $\xi|_{I_i}(t_i) = \xi|_{I_{i+1}}(t_i)$, $1 \leq i \leq n - 2$, we readily conclude that (37)

$$|\xi(t_n)|^2 \leq \sum_{i=1}^{n-1} \frac{A_{i+1}}{1 - 2A_{i+1}}|\xi(t_i)|^2 + \sum_{i=1}^n \frac{2(1 + K_1k_i)t_i}{1 - 2A_i}||\xi||^2_{L^2(I_i)}$$

$$+ \sum_{i=1}^n \frac{L_2}{1 - 2A_i}\|e(\theta(t_i))\|^2_{L^2(I_i)} + \sum_{i=1}^{n-1} \frac{K_1k_{i+1}t_i}{1 - 2A_{i+1}}\|e\|^2_{L^2(0,t_{i+1})}$$

$$+ \sum_{i=1}^n \frac{K_2L_4}{1 - 2A_i}\left(\int_0^{\theta(t_i)} |e(s)| ds \right)^2_{L^2(I_i)}$$

$$\leq \frac{3L_1 + L_2 + \frac{3K_1L_3k}{\sqrt{2}} + K_1L_3 + K_2L_4 \sum_{i=1}^{n-1} k_{i+1}|\xi(t_i)|^2}{1 - \gamma}$$

$$\leq \frac{2(1 + K_1k_i)t_i}{1 - \gamma}||\xi||^2_{L^2(0,t_n)} + \frac{L_2}{1 - \gamma}\|e(\theta(t_i))\|^2_{L^2(0,t_n)}$$

$$+ \frac{K_1L_3}{1 - \gamma} \sum_{i=1}^{n-1} k_{i+1}t_i\|e\|^2_{L^2(0,t_{i+1})} + \frac{K_2L_4}{1 - \gamma}\left(\int_0^{\theta(t_i)} |e(s)| ds \right)^2_{L^2(0,t_n)}.$$
where the constant $C > 0$ depends on $q_0, \bar{K}_1, \bar{K}_2, L_1, L_2, L_3, L_4, \gamma$ and $t_n$. Inserting (40) into (35), and then using the estimates (38) and (39), we obtain

$$
\| \xi \|_{L^2(I_n)}^2 \leq C e^{C t_n} k_n \left( \| \eta \|_{L^2(I_n)}^2 + \| e \|_{L^2(I_n)}^2 \right) + C k_n \| \eta \|_{L^2(I_n)}^2 + C k_n \| e(\theta(t)) \|_{L^2(I_n)}^2 + C k_n \| e(s) \|_{L^2(I_n)}^2 \leq C k_n \| \eta \|_{L^2(I_n)}^2 + C k_n \| \xi \|_{L^2(I_n)}^2,
$$

(41)

Assume that $k_n$ is sufficiently small, then (41) can be rewritten as

$$
\| \xi \|_{L^2(I_n)}^2 \leq C k_n \| \eta \|_{L^2(I_n)}^2 + C k_n \| \xi \|_{L^2(I_n-1)}^2,
$$

or equivalently,

$$
\frac{\| \xi \|_{L^2(I_n)}^2}{k_n} \leq C \| \eta \|_{L^2(I_n)}^2 + C \sum_{i=1}^{n-1} k_i \frac{\| \xi \|_{L^2(I_i)}^2}{k_i}.
$$

(42)

Then, we apply Lemma 3.3 to (42) get

$$
\frac{\| \xi \|_{L^2(I_n)}^2}{k_n} \leq C \| \eta \|_{L^2(I_n)}^2 \exp \left( C \sum_{i=1}^{n-1} k_i \right) \leq C e^{C t_n-1} \| \eta \|_{L^2(I_n)}^2,
$$

which leads to

$$
\| \xi \|_{L^2(I_n)}^2 \leq C k_n \| \eta \|_{L^2(I_n)}^2.
$$

(43)

Summing up (43) over all element $I_i, 1 \leq i \leq n$, gives

$$
\| \xi \|_{L^2(I_n)}^2 \leq C \sum_{i=1}^{n} k_i \| \eta \|_{L^2(I_n)}^2 \leq C \| \eta \|_{L^2(I_n)}^2 \sum_{i=1}^{n} k_i \leq C t_n \| \eta \|_{L^2(I_n)}^2.
$$

This completes the proof of (29).

By choosing $\varphi = \xi'$ in (28) we find that

$$
\| \xi' \|_{L^2(I_n)}^2 \leq L_1 \| e \|_{L^2(I_n)}^2 \| \xi' \|_{L^2(I_n)} + L_2 \| e(\theta(t)) \|_{L^2(I_n)} \| \xi' \|_{L^2(I_n)} + \bar{K}_1 \| e(s) \|_{L^2(I_n)} \| \xi' \|_{L^2(I_n)} + \bar{K}_2 \| e(s) \|_{L^2(I_n)} \| \xi' \|_{L^2(I_n)},
$$

which implies

$$
\| \xi \|_{H^1(I_n)} \leq L_1 \| e \|_{L^2(I_n)} + L_2 \| e(\theta(t)) \|_{L^2(I_n)} + \bar{K}_1 \| e(s) \|_{L^2(I_n)} + \bar{K}_2 \| e(s) \|_{L^2(I_n)}.
$$
Iterating this estimate, then using (48), (39), and (29) we conclude that
\[ \|\xi\|^2_{L^2((0,t_n))} \leq C\left(\|e\|^2_{L^2((0,t_n))} + \|e(\theta(t))\|^2_{L^2((0,t_n))} + \int_0^t |e(s)| ds \right) \leq C\|\xi\|^2_{L^2((0,t_n))} + C\|\eta\|^2_{L^2((0,t_n))}, \]
which implies (30). Here, we have used the fact that
\[ \|\xi\|^2_{L^2((0,t_n))} \leq C\|\xi\|^2_{L^2((0,t_n))} + C\|\eta\|^2_{L^2((0,t_n))}. \]
Finally, combining (40) and (29) we obtain
\[ |\xi(t_n)|^2 \leq Ce^{Ct_n}\left(\|\eta\|^2_{L^2((0,t_n))} + \|\xi\|^2_{L^2((0,t_n))}\right) \leq C\|\eta\|^2_{L^2((0,t_n))}, \]
This ends the proof of (31). \(\square\)

We next bound the derivative of \(\xi\) as follows.

**Lemma 3.5.** For \(1 \leq n \leq N\), there holds
\[ \int_{I_n} |\xi'|^2(t - t_{n-1}) dt \leq Ck_n\|\eta\|^2_{L^2((0,t_n))}, \]
where the constant \(C > 0\) solely depends on \(q_0, L_1, L_2, L_3, L_4, K_1, K_2,\) and \(t_n\).

**Proof.** By selecting \(\varphi = \Pi^{n-1}(t - t_{n-1})\xi'\) in (28), we deduce that
\[ \int_{I_n} (t - t_{n-1})|\xi'|^2 dt = \int_{I_n} (f(t,u(t),u(\theta(t))) - f(t,U(t),U(\theta(t))))\Pi^{n-1}(t - t_{n-1})\xi' dt + \int_{I_n} (\nu u - \nu U)\Pi^{n-1}(t - t_{n-1})\xi' dt + \int_{I_n} (\nu_\theta u - \nu_\theta U)\Pi^{n-1}(t - t_{n-1})\xi' dt \leq L_1 \int_{I_n} |e| \cdot |\Pi^{n-1}(t - t_{n-1})\xi'| dt + L_2 \int_{I_n} |e(\theta(t))| \cdot |\Pi^{n-1}(t - t_{n-1})\xi'| dt + K_1 L_3 \int_{I_n} \left( \int_0^t |e(s)| ds \right) |\Pi^{n-1}(t - t_{n-1})\xi'| dt + K_2 L_4 \int_{I_n} \left( \int_0^{\theta(t)} |e(s)| ds \right) |\Pi^{n-1}(t - t_{n-1})\xi'| dt \]
\[ := A_1 + A_2 + A_3 + A_4. \]
Thanks to the Cauchy-Schwarz inequality and the \(L^2\)-stability of \(\Pi^{n-1}\), we have
\[ A_1 \leq L_1 \|e\|_{L^2(I_n)} \|t - t_{n-1}\|_{L^2(I_n)} \]
\[ \leq L_1 k_n^{1/2} \|e\|_{L^2(I_n)} \left\{ \int_{I_n} (t - t_{n-1})|\xi'|^2 dt \right\}^{1/2}. \]
Similarly, by [35], [41] and (39), we readily find that
\[
A_2 \leq L_2 |e(\theta(t))|_{L^2(I_n)} |(t - t_{n-1})\xi'|_{L^2(I_n)}
\]
\[
(48)
\]
\[
\leq \frac{L_2}{\sqrt{q_0}} k_n^{\frac{1}{2}} |e|_{L^2(0, t_n)} \left\{ \int_{I_n} (t - t_{n-1})|\xi'|^2 dt \right\}^\frac{1}{2},
\]
\[
A_3 \leq \tilde{K}_1 L_3 \int_0^t |e(s)|_{L^2(I_n)} \|(t - t_{n-1})\xi'|_{L^2(I_n)}
\]
\[
(49)
\]
\[
\leq \frac{\tilde{K}_1 L_3 t_n}{\sqrt{2}} k_n^{\frac{1}{2}} |e|_{L^2(0, t_n)} \left\{ \int_{I_n} (t - t_{n-1})|\xi'|^2 dt \right\}^\frac{1}{2},
\]
and
\[
A_4 \leq \tilde{K}_2 L_4 \int_0^t |e(s)|_{L^2(I_n)} \|(t - t_{n-1})\xi'|_{L^2(I_n)}
\]
\[
(50)
\]
\[
\leq \frac{\tilde{K}_2 L_4 t_n}{\sqrt{2}} k_n^{\frac{1}{2}} |e|_{L^2(0, t_n)} \left\{ \int_{I_n} (t - t_{n-1})|\xi'|^2 dt \right\}^\frac{1}{2}.
\]
Hence, combining (16)–(50) and (29) yields
\[
\left\{ \int_{I_n} |\xi'|^2 (t - t_{n-1}) dt \right\}^\frac{1}{2} \leq C k_n^{\frac{1}{2}} |e|_{L^2(0, t_n)} \leq C k_n^{\frac{1}{2}} \left( \|\eta\|_{L^2(0, t_n)} + \|\xi\|_{L^2(0, t_n)} \right)
\]
\[
\leq C k_n^{\frac{1}{2}} \|\eta\|_{L^2(0, t_n)}.
\]
This implies the assertion.

We also need the following inverse inequality (cf. [9]).

**Lemma 3.6.** On each interval \( I_n \) there holds
\[
\| \varphi \|^2_{L^\infty(I_n)} \leq C \left( \log(r_n + 1) \int_{I_n} \varphi'(t)^2 (t - t_{n-1}) dt + \varphi(t_n)^2 \right)
\]
for any \( \varphi \in P_{r_n}(I_n) \), where the constant \( C > 0 \) is independent of \( k_n \) and \( r_n \). Moreover, the estimate cannot be improved asymptotically as \( r_n \to \infty \).

The following results state abstract error bounds of the CPG method.

**Theorem 3.2.** Let \( u \) be the exact solution of (1) and \( U \) be the \( h-p \) CPG approximation defined by (2). For \( k \) sufficiently small, there holds
\[
\| u - U \|_{L^2(I)} \leq C \| u - \mathcal{I} u \|_{L^2(I)},
\]
\[
(51)
\]
\[
\| u - U \|_{H^1(I)} \leq C \| u - \mathcal{I} u \|_{H^1(I)},
\]
\[
(52)
\]
\[
\| u - U \|_{L^\infty(I)} \leq C \left( 1 + k \log(r + 1) \right) \frac{1}{2} \| u - \mathcal{I} u \|_{L^\infty(I)},
\]
\[
(53)
\]
where \( r = \max \{ r_n \}_{n=1}^N \) and the constants \( C > 0 \) solely depend on \( q_0, L_1, L_2, L_3, L_4, \tilde{K}_1, \tilde{K}_2, \) and \( T \).

**Proof.** With the aid of (29) we get
\[
\| u - U \|_{L^2(I)} \leq \| \eta \|_{L^2(I)} + \| \xi \|_{L^2(I)} \leq C \| \eta \|_{L^2(I)},
\]
which completes the proof of (51). Similarly, by (30) we obtain
\[
\| u - U \|_{H^1(I)} \leq \| \eta \|_{H^1(I)} + \| \xi \|_{H^1(I)} \leq \| \eta \|_{H^1(I)} + C \| \eta \|_{L^2(I)} \leq C \| \eta \|_{H^1(I)},
\]
which implies (52).
By using (31) and employing Lemmas 3.5 and 3.6, we deduce that
\[
\|u - U\|_{L^\infty(I)}^2 \leq 2\|\eta\|_{L^\infty(I)}^2 + 2 \max_{1 \leq n \leq N} \|\xi\|_{L^\infty(I_n)}^2 \leq 2 \|\eta\|_{L^\infty(I)}^2 + C \max_{1 \leq n \leq N} \left\{ \log(r_n + 1) \int_I |\xi'(t)|^2 (t - t_{n-1}) dt + |\xi(t_n)|^2 \right\} \leq 2 \|\eta\|_{L^\infty(I)}^2 + C \max_{1 \leq n \leq N} \left\{ k_n \log(r_n + 1) \|\eta\|_{L^2(I_n)}^2 + \|\eta\|_{L^2(I_n)}^2 \right\} \leq 2 \|\eta\|_{L^\infty(I)}^2 + Ck \log(r + 1) \|\eta\|_{L^2(I)}^2 + C \|\eta\|_{L^2(I)}^2 \]

Moreover, if \( u \in H^1(I) \) satisfies \( u|_{I_n} \in H^{n, n+1}(I_n) \) for \( s_{0,n} \geq 0 \), then \( C \) sufficiently small, there holds
\[
\|u - U\|_{L^2(I)}^2 = \sum_{n=1}^{N} \left( \frac{k_n}{2} \right)^{2s+n} \Gamma(r_n + 1 - s_n) \|u\|_{H^{n+1}(I_n)}^2 \],
\[
|u - U|_{H^1(I)}^2 = \sum_{n=1}^{N} \left( \frac{k_n}{2} \right)^{2s+n} \Gamma(r_n + 1 - s_n) \|u\|_{H^{n+1}(I_n)}^2 \]

for any real \( s_n \geq 0 \), \( 0 \leq s_n \leq \min\{r_n, s_{0,n}\} \).
Moreover, if \( u \in H^1(I) \) satisfies \( u|_{I_n} \in W^{s_{0,n}+1, \infty}(I_n) \) for \( s_{0,n} \geq 0 \), there holds
\[
\|u - U\|_{L^\infty(I)}^2 \leq C(1 + k \log(r + 1)) \max_{1 \leq n \leq N} \left\{ \left( \frac{k_n}{2} \right)^{2s_n+2} \Gamma(r_n + 1 - s_n) \|u\|_{W^{s+n+1, \infty}(I_n)}^2 \right\} \]

for any real \( s_n \geq 0 \), \( 0 \leq s_n \leq \min\{r_n, s_{0,n}\} \).

**Proof.** The assertions follow readily from Theorem 3.2 and Lemma 3.2. 

**Remark 3.1.** These estimates show that the error bounds are explicit with respect to the time steps \( k_n \), the approximation order \( r_n \), and the regularity of the exact solution \( s_n \).

From the error bounds in Theorem 3.3, the following convergence rates can be obtained for the \( h \)- and \( p \)-version of the CPG method.

**Corollary 3.1.** Let \( r_n = r, 1 \leq n \leq N \) and \( \mathcal{T}_h \) be a quasi-uniform mesh in \( I \). If \( u \in H^{s+1}(I) \) for \( s \geq 0 \), then
\[
\|u - U\|_{L^2(I)}^2 \leq C\left( \frac{\min\{s,r\} + 1}{p+1} \right)^{p+1} \|u\|_{H^{s+1}(I)},
\]
\[
|u - U|_{H^1(I)} \leq C\left( \frac{\min\{s,r\}}{p} \right)^p \|u\|_{H^{s+1}(I)}.
\]
Moreover, if \( u \in W^{s+1,\infty}(I) \), there holds

\[
\|u - U\|_{L^\infty(I)} \leq C \left(1 + k \log(r + 1)\right)^{\frac{1}{2}} \frac{\min\{s,r\} + 1}{r^s} \|u\|_{W^{s+1,\infty}(I)}.
\]

**Proof.** The assertions follows from Theorem 3.3 and Stirling’s formula. \( \square \)

**Remark 3.2.** These estimates show that the \( h \)-version CPG method converges either as the time step \( k \) is decreased or as the polynomial degrees \( r \) is increased. Moreover, the \( p \)-version (with fixed time partition) can yields arbitrarily high-order algebraic convergence rates (i.e., spectral convergence) if the solution \( u \) is smooth enough. Moreover, it can be proved that the \( p \)-version converges exponentially if \( u \) is analytic on \([0,T]\) (see, for instance, [11]).

4. Numerical experiments

In this section, we illustrate the performance of the \( h \)-version of the CPG method for the following VFIDE:

\[
\begin{aligned}
\frac{d}{dt} u(t) &= g(t) + e^{-u(t)} + e^{-t} e^{-u(\theta(t))} + \int_0^t e^{s-t}(u(s) + e^{-u(s)}) ds \\
&\quad + \int_0^{\theta(t)} e^{s-t}(u(s) + e^{-u(s)}) ds, \quad t \in [0,1], \\
\end{aligned}
\]

with \( g(t) = -\ln(t + e) + 2e^{-t} - e^{\theta(t)-t} \ln(\theta(t) + e) - \frac{e^{-t}}{\theta(t) + e} \) and \( \theta(t) = \frac{1}{8} \sin(t) \) such that the exact solution \( u(t) = \ln(t + e) \).

![Figure 1. \( L^\infty \)-errors of the \( h \)-version.](image)

We begin by considering the behaviour of the \( h \)-version of the CPG method on uniform time partitions for problem (54). The \( L^\infty \)-errors are shown in Fig. 1. Obviously, the straight error curves correspond to algebraic convergence in the step-size \( k \), for each polynomial degree \( r \). Moreover, we list the \( L^2 \), \( H^1 \)-(seminorm), and \( L^\infty \)-errors of the \( h \)-version CPG method in Table 1; the convergence rates confirm the sharpness prediction in Corollary 3.1.

In Fig. 2 we present the \( L^\infty \)-errors of the \( p \)-version of the CPG method. The results show that exponential rates of convergence are achieved for each fixed uniform time partitions. In addition, we note that the global \( L^\infty \)-error of \( 10^{-15} \) can be obtained with less than 15 degrees of freedom for the \( p \)-version. However, this is not the case for the \( h \)-version as shown in Fig. 1. This implies that, for smooth
solution it is advantageous to increase $r$ and keep $k$ fixed ($p$-version of the CPG method) rather than to reduce $k$ for $r$ fixed ($h$-version of the CPG method).

5. Concluding Remarks

In this paper, we have presented an $h$-$p$ version of the CPG method for the nonlinear VFIDEs with vanishing delays. We have proved that the CPG scheme is well-defined as long as the time steps are sufficiently small. Moreover, we have obtained a priori error bounds in the $L^2$-, $H^1$- and $L^\infty$-norms that are explicit with respect to the local time steps, the local approximation orders, and the local regularity of the exact solutions. Extensions of the analysis presented herein to the $h$-$p$ version of the CPG method might be possible for VFIDEs with weakly singular kernels by following along the lines of this paper, in conjunction with our recent work [20] for Volterra integro-differential equations with weakly singular kernels. This will be a topic for our future research.

Acknowledgments

This research is supported by the NSF of China (Nos. 11771298 and 11301343) and the NSF of Shanghai (No. 15ZR1430900).

References


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