A NOVEL ADAPTIVE FINITE VOLUME METHOD FOR ELLIPTIC EQUATIONS

YANHUI ZHOU AND QINGSONG ZOU

Abstract. In this paper, we propose a novel adaptive finite volume method (AFVM) for elliptic equations. As a standard adaptive method, a loop of our method involves four steps: Solve → Estimate → Mark → Refine. The novelty of our method is that we do not have the traditional “completion” procedure in the Refine step. To guarantee the conformity, a triangular element with a hanging node is treated as a quadrilateral element, and the corresponding function space consists of the bilinear functions. The optimal computational complexity of our AFVM is validated by numerical examples.

Key words. Adaptive finite volume method, hanging nodes, hybrid meshes, error analysis.

1. Introduction

The finite volume method (FVM) is a popular numerical tool for partial differential equations (PDEs), cf. [1, 2, 5, 13–18, 20–22, 24–26, 28, 29, 31–33, 37]. Recently, the adaptive finite volume method (AFVM) attracts a lot of attention, see [6, 8, 11, 12, 19, 23, 27, 30, 36]. Especially the a posteriori error estimator has been studied in many papers, see [36] for the hierarchical type and [6, 11, 12, 23, 27, 30] the residual type error estimators.

In this paper, we design a novel AFVM for elliptic equations. Unlike the previous works which pay a lot of attention on the construction of a posteriori error estimators for the FVM, here we construct our novel adaptive method modifying the adaptive strategy. It is known that an iteration of a standard adaptive method involves four steps: Solve → Estimate → Mark → Refine. In particular, in the Refine step, after having refined the mesh according to the previous marking step, we should refine more elements to eliminate the so-called “hanging nodes”. The novelty of our method is that we do not have the traditional “completion” procedure in the Refine step which allow hanging nodes. In order to guarantee the conformity, a triangular element with one hanging node will not be divided if the edge with the hanging node is not a base of the triangle. We consider this triangle with the hanging node as a quadrilateral. We only divide a triangular element has a hanging node on the base, or has more than one hanging nodes on edges. As a consequence, our meshes in AFVM consist of hybrid triangular and quadrilateral elements, to ensure the conformity of our trial function of the finite volume scheme in the Solve step, we let the trial function on triangular element be constructed as the linear function and on quadrilateral element (a triangular element with a hanging node) as the isoparametric bilinear. In other words, we define the trial function as

\[
\begin{align*}
\text{linear function,} & \quad \text{if a triangular element has no hanging node} \\
\text{isoparametric bilinear function,} & \quad \text{if a triangular element has a hanging node}
\end{align*}
\]


Received by the editors March 5, 2017 and, in revised form, June 2, 2017.

2000 Mathematics Subject Classification. 35R35, 49J40, 60G40.
One may naturally ask a question: how about the actual significance of the novel AFVM? It is known that to keep the conformity, in the Refine step, one often needs an additive “completion” procedure [3] to eliminate the hanging nodes. Our numerical experiments show that the novel AFVM decrease the steps of bisection for the conformity compare to the standard AFVM which needs traditional “completion” procedure. Moreover, our AFVM possesses the local conservation property. Furthermore, suppose $u$ is the exact solution of the elliptic equation, our numerical results show that

$$|u - u_k| \leq C_1 N_k^{-1/2} \text{ and } ||u - u_k||_0 \leq C_2 N_k^{-1},$$

where $u_k$ and $N_k$ are the FVM solution and the number of elements of the mesh to $k$-th iteration, $C_1$ and $C_2$ are two constants. We note that in our numerical example, the optimal convergence order of $H^1$ and $L^2$ errors can be obtained even if $u \in H^{1+\frac{4}{p}+\varepsilon}(\Omega)$ for all $\varepsilon > 0$ and $u \notin H^2(\Omega)$, and the domain $\Omega$ is not convex.

The main idea of our AFVM can be applied to general elliptic equations. To illustrate the basic idea, we focus on the model problem

\begin{equation}
-\nabla \cdot (\alpha \nabla u) = f \text{ in } \Omega,
\end{equation}

\begin{equation}
u = 0 \text{ on } \partial \Omega,
\end{equation}

where $\Omega \subset \mathbb{R}^2$ is a convex polygon domain, $\alpha$ is a piecewise continuous function that bounded below: There exists a positive constant $\alpha_0 > 0$ such that $\alpha(x) \geq \alpha_0$ for almost all $x \in \Omega$, and $f$ is a real valued function defined on $\Omega$. The stability analysis of our finite volume scheme is a routine under the framework of linear on triangular element [29] and isoparametric bilinear on quadrilateral element [25], and we obtain the optimal convergence rate of $H^1$ and $L^2$ norms.

The rest of this paper is organized as follows. In the next section we introduce a FVM on hybrid triangular and quadrilateral meshes. The optimal convergence properties are studied both in $H^1$ and $L^2$ spaces. Our novel AFVM is presented in Section 3. Numerical examples are provided in Section 4 to validate that our novel AFVM has optimal computational cost and the theoretical results of our FVM.

We end this section with some notations. For an integer $m \geq 0$ and $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ denote the standard Sobolev spaces of functions that have generalized derivatives up to order $m$ in $L^p(\Omega)$. The norm (or semi-norm) is defined by $\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_p^p\right)^{1/p}$ for $1 \leq p < \infty$, and with the standard modification for $p = \infty$. $H^m(\Omega) := W^{m,2}(\Omega)$ and $H^1_0(\Omega)$ denote the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial \Omega$. For simplicity, in the rest of the paper we will omit the subscript index when $p = 2$ and the domain index $\Omega$ if needed. Furthermore, $(\cdot, \cdot)$ denotes the standard $L^2(\Omega)$-inner product. To avoid writing constants repeatedly, “$A \lesssim B$” means that $A$ can be bounded by $B$ multiplied by a constant which is independent of the parameters which $A$ and $B$ may depend on, “$A \gtrsim B$” means that $B$ can be bounded by $A$. “$A \sim B$” means “$A \lesssim B$” and “$B \lesssim A$”.

2. A finite volume method on hybrid meshes

2.1. A hybrid triangular and quadrilateral mesh. We partition $\Omega$ into a mesh $\mathcal{T}_h$ consisting of a finite number of triangles and convex quadrilaterals, where $h$ is the largest diameter of all triangles and quadrilaterals, and we call $\mathcal{T}_h$ the primal mesh of $\Omega$, see Fig.1. We denote by $\mathcal{N}_h$ the set of all vertices of $\mathcal{T}_h$, and let $\mathcal{N}_h^\circ = \mathcal{N}_h \setminus \partial \Omega$ be the set of all interior vertices.
Suppose \( P_i (i = 1, 2, 3, 4) \) are four vertices of quadrilateral \( \tau_Q \in \mathcal{T}_h \), \( M_i \) is the midpoint of the edge \( \overline{P_i P_{i+1}} \) (\( i = 1, 2, 3, 4 \)). We denote \( Q \) as the intersection point of \( \overline{M_1M_3} \) and \( \overline{M_2M_4} \), and call point \( Q \) as the averaging center of \( \tau_Q \). We denote \( S_Q \) and \( d_Q \) as the area of \( \tau_Q \) and the length \( |\overrightarrow{P_1P_2} + \overrightarrow{P_3P_4}| \) respectively. We note that the length \( d_Q \) equals to the distance vector of the diagonal midpoints of \( \tau_Q \). Moreover, if \( \tau_Q \) is a parallelogram, then \( d_Q = 0 \).

We call a partition \( \mathcal{T}_h \) conform if different triangles and quadrilaterals in \( \mathcal{T}_h \) have no common interior points, and a vertex of any triangle or quadrilateral does not lie on the interior of an edge of any other triangle or quadrilateral.

A partition \( \mathcal{T}_h \) is called shape regular provided the following conditions are satisfied:

(i) For each triangle \( \tau_T \in \mathcal{T}_h \), there exists a positive constant \( c_1 \) independent of \( h \) such that \( \frac{\rho_T}{h_T} \leq c_1 \), where \( h_T \) is the largest diameter of \( \tau_T \) and \( \rho_T \) is the maximum diameter of circles contained in \( \tau_T \).

(ii) For each quadrilateral \( \tau_Q \in \mathcal{T}_h \), the ratio of the maximal edge to minimal edge lengths is bounded by a constant, and the four angles are bigger than \( \theta_0 \) and smaller than \( \pi - \theta_0 \) with some \( \theta_0 > 0 \).

We call a partition \( \mathcal{T}_h \) quasi uniform if \( \mathcal{T}_h \) is shape regular and satisfies the following conditions:

(i) For each triangle \( \tau_T \in \mathcal{T}_h \), there exists a positive constant \( c_2 \) independent of \( h \) such that \( \frac{\rho_T}{h_T} \leq c_2 \).

(ii) For each quadrilateral \( \tau_Q \in \mathcal{T}_h \), \( S_Q \sim h^2 \), \( d_Q \lesssim h^2 \).

2.2. A finite volume scheme on hybrid meshes. Our finite volume scheme is based on the framework of Petrov-Galerkin method. We first construct the trial function space. For any quadrilateral \( \tau_Q \in \mathcal{T}_h \), there exists a unique invertible bilinear transformation \( F_{\tau_Q} \) maps the reference unit square \( \widehat{\tau_Q} = [0,1]^2 \) to \( \tau_Q \) (cf. [4]). With respect to \( \mathcal{T}_h \), we define the trial function space

\[
U_h = \{ v_h \in C(\Omega) : \quad v_h|_{\tau_T} \in \mathcal{P}_1, \quad v_h|_{\tau_Q} \in \mathcal{Q}_1(\widehat{\tau_Q}) \circ F_{\tau_Q}^{-1}, \quad \forall \tau_T \in \mathcal{T}_h, \quad \forall \tau_Q \in \mathcal{T}_h : v_h|_{\partial \Omega} = 0 \},
\]

where \( \mathcal{P}_1 \) and \( \mathcal{Q}_1 \) are the sets of all polynomials and bi-polynomials of degree no more than 1 respectively, \( \tau_T \) and \( \tau_Q \) are the triangular and quadrilateral element of \( \mathcal{T}_h \), \( F_{\tau_Q}^{-1} \) denotes the inverse of the transformation \( F_{\tau_Q} \), and \( \mathcal{Q}_1(\widehat{\tau_Q}) \circ F_{\tau_Q}^{-1} \) is a composite function. From the definition of \( U_h \), we have

\[
\dim U_h = \#\Lambda_h^0,
\]
Now, we proceed to present the dual partition of $\Omega$ and the corresponding test function space. The dual partition is another mesh of $\Omega$, and the test function space is based on the dual mesh. The dual partition of $\Omega$ is constructed as follows. For each vertex $P$ of $\mathcal{T}_h$ (see Fig. 2), the point $P$ is a common vertex of $\triangle PP_3P_4$, $\triangle PP_4P_5$, $\triangle PP_5P_6$, and $Q_1, Q_4$ be the averaging centers of $PP_1P_2, PP_5P_6$; and $Q_2, Q_3, Q_5$ be the barycenters of $\triangle PP_3P_4$, $\triangle PP_4P_5$, $\triangle PP_5P_6$. Then we connect points $M_1, M_3, M_4, M_5, M_7$ be the midpoints of edges $PP_1, PP_3, PP_5, PP_7$. Then we connect points $M_1, M_3, M_4, M_5, M_7, Q_5, M_1$ successively to obtain a polygonal domain $V_P$ surrounding point $P$, and call $V_P$ a control volume with respect to the vertex $P$. Let $\mathcal{T}'_h$ be a dual partition of $\Omega$ if $\mathcal{T}'_h$ is constructed by all control volumes $V_P$, $P \in \mathcal{N}_h$. In other words, we define

$$\mathcal{T}'_h = \{V_P : P \in \mathcal{N}_h\}.$$ 

The corresponding test function space $V_h$ contains all the piecewise constant functions with respect to $\mathcal{T}'_h$ defined as

$$V_h = \text{Span}\{\psi_{V_P} : P \in \mathcal{N}_h\},$$

where $\psi_{V_P}$ is the characteristic function on $V_P$. Similar to (4), we have

$$(5) \quad \dim V_h = \#\mathcal{N}_h^0.$$ 

It follows from (4) and (5) that

$$(6) \quad \dim U_h = \dim V_h = \#\mathcal{N}_h^0.$$ 

We are now ready to describe the finite volume scheme on hybrid meshes. The finite volume solution of Eqs. (1) and (2) is a function $u_h \in U_h$ which satisfies the following conservation law

$$(7) \quad - \int_{\partial V_P} \alpha \frac{\partial u_h}{\partial n} \, ds = \int_{V_P} f \, dx \, dy$$

on each control volume $V_P$, $P \in \mathcal{N}_h^0$, where $n$ is the unit outward normal on the boundary $\partial V_P$. Let $w_h \in V_h$, and $w_h$ can be written as

$$w_h = \sum_{P \in \mathcal{N}_h^0} w_P \psi_{V_P},$$
where the coefficients \( w_P, P \in \mathcal{N}_h^0 \) of \( w_h \) are constants. Multiplying (7) with \( w_P \) and then summing up for all \( P \in \mathcal{N}_h^0 \), then we obtain

\[
- \sum_{P \in \mathcal{N}_h^0} w_P \int_{\partial V_P} \alpha \frac{\partial u_h}{\partial n} \, ds = \int_{\Omega} f w_h \, dx dy.
\]

Define the finite volume scheme bilinear form for all \( v \in H^1_0(\Omega), w_h \in V_h \) as

\[
a_h(v, w_h) = - \sum_{P \in \mathcal{N}_h^0} w_P \int_{\partial V_P} \alpha \frac{\partial u_h}{\partial n} \, ds.
\]

The finite volume scheme for solving Eqs. (1) and (2) reads as: Find \( u_h \in U_h \) such that

\[
(8) \quad a_h(u_h, w_h) = (f, w_h), \; \forall w_h \in V_h.
\]

2.3. Stability and convergence. From (6) we define a linear bijective mapping \( \Pi_h \): \( v_h \in U_h \rightarrow V_h \) such that

\[
\Pi_h v_h(P) = v_h(P), \; \forall P \in \mathcal{N}_h.
\]

**Theorem 2.1.** Suppose \( T_h \) is a conform and shape regular hybrid mesh, and the coefficient \( \alpha \) is a piecewise constant with respect to \( T_h \), then

\[
(9) \quad a_h(v_h, \Pi v_h) \gtrsim |v_h|_{1, \Omega}^2, \; \forall v_h \in U_h.
\]

Consequently, Let \( u \in H^1_0(\Omega) \cap H^2(\Omega) \) be the exact solution of (1) and (2), \( u_h \in U_h \) be the numerical solution of (8). We have

\[
(10) \quad |u - u_h|_1 \lesssim h |u|_2.
\]

Furthermore, if \( T_h \) is a quasi uniform hybrid mesh and \( f \in H^1(\Omega), \alpha \in W^{2, \infty}(\Omega) \), then

\[
(11) \quad \|u - u_h\|_0 \lesssim h^2 \|f\|_1.
\]

**Proof.** We have the coercivity of triangular [29] and quadrilateral [25] meshes respectively, then summing up them we obtain (9). The \( H^1 \) error estimate (10) is a routine by (9). Similar to the \( L^2 \) error estimate on triangular [13] and quadrilateral [21] meshes, (11) can be proved by Aubin-Nitsche technique [4, 9]. \( \square \)

**Remark 2.1.** In the Section 4, numerical example 4.1 shows that we also obtained the optimal convergence rate of \( L^2 \) norm even if \( T_h \) is not quasi uniform; examples 4.2 and 4.3 show that the above FVM is stable even if \( T_h \) is not shape regular.

Now, we are ready to introduce a novel hybrid mesh. In Fig.3, there are three triangles \( \triangle P_1P_2P_4, \triangle PP_2P_3 \) and \( \triangle PP_3P_4 \), it is a triangular mesh. However, this mesh can not ensure the conformity since there exists a hanging node \( P \). To guarantee the conformity, we consider this triangular grid as a novel hybrid mesh when we treat \( \triangle P_1P_2P_4 \) with the hanging node \( P \) as a quadrilateral \( P_1P_2P_2P \). This new feature is great useful to the following adaptive finite volume method since it allow hanging nodes.
3. A novel adaptive finite volume method

In this section, we present a novel adaptive finite volume method (AFVM) for (1) and (2). Our AFVM is a loop involves: Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine. The novelty of our method is that we do not have the traditional “completion” procedure in the Refine step. Let $T_k$ be the $k$-th iteration mesh of $\Omega$, and suppose $T_1$ is a triangular mesh. For convenience, we denote $u_k := u_{T_k}$, $\eta_k(u_k, \tau)_{\tau \in T_k}$, $M_k := M_{T_k}$, and $N_k$ as the number of elements of the mesh $T_k$.

Firstly, we discuss the Solve subroutine. For the mesh $T_k$, the subroutine $u_k = \text{Solve}(T_k)$ outputs the finite volume solution $u_k$ of (1) and (2). We note that $T_k$ ($k \geq 2$) may be hybrid triangular and quadrilateral meshes, since in the Refine step there produce quadrilateral element (a triangular element with a hanging node). Therefore, the trial function space of mesh $T_k$ ($k \geq 2$) should be linear function on triangular element and bilinear on quadrilateral, which is different from $T_1$. For example, in Fig.3, it should be linear on triangular elements $\triangle P_1P_2P_3$ and $\triangle P_2P_3P_4$, and bilinear on quadrilateral $P_1P_2P_3P_4$. Moreover, for the mesh $T_k$ ($k \geq 2$), the trial function space can be rewritten as

$$U_h = \{ v_h \in C(\Omega) : \begin{cases} v_h|_{\tau} \in P_1, & \text{if } \tau \text{ has no hanging node} \\ v_h|_{\tau} \in Q_1(\hat{\tau}) \circ F^{-1}_{\tau}, & \text{if } \tau \text{ has a hanging node} \end{cases}, \forall \tau \in T_h \}.$$ 

Secondly, we present gradient recovery error estimator for the finite volume solution $u_k$ and the corresponding mesh $T_k$. Define $G : u_k \in U_h \rightarrow G_{u_k} \in (W_h)^2$ be a Zienkiewicz-Zhu [34, 35] type gradient recovery operator, such that for arbitrary vertex $P \in N_h$

$$G_{u_k}(P) = \frac{1}{|w_P|} \iint_{w_P} \nabla u_k \, dx \, dy,$$

where

$$W_h := \{ v_h \in C(\Omega) : \begin{cases} v_h|_{\tau} \in P_1, & \text{if } \tau \text{ has no hanging node} \\ v_h|_{\tau} \in Q_1(\hat{\tau}) \circ F^{-1}_{\tau}, & \text{if } \tau \text{ has a hanging node} \end{cases}, \forall \tau \in T_h \},$$

and $|w_P|$ is the area of $w_P$. Then the subroutine

$$\eta_k(u_k, \tau)_{\tau \in T_k} = \text{Estimate}(T_k, u_k)$$

outputs the gradient recovery local indicator defined by

$$\eta_k(u_k, \tau) := \| G_{u_k} - \nabla u_k \|_{0, \tau}, \forall \tau \in T_k.$$
Thirdly, we use the marking strategy proposed by Dörfler [10] to mark $T_k$. For arbitrary subset $S$ of the mesh $T_k$, we define

$$\eta^2_k(u_k, S) := \sum_{\tau \in S} \eta^2_k(u_k, \tau)$$

as the global error estimator on $S$. Let the local indicators $\eta_k(u_k, \tau), \tau \in T_k$ ordered from the largest to the smallest according to their size, then we choose a subset $M_k \subset T_k$ with minimal cardinality such that

$$\eta^2_k(u_k, M_k) \geq \theta \eta^2_k(u_k, T_k),$$

where $\theta \in (0, 1)$ is a given constant. In short, the subroutine

$$M_k = \text{Mark}(\eta_k(u_k, \tau), \tau \in T_k, \theta)$$

outputs the subset $M_k$ which satisfies (12).

In the last Refine step, we use the longest edge bisection [7]. There are three procedures to accomplish this step. Firstly, marking the subset $M_k$ for reducing the error. Secondly, marking edges for the conformity. Finally, refining the mesh $T_k$. We include two parts to interpret them below. The first part illustrate how to implement them in the first loop of our AFVM. The second part describe the difference between the first and remainder loops of AFVM in the Refine step.

For each triangular element $\tau \in T_1$, we label the longest edge of $\tau$ as base or refinement edge. The opposite vertex of the base is called peak. Firstly, for each triangle $\tau \in M_1 \subset T_1$, we mark the base, and the triangle is bisected to two new children triangles by connecting the peak to the midpoint of the base for reducing the error, and connect the midpoint of the base to the midpoint of the marked edges for the conformity (Fig.4). After all the triangles of subset $M_1$ are refined, there need more bisections to eliminate the hanging nodes for conformity in general. Our strategy is that for each triangle $\tau \in T_1 \setminus M_1$, we divide it as above if there has a hanging node on the base; otherwise we mark the base if and only if the two shortest edges of $\tau$ are marked, and then divide the triangle to four new children triangles by connecting the peak and the midpoints of the two shortest edges to the midpoint of the base for conformity respectively; if there is only one hanging node which not on the base, we consider the triangle $\tau$ with the hanging node as a quadrilateral and need not completion. For example, in Fig.5(a)(b), we treat the triangle $p(1)p(2)p(3)$ with the hanging node $p(4)$ as a quadrilateral; and in Fig.5(c) we mark the base $p(2)p(3)$ since the two shortest edges $p(1)p(2)$ and $p(1)p(3)$ being marked. We mention that in Fig.3 the dashed line shows the control volume $V_P$ of the hanging node $P$, and the contribution of quadrilateral $P_2P_1P_3P$ is $\Delta Q_1M_2M_4$, which is different from Fig.2.

After the first loop of AFVM, there should generate quadrilateral elements (a triangular element with a hanging node) in mesh $T_k$ ($k \geq 2$). Therefore, we need to refine triangular and quadrilateral elements for reducing the error and ensuring
Figure 5. Divide a triangle $p(1)p(2)p(3)$ for the conformity that allow hanging node.

Figure 6. Divide a quadrilateral $p(1)p(2)p(3)p(4)$ for reducing the error.

Figure 7. Divide a quadrilateral $p(1)p(2)p(3)p(4)$ for the conformity that allow hanging node.

the conformity in the \textit{Refine} step. The unique difference is how to divide a quadrilateral element compare to the first loop. Now, we proceed to present it. The concepts of \textit{base} and \textit{peak} of a quadrilateral are similar to a triangle, for example in Fig.6, vertex $p(1)$ and edge $p(3)p(4)$ are the peak and base to the quadrilateral $p(1)p(2)p(3)p(4)$. For each quadrilateral $\tau \in \mathcal{M}_k \subset \mathcal{T}_k$ ($k \geq 2$), we mark the base, and the quadrilateral is divided to three new children triangles by connecting the peak and hanging node to the midpoint of the base for reducing the error (Fig.6(a)); for conformity we connect the midpoint $p(5)$ of the base to the midpoint $p(6)$ if the edge $p(1)p(4)$ is marked (Fig.6(b)), and treat the hanging node as a new vertex if $p(1)p(2)$ or $p(2)p(3)$ is marked (Fig.6(c)(d)) and need not completion. For each quadrilateral $\tau \in \mathcal{T}_k \setminus \mathcal{M}_k$ ($k \geq 2$), in order to ensure the conformity, we mark the base as long as one edge of $\tau$ is marked, and connect the hanging node $p(2)$ to the midpoint of the base $p(3)p(4)$ (Fig.7); we connect the midpoint of the edge $p(1)p(4)$ to the midpoint of the base $p(3)p(4)$ if $p(1)p(4)$ is marked (Fig.7(a)), and treat the hanging node of $p(1)p(2)$ or $p(2)p(3)$ as a new vertex if it is marked (Fig.7(b)(c)) and need not completion. Then, the subroutine

$$\mathcal{T}_{k+1} = \text{Refine}(\mathcal{M}_k, \mathcal{T}_k)$$

outputs hybrid triangular and quadrilateral mesh $\mathcal{T}_{k+1}$ which allow hanging nodes.

In short, the novelty of our AFVM is that we do not have the traditional “completion” procedure in the \textit{Refine} step. To ensure the conformity, a triangular element with a hanging node is treated as a quadrilateral element.
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To illustrate the practicability and effectiveness of our AFVM, we are interested in the following type result

\[ |u - u_k|_1 \lesssim N_k^{-1/2} \text{ and } \|u - u_k\|_0 \lesssim N_k^{-1}. \]

The numerical results in the next section show that the optimal convergence rate with the above norms can be obtained, even if the exact solution \( u \) has lower regularity and with the \( \Omega \) non convex.

4. Numerical Results

In this section, we present example 4.1 to validate our above theoretical results of FVM, examples 4.2 and 4.3 to the AFVM.

**Example 4.1.** We consider the problem (1), (2) with \( \Omega = [0,1]^2 \) and \( \alpha = 1 \). We choose the exact solution as

\[ u(x,y) = \sin(\pi x) \sin(\pi y). \]

Two kinds of hybrid triangular and quadrilateral meshes are implemented. The first kind of meshes are hybrid triangular and rectangular (Fig.8(a)), the second kind of meshes are hybrid triangular and trapezoidal (Fig.8(b)). The numerical results are presented in Table 1 and Table 2. In these two tables, the first column \( N \) indicates the number of elements along the \( x \)-direction. For example, in Fig.8(a) \( N = 4 \), and in Fig.8(b) \( N = 8 \). We see that the optimal accuracy order can be obtained under \( L^2 \) and \( H^1 \) norms which are consistent with our theoretical results. Furthermore, we also give the accuracy order under \( L^\infty \) and \( W^{1,\infty} \) norms. We mention that the second kind of meshes are not quasi uniform, since the distance vector of the diagonal midpoints of the trapezoids are \( O(h) \), not \( O(h^2) \).

**Example 4.2.** We consider the elliptic equation

\[ -\nabla \cdot (\alpha \nabla u) = f \]

on the domain \( \Omega = \{ -1 \leq x \leq 1, -1 \leq y \leq 1 \} \). We choose \( \alpha = 1 \) and the exact solution as

\[ u(x,y) = e^{-(x^2+y^2)/0.01}. \]

We note that the boundary condition is inhomogeneous for this equation.

We implement our AFVM to this problem, choose the initial triangulation \( T_1 \) as in Fig.9(a), the iteration number \( n = 30 \) and parameter \( \theta = 0.3 \). Fig.9(b) shows...
Table 1. Error and convergence order on hybrid triangular and rectangular meshes.

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<th>$H^1$ error</th>
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<td>0.97</td>
<td>8.0e-003</td>
<td>1.85</td>
<td>2.6e-001</td>
<td>0.99</td>
</tr>
<tr>
<td>32</td>
<td>6.9e-004</td>
<td>1.98</td>
<td>8.8e-002</td>
<td>0.98</td>
<td>2.1e-003</td>
<td>1.91</td>
<td>1.4e-001</td>
<td>0.91</td>
</tr>
<tr>
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<td>4.4e-002</td>
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<td>7.0e-002</td>
<td>0.97</td>
</tr>
<tr>
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<td>4.3e-005</td>
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<td>2.2e-002</td>
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<td>1.98</td>
<td>3.5e-002</td>
<td>0.99</td>
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</table>

Table 2. Error and convergence order on hybrid triangular and trapezoidal meshes.

<table>
<thead>
<tr>
<th>N</th>
<th>$L^2$ error</th>
<th>Order</th>
<th>$H^1$ error</th>
<th>Order</th>
<th>$L^\infty$ error</th>
<th>Order</th>
<th>$W^{1,\infty}$ error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.0e-001</td>
<td></td>
<td>1.1e+000</td>
<td></td>
<td>2.5e-001</td>
<td></td>
<td>1.9e+000</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2.8e-002</td>
<td>1.89</td>
<td>5.5e-001</td>
<td>0.94</td>
<td>7.5e-002</td>
<td>1.74</td>
<td>1.0e+000</td>
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<tr>
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<td>2.8e-001</td>
<td>0.98</td>
<td>2.0e-002</td>
<td>1.94</td>
<td>5.1e-001</td>
<td>0.98</td>
</tr>
<tr>
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<td>1.8e-003</td>
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<td>1.4e-001</td>
<td>1.00</td>
<td>5.0e-003</td>
<td>1.97</td>
<td>2.6e-001</td>
<td>0.99</td>
</tr>
<tr>
<td>64</td>
<td>4.4e-004</td>
<td>2.00</td>
<td>7.0e-002</td>
<td>1.00</td>
<td>1.2e-003</td>
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<td>1.3e-001</td>
<td>1.00</td>
</tr>
<tr>
<td>128</td>
<td>1.1e-004</td>
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<td>3.5e-002</td>
<td>1.00</td>
<td>3.1e-004</td>
<td>2.00</td>
<td>6.4e-002</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Figure 9. Original and adaptive meshes for Example 4.2.

The adaptive mesh after 30 iterative steps. In Fig.10(a), the horizontal coordinate indicate the quantity $\log N_k$, while the vertical coordinate present the logarithm of errors; we depict $\log \|u - u_k\|_{0,\mathcal{T}_k}$ by the solid curve with ‘◦’, $\log \|u - u_k\|_{\infty,\mathcal{T}_k}$ by the solid curve with ‘∗’, and $-\log N_k$ by the solid curve with ‘□’. In Fig.10(b), the horizontal coordinate indicate the quantity $\log \sqrt{N_k}$, we depict $\log |u - u_k|_{1,\mathcal{T}_k}$ by the solid curve with ‘◦’, and $-\log \sqrt{N_k}$ by the solid curve with ‘□’. We observe that $\|u - u_k\|_{0,\mathcal{T}_k}$ and $\|u - u_k\|_{\infty,\mathcal{T}_k}$ decay with $N_k^{-1}$, and $|u - u_k|_{1,\mathcal{T}_k}$ decay with $N_k^{-0.5}$. Therefore, the optimal convergence order of $L^2$, $H^1$ and $L^\infty$ norms can be obtained by using our AFVM. Moreover, the numerical results show that the Solve step is stable even if $\mathcal{T}_k$ is not shape regular, since there is an angle equal to $\pi$ for the quadrilateral element (a triangular element with a hanging node).
A NOVEL ADAPTIVE FINITE VOLUME METHOD

Figure 10. The $L^2$, $L^\infty$ and $H^1$ errors of our AFVM for Example 4.2.

Table 3. The number of elements of each iteration for Example 4.2.

<table>
<thead>
<tr>
<th>$S$</th>
<th>NAFVM</th>
<th>SAFVM</th>
<th>$S$</th>
<th>NAFVM</th>
<th>SAFVM</th>
<th>$S$</th>
<th>NAFVM</th>
<th>SAFVM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>8</td>
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<td>62</td>
<td>72</td>
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<td>384</td>
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<tr>
<td>2</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>68</td>
<td>119</td>
<td>22</td>
<td>496</td>
<td>1100</td>
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<tr>
<td>3</td>
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<td>18</td>
<td>13</td>
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<td>184</td>
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<td>638</td>
<td>434</td>
</tr>
<tr>
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<td>20</td>
<td>26</td>
<td>14</td>
<td>84</td>
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<td>24</td>
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<td>1854</td>
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<tr>
<td>5</td>
<td>26</td>
<td>34</td>
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<td>102</td>
<td>234</td>
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<td>2478</td>
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<tr>
<td>6</td>
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<td>42</td>
<td>16</td>
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<td>274</td>
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<td>1476</td>
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<td>28</td>
<td>2676</td>
<td>6186</td>
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<td>60</td>
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<td>240</td>
<td>500</td>
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<td>3610</td>
<td>8488</td>
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<tr>
<td>10</td>
<td>56</td>
<td>66</td>
<td>20</td>
<td>308</td>
<td>656</td>
<td>30</td>
<td>4996</td>
<td>11650</td>
</tr>
</tbody>
</table>

To illustrate the advantage of our AFVM compare to the standard AFVM need traditional “completion” procedure. We also implement the standard AFVM to example 4.2, the choice of the parameters of these two AFVMs are same in each iteration, and the unique difference is that the standard AFVM needs “completion” procedure compare to our AFVM. We list the number of elements in each iteration of the two AFVMs in Table 3. In this table, $S$ indicates the iterative steps, $N$ indicates the number of elements in each iteration. The NAFVM indicates our novel AFVM, and SAFVM indicates the standard AFVM. It shows that our AFVM decrease the steps of bisection for the conformity compare to the standard AFVM.

Example 4.3. We consider the elliptic equation on the L-shape domain $\Omega = \{ -1 \leq x \leq 1, -1 \leq y \leq 1 \} \setminus \{ 0 \leq x \leq 1, -1 \leq y \leq 0 \}$ with $\alpha = 1$ and the exact solution as

$$u(x, y) = r^{\frac{\theta}{3}} \sin \left( \frac{2\theta}{3} \right) ,$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$ and $\theta = \arctan \frac{y}{x}$. A direct calculation yields the right-hand side function $f = 0$. We note that the domain $\Omega$ is not convex and the boundary condition is inhomogeneous for this equation.

We implement our AFVM to this problem, choose the initial triangulation $T_1$ as in Fig.11(a), the iteration number $n = 30$ and parameter $\theta = 0.3$. Fig.11(b)
Figure 11. Original and adaptive meshes for Example 4.3.

Table 4. Error and convergence order of our AFVM for Example 4.3.

<table>
<thead>
<tr>
<th>S</th>
<th>$N_k$</th>
<th>$L^2$ error</th>
<th>Order</th>
<th>$H^1$ error</th>
<th>Order</th>
<th>$L^\infty$ error</th>
<th>Order</th>
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</thead>
<tbody>
<tr>
<td>26</td>
<td>2342</td>
<td>2.6e-004</td>
<td>1.82</td>
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<td>1.01</td>
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</table>

Figure 12. The $L^2$, $L^\infty$ and $H^1$ errors of our AFVM for Example 4.3.

shows the adaptive mesh after 30 iterative steps. The partial numerical results are presented in Table 4. Fig.12 shows the complete results of the true error with respect to $N_k$. We observe that $\|u - u_k\|_{0,\mathcal{T}_k}$ and $|u - u_k|_{1,\mathcal{T}_k}$ decay with $N_k^{-1}$ and $N_k^{-0.5}$ respectively. Note that for linear FVM on triangle or bilinear FVM on quadrilateral, the convergence rate for $\|u - u_h\|_0$ and $|u - u_h|_1$ are $O(h^2) \sim O(N^{-1})$ and $O(h) \sim O(N^{-0.5})$ for uniform meshes, if $u$ is sufficiently regular. In our numerical example, $u \in H^{1+\frac{\varepsilon}{2}-1}(\Omega)$ for all $\varepsilon > 0$ and $u \notin H^{2}(\Omega)$, and the domain $\Omega$ is not convex, we could not obtain these results if we use uniform meshes. However, the optimal convergence order of $L^2$ and $H^1$ norms can be obtained by using our
Table 5. The number of elements of each iteration for Example 4.3.

<table>
<thead>
<tr>
<th>S</th>
<th>N</th>
<th>S</th>
<th>N</th>
<th>S</th>
<th>N</th>
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</table>

AFVM. From Table 4 and Fig. 12(a), we also observe that the convergence order of \( ||u - u_k||_{\infty, \mathcal{T}_k} \) is bigger than 1. We also implement the standard AFVM to example 4.3, and the results are listed in Table 5.

Acknowledgments

The work was supported in part by the following Grants: the special project High performance computing of National Key Research and Development Program 2016YFB0200604, NSFC 11571384, Guangdong Provincial NSF 2014A030313179, the Fundamental Research Funds for the Central Universities 16lgjce80.

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