INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 6, Pages 842–868

VARIABLE TIME-STEP θ-SCHEME FOR NONLINEAR SECOND ORDER EVOLUTION INCLUSION

KRZYSZTOF BARTOSZ

Abstract. We deal with a multivalued second order dynamical system involving a Clarke subdifferential of a locally Lipschitz functional. We apply a time discretization procedure to construct a sequence of solutions to a family of the approximate problems and show its convergence to a solution of the exact problem as the time step size vanishes. We consider a nonautonomous problem in which both the viscosity and the multivalued operators depend on time explicitly. The time discretization method we use, is the θ-scheme with $\theta \in [\frac{1}{2}, 1]$, thus, in particular, the Crank-Nicolson scheme and the implicit Euler scheme are included. We apply our result to a class of dynamic hemivariational inequalities.

Key words. Clarke subdifferential, hemivariational inequality, second order inclusion, time discretization, numerical methods.

1. Introduction

In this paper we deal with a time discretization method for a second order, dynamic subdifferential inclusion, involving nonlinear, time dependent viscosity operator and a multivalued term that is a Clarke subdifferential of a locally Lipschitz continuous function that is possibly nonmonotone and nonsmooth.

Various types of Clarke subdifferential inclusions, formulated often in an alternative way, by means of hemivariational inequalities (HVIs), are motivated by numerous physical phenomena, in which contact problems in mechanics play a leading role. Indeed, the number of applications became a main impulse for research in this field. After first results of Clarke and Panagiotopoulos (see [10, 29]), the theory of HVIs has been developed by Miettinen, Migórski, Motreanu, Naniewicz, Panagiotopulos and Ochal (see [21, 22, 23, 24, 25, 27, 28, 30]). Currently many authors devote their attention either to the theory of HVIs (see for instance [8, 9, 14, 19]) or to its applications in modelling of contact problems in mechanics (see [4, 5]). For the present state of the art we refer to [6, 26]. In spite of an impressive progress of the theory, there are still relatively few results concerning numerical methods for HVIs and many problems still remain open in this field. In particular, the dynamic development of computational devices, allows to implement increasingly complex mathematical models. Due to this fact, the numerical results become more and more needed in case of HVIs as well. Haslinger and Miettinen were the first to apply the Finite Element Methods for problems modelled by HVIs (see [15, 20]). As for the time discretization methods in HVIs, the strong results concerning parabolic problems, were obtained by Kalita et al. in [7, 16, 17, 18]. Similar methods have been used by Liu, Peng and Xiao in [31, 32, 33] in the case of evolution HVIs with doubly nonlinear operators. In particular the second order HVIs has been studied in [33] in the framework of the Gelfand triple $V \subset H \subset V^*$, where the multivalued term is defined on the space H. In this paper, the multivalued term is defined on another Banach space U such that there exists a linear, continuous and compact operator $\iota : V \to U$. In applications $U = L^p(\Gamma; \mathbb{R}^d)$, where Γ is contained in the

Received by the editors December 13, 2015.
2000 *Mathematics Subject Classification.* 35G55, 74H15, 74H20, 65J15, 35R70.

boundary of the set $\Omega \subset \mathbb{R}^d$. It allows to apply our result to HVIs arising from nonmonotone and nonsmooth contact problems in mechanics. For other recent results concerning numerical methods for static or dynamic HVIs, we refer for instance to [2, 3, 11, 12, 13, 35].

In this work, we deal with a numerical analysis of dynamic, second order inclusion, which is based on time semidiscrete θ-scheme. To this end, we apply a technique, that was used in [18] in a study of parabolic problems. We apply our result to a class of dynamic boundary HVIs. In several ways, our paper improves known existence results in this area. One of basic applications of HVI's is mathematical modelling of a behaviour of physical body which occupies a region $\Omega \in \mathbb{R}^d$ and stays in a contact with a foundation. It is usually assumed, that the body is clamped on a part of boundary $\Gamma_D \subset \partial\Omega$. Our result allows to skip this restriction (see Section 7). It generalizes also [22] by removing a smallness assumption for $p = 2$ (see Remark 32 for details). Moreover, not only we provide the existence result, but we also construct a sequence of functions, which approximate the solution of the exact problem. From this point of view, our method is not only constructive but can be used in computer implementation. Finally, the present paper generalizes also the result obtained by the author in [6] for the autonomous case.

The rest of the paper is organized as follows. In Section 2, we introduce the notations and definitions used in the paper and present several auxiliary propositions. In Section 3, we formulate an abstract second order subdifferential inclusion and describe assumptions on the data of the problem. We also provide two crucial lemmas concerning properties of the Nemytskii operator corresponding to the viscosity operator A. In Section 4, we formulate an auxiliary discrete problem and provide its solvability. Based on this, we construct a sequence of piecewise constant and piecewise linear functions, for which we derive a-priori estimates and, using the reflexivity of the spaces, we obtain a weak convergence result. Finally, we provide the main existence result, namely, we show, that the limit function is a solution of the exact problem. In particular, this provides a constructive proof of the existence for the problem. In Section 5, we use the (S_+) property of the viscosity operator in order to obtain a strong convergence result. In Section 6, we apply the abstract result to a class of boundary HVIs arising from contact problems in mechanics. In Section 7, we deal with the non-clamped dynamic contact problem modelled by HVI's.

2. Preliminaries

In this section we recall some definitions and propositions which we will refer to in the sequel. We start with the definitions of Clarke directional derivative and Clarke subdifferential. Let X be a real Banach space, X^* its dual and let $J\colon X \to \mathbb{R}$ be a locally Lipschitz functional.

Definition 1. *Generalized directional derivative in the sense of Clarke at the point $x \in X$ in the direction $v \in X$, is defined by*

$$J^0(x,v) = \limsup_{y\to x,\lambda\searrow 0} \frac{J(y+\lambda v) - J(y)}{\lambda}. \tag{1}$$

Definition 2. *Clarke subdifferential of J at the point $x \in X$ is defined by*

$$\partial J(x) = \{\xi \in X^* \mid J^0(x,v) \geq \langle \xi, v\rangle_{X^*\times X} \ for\ all\ v \in X\}.$$

Now we pass to the definition of a pseudomonotone operator.

Definition 3. *(see* [36]*, Chapter 27) A single valued operator* $A\colon X \to X^*$ *is called pseudomonotone, if for any sequence* $\{v_n\}_{n=1}^{\infty} \subset X$ *such that* $v_n \to v$ *weakly in* X *and* $\limsup_{n\to\infty}\langle Av_n, v_n - v\rangle \leq 0$ *we have* $\langle Av, v - y\rangle \leq \liminf_{n\to\infty}\langle Av_n, v_n - y\rangle$ *for every* $y \in X$*.*

In the sequel we will use the following proposition.

Proposition 4. *If the operator* $A : X \to X^*$ *is linear and monotone, then it is pseudomonotone.*

Proposition 4 follows from Proposition 27.6 in [36] and the fact, that each linear operator is hemicontinuous.

Definition 5. *An operator* $A\colon X \to X^*$ *is called to be of type* $(S)_+$ *if* $v_n \to v$ *weakly in* X *and* $\limsup_{n\to\infty}\langle Av_n, v_n - v\rangle \leq 0$ *imply that* $v_n \to v$ *strongly in* X*.*

Definition 6. *Let* X *be a real Banach space. The multivalued operator* $A\colon X \to 2^{X^*}$ *is called pseudomonotone if the following conditions hold*

1) *A has values which are nonempty, bounded, closed and convex,*
2) *A is upper semicontinuous from every finite dimensional subspace of* X *into* X^* *furnished with weak topology,*
3) *if* $v_n \to v$ *weakly in* X *and* $v_n^* \in A(v_n)$ *is such that* $\limsup_{n\to\infty}\langle v_n^*, v_n - v\rangle \leq 0$ *then for every* $y \in X$ *there exists* $u(y) \in A(v)$ *such that* $\langle u(y), v - y\rangle \leq \liminf_{n\to\infty}\langle v_n^*, v_n - y\rangle$*.*

The following result can be found, for example, in [26] (see Proposition 3.58).

Proposition 7. *Let* X *be a real reflexive Banach space, and assume that* $A : X \to 2^{X^*}$ *satisfies the following conditions*

1) *for each* $v \in X$ *we have that* $A(v)$ *is a nonempty, closed and convex subset of* X^**,*
2) *A is bounded, i.e., it maps bounded sets into bounded ones,*
3) *if* $v_n \to v$ *weakly in* X *and* $v_n^* \to v^*$ *weakly in* X^* *with* $v_n^* \in A(v_n)$ *and if* $\limsup_{n\to\infty}\langle v_n^*, v_n - v\rangle \leq 0$*, then* $v^* \in A(v)$ *and* $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle$*.*

Then the operator A is pseudomonotone.

Next we recall Proposition 5.6 of [6].

Proposition 8. *Let* X *be a reflexive Banach space and* $A\colon X \to 2^{X^*}$ *be a pseudomonotone operator. Then for a given* $v_0 \in X$ *and* $\lambda > 0$ *the operator* $M : X \to 2^{X^*}$ *defined by* $M(v) = A(v_0 + \lambda v)$ *for all* $v \in X$ *is also pseudomonotone.*

We also recall a well known property for the sum of two pseudomonotone operators.

Proposition 9. *Let* X *be a reflexive Banach space. If* $A_1, A_2 : X \to 2^{X^*}$ *are pseudomonotone then so is* $A_1 + A_2$*.*

In what follows we introduce the notion of coercivity.

Definition 10. *Let* X *be a real Banach space and* $A\colon X \to 2^{X^*}$ *be an operator. We say that A is coercive if either* $D(A)$ *is bounded or* $D(A)$ *is unbounded and*

$$\lim_{\|v\|_X\to\infty \; v\in D(A)} \frac{\inf\{\langle v^*, v\rangle_{X^*\times X} \mid v^* \in Av\}}{\|v\|_X} = +\infty.$$

The following is the main surjectivity result for multivalued pseudomonotone and coercive operator.

Proposition 11. *Let X be a real, reflexive Banach space and $A\colon X \to 2^{X^*}$ be pseudomonotone and coercive. Then A is surjective, i.e., for all $b \in X^*$ there exists $v \in X$ such that $Av \ni b$.*

Let X be a Banach space and $T > 0$. We introduce the space $BV(0,T;X)$ of functions of bounded total variation on $[0,T]$. Let π denotes any finite partition of $[0,T]$ by a family of disjoint subintervals $\{\sigma_i = (a_i, b_i)\}$ such that $[0,T] = \cup_{i=1}^n \overline{\sigma}_i$. Let $\mathcal{F}$ denotes the family of all such partitions. Then, for a function $x\colon [0,T] \to X$ and for $1 \le q < \infty$, we define a seminorm

$$\|x\|^q_{BV^q(0,T;X)} = \sup_{\pi \in \mathcal{F}} \left\{ \sum_{\sigma_i \in \pi} \|x(b_i) - x(a_i)\|^q_X \right\},$$

and the space

$$BV^q(0,T;X) = \{x\colon [0,T] \to X |\ \|x\|_{BV^q(0,T;X)} < \infty\}.$$

For $1 \le p \le \infty$, $1 \le q < \infty$ and Banach spaces X, Z such that $X \subset Z$, we introduce a vector space

$$M^{p,q}(0,T;X,Z) = L^p(0,T;X) \cap BV^q(0,T;Z).$$

Then $M^{p,q}(0,T;X,Z)$ is also a Banach space with the norm given by $\|\cdot\|_{L^p(0,T;X)} + \|\cdot\|_{BV^q(0,T;Z)}$.

The following proposition will play the crucial role for the convergence of the Rothe functions which will be constructed later. For its proof, we refer to [16].

Proposition 12. *Let $1 \le p, q < \infty$. Let $X_1 \subset X_2 \subset X_3$ be real Banach spaces such that X_1 is reflexive, the embedding $X_1 \subset X_2$ is compact and the embedding $X_2 \subset X_3$ is continuous. Then the embedding $M^{p,q}(0,T;X_1;X_3) \subset L^p(0,T;X_2)$ is compact.*

The following version of Aubin-Celina convergence theorem (see [1]) will be used in what follows.

Proposition 13. *Let X and Y be Banach spaces, and $F\colon X \to 2^Y$ be a multifunction such that*

(a) *the values of F are nonempty, closed and convex subsets of Y,*
(b) *F is upper semicontinuous from X into $w - Y$.*

Let $x_n\colon (0,T) \to X$, $y_n\colon (0,T) \to Y$, $n \in \mathbb{N}$, be measurable functions such that x_n converges almost everywhere on $(0,T)$ to a function $x\colon (0,T) \to X$ and y_n converges weakly in $L^1(0,T;Y)$ to $y : (0,T) \to Y$. If $y_n(t) \in F(x_n(t))$ for all $n \in \mathbb{N}$ and almost all $t \in (0,T)$, then $y(t) \in F(x(t))$ for a.e. $t \in (0,T)$.

In the forthcoming sections we will often use the following inequalities

$$ab \le \frac{\varepsilon^p a^p}{p} + \frac{b^q}{\varepsilon^q q}, \quad \text{or} \quad ab \le \varepsilon a^p + q^{-1}(\varepsilon p)^{-\frac{q}{p}} b^q, \tag{2}$$

for all $a, b \ge 0$, $\varepsilon > 0$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and

$$(a_1 + a_2 + ... + a_n)^p \le C(p)(a_1^p + a_2^p + ... + a_n^p), \tag{3}$$

for all $a_1, ..., a_n \ge 0$, $p > 0$, $n \in \mathbb{N}$, and $C(p) = 1$ for $p \in (0,1]$, $C(p) = n^{p-1}$ for $p > 1$.

3. Problem formulation

Let V be a real, reflexive and separable Banach space, V^* its dual and H a real, separable Hilbert space. Identifying H with its dual we consider an evolution triple $V \subset H \subset V^*$ with dense, continuous and compact embeddings. We denote by $\langle \cdot, \cdot \rangle$ the duality of V and V^*, by $(\cdot, \cdot)$ the scalar product in H. Let $i : V \to H$ be an embedding operator (for $v \in V$ we will denote $iv \in H$ again by v). For all $u \in H$ and $v \in V$ we have $\langle u, v \rangle = (u, v)$. The norms in V and H we denote by $\| \cdot \|$ and $| \cdot |$ respectively. We also introduce a reflexive Banach space U and a linear, continuous operator $\iota : V \to U$. By $\|i\|$ and $\|\iota\|$ we denote the norms $\|i\|_{\mathcal{L}(V,H)}$ and $\|\iota\|_{\mathcal{L}(V,U)}$, respectively. For $T > 0$ and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, we define the spaces $\mathcal{V} = L^p(0,T;V)$, $\mathcal{V}^* = L^q(0,T;V^*)$, $\mathcal{H} = L^2(0,T;H)$, $\mathcal{U} = L^p(0,T;U)$, $\mathcal{U}^* = L^q(0,T;U^*)$ and $\mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\}$, where v' denotes the time derivative of v understood in the sense of distributions. We consider two parallel problems denoted by (P^1) and (P^2) which read as follows

$$
(P^m) \quad \begin{cases} \text{Find } \ (u, w, \eta) \in \mathcal{V} \times \mathcal{W} \times \mathcal{U} \ \text{ such that} & \\ u'(t) = w(t) & \text{for a.e. } t \in (0,T), \\ w'(t) + A(t, w(t)) + Bu(t) + \iota^* \eta(t) = f(t) & \text{for a.e. } t \in (0,T), \\ \eta(t) \in M(t, \iota z(t)) & \text{for a.e. } t \in (0,T), \\ u(0) = u_0, \quad w(0) = u_1, & \end{cases}
$$

where $m = 1, 2$, $A : [0,T] \times V \to V^*$, $B : V \to V^*$, $M : [0,T] \times U \to 2^{U^*}$, $f \in \mathcal{V}^*$, and we put $z = u$ in Problem (P^1) while $z = u'$ in Problem (P^2), respectively.
We impose the following hypotheses on the data of Problems (P^1) and (P^2).

$H(A)$: The operator $A : [0,T] \times V \to V^*$ satisfies
- (i) $A(\cdot, v)$ is measurable on $[0,T]$ for all $v \in V$,
- (ii) $A(t, \cdot)$ is pseudomonotone for every for a.e. $t \in (0,T)$,
- (iii) $\langle A(t,v), v \rangle \geq \alpha \|v\|^p - \beta |v|^2$ for a.e. $t \in (0,T)$, for all $v \in V$ with $\alpha > 0$, $\beta \geq 0$,
- (iv) $\|A(t,v)\|_{V^*}^q \leq c_A(1 + \|v\|^p)$ for a.e. $t \in (0,T)$, for all $v \in V$ with $c_A > 0$.

$H(B)$: The operator $B : V \to V^*$ is bounded, linear, monotone and symmetric, i.e., $B \in \mathcal{L}(V, V^*)$, $\langle Bv, v \rangle \geq 0$ for all $v \in V$, $\langle Bv, w \rangle = \langle Bw, v \rangle$ for all $v, w \in V$.

$H(M)$: The multivalued operator $M : [0,T] \times U \to 2^{U^*}$ satisfies
- (i) $M(\cdot, u)$ is measurable for all $u \in U$,
- (ii) the set $M(t,u)$ is nonempty, convex and weakly$*$ compact in U^* for all $u \in U$ and a.e. $t \in (0,T)$,
- (iii) the mapping $M(t, \cdot)$ is upper semicontinuous from the strong topology of U into weak topology of U^* for a.e. $t \in (0,T)$,
- (iv) M satisfies the growth condition $\|\eta\|_{U^*} \leq c_M(1 + \|z\|_U^{p-1})$ for all $z \in U$, $\eta \in M(t,z)$ a.e. $t \in (0,T)$ with $c_M \geq 0$.

$H(\iota)$: The operator $\iota \in \mathcal{L}(V,U)$ is compact and its associated Nemytskii operator $\overline{\iota} : M^{2,2}(0,T;V,V^*) \to \mathcal{U}$ defined by $(\overline{\iota} v)(t) = \iota(v(t))$ is also compact.

$H(U)$: The space U satisfies: for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that $\|\iota u\|_U \leq \varepsilon \|u\| + C(\varepsilon)|u|$ for all $u \in V$.

H_0: $f \in \mathcal{V}^*$, $u_0 \in V$, $u_1 \in H$.

Let $\mathcal{A}\colon \mathcal{V} \to \mathcal{V}^*$ be a Nemytskii operator corresponding to A defined by $(\mathcal{A}v)(t) = A(t, v(t))$ for all $t \in (0,T)$, $v \in \mathcal{V}$. We provide two results concerning properties of the operator $\mathcal{A}$.

LEMMA 14. *Assume that $H(A)$ holds and a sequence $\{v_n\} \subset \mathcal{V}$ satisfies: v_n is bounded in $M^{p,q}(0,T;V,V^*)$, $v_n \to v$ weakly in $\mathcal{V}$ and* $\limsup_{n\to\infty} < \mathcal{A}v_n, v_n - v > \mathcal{V} \le 0$. *Then $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$.*

LEMMA 15. *Assume that $H(A)$ holds and, moreover, $A(t,\cdot)$ is of type $(S)_+$ for a.e. $t \in (0,T)$. Then the operator $\mathcal{A}$ is of type $(S)_+$ with respect to the space $M^{p,q}(0,T;V,V^*)$, i.e., for any sequence $\{v_n\} \subset \mathcal{V}$ such that v_n is bounded in $M^{p,q}(0,T;V,V^*)$, $v_n \to v$ weakly in $\mathcal{V}$ and* $\limsup_{n\to\infty} \langle \mathcal{A}v_n, v_n - v\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0$ *we have $v_n \to v$ strongly in $\mathcal{V}^*$.*

The proofs of Lemmas 14 and 15 can be obtained by standard techniques. For details, see comments on Lemma 2 and 3 in [18].

4. The Rothe problem

In this section we consider a semi discrete approach to Problem (P^m), $m = 1, 2$, known as Rothe method. To this end, we define the system of grids indexed by $n \in \mathbb{N}$

$$\mathcal{T}_n = \{0 = t_n^0 < t_n^1 < \cdots < t_n^{N_n} = T\}.$$

We define $\tau_n^k = t_n^k - t_n^{k-1}$ for $k \in \{1,\ldots,N_n\}$ and we use the notation $\tau_n^{max} = \max_{k=1,\ldots,N_n}\{\tau_n^k\}$ and $\tau_n^{min} = \min_{k=1,\ldots,N_n}\{\tau_n^k\}$. We need the following regularity assumption on the time grid.

$H(t)$: the sequence of time grids satisfies

(i) $\lim_{n\to\infty} \tau_n^{max} = 0$,

(ii) there exists a constant $K > 0$ such that $\tau_n^{max} \le K\tau_n^{min}$ for all $n \in \mathbb{N}$.

Moreover, in the analysis of each of Problems (P^1) and (P^2) we will use the following additional assumptions.

H_1: $c_M 3^{p-2}(KT)^{p-1}\|\iota\|^p < \alpha$.
H_2: $c_M\|\iota\|^p < \alpha$.

For a reflexive Banach space X and $s \ge 1$ we introduce the operator $\pi_n^{s,X}: L^s(0,T;X) \to L^s(0,T;X)$, defined by

$$(\pi_n^{s,X}(v))(t) = \frac{1}{\tau_n^k}\int_{t_n^{k-1}}^{t_n^k} v(t)\,dt \quad \text{for } t \in (t_n^{k-1}, t_n^k),\ k = 1,2,...,N_n, \quad \text{for all } v \in L^s(0,T;X).$$

We define $A_n^k : V \to V^*$, $f_n^k \in V^*$, for $k = 1,...,N_n$ by

$$A_n^k(u) = \frac{1}{\tau_n^k}\int_{t_n^{k-1}}^{t_n^k} A(t,u)\,dt \text{ for all } u \in V,$$

$$f_n^k = \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} f(t)\, dt,$$

respectively. Moreover, we define the multivalued operator $M_n^k : U \to 2^{U^*}$ for $k = 1, ..., N_n$ by the following relation between its argument $z \in U$ and the element $\eta \in U^*$ of its value $M_n^k(z) \subset U^*$:

$$\eta \in M_n^k(z) \Leftrightarrow \text{there exists } \zeta_n^k : (t_n^{k-1}, t_n^k) \to U^*, \text{ such that } \eta = \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} \zeta_n^k(t)\, dt,$$

and $\zeta_n^k(t) \in M(t, z)$ for a.e. $t \in (t_n^{k-1}, t_n^k)$.

We approximate the initial conditions u_0 and u_1 by elements of V. Namely, let $\{u_{0n}\}, \{u_{1n}\} \subset V$ be sequences such that $u_{0n} \to u_0$ in V and $u_{1n} \to u_1$ in H as $n \to \infty$ and $\|u_{1n}\| \leq C/\sqrt{\tau_n}$ for some constant $C > 0$.

For a given $\Theta \in [0, 1]$ we formulate the following Rothe problem

$$(P_n^m) \quad \begin{cases} \text{Find sequences } \{u_n^k\}_{k=1}^{N_n} \subset V,\ \{w_n^k\}_{k=1}^{N_n} \subset V,\ \{\eta_n^k\}_{k=1}^{N_n} \subset U^* \text{ such that} \\ \frac{1}{\tau_n^k}(u_n^k - u_n^{k-1}) = w_n^{k-1+\Theta}, \qquad \text{for } k = 1, \ldots, N_n, \\ \frac{1}{\tau_n^k}(w_n^k - w_n^{k-1}) + A_n^k(w_n^{k-1+\Theta}) + B(u_n^{k-1+\Theta}) + \iota^* \eta_n^k = f_n^k, \\ \qquad \text{for } k = 1, \ldots, N_n, \\ \eta_n^k \in M_n^k(\iota z_n^{k-1+\Theta}), \qquad \text{for } k = 1, \ldots, N_n, \\ u_n^0 = u_{0n}, \quad w_n^0 = u_{1n}, \end{cases}$$

where $w_n^{k-1+\Theta} = \Theta w_n^k + (1 - \Theta) w_n^{k-1}$, $u_n^{k-1+\Theta} = \Theta u_n^k + (1 - \Theta) u_n^{k-1}$ and we put $z_n^{k-1+\Theta} = u_n^{k-1+\Theta}$ in Problem (P_n^1) while $z_n^{k-1+\Theta} = w_n^{k-1+\Theta}$ in Problem (P_n^2). Now we provide a theorem on existence of solution for both Problems (P_n^m), $m = 1, 2$.

Theorem 16. *Let the assumptions $H(A)$, $H(B)$, $H(M)$ and $H(t)$ hold. Then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ Problem (P_n^1) has a solution. Assume moreover that $H(U)$ holds, and, if $p > 2$, then also H_2 holds. Then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ Problem (P_n^2) has a solution.*

The proof of Theorem 16, will be preceded by several lemmas.

Lemma 17. *Let the assumptions $H(A)$ hold. Then the operators A_n^k are pseudomonotone for all $n \in \mathbb{N}$ and $k \in \{1, ..., N_n\}$.*

Lemma 18. *Let the assumptions $H(M)$ hold. Then the multivalued operators defined by $V \ni v \to \iota^* M_n^k(\iota v) \in 2^{V^*}$ are pseudomonotone for all $n \in \mathbb{N}$ and $k \in \{1, ..., N_n\}$.*

The proofs of Lemmas 17 and 18 follow the line of the proofs of Lemmas 1 and 4 in [18], respectivelly.

Now we define two multivalued operators $\tilde{T}_n^k : [0, \infty) \times V \times V \to 2^{V^*}$ and $\overline{T}_n^k : V \to 2^{V^*}$ by

$\tilde{T}_n^k(\lambda, v, u) = i^* i u + \Theta \tau_n^k A_n^k u + \Theta^2 (\tau_n^k)^2 B u + \Theta \tau_n^k \iota^* M_n^k(\iota v + \lambda \iota u),$
for all $\lambda \geq 0, v, u \in V,\ \ k = 1, \ldots, N_n$.

$\overline{T}_n^k u = i^* i u + \Theta \tau_n^k A_n^k u + \Theta^2 (\tau_n^k)^2 B u + \Theta \tau_n^k \iota^* M_n^k \iota u,$ for all $u \in V,\ \ k = 1, \ldots, N_n$.

Now we formulate two lemmas concerning surjectivity of operators $\tilde{T}_n^k$ and $\overline{T}_n^k$.

LEMMA 19. *Let the assumptions $H(A)$, $H(B)$ and $H(M)$ hold. Then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ and for all $v \in V$ the operator $\tilde{T}_n^k(\lambda, v, \cdot) : V \to 2^{V^*}$ is surjective.*

Proof Let us fix $v \in V$. In order to prove the surjectivity of $\tilde{T}_n^k(\lambda, v, \cdot)$ we apply Proposition 11. To this end, we will show, that for all $\lambda > 0$, operator $\tilde{T}_n^k(\lambda, v, \cdot)$ is pseudomonotone and for λ small enough it is coercive. By Lemma 18 and Proposition 8, we claim, that operator

$$V \ni u \to \Theta\tau_n^k \iota^* M_n^k(\iota v + \lambda\iota u) \in 2^{V^*}$$

is pseudomonotone. By Lemma 17, Proposition 4 and Proposition 9, opertor $\tilde{T}_n^k(\lambda, v, \cdot)$ is pseudomonotone. In order to show that it is coercive, we observe, that for all $u \in V$ we have

$$\left\langle \tilde{T}_n^k u, u \right\rangle = |u|^2 + \Theta\tau_n^k \left\langle A_n^k u, u \right\rangle + \Theta^2 (\tau_n^k)^2 \left\langle Bu, u \right\rangle + \Theta\tau_n^k \left\langle \eta_n^k, \iota u \right\rangle_{U^* \times U},$$

where $\eta_n^k \in M_n^k \iota(\iota v + \lambda\iota u)$. Thus, by $H(M)(iv)$, we have

$$\begin{aligned} (4) \quad & \left\langle \eta_n^k, \iota u \right\rangle_{U^* \times U} \geq -c_M(1 + \|\iota v + \lambda\iota u\|_U^{p-1})\|\iota u\|_U \geq \\ & -(c_M C(p)\lambda^{p-1}\|\iota\|^p + 2\varepsilon)\|u\|^p - (p\varepsilon)^{-q/p} q^{-1} c_M^q (\|\iota\|^q + C(p)^q \|\iota\|^{pq} \|v\|^p), \end{aligned}$$

for all $\varepsilon > 0$, where $C(p) = 1$ if $1 < p \leq 2$ and $C(p) = 2^{p-1}$ if $p > 2$. Let us take $\lambda_0 := \alpha^{1/(p-1)}[c_M C(p)\|\iota\|^p]^{1/1-p}$. Then, for all $0 < \lambda < \lambda_0$, there exists $\varepsilon > 0$, such that $\alpha - \lambda^{p-1} c_M C(p)\|\iota\|^p - 2\varepsilon > 0$, which together with $H(A)(iii)$ and (4) implies that $\tilde{T}_n^k u$ is coercive. This completes the proof. ■

LEMMA 20. *Let the assumptions $H(A)$, $H(B)$, $H(M)$, $H(U)$ and $H(t)$ hold. Moreover, if $p > 2$, we assume also H_2. Then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ operator $\overline{T}_n^k$ is surjective for $k = 1, ..., N_n$.*

Proof. As in the proof of Lemma 19 we will show that operator $\overline{T}_n^k$ is pseudomonotone and coercive. From Lemma 17, Lemma 18, Proposition 4 and Proposition 9 we conclude that $\overline{T}_n^k$ is pseudomonotone. It remains to show that it is coercive. To this end, we take $u \in V$ and estimate

$$(5) \quad \left\langle \overline{T}_n^k u, u \right\rangle = |u|^2 + \Theta\tau_n^k \left\langle A_n^k u, u \right\rangle + \Theta^2 (\tau_n^k)^2 \left\langle Bu, u \right\rangle + \Theta\tau_n^k \left\langle \eta_n^k, \iota u \right\rangle_{U^* \times U},$$

where $\eta_n^k \in M_n^k \iota u$. By $H(M)(iv)$, we get

$$(6) \quad \left\langle \eta_n^k, \iota u \right\rangle_{U^* \times U} \geq -c_M(1 + \|\iota u\|_U^{p-1})\|\iota u\|_U.$$

Assume first, that $p \in (1, 2]$. Using (2) with $\varepsilon = 1$ and $H(U)$, we obtain

$$\begin{aligned} & (1 + \|\iota u\|_U^{p-1})\|\iota u\|_U \leq \left(1 + \frac{1}{p}\right) \|\iota u\|_U^p + \frac{1}{q} \leq \left(1 + \frac{1}{p}\right) (\varepsilon\|u\| + C(\varepsilon)|u|)^p + \frac{1}{q} \\ (7) \quad & \leq \left(1 + \frac{1}{p}\right) 2^{p-1} \left(\varepsilon^p \|u\|^p + C(\varepsilon)^p |u|^p\right) + \frac{1}{q}. \end{aligned}$$

Using $H(A)(iii)$, $H(B)$ and (7) with $\varepsilon = \overline{\varepsilon} := \frac{1}{2}\left(\frac{\alpha p}{(1+p)C_M}\right)^{\frac{1}{p}}$, from (5) we get

$$(8) \quad \left\langle \overline{T}_n^k u, u \right\rangle \geq \left(1 - \Theta\tau_n^k \beta\right) |u|^2 + \frac{\alpha}{2}\Theta\tau_n^k \|u\|^p - \Theta\tau_n^k \overline{C}|u|^p - \frac{\Theta\tau_n^k C_M}{q},$$

where $\overline{C} = 2^{p-1}C_M C(\overline{\varepsilon})^p \left(1+\frac{1}{p}\right)$. Since $p \leq 2$, we have $|u|^p \leq |u|^2 + 1$ for all $u \in V$. Then from (8) we deduce

$$\left\langle \overline{T}_n^k u, u \right\rangle \geq \left(1 - \Theta\tau_n^k(\beta + \overline{C})\right)|u|^2 + \Theta\tau_n^k\|u\|^p - \Theta\tau_n^k\overline{C} - \frac{\Theta\tau_n^k C_M}{q}. \tag{9}$$

Taking n_0 such that $\tau_n^{max} < \Theta^{-1}(\beta + \overline{C})^{-1}$ for all $n > n_0$, it follows from (9), that $\overline{T}_n^k$ is coercive. If $p > 2$, then the coercivity of $\overline{T}_n^k$ follows directly from $H(A)(iii)$, (5), (6) and H_2. By Proposition 11 we conclude that $\overline{T}_n^k$ is surjective. ■

We pass to the proof of the existence result.

Proof of Theorem 16. Let $u_n^{k-1}, w_n^{k-1} \in V$, $\eta_n^{k-1} \in U^*$, $k = 1, ..., N-1$ be given. In order to provide the solvability of Problems (P_n^m), $m = 1, 2$, we need to show that there exist $u_n^{k-1+\Theta}$, $w_n^{k-1+\Theta}$, w_n^k and u_n^k, which satisfy the relations formulated in the definition of Problem (P_n^m), $m = 1, 2$.
We deal with Problem (P_n^1) first. Let λ_0 be the constant from Lemma 19 and let $n_0 \in \mathbb{N}$ be such that for all $n > n_0$ we have $\tau_n^{max} < \lambda_0/\Theta$. It implies that $\tau_n^k\Theta < \lambda_0$ and by Lemma 19 it follows that operator $\tilde{T}_n^k(\tau_n^k\Theta, u_n^{k-1})$ is surjective. Thus, in particular, there exists $w_n^{k-1+\Theta}$, such that

$$\begin{aligned} &w_n^{k-1+\Theta} + \Theta\tau_n^k A_n^k(w_n^{k-1+\Theta}) + \Theta^2(\tau_n^k)^2 B w_n^{k-1+\Theta} + \Theta\tau_n^k \iota^* M_n^k(\iota u_n^{k-1} \\ &\quad + \tau_n^k\Theta\iota w_n^{k-1+\Theta}) \ni \Theta\tau_n^k f_n^k + w_n^{k-1} - \Theta\tau_n^k B u_n^{k-1}. \end{aligned} \tag{10}$$

Defining $w_n^k = \frac{1}{\Theta}\left(w_n^{k-1+\Theta} + (\Theta-1)w_n^{k-1}\right)$ we write (10) in an equivalent form

$$\frac{1}{\tau_n^k}\left(w_n^k - w_n^{k-1}\right) + A_n^k(w_n^{k-1+\Theta}) + B\left(\Theta\tau_n^k w_n^{k-1+\Theta} + u_n^{k-1}\right) + \iota^*\eta_n^k \ni f_n^k, \tag{11}$$

where $\eta_n^k \in M_n^k(\iota u_n^{k-1} + \tau_n^k\Theta\iota w_n^{k-1+\Theta})$. Defining $u_n^{k-1+\Theta} = \tau_n^k\Theta w_n^{k-1+\Theta} + u_n^{k-1}$ and $u_n^k = \frac{1}{\Theta}\left(u_n^{k-1+\Theta} + (\Theta-1)u_n^{k-1}\right)$ we obtain the solution of Problem (P_n^1).
Now we deal with problem (P_n^2). By Lemma 20 there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ operator $\overline{T}_n^k$ is surjective. Hence, there exists $w_n^{k-1+\Theta}$ such that

$$\begin{aligned} &w_n^{k-1+\Theta} + \Theta\tau_n^k A_n^k(w_n^{k-1+\Theta}) + \Theta^2(\tau_n^k)^2 B w_n^{k-1+\Theta} + \Theta\tau_n^k \iota^* M_n^k(\iota w_n^{k-1+\Theta}) \\ &\ni \Theta\tau_n^k f_n^k + w_n^{k-1} - \Theta\tau_n^k B u_n^{k-1}. \end{aligned}$$

Defining w_n^k, $u_n^{k-1+\Theta}$ and u_n^k as above, we obtain the solution of Problem (P_n^2).
This completes the proof of the theorem. ■

Now we give a lemma on a priori estimates for a solution of Problem (P_n^m), $m = 1, 2$.

Lemma 21. *Let the assumptions $H(A)$, $H(B)$, $H(M)$, $H(U)$, H_0 and $H(t)$ hold and let $\Theta \in [\frac{1}{2}, 1]$. Moreover, if $p > 2$ we assume hypotheses H_1 (and H_2, respectively). If the triple $\left(\{u_n^k\}_{k=0}^{N_n}, \{w_n^k\}_{k=0}^{N_n}, \{\eta_n^k\}_{k=0}^{N_n}\right)$ is a solution of Problem (P_n^1) (and (P_n^2), respectively), then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have*

$$\max_{k=1,\ldots,N_n} |w_n^k|^2 + \sum_{k=1}^{N_n} \tau_n^k \|w_n^{k-1+\Theta}\|^p \leq const, \tag{12}$$

$$\max_{k=1,\ldots,N_n} \|u_n^k\| \leq const, \tag{13}$$

where const denotes a positive constant independent of n.

Proof. We test the inclusion in Problem (P_n^m), $m = 1, 2$ by $w_n^{k-1+\Theta}$ and obtain

$$(w_n^k - w_n^{k-1}, w_n^{k-1+\Theta}) + \tau_n^k \left\langle A_n^k w_n^{k-1+\Theta}, w_n^{k-1+\Theta} \right\rangle + \tau_n^k \left\langle B u_n^{k-1+\Theta}, w_n^{k-1+\Theta} \right\rangle$$

$$+ \tau_n^k \left\langle \eta_n^k, \iota w_n^{k-1+\Theta} \right\rangle_{U^* \times U} = \tau_n^k \left\langle f_n^k, w_n^{k-1+\Theta} \right\rangle, \tag{14}$$

where $\eta_n^k \in M_n^k(z_n^{k-1+\Theta})$, and $z_n^{k-1+\Theta} = u_n^{k-1+\Theta}$ in Problem (P_n^1), while $z_n^{k-1+\Theta} = w_n^{k-1+\Theta}$ in Problem (P_n^2). Using the properties of scalar product in Hilbert space, $H(B)$ and the fact, that $\Theta \geq \frac{1}{2}$, we obtain

$$\begin{aligned} (w_n^k - w_n^{k-1}, w_n^{k-1+\Theta}) &= \frac{1}{2} \left(|w_n^k|^2 - |w_n^{k-1}|^2 + (2\Theta - 1)|w_n^k - w_n^{k-1}|^2 \right) \\ &\geq \frac{1}{2}|w_n^k|^2 - \frac{1}{2}|w_n^{k-1}|^2, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \left\langle B u_n^{k-1+\Theta}, w_n^{k-1+\Theta} \right\rangle =& \frac{1}{2} \left\langle B u_n^k, u_n^k \right\rangle - \frac{1}{2} \left\langle B u_n^{k-1}, u_n^{k-1} \right\rangle \\ &+ \frac{1}{2}(2\Theta - 1) \left\langle B(u_n^k - u_n^{k-1}), u_n^k - u_n^{k-1} \right\rangle \\ \geq& \frac{1}{2} \left\langle B u_n^k, u_n^k \right\rangle - \frac{1}{2} \left\langle B u_n^{k-1}, u_n^{k-1} \right\rangle. \end{aligned} \tag{16}$$

By (2) for $\delta > 0$, we have

$$\left\langle f_n^k, w_n^{k-1+\Theta} \right\rangle \leq \delta \|w_n^{k-1+\Theta}\|^p + \frac{1}{q}(\delta p)^{-\frac{q}{p}} \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} \|f(t)\|_{V^*}^q \, dt. \tag{17}$$

In order to estimate $\left\langle \eta_n^k, \iota w_n^{k-1+\Theta} \right\rangle_{U^* \times U}$ we proceed in two steps. First, we deal with Problem (P_n^1), namely, we have $\eta_n^k \in M_n^k(u_n^{k-1+\Theta})$. From $H(M)(iv)$ and the relation

$$u_n^k = u_n^0 + \sum_{i=1}^{k} \tau_n^i w_n^{i-1+\Theta} \quad \text{for } k = 1, ..., N_n, \tag{18}$$

we obtain

$$\begin{aligned} &\left\langle \eta_n^k, \iota w_n^{k-1+\Theta} \right\rangle_{U^* \times U} \\ \geq& - c_M (1 + \|\iota u_n^{k-1+\Theta}\|_U^{p-1}) \|\iota w_n^{k-1+\Theta}\|_U \\ \geq& - c_M (1 + \|\iota u_n^{k-1} + \Theta \tau_n^k \iota w_n^{k-1+\Theta}\|_U^{p-1}) \|\iota w_n^{k-1+\Theta}\|_U \\ \geq& - c_M \left(1 + \left\| \iota u_n^0 + \sum_{i=1}^{k-1} \tau_n^i \iota w_n^{i-1+\Theta} + \Theta \tau_n^k \iota w_n^{k-1+\Theta} \right\|_U^{p-1} \right) \|\iota w_n^{k-1+\Theta}\|_U \\ \geq& - c_M \left(1 + C(p) \|\iota u_n^0\|_U^{p-1} \right) \|\iota w_n^{k-1+\Theta}\|_U \\ &- c_M C(p) \left(\Theta \tau_n^k \right)^{p-1} \|\iota\|^p \|w_n^{k-1+\Theta}\|^p \\ &- c_M C(p) \left\| \sum_{i=1}^{k-1} \tau_n^i \iota w_n^{i-1+\Theta} \right\|_U^{p-1} \|\iota w_n^{k-1+\Theta}\|_U, \end{aligned} \tag{19}$$

where $C(p) = 1$ if $p \in (1,2]$ and $C(p) = 3^{p-2}$ if $p > 2$. As for the first term of right hand side we estimate

$$c_M \left(1 + C(p)\|\iota u_n^0\|_U^{p-1}\right) \|\iota w_n^{k-1+\Theta}\|_U \le \varepsilon \|w_n^{k-1+\Theta}\|^p + C_1(\varepsilon), \tag{20}$$

for $\varepsilon > 0$ and $C_1(\varepsilon) = \frac{1}{q}(p\varepsilon)^{-\frac{q}{p}}\|\iota\|^q(1 + C(p)\|\iota u_n^0\|^{p-1})^q$. Moreover, using (3), we obtain the following auxiliary estimate

$$\begin{aligned}
&\sum_{k=1}^{l} \tau_n^k \left\| \sum_{i=1}^{k-1} \tau_n^i \iota w_n^{i-1+\Theta} \right\|_U^{p-1} \|\iota w_n^{k-1+\Theta}\|_U \\
&\le \sum_{k=1}^{l} \tau_n^k \|\iota w_n^{k-1+\Theta}\|_U \left(\sum_{i=1}^{l} \tau_n^i \left\|\iota w_n^{i-1+\Theta}\right\|_U \right)^{p-1} \\
&\le \left(\sum_{k=1}^{l} \tau_n^k \left\|\iota w_n^{k-1+\Theta}\right\|_U \right)^p \le (N_n \tau_{max})^{p-1} \sum_{k=1}^{l} \tau_n^k \left\|\iota w_n^{k-1+\Theta}\right\|_U^p,
\end{aligned} \tag{21}$$

for $l = 2, ..., N_n$. By the convexity of the norm and the quadratic function we have

$$\sum_{k=1}^{l} \tau_n^k |w_n^{k-1+\Theta}|^2 \le \tau_n^{max} \left((1-\Theta)|w_n^0|^2 + \sum_{k=1}^{l-1} |w_n^k|^2 + \Theta |w_n^l|^2 \right). \tag{22}$$

Summing up (14) with $k = 1, ...l$ and taking into account $H(A)(iii)$, $H(B)$, (15)-(17), (19)-(22) and $H(t)$, we get

$$\begin{aligned}
&\left(\frac{1}{2} - \beta\Theta\tau_n^{max}\right) |w_n^l|^2 + \frac{1}{2}\tau_n^l \left\langle B u_n^l, u_n^l \right\rangle \\
&+ \left(\alpha - \delta - c_M C(p)(\Theta\tau_n^k)^{p-1}\|\iota\|^p - \varepsilon\right) \sum_{k=1}^{l} \tau_n^k \|w_n^{k-1+\Theta}\|^p \\
\le& \frac{1}{2}\tau_n^0 \left\langle B u_n^0, u_n^0 \right\rangle + \left(\frac{1}{2} + \beta\tau_n^{max}(1-\Theta)\right) |w_n^0|^2 + \frac{1}{q}(\delta p)^{-\frac{q}{p}} \|f\|_{\mathcal{V}^*}^q \\
&+ c_M C(p)(KT)^{p-1} \sum_{k=1}^{l} \tau_n^k \left\|\iota w_n^{k-1+\Theta}\right\|_U^p + \beta \sum_{k=1}^{l-1} \tau_n^{max} |w_n^k|^2 + C_1(\varepsilon) T.
\end{aligned} \tag{23}$$

If $1 < p \le 2$, we use $H(U)$ and obtain

$$\|\iota w_n^{k-1+\Theta}\|_U^p \le 2^{p-1}\epsilon^p \|w_n^{k-1+\Theta}\|^p + 2^{p-1} C(\epsilon)^p (|w_n^{k-1+\Theta}|^2 + 1). \tag{24}$$

Concerning it in (23), and using (22), we get

$$\begin{aligned}&\left(\frac{1}{2}-\Theta\tau_n^{max}(\beta+c_MC(p)(2KT)^{p-1}C(\epsilon)^p)\right)|w_n^l|^2\\&+\left(\alpha-\delta-c_MC(p)(\Theta\tau_n^k)^{p-1}\|\iota\|^p-\varepsilon-c_MC(p)(2KT)^{p-1}\epsilon^p\right)\sum_{k=1}^{l}\tau_n^k\|w_n^{k-1+\Theta}\|^p\\&+\frac{1}{2}\tau_n^l\left\langle Bu_n^l,u_n^l\right\rangle\\\leq&\frac{1}{2}\tau_n^0\left\langle Bu_n^0,u_n^0\right\rangle+\frac{1}{q}(\delta p)^{-\frac{q}{p}}\|f\|_{\mathcal{V}^*}^q+T2^{p-1}C(\epsilon)^p\\&+\left(\frac{1}{2}+\tau_n^{max}(1-\Theta)(\beta+c_MC(p)(2KT)^{p-1}C(\epsilon)^p)\right)|w_n^0|^2\\&+(\beta+c_MC(p)(2KT)^{p-1}C(\epsilon)^p)\sum_{k=1}^{l-1}\tau_n^{max}|w_n^k|^2+C_1(\varepsilon)T.\end{aligned}$$

Taking n_0, such that for all $n>n_0$, $\tau_n^{max}<\min\{\frac{1}{2M\Theta},\left(\frac{\alpha}{\|\iota\|^pc_MC(p)}\right)^{\frac{1}{p-1}}\Theta^{-1}\}$, where $M:=\beta+c_MC(p)(2KT)^{p-1}C(\epsilon)^p$, we can find ε and ϵ small enough to apply the Gronwall lemma and obtain (12).
If $p>2$, we apply the inequality $\|\iota w_n^{k-1+\Theta}\|_U^p\leq\|\iota\|^p\|w_n^{k-1+\Theta}\|^p$ to the right hand side of (23). Then, using H_1 we can apply the Gronwall lemma to (23), provided $\tau_n^{max}\leq\min\{\frac{1}{2\beta\Theta},\Theta^{-1}\left(\alpha(c_MC(p)\|\iota\|^p)^{-1}-(KT)^{p-1}\right)^{\frac{1}{p-1}}\}$ and δ, ε small enough.
Now we deal with Problem (P_n^2), namely, we have $\eta_n^k\in M_n^k(w_n^{k-1+\Theta})$. From $H(M)(iv)$ and (2), we have

$$\left\langle\eta_n^k,\iota w_n^{k-1+\Theta}\right\rangle_{U^*\times U}\geq-c_M\|\iota w_n^{k-1+\Theta}\|_U^p-\varepsilon\|w_n^{k-1+\Theta}\|^p-C_2(\varepsilon),\tag{25}$$

where $C_2(\varepsilon)=q^{-1}(\varepsilon p)^{-\frac{q}{p}}c_M^q\|\iota\|^q$. Summing up (14) with $k=1,...,l$, exploiting $H(A)(iii)$, $H(B)$, (15)-(17), (25), and the fact, that $\Theta\geq\frac{1}{2}$, we get

$$\begin{aligned}&\left(\frac{1}{2}-\beta\Theta\tau_n^{max}\right)|w_n^l|^2+(\alpha-\delta-\varepsilon)\sum_{k=1}^{l}\tau_n^k\|w_n^{k-1+\Theta}\|^p+\frac{1}{2}\tau_n^l\left\langle Bu_n^l,u_n^l\right\rangle\\&\leq\frac{1}{2}\tau_n^0\left\langle Bu_n^0,u_n^0\right\rangle+\frac{1}{2}|w_n^0|^2+\frac{1}{q}(\delta p)^{-\frac{q}{p}}\|f\|_{\mathcal{V}^*}^q+c_M\sum_{k=1}^{l}\tau_n^k\left\|\iota w_n^{k-1+\Theta}\right\|_U^p\\&+\beta\tau_n^{max}(1-\Theta)|w_n^0|^2+\beta\sum_{k=1}^{l-1}\tau_n^{max}|w_n^k|^2+C_2(\varepsilon)T.\end{aligned}\tag{26}$$

If $1<p\leq2$, we use (24), together with (22) for the right hand side of (26) and apply the Gronwall lemma provided $\tau_n^{max}<\frac{1}{2M\Theta}$, where $M:=\beta+c_M2^{p-1}C(\epsilon)^p$, $\epsilon<\alpha^{\frac{1}{p}}c_M^{-\frac{1}{p}}2^{\frac{1-p}{p}}$ and δ, ε satisfy $\delta+\varepsilon<\alpha-c_M2^{p-1}\epsilon^p$. If $p\geq2$, we use H_2 and apply the Gronwall lemma provided $\tau_n^{max}<\frac{1}{2\beta\Theta}$ and $\delta+\varepsilon<\alpha-c_M\|\iota\|^p$.

In order to prove (13), we note that from (18) and the Hölder inequality, for $k = 1, ..., N_n$, we get

$$\begin{aligned}\|u_n^k\| \leq & \|u_n^0\| + \sum_{i=1}^{k} \tau_n^i \|w_n^{i-1+\Theta}\| \\ = & \|u_n^0\| + \sum_{i=1}^{k} (\tau_n^i)^{\frac{1}{q}} (\tau_n^i)^{1-\frac{1}{q}} \|w_n^{i-1+\Theta}\| \\ \leq & \|u_n^0\| + \left(\sum_{i=1}^{k} \tau_n^i \right)^{\frac{1}{q}} \left(\sum_{i=1}^{k} (\tau_n^i)^{p-\frac{p}{q}} \|w_n^{i-1+\Theta}\|^p \right)^{\frac{1}{p}} \\ \leq & \|u_n^0\| + T^{\frac{1}{q}} \left(\sum_{i=1}^{N_n} \tau_n^i \|w_n^{i-1+\Theta}\|^p \right)^{\frac{1}{p}}.\end{aligned}$$

Since the last term is bounded by (12), we deduce (13), which completes the proof. ■

LEMMA 22. *Let $\Theta \in [\frac{1}{2}, 1]$ and the assumptions $H(A)$, $H(B)$, $H(M)$, $H(U)$, H_0 and $H(t)$ hold. Moreover, if $p > 2$ we assume hypotheses H_1 (and H_2, respectively). If the triple $\left(\{u_n^k\}_{k=0}^{N_n}, \{w_n^k\}_{k=0}^{N_n}, \{\eta_n^k\}_{k=0}^{N_n}\right)$ is a solution of Problem (P_n^1) (and (P_n^2), respectively) then we have*

$$\sum_{k=1}^{N_n} \tau_n^k \|A_n^k w_n^{k-1+\Theta}\|_{V^*}^q \leq const, \tag{27}$$

$$\sum_{k=1}^{N_n} \tau_n^k \|\eta_n^k\|_{U^*}^q \leq const, \tag{28}$$

$$\sum_{k=1}^{N_n} \tau_n^k \left\| \frac{w_n^k - w_n^{k-1}}{\tau_n^k} \right\|_{V^*}^q \leq const, \tag{29}$$

where const denotes a positive constant independent of n.

Proof. By the Jensen inequality and $H(A)(iv)$, we get

$$\|A_n^k w_n^{k-1+\Theta}\|_{V^*}^q \leq \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} \|A(t, w_n^{k-1+\Theta})\|^q \, dt \leq c_A(1 + \|w_n^{k-1+\Theta}\|^p), \tag{30}$$

which together with (12) gives (27). Next, using $H(M)(iv)$ we obtain

$$\sum_{k=1}^{N_n} \tau_n^k \|\eta_n^k\|_{U^*}^q \leq c_M^q 2^{q-1} T + c_M^q 2^{q-1} \|\iota\|^p \sum_{k=1}^{N_n} \tau_n^k \|z_n^{k-1+\Theta}\|^p,$$

and, conclude that (28) follows either from (13) in case of Problem (P_n^1) or from (12) in case of Problem (P_n^2).

Now we pass to the proof of (29). From the formulation of Problem (P_n^m), $m = 1, 2$, we get

$$\sum_{k=1}^{N_n} \tau_n^k \left\| \frac{w_n^k - w_n^{k-1}}{\tau_n^k} \right\|_{V^*}^q$$

$$\leq C \sum_{k=1}^{N_n} \tau_n^k \left(\|A_n^k w_n^{k-1+\Theta}\|_{V^*}^q + \|B u_n^{k-1+\Theta}\|_{V^*}^q + \|\iota^* \eta_n^k\|_{V^*}^q + \|f_n^k\|_{V^*}^q \right), \tag{31}$$

where $C > 0$ is independent of n. Using the convexity of the function $\|\cdot\|^q$ and (13) we estimate

$$\begin{aligned}\sum_{k=1}^{N_n} \tau_n^k \|B u_n^{k-1+\Theta}\|_{V^*}^q \leq & \|B\|_{\mathcal{L}(V,V^*)} \sum_{k=1}^{N_n} \tau_n^k \|u_n^{k-1+\Theta}\|_V^q \\ \leq & \|B\|_{\mathcal{L}(V,V^*)} \sum_{k=1}^{N_n} \tau_n^k \left(\Theta \|u_n^k\|^q + (1-\Theta)\|u_n^{k-1}\|^q \right) \\ \leq & \|B\|_{\mathcal{L}(V,V^*)} T \max_{k=0,\dots,N} \|u_n^k\|^q. \end{aligned} \tag{32}$$

Finally using the Jensen inequality we get

$$\sum_{k=1}^{N_n} \tau_n^k \|f_n^k\|_{V^*}^q \leq \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \|f(t)\|_{V^*}^q \, dt = \int_0^T \|f(t)\|_{V^*}^q \, dt = \|f\|_{\mathcal{V}^*}^q. \tag{33}$$

Using (27), (28), (32) and (33), we obtain (29) directly from (31). The proof is complete. ■

Let the triple $\left(\{u_n^k\}_{k=0}^{N_n}, \{w_n^k\}_{k=0}^{N_n}, \{\eta_n^k\}_{k=0}^{N_n}\right)$ be a solution of Problem (P_n^1) or (P_n^2). For a fixed $n \in \mathbb{N}$ we define piecewise constant and piecewise linear functions $\overline{u}_n$, $\overline{w}_n$, $\hat{u}_n$, $\hat{w}_n \colon [0,T] \to V$ and $\overline{\eta}_n \colon [0,T] \to U^*$ by

$$\overline{u}_n(t) = u_n^{k-1+\Theta} \quad \text{for} \quad t \in (t_n^{k-1}, t_n^k], \quad k = 1, ..., N_n, \quad \text{and} \quad \overline{u}_n(0) = u_n^0, \tag{34}$$

$$\hat{u}_n(t) = u_n^{k-1} + \frac{u_n^k - u_n^{k-1}}{\tau_n^k}(t - t_n^{k-1}) \quad \text{for} \quad t \in [t_n^{k-1}, t_n^k], \quad k = 1, ..., N_n, \tag{35}$$

$$\overline{w}_n(t) = w_n^{k-1+\Theta} \quad \text{for} \quad t \in (t_n^{k-1}, t_n^k], \quad k = 1, ..., N_n, \quad \text{and} \quad \overline{w}_n(0) = w_n^0, \tag{36}$$

$$\hat{w}_n(t) = w_n^{k-1} + \frac{w_n^k - w_n^{k-1}}{\tau_n^k}(t - t_n^{k-1}) \quad \text{for} \quad t \in [t_n^{k-1}, t_n^k], \quad k = 1, ..., N_n, \tag{37}$$

$$\overline{\eta}_n(t) = \eta_n^k \quad \text{for} \quad t \in (t_n^{k-1}, t_n^k], \quad k = 1, ..., N_n, \quad \text{and} \quad \overline{\eta}_n(0) = \eta_n^1. \tag{38}$$

We introduce the Nemytskii operators $\mathcal{A}$, $\mathcal{B} \colon \mathcal{V} \to \mathcal{V}^*$ and $\bar{\iota} : \mathcal{V} \to \mathcal{U}$ defined by $(\mathcal{A}v)(t) = A(t, v(t))$, $(\mathcal{B}v)(t) = Bv(t)$, $(\bar{\iota}v)(t) = \iota v(t)$ for a.e. $t \in [0,T]$ for all $v \in \mathcal{V}$. Since $\left(\{u_n^k\}_{k=0}^{N_n}, \{w_n^k\}_{k=0}^{N_n}, \{\eta_n^k\}_{k=0}^{N_n}\right)$ solves Problem (P_n^m), $m = 1, 2$, it follows that the functions $\overline{u}_n,\ \overline{w}_n,\ \hat{u}_n,\ \hat{w}_n$ and $\overline{\eta}_n$ satisfy

(39)
$$\begin{cases} \hat{u}_n'(t) = \overline{w}_n(t), & \text{for a.e. } t \in (0,T),\\ \hat{w}_n'(t) + \left(\pi_n^{q,V^*}\mathcal{A}\overline{w}_n\right)(t) + \left(\mathcal{B}\overline{u}_n\right)(t) + (\overline{\iota}^*\overline{\eta}_n)(t) = (\pi_n^{q,V^*}(f))(t), & \text{for a.e. } t \in (0,T),\\ \overline{\eta}_n(t) \in M_n^k((\overline{\iota}\,\overline{z}_n)(t)),\ \text{for a.e. } t \in (t_n^{k-1}, t_n^k], & \text{for } k = 1, ..., N_n,\\ \overline{u}_n(0) = u_n^{\Theta},\ \hat{u}_n(0) = u_n^0,\ \overline{w}_n(0) = w_n^{\Theta},\ \hat{w}_n(0) = w_n^0, \end{cases}$$

where $\overline{\iota}^* : \mathcal{U}^* \to \mathcal{V}^*$ denotes the adjoint operator to $\overline{\iota}$ and $\overline{z}_n(t) = \overline{u}_n(t)$ in Problem (P_n^1) and $\overline{z}_n(t) = \overline{w}_n(t)$ in Problem (P_n^2). Now we formulate a lemma on a priori estimates for solution to (39).

LEMMA 23. *Let $\Theta \in [\frac{1}{2}, 1]$ and the assumptions $H(A)$, $H(B)$, $H(M)$, $H(U)$, H_0 and $H(t)$ hold. Moreover, if $p > 2$ we assume hypotheses H_1 (and H_2, respectively). Let the triple $\left(\{u_n^k\}_{k=0}^{N_n}, \{w_n^k\}_{k=0}^{N_n}, \{\eta_n^k\}_{k=0}^{N_n}\right)$ be a solution of Problem (P_n^1) (and (P_n^2), respectively) and the functions $\overline{u}_n,\ \overline{w}_n,\ \hat{u}_n,\ \hat{w}_n\colon [0,T] \to V$ and $\overline{\eta}_n\colon [0,T] \to U^*$ be defined by (34)-(38). Then the following estimates hold*

$$\|\overline{u}_n\|_{L^\infty(0,T;V)} \le const, \tag{40}$$
$$\|\overline{w}_n\|_{L^\infty(0,T;H)} \le const, \tag{41}$$
$$\|\hat{u}_n\|_{C(0,T;V)} \le const, \tag{42}$$
$$\|\hat{w}_n\|_{C(0,T;H)} \le const, \tag{43}$$
$$\|\overline{w}_n\|_{\mathcal{V}} \le const, \tag{44}$$
$$\|\pi_n^{q,V^*}(\mathcal{A}\overline{w}_n)\|_{\mathcal{V}^*} \le const, \tag{45}$$
$$\|\overline{\eta}_n\|_{\mathcal{U}^*} \le const, \tag{46}$$
$$\|\hat{w}_n'\|_{\mathcal{V}^*} \le const, \tag{47}$$
$$\|\overline{w}_n\|_{M^{p,q}(0,T;V,V^*)} \le const, \tag{48}$$
$$\|\overline{u}_n\|_{M^{p,q}(0,T;V,V^*)} \le const, \tag{49}$$

where const denotes a positive constant independent of n.

Proof. Since the sequence $\{u_n^0\}$ converges weakly to u_0, it is bounded in V. We also recall, that $w_n^0 = u_{1n} \to u_1$ strongly in H, so the sequence $\{w_n^0\}$ is bounded in H. Thus (40) and (41) follow directly from (13) and (12), respectively. Moreover, we observe, that for $t \in [t_n^{k-1}, t_n^k],\ k = 1, ...N_n$, the value $\hat{u}_n(t)$ is a convex combination of elements u_n^{k-1} and u_n^k. Using the convexity of the norm and (13) again, we derive (42). Using the same arguments, we obtain (43) from (12). In order to prove (44), it is enough to observe that $\|\overline{w}_n\|_{\mathcal{V}}^p = \sum_{k=1}^{N_n} \tau_n^k \|w_n^{k-1+\Theta}\|^p$ and use (12). In order to derive (45), we first observe that

$$\left(\pi_n^{q,V^*}(\mathcal{A}\overline{w}_n)\right)(t) = A_n^k w_n^{k-1+\Theta},$$

for $t \in (t_n^{k-1}, t_n^k)$, $k = 1, ..., N_n$. Thus we get

$$\|\pi_n^{q,V^*}(\mathcal{A}\overline{w}_n)\|_{\mathcal{V}^*}^q = \int_0^T \| \left(\pi_n^{q,V^*}(\mathcal{A}\overline{w}_n)\right)(t)\|_{\mathcal{V}^*}^q \, dt$$

$$= \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \| \left(\pi_n^{q,V^*}(\mathcal{A}\overline{w}_n)\right)(t)\|_{\mathcal{V}^*}^q \, dt$$

$$(50) \qquad = \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \left\|A_n^k w_n^{k-1+\Theta}\right\|_{\mathcal{V}^*}^q \, dt = \sum_{k=1}^{N_n} \tau_n^k \left\|A_n^k w_n^{k-1+\Theta}\right\|_{\mathcal{V}^*}^q .$$

From (27) and (50), we get (45). Since $\|\overline{\eta}_n\|_{\mathcal{U}^*}^q = \sum_{k=1}^{N_n} \tau_n^k \left\|\eta_n^k\right\|_{\mathcal{U}^*}^q$, we obtain (46) from (28). Similarly, we observe that

$$(51) \qquad \|\hat{w}_n'\|_{\mathcal{V}^*}^q = \sum_{k=1}^{N_n} \tau_n^k \left\| \frac{w_n^k - w_n^{k-1}}{\tau_n^k} \right\|_{V^*}^q ,$$

and obtain (47) from (29). Now we pass to proof of (48). Taking into account (44), it is enough to estimate the seminorm $\|\overline{w}_n\|_{BV^q(0,T;V^*)}$. Since the function $\overline{w}_n$ is piecewise constant, it will be measured by means of jumps between elements of sequence $\{w_n^{k-1+\Theta}\}_{k=1}^{N_n}$. Namely, let $\{k_j\}_{j=1}^{M_n} \subset \{1, ..., N_n\}$ be an increasing sequence of numbers such that $k_1 = 1$, $k_{M_n} = N_n$ and

$$(52) \qquad \|\overline{w}_n\|_{BV^q(0,T;V^*)} = \sum_{j=1}^{M_n-1} \|w_n^{k_{j+1}-1+\Theta} - w_n^{k_j-1+\Theta}\|_{V^*}^q.$$

For a fixed $j = 1, ..., M_n - 1$ we obtain

$$(53) \qquad \left\|w_n^{k_{j+1}-1+\Theta} - w_n^{k_j-1+\Theta}\right\|_{V^*}^q$$

$$= \left\|w_n^{k_{j+1}-1+\Theta} - w_n^{k_{j+1}-2+\Theta} + w_n^{k_{j+1}-2+\Theta} - ... + w_n^{k_j+\Theta} - w_n^{k_j-1+\Theta}\right\|_{V^*}^q$$

$$\leq (k_{j+1} - k_j)^{q-1} \sum_{l=k_j}^{k_{j+1}-1} \left\|w_n^{l+\Theta} - w_n^{l-1+\Theta}\right\|_{V^*}^q$$

$$\leq (N_n - 1)^{q-1} \sum_{l=k_j}^{k_{j+1}-1} \left\|w_n^{l+\Theta} - w_n^{l-1+\Theta}\right\|_{V^*}^q .$$

Combining (52) with (53) and, using the convexity of the function $V^* \ni v \to \|v\|_{V^*}^q$, we get

$$\|\overline{w}_n\|_{BV^q(0,T;V^*)} \leq N_n^{q-1} \sum_{l=1}^{N_n-1} \left\|w_n^{l+\Theta} - w_n^{l-1+\Theta}\right\|_{V^*}^q$$

$$= N_n^{q-1} \sum_{l=1}^{N_n-1} \left\|\Theta w_n^{l+1} + (1-\Theta) w_n^l - (\Theta w_n^l + (1-\Theta) w_n^{l-1})\right\|_{V^*}^q$$

$$= N_n^{q-1} \sum_{l=1}^{N_n-1} \left\|\Theta (w_n^{l+1} - w_n^l) + (1-\Theta)(w_n^l - w_n^{l-1})\right\|_{V^*}^q$$

$$\leq N_n^{q-1} \sum_{l=1}^{N_n-1} \left(\left\| \Theta(w_n^{l+1} - w_n^l) \right\|_{V^*}^q + (1-\Theta) \left\| (w_n^l - w_n^{l-1}) \right\|_{V^*}^q \right)$$

$$\leq N_n^{q-1} \sum_{l=1}^{N_n} \left\| w_n^l - w_n^{l-1} \right\|_{V^*}^q = N_n^{q-1} \sum_{l=1}^{N_n} (K\tau_n^{min})^{q-1} \tau_n^k \left\| \frac{w_n^l - w_n^{l-1}}{\tau_n^k} \right\|_{V^*}^q$$

$$\leq (TK)^{q-1} \sum_{l=1}^{N_n} \tau_n^k \left\| \frac{w_n^l - w_n^{l-1}}{\tau_n^k} \right\|_{V^*}^q . \tag{54}$$

Using (29), (44) and (54), we obtain (48). Analogously to (54), we obtain

$$\|\overline{u}_n\|_{BV^q(0,T;V^*)} \leq (TK)^{q-1} \sum_{l=1}^{N_n} \tau_n^k \left\| \frac{u_n^l - u_n^{l-1}}{\tau_n^k} \right\|_{V^*}^q$$
$$\leq (TK)^{q-1} \|i\| \|\overline{w}_n\|_{L^q(0,T;H)}, \tag{55}$$

which together with (41) gives (49). This completes the proof of the lemma. □

Now we formulate the theorem concerning the convergence of functions defined by (34)-(38) and the solvability of Problem (P^m), $m = 1, 2$.

THEOREM 24. *Let $\Theta \in [\frac{1}{2}, 1]$ and the assumptions $H(A)$, $H(B)$, $H(M)$, $H(U)$, H_0 and $H(t)$ hold. Moreover, if $p > 2$ we assume H_1 (and H_2, respectively). Let the triple $\left(\{u_n^k\}_{k=0}^{N_n}, \{w_n^k\}_{k=0}^{N_n}, \{\eta_n^k\}_{k=0}^{N_n} \right)$ be a solution of Problem (P_n^1) (and (P_n^2), respectively) and the functions $\overline{u}_n$, $\overline{w}_n$, $\hat{u}_n$, $\hat{w}_n \colon [0,T] \to V$ and $\overline{\eta}_n \colon [0,T] \to U^*$ be defined by (34)-(38). Then, there exist $u \in \mathcal{V}$, $w \in \mathcal{W}$, $\eta \in \mathcal{U}$ and $\xi \in \mathcal{V}^*$ such that for a subsequence still enumerated by n, the following convergences hold*

$$\overline{u}_n \to u \ \text{ weakly} * \ \text{ in } L^\infty(0,T;V), \tag{56}$$
$$\hat{u}_n \to u \ \text{ weakly} * \ \text{ in } L^\infty(0,T;V), \tag{57}$$
$$\overline{w}_n \to w \ \text{ weakly} * \ \text{ in } L^\infty(0,T;H) \text{ and weakly in } \mathcal{V}, \tag{58}$$
$$\hat{w}_n \to w \ \text{ weakly} * \ \text{ in } L^\infty(0,T;H), \tag{59}$$
$$\hat{w}_n' \to w' \ \text{ weakly in } \mathcal{V}^*, \tag{60}$$
$$\pi_n^{q,V^*}(\mathcal{A}\overline{w}_n) \to \xi \ \text{ weakly in } \mathcal{V}^*, \tag{61}$$
$$\overline{\eta}_n \to \eta \ \text{ weakly in } \mathcal{U}^*, \tag{62}$$
$$\bar{\iota}\,\overline{w}_n \to \bar{\iota} w \ \text{ strongly in } \mathcal{U}, \tag{63}$$
$$\bar{\iota}\,\overline{u}_n \to \bar{\iota} u \ \text{ strongly in } \mathcal{U}. \tag{64}$$

Moreover, the triple (u, w, η) is a solution of Problem (P^m), $m = 1, 2$.

Proof. The existence of limits required in (56)-(62) follows directly from the bounds (40)-(47) obtained in Lemma 23. However, we still need to show, that $\overline{u}_n$ and $\hat{u}_n$ converge to the same element, as well as for the sequences $\overline{w}_n$ and $\hat{w}_n$. To this end, we estimate

$$\begin{aligned}\|\hat{u}_n - \overline{u}_n\|_{\mathcal{V}}^p &= \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \|\hat{u}_n(t) - \overline{u}_n(t)\|^p \, dt \\ &= \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \left\| \left(\frac{t - t_n^{k-1}}{\tau_n^k} - \Theta \right) (u_n^k - u_n^{k-1}) \right\|^p dt \\ &= \sum_{k=1}^{N_n} \int_{-\Theta}^{1-\Theta} |s|^p \left\| u_n^k - u_n^{k-1} \right\|^p \tau_n^k \, ds \\ &= \frac{1}{p+1} \left[\Theta^{p+1} + (1-\Theta)^{p+1} \right] \sum_{k=1}^{N_n} \tau_n^k \left\| \frac{u_n^k - u_n^{k-1}}{\tau_n^k} \right\|^p \left(\tau_n^k\right)^p \, ds \\ &\le (\tau_n^{max})^p \, \frac{1}{p+1} \left[\Theta^{p+1} + (1-\Theta)^{p+1} \right] \sum_{k=1}^{N_n} \tau_n^k \|w_n^{k-1+\Theta}\|^p.\end{aligned}$$

Thus, using (12) and $H(t)(i)$, we conclude that

$$\|\hat{u}_n - \overline{u}_n\|_{\mathcal{V}} \to 0 \text{ as } n \to \infty, \tag{65}$$

so both sequences have the same limit. Similarly, we estimate

$$\|\hat{w}_n - \overline{w}_n\|_{\mathcal{V}^*}^q \le (\tau_n^{max})^q \, \frac{1}{q+1} \left[\Theta^{q+1} + (1-\Theta)^{q+1} \right] \sum_{k=1}^{N_n} \tau_n^k \left\| \frac{w_n^k - w_n^{k-1}}{\tau_n^k} \right\|_{V^*}^q .$$

Using (29) and $H(t)(i)$, again we see that $\|\hat{w}_n - \overline{w}_n\|_{\mathcal{V}^*} \to 0$ as $n \to \infty$, so the limits of both sequences coincide. The convergences (63) and (64) follow directly from (48) and (49), and the hypotheses $H(\iota)$. In the remaining part of the proof we show that (u, w, η) solves Problem (P^m). To this end we will pass to the limit in (39) as $n \to \infty$ and see that the limit relation fits the formulation of Problem (P^m). Since $\hat{u}_n' = \overline{w}_n(t)$, we conclude from (57) and (58) that

$$u'(t) = w(t) \text{ for a.e. } t \in (0, T). \tag{66}$$

It follows from $H(B)$ that the Nemytskii operator $\mathcal{B}$ is continuous and, since it is linear, it is also weakly continuous. Thus, from (56) we see that

$$\mathcal{B}\overline{u}_n \to \mathcal{B}\overline{u} \text{ weakly in } \mathcal{V}^* \text{ as } n \to \infty. \tag{67}$$

It is well known that

$$\pi_n^{q,V^*}(f) \to f \text{ strongly in } \mathcal{V}^* \text{ with } n \to \infty. \tag{68}$$

From (62), it is clear that

$$\overline{\iota}^* \overline{\eta}_n \to \overline{\iota}^* \eta \text{ weakly in } \mathcal{V}^*. \tag{69}$$

It remains to pass to the limit with the term $\pi_n^{q,V^*}(\mathcal{A}\overline{w}_n)$ in (39). It will be preceded by several auxiliary results, derived in what follows. First, by the direct calculation, we obtain

$$\langle \hat{w}_n', \hat{w}_n - \overline{w}_n \rangle_{\mathcal{V}^* \times \mathcal{V}} = \left(\frac{1}{2} - \Theta \right) \sum_{k=1}^{N_n} |w_n^k - w_n^{k-1}|^2 \le 0. \tag{70}$$

Using (70), we have

$$\begin{aligned}\langle \hat{w}_n', w - \overline{w}_n \rangle_{\mathcal{V}^* \times \mathcal{V}} &= \langle \hat{w}_n', w \rangle_{\mathcal{V}^* \times \mathcal{V}} - \langle \hat{w}_n', \hat{w}_n \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \hat{w}_n', \hat{w}_n - \overline{w}_n \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\le \langle \hat{w}_n', w \rangle_{\mathcal{V}^* \times \mathcal{V}} - \langle \hat{w}_n', \hat{w}_n \rangle_{\mathcal{V}^* \times \mathcal{V}} .\end{aligned}$$

Thus, by (60), we get

$$\limsup_{n\to\infty} \langle \hat{w}_n', w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}} \le \langle w', w \rangle_{\mathcal{V}^*\times\mathcal{V}} - \liminf_{n\to\infty} \langle \hat{w}_n', \hat{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}}$$

$$= \frac{1}{2}\left(|w(T)|^2 - |w(0)|^2 - \liminf_{n\to\infty} |\hat{w}_n(T)|^2 + \limsup_{n\to\infty} |\hat{w}_n(0)|^2 \right). \tag{71}$$

We observe that (59) implies $\hat{w}_n \to w$ weakly in $\mathcal{V}^*$. Taking into account (60) and the fact that the embedding $\{w \in \mathcal{V}^* \,|w' \in \mathcal{V}^*\} \subset C(0,T;V^*)$ is continuous, we claim that $\hat{w}_n \to w$ weakly in $C(0,T;V^*)$. Thus, in particular, we have $\hat{w}_n(T) \to w(T)$ weakly in V^*. On the other hand, from (43), it follows that $\hat{w}_n(T)$ is bounded in H, so for a subsequence still denoted by n, we may assume that $\hat{w}_n(T) \to h$ weakly in H (and, in consequence, weakly in V^*), where $h \in H$. From uniqueness of the limit, we have $\hat{w}_n(T) \to w(T)$ weakly in H. Since the norm is weakly lower semicontinuous, we have

$$|w(T)|^2 \le \liminf_{n\to\infty} |\hat{w}_n(T)|^2. \tag{72}$$

Analogously, we have $\hat{w}_n(0) \to w(0)$ weakly in H. On the other hand, we know, that $\hat{w}_n(0) = w_n^0 = u_{1n} \to u_1$ strongly in H and in consequence $\hat{w}_n(0) \to u_1$ weakly in H. Since the weak limit is unique, we have

$$w(0) = u_1, \text{ and } \hat{w}_n(0) \to w(0) \text{ in } H. \tag{73}$$

Combining (71)-(73), we obtain

$$\limsup_{n\to\infty} \langle \hat{w}_n', w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0. \tag{74}$$

Next, we observe that

$$\langle \mathcal{B}\overline{u}_n, w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle \mathcal{B}\hat{u}_n - \mathcal{B}u, w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}}$$

$$+ \langle \mathcal{B}u, w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathcal{B}(\overline{u}_n - \hat{u}_n), w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{75}$$

From (58) and (65), it follows that

$$\lim_{n\to\infty} \left(\langle \mathcal{B}u, w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathcal{B}(\overline{u}_n - \hat{u}_n), w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}} \right) = 0. \tag{76}$$

Using monotonicity of B, we obtain

$$\langle \mathcal{B}\hat{u}_n - \mathcal{B}u, w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle \mathcal{B}\hat{u}_n - \mathcal{B}u, u' - \hat{u}_n' \rangle_{\mathcal{V}^*\times\mathcal{V}} =$$

$$\frac{1}{2}\langle B(u(0) - \hat{u}_n(0)), u(0) - \hat{u}_n(0) \rangle - \frac{1}{2}\langle B(u(T) - \hat{u}_n(T)), u(T) - \hat{u}_n(T) \rangle$$

$$\le \frac{1}{2}\|B\|_{\mathcal{L}(V,V^*)}\|u(0) - \hat{u}_n(0)\|^2. \tag{77}$$

Since $\hat{u}_n' = \overline{w}_n$, from (57) and (58), it is clear that $\hat{u}_n \to u$ and $\hat{u}_n' \to u'$ both weakly in $\mathcal{V}$. Since the embedding $\{u \in \mathcal{V} \,|u' \in \mathcal{V}\} \subset C(0,T;V)$ is continuous, it results that $\hat{u}_n \to u$ weakly in $C(0,T;V)$ and, in particular, $\hat{u}_n(0) \to u(0)$ weakly in V. On the other hand, by assumption, we have $\hat{u}_n(0) = u_n^0 = u_{0n} \to u_0$ in V. Since the limit is unique, we have

$$u(0) = u_0, \text{ and } \hat{u}_n(0) \to u(0) \text{ in } V. \tag{78}$$

From (75)-(78), we get

$$\limsup_{n\to\infty} \langle \mathcal{B}\overline{u}_n, w - \overline{w}_n \rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0. \tag{79}$$

It follows from (39), (63), (68), (74) and (79), that

$$\limsup_{n\to\infty} \left\langle \pi_n^{q,V^*}(\mathcal{A}\overline{w}_n), \overline{w}_n - w \right\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0. \tag{80}$$

Now, we calculate

$$\begin{aligned}(81)\qquad &\left\langle \mathcal{A}\overline{w}_n - \pi_n^{q,V^*}(\mathcal{A}\overline{w}_n), \overline{w}_n \right\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &= \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \left\langle A(t, w_n^{k-1+\Theta}) - \frac{1}{\tau_n^k}\int_{t_n^{k-1}}^{t_n^k} A(s, w_n^{k-1+\Theta})ds, w_n^{k-1+\Theta} \right\rangle dt \\ &= \sum_{k=1}^{N_n} \left\langle \int_{t_n^{k-1}}^{t_n^k} A(t, w_n^{k-1+\Theta})dt - \int_{t_n^{k-1}}^{t_n^k} A(s, w_n^{k-1+\Theta})ds, w_n^{k-1+\Theta} \right\rangle dt = 0.\end{aligned}$$

Next, for any $\zeta \in \mathcal{V}$, the direct calculation gives

(82)

$$\begin{aligned}&\left\langle \mathcal{A}\overline{w}_n - \pi_n^{q,V^*}(\mathcal{A}\overline{w}_n), \zeta \right\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &= \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \left\langle A(t, w_n^{k-1+\Theta}) - \frac{1}{\tau_n^k}\int_{t_n^{k-1}}^{t_n^k} A(s, w_n^{k-1+\Theta})ds, \zeta(t) \right\rangle dt \\ &= \sum_{k=1}^{N_n} \left(\int_{t_n^{k-1}}^{t_n^k} \left\langle A(t, w_n^{k-1+\Theta}), \zeta(t) \right\rangle dt - \int_{t_n^{k-1}}^{t_n^k} \left\langle \frac{1}{\tau_n^k}\int_{t_n^{k-1}}^{t_n^k} A(s, w_n^{k-1+\Theta})ds, \zeta(t) \right\rangle dt \right) \\ &= \sum_{k=1}^{N_n} \left(\int_{t_n^{k-1}}^{t_n^k} \left\langle A(t, w_n^{k-1+\Theta}), \zeta(t) \right\rangle dt - \left\langle \int_{t_n^{k-1}}^{t_n^k} A(s, w_n^{k-1+\Theta})ds, \frac{1}{\tau_n^k}\int_{t_n^{k-1}}^{t_n^k} \zeta(t)dt \right\rangle \right) \\ &= \sum_{k=1}^{N_n} \left(\int_{t_n^{k-1}}^{t_n^k} \left\langle A(t, w_n^{k-1+\Theta}), \zeta(t) \right\rangle dt - \int_{t_n^{k-1}}^{t_n^k} \left\langle A(s, w_n^{k-1+\Theta}), (\pi_n^{p,V}(\zeta))(t) \right\rangle ds \right) \\ &= \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \left\langle A(t, w_n^{k-1+\Theta}), \zeta(t) - (\pi_n^{p,V}(\zeta))(t) \right\rangle dt = \left\langle \mathcal{A}\overline{w}_n, \zeta - \pi_n^{p,V}(\zeta) \right\rangle_{\mathcal{V}^*\times\mathcal{V}}.\end{aligned}$$

From $H(A)(iv)$ and (44), it follows, that the sequence $\mathcal{A}\overline{w}_n$ is bounded in $\mathcal{V}^*$. Moreover, it is clear that $\pi_n^{p,V}(\zeta) \to \zeta$ in $\mathcal{V}$. Thus, from (82), we have

$$(83)\qquad \mathcal{A}\overline{w}_n - \pi_n^{q,V^*}(\mathcal{A}\overline{w}_n) \to 0 \text{ weakly in } \mathcal{V}^* \text{ as } n \to \infty.$$

From (81) and (83), we get

$$(84)\qquad \left\langle \mathcal{A}\overline{w}_n - \pi_n^{q,V^*}(\mathcal{A}\overline{w}_n), \overline{w}_n - w \right\rangle_{\mathcal{V}^*\times\mathcal{V}} \to 0 \text{ as } n \to \infty.$$

It follows from (80) and (84), that

$$\begin{aligned}&\limsup_{n\to\infty} \langle \mathcal{A}\overline{w}_n, \overline{w}_n - w\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \limsup_{n\to\infty} \left\langle \pi_n^{q,V^*}(\mathcal{A}\overline{w}_n), \overline{w}_n - w \right\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ (85)\qquad &+ \lim_{n\to\infty} \left\langle \mathcal{A}\overline{w}_n - \pi_n^{q,V^*}(\mathcal{A}\overline{w}_n), \overline{w}_n - w \right\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0.\end{aligned}$$

From (48), (58), (85) and Lemma 14, we get $\mathcal{A}\overline{w}_n \to \mathcal{A}w$ weakly in $\mathcal{V}^*$. Combining it with (83), we obtain

$$(86)\qquad \pi_n^{q,V^*}(\mathcal{A}\overline{w}_n) \to \mathcal{A}w \text{ weakly in } \mathcal{V}^*.$$

Passing to the limit in the second equation in (39), and using (67)-(69) and (86), we conclude that the limit function w satisfies

$$(87)\qquad w' + \mathcal{A}w + \mathcal{B}w + \overline{\iota}^*\eta = f.$$

Now we pass to the limit with the inclusion in (39). From the definition of M_n^k, it follows that for a fixed $n \in \mathbb{N}$ and for all $k = 1, ..., N_n$ there exists $\zeta_n^k : (t_n^{k-1}, t_n^k) \to \mathcal{U}^*$ such that $\eta_n^k = \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} \zeta_n^k(t)\, dt$ and $\zeta_n^k(t) \in M(t, (\overline{\iota}\, \overline{z}_n)(t))$ for a.e. $t \in (t_n^{k-1}, t_n^k)$, where $\overline{z}_n(t) = \overline{u}_n(t)$ in Problem (P^1) and $\overline{z}_n(t) = \overline{w}_n(t)$ in Problem (P^2). We define the function $\zeta_n : [0, T] \to \mathcal{U}^*$ by $\zeta_n(t) = \zeta_n^k(t)$ for all $t \in (t_n^{k-1}, t_n^k)$ for $k = 1, ..., N_n$. Thus we have

$$\zeta_n(t) \in M(t, (\overline{\iota}\, \overline{z}_n)(t)) \text{ for a.e. } t \in (0, T). \tag{88}$$

By (40), (44), $H(M)(iv)$, $H(\iota)$ and H_{aux} we conclude that the sequence ζ_n remains bounded in $\mathcal{U}^*$. So, for a subsequence, we have

$$\zeta_n \to \zeta \text{ weakly in } \mathcal{U}^* \text{ as } n \to \infty. \tag{89}$$

Using (63), (64), (88), (89), $H(M)(ii) - (iii)$, and the fact, that every weakly$*$ compact set is closed, we can apply Proposition 13, and conclude that

$$\zeta(t) \in M(t, (\overline{\iota}\, z)(t)) \text{ for a.e. } t \in (0, T), \tag{90}$$

where $z = u$ in Problem (P^1) and $z = w$ in Problem (P^2). We define $\overline{\zeta}_n = \pi_n^{q, U^*}(\zeta)$ and we will show that

$$\overline{\zeta}_n - \overline{\eta}_n \to 0 \text{ weakly in } \mathcal{U}^*. \tag{91}$$

To this end, for any $\lambda \in \mathcal{U}$, by the Foubini theorem, we have

(92)

$$\begin{aligned}
&\left\langle \overline{\zeta}_n - \overline{\eta}_n, \lambda \right\rangle_{\mathcal{U}^* \times \mathcal{U}} = \int_0^T \left\langle \overline{\zeta}_n(t) - \overline{\eta}_n(t), \lambda(t) \right\rangle_{U^* \times U} \, dt \\
&= \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \left\langle \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} \zeta(s)\, ds - \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} \zeta_n(s)\, ds, \lambda(t) \right\rangle_{U^* \times U} dt \\
&= \sum_{k=1}^{N_n} \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} \int_{t_n^{k-1}}^{t_n^k} \langle \zeta(s) - \zeta_n(s), \lambda(t) \rangle_{U^* \times U} \, ds\, dt \\
&= \sum_{k=1}^{N_n} \int_{t_n^{k-1}}^{t_n^k} \left\langle \zeta(s) - \zeta_n(s), \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} \lambda(t)\, dt \right\rangle_{U^* \times U} ds = \left\langle \zeta - \zeta_n, \pi_n^{q,U}(\lambda) \right\rangle_{\mathcal{U}^* \times \mathcal{U}}.
\end{aligned}$$

Since $\pi_n^{q,U}(\lambda) \to \lambda$ strongly in $\mathcal{U}$, we obtain (91) from (89) and (92). Moreover, since $\overline{\zeta}_n \to \zeta$ strongly in $\mathcal{U}^*$, we conclude from (91) that $\overline{\eta}_n \to \zeta$ weakly in $\mathcal{U}$. Since the weak limit is unique, we have $\zeta = \eta$ and, from (90) we derive

$$\eta(t) \in M(t, (\overline{\iota}\, z)(t)) \text{ for a.e. } t \in (0, T). \tag{93}$$

Summarizing, from (66), (73), (78), (87), and (93) it follows, that the triple (u, w, η) solves Problem (P^m), $m = 1, 2$, which completes the proof. □

5. Strong convergence

In this section we deal with a strong convergence result of the Rothe method. To this end, we impose more restrictive assumption on the operator A.

$H(A)_1$: The operator $A : [0, T] \times V \to V^*$ satisfies: $A(t, \cdot)$ is of type $(S)_+$ for a.e. $t \in (0, T)$.

Now we formulate the following theorem.

THEOREM 25. *Let all assumptions of Theorem 24 hold and, moreover, operator A satisfies $H(A)_1$. Then $\hat{u}_n \to u$ strongly in $W^{1,p}(0,T;V)$ as $n \to \infty$.*

Proof. From (48), (58) and (85), we know that $\overline{w}_n$ is bounded in $M^{p,q}(0,T;V,V^*)$, $\overline{w}_n \to w$ weakly in $\mathcal{V}$ and $\limsup_{n\to\infty} \langle \mathcal{A}\overline{w}_n, \overline{w}_n - w\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0$. Thus, by Lemma 15, we have $\overline{w}_n \to w$ strongly in $\mathcal{V}$ as $n \to \infty$. Since $\hat{u}_n' = \overline{w}_n$ and $u' = w$, we also have for $C > 0$

$$\begin{aligned}\|\hat{u}_n - u\|_{\mathcal{V}}^p &= \int_0^T \|\hat{u}_n(t) - u(t)\|^p dt \\ &\le C \int_0^T \left(\|\hat{u}_n(0) - u(0)\|^p + \int_0^t \|\overline{w}_n(s) - w(s)\|^p ds \right) dt \\ &\le C(T\|u_{0n} - u_0\|^p + T\|\overline{w}_n - w\|_{\mathcal{V}}^p) \to 0 \text{ with } n \to \infty.\end{aligned}$$

It follows that $\hat{u}_n \to u$ strongly in $W^{1,p}(0,T;V)$, which completes the proof. □

COROLLARY 26. *Let all assumptions of Theorem 24 hold. Assume moreover, that operator A satisfies*

$$\langle A(t,v) - A(t,u), v - u\rangle \ge \gamma_1 \|v - u\|^2 - \gamma_2 |v - u|^2, \tag{94}$$

for all $u, v \in V$, a.e. $t \in (0,T)$, with $\gamma_1 > 0$, $\gamma_2 \ge 0$. Then $\hat{u}_n \to u$ strongly in $W^{1,p}(0,T;V)$ as $n \to \infty$.

Proof. We will show, that operator A satisfies $H(A)_1$. Let us fix $t \in (0,T)$, for which (94) holds true and let us consider a sequence $\{v_n\} \subset V$, such that $v_n \to v$ weakly in V. Assume that $\limsup_{n\to\infty} \langle A(t,v_n), v_n - v\rangle \le 0$. By (94), we get

$$\begin{aligned}&\lim_{n\to\infty} \gamma_1 \|v_n - v\|^2 \le \limsup_{n\to\infty} \langle A(t,v_n), v_n - v\rangle \\ &\quad - \lim_{n\to\infty} \langle A(t,v), v_n - v\rangle + \gamma_2 \lim_{n\to\infty} |v_n - v|^2 \le \gamma_2 \lim_{n\to\infty} |v_n - v|^2.\end{aligned} \tag{95}$$

Since the embedding $V \subset H$ is compact, then, from weak convergence $v_n \to v$ in V, it follows that $\lim_{n\to\infty} |v_n - v| = 0$ and, from (95), also $\lim_{n\to\infty} \gamma_1 \|v_n - v\| = 0$. We conclude that the operator $A(t,\cdot)$ is of class $(S)_+$ for a.e. $t \in (0,T)$, and we can apply Theorem 25 to deduce the thesis. □

6. Boundary hemivariational inequalities

In this section we apply our result of Theorem 24 to a class of boundary hemivariational inequalities.

We restrict ourselves to the case $p \ge 2$. Let Ω be an open bounded subset of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ and let Γ be an open subset of $\partial\Omega$ with positive surface measure. According to the notation introduced at the beginning of Section 3, we define the following spaces. Let $V = \{v \in W^{1,p}(\Omega;\mathbb{R}^d) | \ \gamma v = 0 \text{ on } \partial\Omega \setminus \Gamma\}$ and $H = L^2(\Omega;\mathbb{R}^d)$, where $\iota : W^{1,p}(\Omega;\mathbb{R}^d) \to U := L^p(\Gamma;\mathbb{R}^d)$ denotes the trace operator. Identifying H with its dual, we have an evolution triple $V \subset H \subset V^*$ with dense, continuous and compact embeddings. Let $T > 0$ and let the spaces $\mathcal{V}$, $\mathcal{H}$, $\mathcal{W}$ and $\mathcal{U}$ be defined as in the Section 3.

We consider the following two types of dynamic hemivariational inequalities.

Problem $\mathbf{HVI_1}$ Find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases}\langle u''(t) + A(t,u'(t)) + Bu(t) - f(t), v\rangle \\ \quad + \int_\Gamma j^0(x,t,\gamma u(t);\gamma v)d\gamma(x) \ge 0 \text{ for all } v \in V \text{ and a.e.}\, t \in (0,T), \\ u(0) = u_0, \ \ u'(0) = u_1.\end{cases}$$

Problem $\mathbf{HVI_2}$ Find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} \langle u''(t) + A(t, u'(t)) + Bu(t) - f(t), v\rangle \\ \quad + \int_\Gamma j^0(x, t, \gamma u'(t); \gamma v) d\gamma(x) \geq 0 \text{ for all } v \in V \text{ and a.e. } t \in (0, T), \\ u(0) = u_0, \ u'(0) = u_1. \end{cases}$$

In the above problems $j^0(x, t, \gamma u(t); \gamma v)$ and $j^0(x, t, \gamma u'(t); \gamma v)$ denote the Clarke directional derivative with respect to the third variable in the direction γv. We impose the following assumptions on the function j.

$H(j)$: $j : \Gamma \times (0, T) \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j(\cdot, \cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^d$ and $j(\cdot, t, 0) \in L^1(\Gamma)$, for a.e. $t \in (0, T)$,

(ii) $j(x, t, \cdot)$ is locally Lipschitz for a.e. $x \in \Gamma$, $t \in (0, T)$,

(iii) $|\eta|_{\mathbb{R}^d} \leq c_j(1 + |\xi|^{p-1}_{\mathbb{R}^d})$ for all $\eta \in \partial j(x, t, \xi)$, a.e. $t \in (0, T)$, $x \in \Gamma$ with $c_j > 0$.

In the assumption $H(j)$ the symbol ∂j denotes the Clarke subdifferential of j with respect to the variable ξ.

Let us define the constant $c_J = c_j 2^{\frac{1}{p}} \max\{1, |\Gamma|^{\frac{1}{q}}\}$, where $|\Gamma|$ denotes $d-1$ dimensional measure of Γ and impose two additional hypotheses.
$\tilde{H}_1$: $c_J 3^{p-2} T^{p-1} \|\iota\|^p < \alpha$.
$\tilde{H}_2$: $c_J \|\iota\|^p < \alpha$.

Now we formulate the theorem on existence of solution to Problems HVI_i, $i = 1, 2$.

THEOREM 27. *Let assumptions $H(A)$, $H(B)$, $H(j)$ and H_0 hold. Moreover, if $p > 2$, we assume the condition $\tilde{H}_1$ (and $\tilde{H}_2$, respectively). Then Problem HVI_1 (and HVI_2, respectively) admits a solution.*

Before passing to the proof of Theorem 27 we define functional $J : (0, T) \times U \to \mathbb{R}$ given by

$$J(t, v) = \int_\Gamma j(x, t, v(x)) d\Gamma, \text{ for all } v \in U, \text{ a.e. } t \in (0, T),$$

and a multivalued operator $M : (0, T) \times U \to 2^{U^*}$ given by $M(t, v) = \partial J(t, v)$ for all $v \in U$, a.e. $t \in (0, T)$. We remark, that under the hypotheses $H(j)$ both J and M are well defined. Now we formulate two auxiliary problems corresponding to Problem HVI_1 and HVI_2 respectively.

Problem $\boldsymbol{Q_1}$ Find $u \in \mathcal{V}$ with $u' \in \mathcal{W}$ and $\eta \in \mathcal{U}^*$ such that

$$\begin{cases} u''(t) + A(t, u'(t)) + Bu(t) + \gamma^* \eta(t) = f(t) \text{ a.e. } t \in (0, T), \\ \eta(t) \in M(t, \gamma u(t)) \text{ a.e. } t \in (0, T), \\ u(0) = u_0, \ u'(0) = u_1. \end{cases}$$

Problem Q_2 Find $u \in \mathcal{V}$ with $u' \in \mathcal{W}$ and $\eta \in \mathcal{U}^*$ such that

$$\begin{cases} u''(t) + A(t, u'(t)) + Bu(t) + \gamma^*\eta(t) = f(t) \ \text{ a.e.}\, t \in (0,T), \\ \eta(t) \in M(t, \gamma u'(t)) \ \text{ a.e.}\, t \in (0,T), \\ u(0) = u_0, \ \ u'(0) = u_1. \end{cases}$$

Now we formulate three lemmas concerning properties of the data of Problems Q_i, $i = 1, 2$.

Lemma 28. *If the assumption $H(j)$ holds, then the multivalued operator M satisfies assumptions $H(M)$ with $c_M = c_J$.*

Proof. The hypothesis $H(M)(i)$ follows from Proposition 3.44 of [26]. The hypotheses $H(M)(ii)$ and $H(M)(iii)$ follow from Proposition 3.23 ((iv) and (vi), respectively) of [26]. Finally $H(M)(iv)$ follows directly from $H(j)(iii)$. □

Lemma 29. *The operator ι satisfies assumption $H(\iota)$.*

Proof. Let $\varepsilon \in (0, \frac{1}{2})$. Then the embedding $i\colon V \to W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d)$ is compact. The trace operator $\gamma_1 : W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d) \to W^{\frac{1}{2}-\varepsilon,p}(\Gamma;\mathbb{R}^d)$ is linear and continuous and, finally, the embedding $j : W^{\frac{1}{2}-\varepsilon,p}(\Gamma;\mathbb{R}^d) \to L^p(\Gamma;\mathbb{R}^d) = U$ is also linear and continuous. Thus $\iota = j \circ \gamma_1 \circ i$ is linear, continuous and compact. Moreover, the spaces $V \subset W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d) \subset V^*$ satisfy assumptions of Proposition 12 so the embedding $M^{p,q}(0,T;V,V^*) \subset L^p(0,T;W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d))$ is compact. Since the embedding $L^p(0,T;W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d)) \subset \mathcal{U}$ is continuous the Nemytskii operator corresponding to ι is compact. □

Lemma 30. *The space U satisfies assumption $H(U)$.*

Proof. As in the proof of Lemma 29 we take $\varepsilon \in (0, \frac{1}{2})$. Since the embedding $V \subset W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d)$ is compact and $W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d) \subset H$ is continuous, we can apply the Ehrling Lemma (cf. Lemma 7.6 of [34]). Thus, for any $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that for all $v \in V$

$$\|v\|_{W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d)} \leq \varepsilon\|v\| + C(\varepsilon)|v|. \tag{96}$$

Using notation from the proof of Lemma 29, we have for all $v \in V$

$$\|\iota v\|_U = \|(j \circ \gamma_1)v\|_U \leq c\|v\|_{W^{1-\varepsilon,p}(\Omega;\mathbb{R}^d)},$$

with $c > 0$. This together with (96) completes the proof. □

Proof of Theorem 27. First, we note that if assumption $H(j)$ holds, then every solution to Problem Q_i is a solution to Problem HVI$_i$, $i = 1, 2$ (see Remark 4 in [22]). Thus, it is enough to establish the solvability of Problems Q_i, $i = 1, 2$. Taking $\Theta \geq \frac{1}{2}$ we consider a time discretization Θ-scheme described in Section 4 based on the uniform partition of the interval $[0, T]$. Then the assumption $H(t)$ is satisfied with the constant $K = 1$. Observe, that in this particular case, assumptions $\tilde{H}_1$ and $\tilde{H}_2$ are equivalent with H_1 and H_2. Using assumptions $H(A)$, $H(B)$ and Lemmas 28-30, we can apply Theorem 24 for Problems Q_i, $i = 1, 2$ and obtain their solutions. The proof is complete. □

Remark 31. *From the Corollary 26, we have a strong convergence of the approximate sequence $\hat{u}_n$ to a solution of Problem HVI$_i$, $i = 1, 2$ provided the operator A satisfies condition (94).*

Remark 32. *The existence result for Problems HVI_i, $i = 1, 2$, has been obtained in* [22] *only for* $p = 2$ *and under the restrictions analogous to assumptions* $\tilde{H}_i$, $i = 1, 2$. *In Theorem 27 we do not assume* $\tilde{H}_1$, $\tilde{H}_2$ *in case* $p = 2$. *Moreover, in* [22], *the operator* A *is assumed to satisfy* $\langle A(t, v), v\rangle \geq \alpha\|v\|^2$ *a.e.* $t \in (0, T)$, *for all* $v \in V$ *with* $\alpha > 0$, *which is more restrictive assumption than* $H(A)(iii)$. *Thus our result is stronger, than the one obtained in* [22].

7. Application to non-clamped dynamic contact problems

In this section, we shortly explain, what is the motivation to relax the classical coercivity assumption by means of less restrictive condition $H(A)(iii)$.
Hemivariational inequalities studied in Section 6 arise from nonmonotone contact problems in mechanics. We refer again to [22] for details and to [26] for the current stay of art. In the typical setting, a physical body occupies a region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and the main unknown in the problem is displacement field $u : \Omega \to \mathbb{R}^d$. One defines also a deformation operator given by $\varepsilon(u) = (\varepsilon_{ij}(u))$, $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, $i, j = 1, ..., d$ and the spaces $H = L^2(\Omega)^d$, $H_1 = \{u \in L^2(\Omega)^d, \mid \varepsilon_{ij}(u) \in L^2(\Omega), i, j = 1, ..., d\}$ with the norms given by $\|u\|_H^2 = \int_\Omega u^2\, dx$, $\|u\|_{H_1}^2 = \int_\Omega u^2\, dx + \int_\Omega \varepsilon(u) : \varepsilon(u)\, dx$. It is also defined viscosity operator $\mathcal{A} : \mathbb{S}^d \to \mathbb{S}^d$, which in particular satisfies the coercivity condition

$$(97) \qquad \mathcal{A}(x, t, \tau) : \tau \geq \alpha\tau : \tau \text{ for all } \tau \in \mathbb{S}^d, \text{ a.e. } (x, t) \in \Omega \times [0, T] \text{ with } \alpha > 0.$$

The typical assumption met in the literature (see [26]) is that $u = 0$ on a part of boundary $\Gamma_D \subset \partial\Omega$, where Γ_D has a positive measure. It means, that the body is clamped at Γ_D. This assumption allows to use the following Korn's inequality $\int_\Omega u^2\, dx \leq C \int_\Omega \varepsilon(u) : \varepsilon(u)\, dx$ for all $u \in V := \{v \in H_1, \mid v = 0 \text{ on } \Gamma_D\}$ with $C > 0$ and introduce a norm $\|u\|_V = \left(\int_\Omega \varepsilon(u) : \varepsilon(u)\, dx\right)^{\frac{1}{2}}$, equivalent with $\|u\|_{H_1}$ for all $u \in V$. The advantage of the new norm is that the operator $A : [0, T] \times V \to V^*$ defined by $\langle A(t, u), v\rangle_{V^* \times V} = \int_\Omega \mathcal{A}(x, t, \varepsilon(u(x)) : \varepsilon(u(x)))\, dx$ is coercive. Namely, from (97), one has $\langle A(t, u), u\rangle_{V^* \times V} \geq \alpha\|u\|_V^2$ for all $t \in [0, T]$, $u \in V$. This property of operator A is exploited in many publications concerning existence result for HVI's as well. We remind, that Theorem 27 allows us to obtain the existence result under more general assumption $H(A)(iii)$. Thus, we can proceed in a different way, namely put $V = H_1$ and observe, that (97) implies $\langle A(t, u), u\rangle_{V^* \times V} \geq \alpha \int_\Omega \varepsilon(u) : \varepsilon(u)\, dx = \alpha\|u\|_V^2 - \alpha\|u\|_H^2$, which is equivalent to $H(A)(iii)$ with $\beta = \alpha$ and allows us to apply our theoretical existence result. In this case, we do not need to change the norm, and the Korn's inequality is useless. In the consequence, we can avoid the assumption, that the body is clamped on the part of boundary.

8. Acknowledgements

- I am deeply grateful to my colleague, Piotr Kalita, for his support, inspiration and useful advices concerning time discretization methods in evolution inclusions.
- Research supported by the Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme under Grant Agreement No. 295118, the National Science Center of Poland under the Maestro Advanced Project no. DEC-2012/06/A/ST1/00262, the project Polonium "Mathematical and Numerical Analysis for Contact Problems with Friction" 2014/15 between the Jagiellonian University and Université de Perpignan Via Domitia, and the

International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under Grant no. W111/7.PR/2012.

References

[1] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, Basel, Berlin (1990).

[2] M. Barboteu, K. Bartosz, W. Han, T. Janiczko, Numerical analysis of a dynamic contact problem with nonmonotone slip-rate dependent friction, SIAM Journal of Numerical Analysis 53 (2015), 527–550.

[3] M. Barboteu, K. Bartosz, P. Kalita, An analytical and numerical approach to a bilateral contact problem with nonmonotone friction, International Journal of Applied Mathematics and Computer Science 23 (2013), 263–276.

[4] M. Barboteu, K. Bartosz, P. Kalita, A. Ramadan, Analysis of a contact problem with normal compliance, finite penetration and nonmonotone slip dependent friction, Communications in Contemporary Mathematics 16 (2014), 1350016 [29 pages], DOI: 10.1142/S0219199713500168.

[5] K. Bartosz, Hemivariational inequality approach to the dynamic viscoelastic sliding contact problem with wear, Nonlinear Analysis 65 (2006), 546–566.

[6] K. Bartosz, Numerical Methods for Evolution Hemivariational Inequalities, Chapter 5 in W. Han, S. Migórski, M. Sofonea, Advances in Variational and Hemivariational Inequalities. Theory, Numerical Analysis and Applications, Advances in Mechanics and Mathematics 33, Springer, New York, 2015.

[7] K. Bartosz, X. Cheng, P. Kalita, Y. Yu, C. Zheng, Rothe method for evolution variational-hemivariational inequalities, Journal of Mathematical Analysis and Applications 423 (2015), 841–862.

[8] S. Carl, Existence and extremal solutions of parabolic variational-hemivariational inequalities, Monatshefte für Mathematik 172 (2013), 29–54.

[9] S. Carl and V. K. Le, Quasilinear parabolic variational inequalities with multi-valued lower order terms, Zeitschrift für Angewandte Mathematik und Physik 65 (2014), 845-864.

[10] F. H. Clarke, Optimization and Nonsmooth Analysis, Wiley–Interscience, New York (1983).

[11] J. Czepiel, P. Kalita, Numerical solution of a variational-hemivariational inequality modelling simplified adhesion of an elastic body, IMA Journal of Numerical Analysis, 35 (2015), 372-393.

[12] M.N. Dao, J. Gwinner, D. Noll, N. Ovcharova, Nonconvex bundle method with application to a delamination problem, arXiv:1401.6807.

[13] J. Gwinner, N. Ovcharova, On Moreau-Yosida approximation and on stability of second-order subdifferentials of Clarke's type, Journal of Convex Analysis, 14(2) (2007), 413-432.

[14] D. Goeleven, D. Motreanu, Y. Dumont, M. Rochdi, Variational and Hemivariational Inequalities: Theory, Methods and Applications, Kluwer Academic Publishers, Boston, Dordrecht, London (2003).

[15] J. Haslinger, M. Miettinen, P.D. Panagiotopoulos, Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999.

[16] P. Kalita, Convergence of Rothe scheme for hemivariational inequalities of parabolic type, International Journal of Numerical Analysis and Modeling 10 (2013), 445–465.

[17] P. Kalita, Regularity and Rothe method error estimates for parabolic hemivariational inequality, Journal of Mathematical Analysis and Applications 389 (2012), 618-631.

[18] P. Kalita, Semidiscrete variable time-step θ-scheme for nonmonotone evolution inclusion, arXiv:1402.3721

[19] Z. Liu, D. Motreanu, A class of variational-hemivariational inequalities of elliptic type, Nonlinearity 23 (2010), 1741–1752.

[20] M. Miettinen, J. Haslinger, Finite element approximation of parabolic hemivariational inequalities, phNumerical Functional Analysis and Optimization 19 (5&6) (1998), 565–585.

[21] M. Miettinen, P. D. Panagiotopoulos, On parabolic hemivariational inequalities and applications, Nonlinear Analysis 35 (1999), 885–915.

[22] S. Migórski, Boundary hemivariational inequalities of hyperbolic type and applications, Journal of Global Optymization 31 (2005), 505–533.

[23] S. Migorski, On the existence of solutions for parabolic hemivariational inequalities, Journal of Computational and Applied Mathematics 129 (2001), 77–87.

[24] S. Migórski, A. Ochal, A unified approach to dynamic contact problems in viscoelasticity, Journal of Elasticity 83 (2006), 247–275.

[25] S. Migorski, A. Ochal, Boundary hemivariational inequality of parabolic type, Nonlinear Analysis Series A: Theory Methods and Applications 57 (2004), 579–596.
[26] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics 26, Springer, New York, (2013).
[27] Z. Naniewicz, P. D. Panagiotopoulos, phMathematical Theory of Hemivariational Inequalities and Applications, Marcel Dekker, Inc., New York, Basel, Hong Kong, (1995).
[28] P. D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer-Verlag, Berlin (1993).
[29] P. D. Panagiotopoulos, Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions, Birkhäuser, Basel (1985).
[30] P.D. Panagiotopoulos, Modelling of nonconvex nonsmooth energy problems: dynamic hemivariational inequalities with impact effects, Journal of Computational and Applied Mathemtaics 63 (1995), 123–138.
[31] Z. Peng, Z. Liu, Evolution hemivariational inequality problems with doubly nonlinear operators, phJournal of Global Optimization 51 (2011), 413-427.
[32] Z. Peng, Z. Liu, X. Liu, Boundary hemivariational inequality problems with doubly nonlinear operators, phMathematische Annalen 356 (2013), 1339-1358.
[33] Z. Peng, C. Xiao, Existence and convergence theorems for evolutionary hemivariational inequalities of second order, phElectronic Journal of Differential Equations 65 (2015), 1-17.
[34] T. Roubiček, Nonlinear Partial Differential Equations with Applications, Birkhäuser Verlag, Basel, Boston, Berlin (2005).
[35] G. Wang, X. Yang, Finite difference approximation of a parabolic hemivariational inequalities arising from temperature control problem, International Journal of Numerical Analysis and Modeling 7 (2010), 108–124.
[36] E. Zeidler, Nonlinear Functional Analysis and its Applications, vol. II/B: Nonlinear Monotone Operators, Springer-Verlag, New York, Berlin, Heidelberg (1990).

Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, 30–348 Krakow, Poland
E-mail: bartosz@ii.uj.edu.pl
URL: http://kos.ii.uj.edu.pl/Bartosz