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A PRIORI ERROR ANALYSIS OF THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE VISCOUS BURGERS-POISSON SYSTEM

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Abstract. In this paper, we propose and analyze the local discontinuous Galerkin method for the viscous Burgers-Poisson system. The proposed method preserves two invariants and hence, yields solutions even for long time. *A priori* error estimates, which are of order $\mathcal{O}(h^{k+1})$, when polynomials of degree $k \geq 1$ are used for approximating solutions are established. Finally, numerical experiments are conducted to confirm our theoretical results.

Key words. Burgers-Poisson system, local discontinuous Galerkin method, *A priori* error estimates.

1. Introduction

We consider the following coupled system of viscous Burgers and Poisson equations: find a pair of solutions (u, ϕ) such that

$$u_t + (\frac{u^2}{2} - \phi)_x - \epsilon u_{xx} = 0, \qquad x \in [0, L] = I, \quad t > 0, \tag{1}$$

$$\phi_{xx} - \phi = u, \tag{2}$$

with $\epsilon > 0$ and periodic boundary conditions:

$$\begin{aligned} &u(t, L) = u(t, 0),\ u_x(t, L) = u_x(t, 0) \text{ and} \\ &\phi(t, L) = \phi(t, 0),\ \phi_x(t, L) = \phi_x(t, 0),\ \text{for } t > 0, \end{aligned} \tag{3}$$

and initial condition:

$$u(0, x) = u_0(x), \qquad x \in I. \tag{4}$$

This problem is one dimensional version of the Navier-Stokes-Poisson system, which often models the transport of charged particles under the influence of the self-consistent electro-static potential as a force arising in the study of collision of dusty plasma, see [7], [9]. This system admits conservation of momentum and L^2 *a priori* bound. Global existence of weak solutions to the Navier-Stokes-Poisson system with large initial data has been proved by Donatelli [6] using Galerkin method and P.L.Lions theory, [13]. Without much difficulty, this theory can be extended to include the global existence of a unique solution for the Burgers-Poisson system (1)-(4).

In recent years, Discontinuous Galerkin (DG) methods are becoming popular due to their flexibility in local mesh adaptivity, element wise conservative property and in taking care of nonuniform degrees of approximation of the solution whose smoothness may exhibit a wide variation over the computational domain. These methods are using completely discontinuous piecewise-polynomials for the numerical solution and the test functions. These schemes are first proposed for solving first order PDEs such as nonlinear conservation laws, [14], [1], [2], [3], [4]. The

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local discontinuous Galerkin (LDG) method is an extension of DG methods for solving higher order PDEs. It was first designed for convection-diffusion equations in [5], and has been extended to other higher order wave equations, including the KdV equation, [19], [16], [11], [17], see, also the recent review paper [18] on the LDG methods for higher order PDEs. The idea of the LDG method is to rewrite higher order equations into a first order system, and then apply DG schemes on the system with appropriate choices of numerical fluxes. Related to our problem, a LDG method was proposed in [12] for the inviscous Burgers-Poisson equation. This scheme preserves the mass and energy of the smooth solution and was proven to be optimal convergence for k even.

In this article, LDG method is applied to the viscous Burgers-Poisson system (1)-(4). Then, it is observed that the semidiscrete system preserves two invariants and as a result, we prove *a priori* bounds in $L^\infty(L^2)$ for the discrete solutions. It is, further, shown that rate of convergence is of order $k+1$ for approximate solution u_h, when polynomial of order k is used to approximate u. The generalized numerical fluxes, which depend on a parameter $\theta \in [0, 1/2]$ are used in the proposed scheme. For $\theta = 1/2$, it is noted that the order of convergence is optimal as in [12] for even degree polynomial degrees. When $\theta \in [0, 1/2)$, optimal error estimates are derived, but with constants in the error analysis explicitely depend on $1/\sqrt{\epsilon}$, where ϵ is a viscosity parameter.

We use standard notation for norms and seminorms in Sobolev spaces. Say for example, for any integer $m \geq 0$, we denote by $H^m(I)$, the Hilbert Sobolev space with norm $\|\cdot\|_m$ and seminorm $|\cdot|_m$. We also use the spaces $L^p(0,T;H^m(I)),\ 1 \leq p \leq \infty$ as the spaces of functions v such that $\int_0^T \|v(s)\|^p_{H^m(I)}\, ds < \infty$. Denote by C a positive generic constant, which does not depend on the mesh parameters, but may vary from context to context in the text.

2. Conservation Properties and *A Priori* Bounds

This section deals with some conservation properties and *a priori* bounds for the viscous Burgers-Poisson system (1)-(4).

Theorem 2.1. *Let (u, ϕ) be a pair of solutions of the coupled system* (1)-(4)*. Then the following conservation property holds:*

$$\int_0^L u(x,t)dx = \int_0^L u_0(x)\ dx. \tag{5}$$

Further, u satisfies

$$\int_0^L |u(x,t)|^2 dx \leq \int_0^L |u_0(x)|^2\ dx. \tag{6}$$

Proof. Integrating equation (1) with respect to space variable x yields

$$\int_0^L u_t\ dx + \int_0^L (\frac{u^2}{2})_x\ dx - \int_0^L \phi_x\ dx - \epsilon \int_0^L u_{xx}\ dx = 0,$$

which can be rewritten using periodic boundary conditions as

$$\frac{d}{dt}\int_0^L u(t,x)\ dx = 0.$$

Integrating above equation with respect to time t yields the equation of conservation of momentum, that is,

$$\int_0^L u(t,x)\ dx = \int_0^L u(0,x)\ dx = \int_0^L u_0(x)\ dx. \tag{7}$$

For (6), multiply equation (2) with term ϕ_x and then integrate it with respect to x from 0 to L. Now use of periodic boundary condition yields $\int_0^L u\phi_x = 0$. We then multiply (1) with the term u and integrate it with respect to x to obtain

$$\int_0^L u_t u \, dx + \int_0^L (\frac{u^2}{2})_x u \, dx - \int_0^L \phi_x u \, dx - \epsilon \int_0^L u_{xx} u \, dx = 0,$$

which can again be rewritten using integration by parts as

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2(t,x) \, dx = -\epsilon \int_0^L (u_x)^2 \, dx.$$

On integrating with respect to time leads to

$$\int_0^L u^2(t,x) \, dx + 2\epsilon \int_0^t \int_0^L (u_x)^2 \, dx \, dt = \int_0^L u_0^2(x) \, dx. \tag{8}$$

Since the second term on the left hand side of (8) is non-negative, hence, it completes the estimate (6). □

Remark 2.1. In stead of periodic boundary conditions, if we use homogeneous Dirichlet boundary conditions, then the conservation of momentum property (5) is not valid, where as the property (6) holds. In case of homogeneous Neumann boundary condition, the conservation property (5) remains valid.

3. LDG Method

In this section, a local discontinuous Galekin method is proposed for approximating solutions of the Burgers-Poisson equation (1)-(2) subject to initial data $u_0(x)$, posed on $I = [0, L]$ with periodic boundary conditions in spatial direction. The resulting semidiscrete scheme also admits conservation of momentum.

To describe the method, the interval I is partitioned into N sub-interval with the partition $0 = x_{1/2}, x_{3/2}, ..., x_{N+1/2} = L$. Let $I_j = [x_{j-1/2}, x_{j+1/2}]$ with mesh size $h_j = x_{j+1/2} - x_{j-1/2}$ for $j = 1, 2, ...N$ and the center of cell be denoted by $x_j = \frac{1}{2}(x_{j-1/2} + x_{j+1/2})$.

Let V_h^k be defined as the space of piece-wise polynomials of degree up to k in each cell I_j, that is,

$$V_h^k = \{v_h : v_h|_{I_j} \in P^k(I_j), j = 1, 2, ...N\}.$$

Since functions belonging to V_h^k are allowed to have discontinuities across the cell interfaces, then for $v_h \in V_h^k$, v_h may have two different values on cell interface and denote $(v_h)_{j+1/2}^-$ and $(v_h)_{j+1/2}^+$, respectively, by the limit values of v_h at $x_{j+1/2}$ from the left and right. Now, set the jump and average across the cell interface as $[v_h] := v_h^+ - v_h^-$ and $\{v_h\} := \frac{v_h^+ + v_h^-}{2}$, respectively. For piece-wise function v with $v|_{I_j} \in H^m(I_j)$, set the discrete H^m-norm as

$$\|v\|_m := \Big(\sum_{j=1}^N \|v\|^2_{H^m(I_j)}\Big)^{1/2} \quad \text{and the seminorm as} \quad |v|_m := \Big(\sum_{j=1}^N \|\frac{d^m v}{dx^m}\|^2_{L^2(I_j)}\Big)^{1/2}.$$

For elements in V_h^k, we have the following inverse property and trace inequality:

(i) Inverse Property. For $v_h \in V_h^k$,

$$\|v_h\|_{L^\infty} \le C\, h^{-1/2}\, \|v_h\| \quad \text{and} \quad \|v_{hx}\| \le C\, h^{-1}\, \|v_h\|. \tag{9}$$

(ii) Trace Inequality. For any $v_h \in V_h^k$,

$$\|v_h\|_{\Gamma_h} \le C\, h^{-1/2}\, \|v_h\|, \tag{10}$$

where

$$\|v_h\|_{\Gamma_h} := \Big(\sum_{j=1}^{N} \Big(|(v_h)_{j+1/2}^-|^2 + |(v_h)_{j+1/2}^+|^2 \Big) \Big)^{1/2}.$$

For LDG method, first rewrite (1)-(2) by introducing two auxiliary variables $w = \sqrt{\epsilon} u_x$ and $p = \phi_x$ as :

$$u_t + (\frac{u^2}{2})_x - p - \sqrt{\epsilon} w_x = 0, \tag{11}$$

$$w - \sqrt{\epsilon} u_x = 0, \tag{12}$$

$$p - \phi_x = 0, \tag{13}$$

$$p_x - \phi = u. \tag{14}$$

Now, the LDG method is to seek $(u_h, p_h, \phi_h, w_h) \in \Big(V_h^k\Big)^4$ such that for $(v, z, \psi, q) \in \Big(V_h^k\Big)^4$

$$\int_{I_j} (u_h)_t v\, dx - \int_{I_j} (\frac{u_h^2}{2}) v_x\, dx + \frac{\widehat{u_h^2 v}}{2}|_{\partial I_j} - \int_{I_j} p_h v\, dx + \sqrt{\epsilon} \int_{I_j} w_h v_x\, dx - \sqrt{\epsilon} \hat{w}_h v|_{\partial I_j} = 0, \tag{15}$$

$$\int_{I_j} w_h z\, dx + \int_{I_j} \sqrt{\epsilon} u_h z_x\, dx - \sqrt{\epsilon} \hat{u}_h z|_{\partial I_j} = 0, \tag{16}$$

$$\int_{I_j} p_h \psi\, dx + \int_{I_j} \phi_h \psi_x\, dx - \hat{\phi}_h \psi|_{\partial I_j} = 0, \tag{17}$$

$$-\int_{I_j} p_h q_x\, dx - \int_{I_j} (\phi_h + u_h) q\, dx + \hat{p}_h q|_{\partial I_j} = 0, \tag{18}$$

$$\int_{I_j} (u_h - u)|_{t=0} v\, dx = 0. \tag{19}$$

Here, the choice of numerical fluxes $\widehat{u_h^2}, \hat{\phi}_h, \hat{p}_h, \hat{u}_h, \hat{w}_h$ are given, respectively, by

$$\widehat{u_h^2} = \frac{1}{3}((u_h^+)^2 + u_h^+ u_h^- + (u_h^-)^2), \tag{20}$$

$$\hat{\phi}_h = \theta \phi_h^+ + (1-\theta)\phi_h^-, \tag{21}$$

$$\hat{p}_h = (1-\theta) p_h^+ + \theta p_h^-, \tag{22}$$

$$\hat{u}_h = \theta u_h^+ + (1-\theta) u_h^-, \tag{23}$$

$$\hat{w}_h = (1-\theta) w_h^+ + \theta w_h^-, \tag{24}$$

where $\theta \in [0, 1/2]$. Note that the numerical fluxes at the endpoints of I are defined using $U_{1/2}^- := U_{N+1/2}^-$ and $U_{N+1/2}^+ := U_{1/2}^+$, where U represents each one of u_h, p_h, ϕ_h or w_h.

With notation:

$$a_j(\psi_h,\chi) := \int_{I_j} \psi_h \chi_x \, dx - \hat{\psi}_h \chi|_{\partial I_j}, \quad \psi_h, \chi \in P^k(I_j), \tag{25}$$

take summation over j from $j=1$ to $j=N$ to arrive for $(v,\psi,q,z) \in (V_h^k)^4$ at

$$((u_h)_t, v) - \sum_{j=1}^{N} \int_{I_j} (\frac{u_h^2}{2}) v_x \, dx + \sum_{j=1}^{N} \frac{\widehat{u_h^2} v}{2}|_{\partial I_j} - (p_h, v) + \sqrt{\epsilon} \sum_{j=1}^{N} a_j(w_h, v) = 0, \tag{26}$$

$$(w_h, z) + \sqrt{\epsilon} \sum_{j=1}^{N} a_j(u_h, z) = 0, \tag{27}$$

$$(p_h, \psi) + \sum_{j=1}^{N} a_j(\phi_h, \psi) = 0, \tag{28}$$

$$-\sum_{j=1}^{N} a_j(p_h, q) - ((\phi_h + u_h), q) = 0, \tag{29}$$

$$((u_h - u)|_{t=0}, v) = \; 0. \tag{30}$$

Below, we discuss some properties of the bilinear form $a_j(\cdot,\cdot)$.

- For $\chi \in P^k(I_j)$,

$$a_j(\chi,\chi) = \int_{I_j} \chi \, \chi_x \, dx - \hat{\chi} \, \chi|_{\partial I_j} = \frac{1}{2}\chi^2|_{\partial I_j} - \hat{\chi} \, \chi|_{\partial I_j}, \tag{31}$$

  and hence, for $\chi \in V_h^k$, it follows that

$$\begin{aligned} a(\chi,\chi) := \sum_{j=1}^{N} a_j(\chi,\chi) &= \sum_{j=1}^{N} \Big(\frac{1}{2}\chi^2|_{\partial I_j} - \hat{\chi} \, \chi|_{\partial I_j} \Big) \\ &= \sum_{j=0}^{N-1} \Big(\hat{\chi} \, [\chi] - \frac{1}{2}[\chi^2] \Big)_{j+1/2} \\ &= -(\frac{1}{2} - \theta) \sum_{j=0}^{N-1} [\chi]_{j+1/2}^2. \end{aligned} \tag{32}$$

- For $z_h, \chi \in P^k(I_j)$, there holds

$$\begin{aligned} a_j(z_h,\chi) &= -\int_{I_j} z_{hx} \, \chi \, dx + z_h \, \chi|_{\partial I_j} - \hat{z}_h \, \chi|_{\partial I_j} \\ &= -a_j(\chi, z_h) + z_h \, \chi \, |_{\partial I_j} - \hat{z}_h \, \chi \, |_{\partial I_j} - \hat{\chi} \, z_h \, |_{\partial I_j}. \end{aligned} \tag{33}$$

3.1. Discrete conservation properties. This subsection focuses on the properties of the numerical solution u_h, namely; conservation of momentum and L^2 bound of the scheme.

Theorem 3.1. *For the LDG scheme* (26)-(30) *with numerical fluxes* (20)-(24) *and* $\theta \in [0,1/2]$*, the following relations hold for all* $t > 0$

$$\int_0^L u_h(t,x)dx = \int_0^L u_h(0,x)dx, \tag{34}$$

(35)
$$\|u_h(t)\|^2 + 2\int_0^t \|w_h(\tau)\|^2\, d\tau + (1-2\theta)\sum_{j=1}^{N}\int_0^t ([\phi_h]^2 + [p_h]^2)_{j+1/2}\, d\tau = \|u_h(0)\|^2.$$

Proof. In order to prove the conservation of momentum equation, that is, (34), choose $v = 1$ in (26) and $\psi = 1$ in (28). Then, add the resulting equations to arrive at
$$\frac{d}{dt}\int_I u_h dx = 0,$$
and, an integration with respect to time t yields the conservation of momentum, that is, (5).

Now to prove (35), choose $v = u_h, \psi = -\phi_h$ and $q = -p_h$ in equations (26),(28) and (29), respectively. Then, add the resulting equations to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2 + \sqrt{\epsilon}\sum_{j=1}^{N} a_j(w_h, u_h) \tag{36}$$
$$= \sum_{j=1}^{N}\frac{u_h^3}{6}|_{\partial I_j} - \sum_{j=1}^{N}\frac{\widehat{u_h^2 u_h}}{2}|_{\partial I_j} + \sum_{j=1}^{N} a_j(\phi_h,\phi_h) - \sum_{j=1}^{N} a_j(p_h,p_h).$$

Using the property (32) of $a_j(\cdot,\cdot)$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2 + \sqrt{\epsilon}\sum_{j=1}^{N} a_j(w_h,u_h) &= \sum_{j=0}^{N-1}(\frac{\widehat{u_h^2}}{2}[u_h] - [\frac{u_h^3}{6}])_{j+1/2} \\ &+ \sum_{j=0}^{N-1}(\hat{\phi}_h[\phi_h] - [\frac{\phi_h^2}{2}])_{j+1/2} - \sum_{j=0}^{N-1}(\hat{p}_h[p_h] - [\frac{p_h^2}{2}])_{j+1/2}.\end{aligned} \tag{37}$$

In order to estimate the second term on the left hand side of (37), we substitute $z = w_h$ in (16) to obtain

$$\int_{I_j} w_h^2 dx + \sqrt{\epsilon}\, a_j(u_h, w_h) = 0. \tag{38}$$

An application of property (33) for the second term on the left hand side of (38) provides

$$\int_{I_j} w_h^2\, dx - \sqrt{\epsilon}\, a_j(w_h,u_h) - \sqrt{\epsilon}\hat{u}_h w_h|_{\partial I_j} - \sqrt{\epsilon}\hat{w}_h(u_h)|_{\partial I_j} + \sqrt{\epsilon}\, u_h w_h|_{\partial I_j} = 0.$$

On summation over j establishes

$$\sqrt{\epsilon}\sum_{j=1}^{N} a_j(w_h,u_h)$$
$$= \|w_h(t)\|^2 - \sqrt{\epsilon}\sum_{j=1}^{N}\hat{u}_h w_h|_{\partial I_j} - \sqrt{\epsilon}\sum_{j=1}^{N}\hat{w}_h u_h|_{\partial I_j} + \sqrt{\epsilon}\sum_{j=1}^{N} u_h w_h|_{\partial I_j}. \tag{39}$$

Using the numerical fluxes (23)-(24), we obtain

$$\begin{aligned}\sqrt{\epsilon}\sum_{j=1}^{N} a_j(w_h,u_h) &= \|w_h(t)\|^2 + \sqrt{\epsilon}\sum_{j=0}^{N-1}\Big((\hat{w}_h[u_h] + \hat{u}_h[w_h])_{j+1/2} - [u_h w_h]_{j+1/2}\Big) \\ &= \|w_h(t)\|^2.\end{aligned} \tag{40}$$

To estimate the terms on right hand side of equation (37), a use of the numerical flux (20) yields

$$\frac{\widehat{u_h^2}}{2}[u_h] - [\frac{u_h^3}{6}] = \frac{1}{6}(((u_h^+)^2 + u_h^+ u_h^- + (u_h^-)^2)(u_h^+ - u_h^-) - ((u_h^+)^3 - (u_h^-)^3)) \\ = 0. \tag{41}$$

A substitution of estimates (40) and (41) in (37) shows

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2 + \|w_h^2(t)\|^2 \\ &= \sum_{j=0}^{N-1}(\hat{\phi}_h[\phi_h] - [\frac{\phi_h^2}{2}])_{j+1/2} - \sum_{j=0}^{N-1}(\hat{p}_h[p_h] - [\frac{p_h^2}{2}])_{j+1/2} \\ &= \sum_{j=0}^{N-1}(\hat{\phi}_h - \{\phi_h\})[\phi_h]_{j+1/2} - \sum_{j=0}^{N-1}(\hat{p}_h - \{p_h\})[p_h]_{j+1/2} \\ &= \sum_{j=0}^{N-1}(\theta - \frac{1}{2})(((\phi_h^+)^2 + (\phi_h^-)^2 - 2\phi_h^-\phi_h^+) + ((p_h^+)^2 + (p_h^-)^2 - 2p_h^- p_h^+))_{j+1/2}. \end{aligned}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2 + \|w_h(t)\|^2 + \sum_{j=0}^{N-1}(\frac{1}{2} - \theta)([\phi_h]^2 + [p_h]^2)_{j+1/2} = 0. \tag{42}$$

On integrating with respect to t, it now follows that

$$\|u_h(t)\|^2 + 2\int_0^t \|w_h\|^2\, d\tau + \sum_{j=0}^{N-1}(1 - 2\theta)\int_0^t ([\phi_h]^2 + [p_h]^2)_{j+1/2} d\tau = \|u_h(0)\|^2.$$

This completes the rest of the proof. □

As a consequence, the following *a priori* bound is derived for the discrete solution u_h and w_h for $0 \leq \theta < 1/2$

$$\|u_h(t)\|^2 + 2\int_0^t \|w_h(\tau)\|^2 d\tau \leq \|u_h(0)\|^2. \tag{43}$$

3.2. Existence, Uniqueness of discrete solutions. In this subsection, we discuss the existence and uniqueness results for the discrete viscous Burgers-Poisson system (17)-(18). Note that the addition of diffusion term does not change the weak formulation of $p - u_x = 0$ and $\phi_x - \phi = u$, which are used to prove the following lemmas (see, [12]). Since proof of the following lemma follows similarly to the proof of Lemmas 3.1, we just state it without proof.

Lemma 3.2. *For the discrete scheme (26)-(30) with the numerical fluxes (20)-(24), the following estimate holds for any* $\theta \in [0, 1]$

$$\|p_h(t)\| + \|\phi_h(t)\| \leq \|u_h(t)\|, \quad t > 0. \tag{44}$$

Below, we sketch of the proof of wellposedness of the discrete problem (26)-(30).

Lemma 3.3. *There exists a unique solution* $\{u_h, w_h, \phi_h, p_h\} \in (V_h^k)^4$ *of the discrete problem (26)-(30) with numerical fluxes (20)-(24) for* $t \in (0, T]$.

Since V_h^k is finite dimensional, the discrete problem (26)-(30) yields a system of non linear ODEs coupled with linear algebraic equations which is known as a DAE system. Moreover, it is of index one as it is easy to check that each of $\{w_h, \phi_h, p_h\}$ can be written explicitely as a function of u_h. On substitution in (26), we, therefore, obtain a system of nonlinear ODEs. An application of Picard's theorem ensures the existence of a local solution u_h, say, in $(0, t_h)$. Since u_h is bounded in $L^\infty(L^2)$-norm, using continuation argument, one proves existence of a unique solution u_h for all $t > 0$. Again use u_h to establish the existence of unique solution $\{w_h, \phi_h, p_h\}$ for $t > 0$.

Now, we discuss some *a priori* error estimates of the viscous Burgers-Poisson system.

4. *A Priori* Error Analysis

In this section, we deal with *a priori* error estimates for the solutions of the viscous Burgers-Poisson system using LDG method.

For our subsequent use, we recall the definition of the global projection and some of its properties from [12].

4.1. Global projection. Let $\omega|_{I_j} \in H^s(I_j)$ with $s \geq k+1$. Define the projection Q_θ as :

$$\int_{I_j} (Q_\theta \omega) v dx = \int_{I_j} \omega v dx, \qquad \forall v \in P^{k-1}, j = 1, 2, ..., N, \tag{45}$$

$$\widehat{Q_\theta \omega}_{j+1/2} = \hat{\omega}_{j+1/2}, \qquad j = 1, 2, ..., N, \tag{46}$$

where

$$\hat{v} = \theta v^+ + (1-\theta) v^-.$$

Note that for $j = N$, we use the periodic extension to define $(Q_\theta \omega)^+_{N+1/2}$. This projection Q_θ satisfying (45)-(46) is uniquely defined for either $\theta \neq \frac{1}{2}$ or $\theta = \frac{1}{2}$ with k even and N odd. For a proof, we refer to Lemma 4.1 of [12]. We now observe that a use of (45)-(46) yields

$$a_j(Q_\theta \omega - \omega, v) = 0 \quad \forall v \in P^{k-1}(I_j). \tag{47}$$

Below, we recall some properties of the projection $Q_\theta \omega$, whose proof can be obtained from Lemmas 4.2-4.3 of [12].

Approximation properties of the global projection. The following approximation properties hold for the global projection Q_θ.

- For $\omega|_{I_j} \in H^{k+1}(I_j)$ for $j = 1, 2, ..., N$, there exists a positive constant $C = C(k, \theta)$, independent of ω such that for $k \geq 0$

$$\|Q_\theta \omega - \omega\| \leq C\, h^{k+1}\, |\omega|_{k+1}, \tag{48}$$

  where h is the maximum size of subintervals I_j and k is the degree of the polynomial.
- For $k \geq 1$, there holds

$$\begin{aligned} \|\omega - Q_\theta \omega\|_{\Gamma_h} :=& \Big(\sum_{j=1}^{N} \Big(|(\omega - Q_\theta \omega)(x^-_{j+1/2})|^2 + |(\omega - Q_\theta \omega)(x^+_{j+1/2})|^2 \Big) \Big)^{1/2} \\ \leq& C\, h^{k+1/2}\, |\omega|_{k+1}. \end{aligned} \tag{49}$$

4.2. Optimal error estimates. This subsection focuses on optimal *a priori* estimates for the LDG scheme (15)-(18).

With the help of the global projections (45)- (46), define

$$\eta_\phi = Q_\theta\phi - \phi_h, \qquad \eta_p = Q_{1-\theta}p - p_h, \qquad \eta_u = Q_\theta u - u_h, \qquad \eta_w = Q_{1-\theta}w - w_h,$$
$$\zeta_\phi = Q_\theta\phi - \phi, \qquad \zeta_p = Q_{1-\theta}p - p, \qquad \zeta_u = Q_\theta u - u, \qquad \zeta_w = Q_{1-\theta}w - w$$

and

$$\hat{\eta}_\phi = \widehat{Q_\theta\phi} - \hat{\phi}_h, \qquad \hat{\eta}_p = \widehat{Q_{1-\theta}p} - \hat{p}_h, \qquad \hat{\eta}_u = \widehat{Q_\theta u} - \hat{u}_h, \quad \hat{\eta}_w = \widehat{Q_{1-\theta}w} - \hat{w}_h$$
$$\hat{\zeta}_\phi = \widehat{Q_\theta\phi} - \phi, \qquad \hat{\zeta}_p = \widehat{Q_{1-\theta}p} - p, \qquad \hat{\zeta}_u = \widehat{Q_\theta u} - u, \qquad \hat{\zeta}_w = \widehat{Q_{1-\theta}w} - w.$$

Then,

$$\begin{aligned} \phi - \phi_h &:= \eta_\phi - \zeta_\phi, \qquad & p - p_h &:= \eta_p - \zeta_p, \\ u - u_h &:= \eta_u - \zeta_u, \qquad & w - w_h &:= \eta_w - \zeta_w, \end{aligned} \tag{50}$$

and

$$\begin{aligned} \phi - \hat{\phi}_h &= \hat{\eta}_\phi - \hat{\zeta}_\phi, \qquad & p - \hat{p}_h &= \hat{\eta}_p - \hat{\zeta}_p, \\ u - \hat{u}_h &= \hat{\eta}_u - \hat{\zeta}_u, \qquad & w - \hat{w}_h &= \hat{\eta}_w - \hat{\zeta}_w. \end{aligned} \tag{51}$$

Since the scheme with fluxes (20)-(24) is consistent, (26)-(29) also hold for solutions (u, p, w, ϕ). Hence, taking the difference, we obtain for $(v, z, \psi, q) \in (V_h^k)^4$ and using the notations (50)-(51) and (47), we arrive at

$$\begin{aligned} ((\eta_u)_t, v) \quad &+ \quad \sqrt{\epsilon}\sum_{j=1}^N a_j(\eta_w, v) = \int_I (\zeta_u)_t v\, dx + \sum_{j=1}^N \int_{I_j} (\frac{u^2}{2} - \frac{u_h^2}{2}) v_x\, dx \\ &- \quad \sum_{j=1}^N \Big(\frac{u^2}{2} - \widehat{\frac{u_h^2}{2}}\Big) v|_{\partial I_j} + (\eta_p, v) - (\zeta_p, v), \end{aligned} \tag{52}$$

$$(\eta_w, z) + \sqrt{\epsilon}\sum_{j=1}^N a_j(\eta_u, z) = (\zeta_w, z), \tag{53}$$

$$(\eta_p, \psi) + \sum_{j=1}^N a_j(\eta_\phi, \psi) = (\zeta_p, \psi), \tag{54}$$

$$\sum_{j=1}^N a_j(\eta_p, q) + (\eta_\phi, q) = (\zeta_\phi, q) - ((\eta_u - \zeta_u), q). \tag{55}$$

In the following Lemma, we estimate η_p, η_ϕ. Although, the proof is similar in spirit to the proof of the Lemma 4.1 of [12], but for completeness we present a brief proof of it.

Lemma 4.1. *Let (u, w, p, ϕ) and (u_h, w_h, p_h, ϕ_h), respectively, be the solution of the problem (11)-(14) and the discrete system (15)-(18) with the choice of fluxes (20)-(24). Assume $\theta \in [0, 1]$ for which Q_θ and $Q_{1-\theta}$ are uniquely defined. Then, the following estimate holds for all $t > 0$*

$$\|\eta_p\| + \|\eta_\phi\| \le 2\Big(\|Q_{1-\theta}p - p\| + \|Q_\theta\phi - \phi\| + \|\ u - u_h\|\Big). \tag{56}$$

Proof. On substitution of $\psi = \eta_p$ and $q = \eta_\phi$ in (54) - (55), respectively, to arrive at

$$\|\eta_p(t)\|^2 + \sum_{j=1}^{N} a_j(\eta_\phi, \eta_p) = (\zeta_p, \eta_p), \tag{57}$$

and

$$\sum_{j=1}^{N} a_j(\eta_p, \eta_\phi) + \|\eta_\phi(t)\|^2 = (\zeta_\phi, \eta_\phi) - ((u - u_h), \eta_\phi). \tag{58}$$

Using property (33), we arrive at

$$\begin{aligned}
\sum_{j=1}^{N} \Big(a_j(\eta_\phi, \eta_p) + a_j(\eta_p, \eta_\phi) \Big) &= \sum_{j=1}^{N} \Big(\eta_\phi \eta_p|_{\partial I_j} - \hat{\eta}_p \; \eta_\phi|_{\partial I_j} - \hat{\eta}_\phi \; \eta_p|_{\partial I_j} \Big) \\
&= \sum_{j=0}^{N-1} \Big(-[\eta_\phi \eta_p] + \hat{\eta}_p \; [\eta_\phi] + \hat{\eta}_\phi \; [\eta_p] \Big)_{j+1/2} \\
&= 0.
\end{aligned} \tag{59}$$

Adding equations (57) - (58), apply property (59) to obtain

$$\|\eta_p\|^2 + \|\eta_\phi\|^2 = (\zeta_p, \eta_p) + (\zeta_\phi, \eta_\phi) - ((u - u_h), \eta_\phi). \tag{60}$$

A use of the Cauchy-Schwarz inequality in (60) yields

$$\begin{aligned}
\|\eta_\phi\|^2 + \|\eta_p\|^2 \quad &\leq \quad \|\zeta_p\| \; \|\eta_p\| + \Big(\|\zeta_\phi\| + \| \; u - u_h\| \Big) \; \|\eta_\phi\| \\
&\leq \Big(\|\zeta_p\| + \|\zeta_\phi\| + \| \; u - u_h\| \Big) \; \Big(\|\eta_p\| + \|\eta_\phi\| \Big).
\end{aligned}$$

Apply $(a + b)^2 \leq 2(a^2 + b^2)$ to complete the rest of the proof. □

Below in Lemma 4.2, we provide an estimate of η_u and η_w. Since its proof is similar to the proof of the Theorem 4.5 in [12], we shall only indicate only the changes.

Lemma 4.2. *Under the assumption of Lemma 4.1, $u \in L^\infty(0, T; H^{k+2})$, $\phi \in L^2(0, T; H^{k+2})$ and for $\theta = 1/2$ with k even and N odd, there exists a positive constant C independent of h such that for all $t \in (0, T]$*

$$\begin{aligned}
&\|\eta_u\|_{L^\infty(0,T;L^2)} + \|\eta_w\|_{L^2(0,T;L^2)} \\
\leq &C \; h^{k+1} \Big(\|u\|_{L^\infty(0,T;H^{k+2})} + \|u_t\|_{L^2(0,T;H^{k+1})} + \|\phi\|_{L^2(0,T;H^{k+2})} \Big).
\end{aligned}$$

Proof. Choose $v = \eta_u$ in (52) and obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta_u(t)\|^2+\sqrt{\epsilon}\sum_{j=1}^{N}a_j(\eta_w,\eta_u)\\
=&((\zeta_u)_t,\eta_u)+\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(u^2-u_h^2)(\eta_u)_x\ dx+\frac{1}{2}\sum_{j=0}^{N-1}(u^2-\widehat{u_h^2})[\eta_u]_{j+1/2}\\
&+(\eta_p,\eta_u)-(\zeta_p,\eta_u)\\
=&((\zeta_u)_t,\eta_u)+\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(u^2-u_h^2)(\eta_u)_x\ dx+\frac{1}{2}\sum_{j=0}^{N-1}(u^2-\{u_h\}^2)[\eta_u]_{j+1/2}\\
&+\frac{1}{2}\sum_{j=0}^{N-1}(\{u_h\}^2-\widehat{u_h^2})[\eta_u]_{j+1/2}+(\eta_p,\eta_u)-(\zeta_p,\eta_u).
\end{aligned}$$

An application of the identity $\frac{a^2}{2}-\frac{b^2}{2}=a(a-b)-\frac{(a-b)^2}{2}$ shows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta_u(t)\|^2+\sqrt{\epsilon}\sum_{j=1}^{N}a_j(\eta_w,\eta_u)\\
=&((\zeta_u)_t,\eta_u)+\sum_{j=1}^{N}\int_{I_j}u(u-u_h)(\eta_u)_x\ dx-\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(u-u_h)^2(\eta_u)_x dx\\
&+\sum_{j=0}^{N-1}u(u-\{u_h\})[\eta_u]_{j+1/2}-\frac{1}{2}\sum_{j=0}^{N-1}(u-\{u_h\})^2[\eta_u]_{j+1/2}\\
&+\frac{1}{2}\sum_{j=0}^{N-1}(\{u_h\}^2-\widehat{u_h^2})[\eta_u]_{j+1/2}+(\eta_p,\eta_u)-(\zeta_p,\eta_u).
\end{aligned}$$

On substitution of $u-\{u_h\}=\{u-u_h\}=\{\eta_u\}-\{\zeta_u\}$, in above equation, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta_u(t)\|^2+\sqrt{\epsilon}\sum_{j=1}^{N}a_j(\eta_w,\eta_u)\\
=&((\zeta_u)_t,\eta_u)+\sum_{j=1}^{N}\int_{I_j}u(\eta_u-\zeta_u)(\eta_u)_x dx-\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(\eta_u-\zeta_u)^2(\eta_u)_x dx\\
&+\sum_{j=0}^{N-1}u(\{\eta_u\}-\{\zeta_u\})[\eta_u]_{j+1/2}-\frac{1}{2}\sum_{j=0}^{N-1}(\{\eta_u\}-\{\zeta_u\})^2[\eta_u]_{j+1/2}\\
&+\frac{1}{2}\sum_{j=0}^{N-1}(\{u_h\}^2-\widehat{u_h^2})[\eta_u]_{j+1/2}+(\eta_p,\eta_u)-(\zeta_p,\eta_u),
\end{aligned}\tag{61}$$

For the second term on the left hand side of (61), we first substitute $z=\eta_w$ in (53) to establish

$$\|\eta_w(t)\|^2+\sqrt{\epsilon}\ \sum_{j=1}^{N}a_j(\eta_u,\eta_w)=(\zeta_w,\eta_w).\tag{62}$$

Adding (62) to (61), a use of the property (33) yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\eta_u(t)\|^2 + \|\eta_w(t)\|^2 &= ((\zeta_u)_t, \eta_u) + (\eta_p, \eta_u) - (\zeta_p, \eta_u) + (\zeta_w, \eta_w) \\
&+ \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5,
\end{aligned} \tag{63}
$$

where,

$$
\begin{aligned}
\tau_1 &= \sum_{j=1}^{N} \int_{I_j} u\eta_u(\eta_u)_x dx + \sum_{j=0}^{N-1} u\{\eta_u\}[\eta_u]_{j+1/2}, \\
\tau_2 &= -\sum_{j=1}^{N} \int_{I_j} u\zeta_u(\eta_u)_x dx - \sum_{j=0}^{N-1} u\{\zeta_u\}[\eta_u]_{j+1/2}, \\
\tau_3 &= -\frac{1}{2}\sum_{j=1}^{N} \int_{I_j} \eta_u^2(\eta_u)_x dx - \frac{1}{2}\sum_{j=0}^{N-1} \{\eta_u\}^2[\eta_u]_{j+1/2}, \\
\tau_4 &= \sum_{j=1}^{N} \int_{I_j} \eta_u\zeta_u(\eta_u)_x dx - \frac{1}{2}\sum_{j=1}^{N} \int_{I_j} \zeta_u^2(\eta_u)_x dx \\
&\quad + \sum_{j=0}^{N-1} \{\eta_u\}\{\zeta_u\}[\eta_u]_{j+1/2} - \frac{1}{2}\sum_{j=0}^{N-1} \{\zeta_u\}^2[\eta_u]_{j+1/2}, \\
\tau_5 &= \frac{1}{2}\sum_{j=0}^{N-1} (\{u_h\}^2 - \widehat{u_h^2})[\eta_u]_{j+1/2},
\end{aligned}
$$

Using projection properties (48)-(49) with the inverse property (9) and trace inequality (10), bounds of $\tau_1, .., \tau_5, \int_I \eta_p\eta_u \ dx, \int_I \zeta_p\eta_u \ dx$ and $\int_I (\zeta_u)_t\eta_u \ dx$ are estimated as in the proof of the Theorem 4.6 in [12]. A use of Young's inequality shows

$$
\begin{aligned}
\frac{d}{dt}\|\eta_u\|^2 + \|\eta_w\|^2 &\le C\Big(h^{2k+2} + \|\eta_u\|^2 + h^{-3/2}\|\eta_u\|^3\Big) \\
&\le C\Big(h^{2k+2} + \|\eta_u\|^2 + h^{-3}\|\eta_u\|^4\Big).
\end{aligned} \tag{64}
$$

Observe that,

$$
\eta_u(0) = \zeta_u(0) + (u_0 - u_h(0)),
$$

where $u_h(0)$ is prepared by using standard L^2 projection of the given data and hence

$$
\|\eta_u(0)\|^2 \le \|\zeta_u(0)\|^2 + \|(u_0 - u_h(0))\|^2 \le Ch^{2k+2}. \tag{65}
$$

On integrating (64) with respect to time t and on using (65), we find that

$$
\begin{aligned}
&\|\eta_u(t)\|^2 + \int_0^t \|\eta_w(s)\|^2 \ ds \\
\le &\|\eta_u(0)\|^2 + C\int_0^t (h^{2k+2} + \|\eta_u(\tau)\|^2 + h^{-3}\|\eta_u(\tau)\|^4) \ d\tau.
\end{aligned}
$$

Setting

$$
|||(\eta_u, \eta_w)(t)|||^2 := \|\eta_u(t)\|^2 + \int_0^t \|\eta_w(s)\|^2 \ ds \tag{66}
$$

and a function Φ as

$$\Phi(t) = h^{2k+2} + \int_0^t \Big(|||(\eta_u, \eta_w)(\tau)|||^2 + h^{-3} |||(\eta_u, \eta_w)(\tau)|||^4 \Big)\, d\tau, \tag{67}$$

we rewrite (66) as

$$|||(\eta_u, \eta_w)(t)|||^2 \leq C\Phi(t). \tag{68}$$

Without loss of generality assume that $|||(\eta_u, \eta_w)(t)||| > 0$, otherwise, we may have to add an arbitrarily small quantity say δ and proceed as in a similar way as describe below and then pass the limit as $\delta \mapsto 0$. Then, note that $0 < \Phi(0) \leq \Phi$ with Φ differentiable. On differentiating $\Phi(t)$ with respect to time t, we obtain

$$\begin{aligned} \Phi'(t) &= |||(\eta_u, \eta_w)(t)|||^2 + h^{-3}\ |||(\eta_u, \eta_w)(t)|||^4 \\ &\leq C\ \Phi(t) + C^2\ h^{-3}(\Phi(t))^2 \\ &\leq C_* \Big(\Phi(t) + h^{-3}\ (\Phi(t))^2 \Big) \quad \text{where } C_* = C\ \max\{1, C\}. \end{aligned}$$

Moreover, $\Phi' > 0$ and hence, Φ is strictly monotonically increasing function which is also positive. An integrating with respect to time t yields

$$\int_0^t \frac{\Phi'(s)}{\Phi(s)\Big(1 + h^{-3}(\Phi(s))\Big)}\, ds \leq \int_0^t C_*\, ds \leq C_* T. \tag{69}$$

Now, we evaluate the integral on the left hand side of (69) exactly and hence, after taking exponential on both sides and using $\Phi(0) = h^{2(k+1)}$, we obtain

$$\frac{\Phi(t)\ (1 + h^{2k-1})}{h^{2(k+1)}\ (1 + h^{-3}(\Phi(t)))} \leq e^{C_* T}.$$

On simplifying

$$\Phi(t)\Big(1 - h^{2k-1}(e^{C_* T} - 1)\Big) \leq e^{C_* T}\ h^{2(k+1)}.$$

For sufficiently small $h > 0$, the term $\Big(1 - h^{2k-1}(e^{C_* T} - 1)\Big)$ can be made greater than equal to $1/2$. Therefore, $\Phi(t) \leq \tilde{C}\ h^{2k+2}$. On substitution in (68) completes the rest of the proof. $\square$

Using Lemma 4.2, approximation properties (48) and triangle inequality, we obtain below one of the main theorems of this section which is valid under the condition that $\theta = 1/2$.

Theorem 4.3. *Let $u \in L^\infty(0, T; H^{k+2}(I)) \cap H^1(0, T; H^{k+1}(I))$, and $\phi \in L^2(0, T; H^{k+2}(I))$, $k \geq 1$, be the smooth solution to (1), for $0 < t < T$. Then for $\theta = 1/2$, k even and N odd, the numerical solutions pair $\{u_h, w_h\}$, obtained from the scheme (15)-(19) and the numerical fluxes (20)-(24) satisfies*

$$\|u - u_h\|_{L^\infty(0,T;L^2(I))} + \|w - w_h\|_{L^2(0,T;L^2(I))} = O(h^{k+1}), \tag{70}$$

where C is a positive depending on T and the data given, but is independent of the maximum mesh size h.

As a consequence of Lemma 4.2 and using Lemma 4.1, we have the following corollary.

Corollary 4.4. *For $\theta = 1/2$ and for k even with N odd, the following estimates hold:*

$$\|p - p_h\|_{L^\infty(0,T;L^2(I))} + \|\phi - \phi_h\|_{L^\infty(0,T;L^2(I))} = \mathcal{O}(h^{k+1}).$$

Proof. Note that using Theorem 4.3 and Lemma 4.1, one observes that

$$\begin{aligned}\|p-p_h\| &\leq \|\zeta_p\|+\|\eta_p\| \\ &\leq 3\|\zeta_p\|+2\|\zeta_\phi\|+2\|u-u_h\| \\ &\leq C\, h^{k+1}.\end{aligned}$$

Similarly, it is easy to show $\|\phi-\phi_h\| \leq Ch^{k+1}$. □

However, several numerical experiments in the next Section indicate that for $\theta \in [0,1/2)$ optimal error estimates in Theorem 4.3 and Corollary 4.4 are valid for both even and odd degree polynomials. To substantiate our claim, we provide some results below.

Theorem 4.5. *Let $u \in L^\infty(0,T;H^{k+2}(I)) \cap H^1(0,T;H^{k+1}(I))$, and $\phi \in L^2(0,T;H^{k+2}(I))$ with $k \geq 1$, be the smooth solution to (1), for $0<t<T$. Then for $\theta \in [0,1/2)$, the numerical solutions pair $\{u_h, w_h\}$, obtained from the scheme (15)-(19) and the numerical fluxes (20)-(24) satisfies*

$$\|u-u_h\|_{L^\infty(0,T;L^2(I))}+\|w-w_h\|_{L^2(0,T;L^2(I))} \leq C(T)\,\epsilon^{-1/2}\,h^{k+1}, \tag{71}$$

where C is a positive depending on T, and the data given, but is independent of the maximum mesh size h. In addition if $\phi \in L^\infty(0,T;H^{k+2}(I))$ $k \geq 1$, then, the following estimates holds:

$$\|p-p_h\|_{L^\infty(0,T;L^2(I))}+\|\phi-\phi_h\|_{L^\infty(0,T;L^2(I))} \leq C(T)\,\epsilon^{-1/2}\,h^{k+1}.$$

For simplicity of exposition, we shall prove the Theorem 4.5, when $\theta=0$ and for other values of $\theta \in (0,1/2)$, the proof goes in a similar lines, provided the following Conjecture 4.6 is valid.

Conjecture 4.6. *Let the pair $\{\eta_u,\eta_w\} \in V_h^k \times V_h^k$ satisfy* (53). *Then, there is a positive constant C, independent h and ϵ, such that*

$$\begin{aligned}\Big(\sum_{j=1}^{N}\Big(\|\eta_{u,x}\|^2_{L^2(I_j)}+|h^{-1/2}[\eta_u]_{j+1/2}|^2\Big)\Big)^{1/2} &\leq C\,\epsilon^{-1/2}\Big(\|\eta_w\|+\|\zeta_w\|\Big) \\ &\leq C\,\epsilon^{-1/2}\Big(\|\eta_w\|+h^{k+1}\Big).\end{aligned} \tag{72}$$

Note that the proof of the above Conjecture is given, when $\theta=1$ which corresponds to our case with $\theta=0$.

For the proof of theorem 4.5, when $\theta=0$, we shall not repeat the arguments stated in the theorem of Lemma 4.2, but briefly indicate below the major differences in the arguments.

Proof of the Theorem 4.5. Since $u-u_h := \eta_u-\zeta_u$ and estimate of ζ_u is known, therefore, it is enough to estimate of η_u. Returning to (63) in the proof of Lemma 4.2 with $\theta=0$, there is hardly any change in the proof of estimates of τ_1, τ_3 and τ_5 and hence,

$$|\tau_1|+|\tau_3|+|\tau_5| \leq C\Big(h^{2(k+1)}+\|\eta_u\|^2+h^{-3/2}\|\eta_u\|^3\Big). \tag{73}$$

For the estimate of τ_2, since $\{\zeta_u\} \neq 0$ when $\theta = 0$, we need to estimate the extra boundary term using the Cauchy-Schwarz inequality, (49) and Conjecture 4.6 as

$$\begin{aligned}
|-\sum_{j=1}^{N} u\,\{\zeta_u\}\,[\eta_u]_{j+1/2}| &\leq C\sum_{j=1}^{N} h^{1/2}\,(|\zeta_u^-|_{j+1/2}| + |\zeta_u^+|_{j+1/2}|)(h^{-1/2}|[\eta_u]_{j+1/2}|)\\
&\leq C\,h^{1/2}\|\zeta_u\|_{\Gamma_h}\,\Big(\sum_{j=1}^{N}(h^{-1/2}|[\eta_u]_{j+1/2}|)^2\Big)^{1/2}\\
&\leq C\,\epsilon^{-1/2}\,\|\zeta_u\|\,\Big(h^{k+1} + \|\eta_w\|\Big)\\
&\leq C\,\epsilon^{-1/2}\,h^{k+1}\,\Big(h^{k+1} + \|\eta_w\|\Big). \qquad (74)
\end{aligned}$$

Using rest of the estimates for τ_2 from Theorem 4.5 of [12], we altogether obtain using the Young's inequality ($ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$, for $a, b \geq 0$ and $\delta > 0$)

$$\begin{aligned}
|\tau_2| &\leq C\,\Big(h^{2(k+1)} + \|\eta_u\|^2 + \epsilon^{-1/2}\,h^{(k+1)}(h^{k+1} + \|\eta_w\|)\Big)\\
&\leq C(\delta)\Big(h^{2(k+1)} + \|\eta_u\|^2 + \epsilon^{-1}\,h^{2(k+1)}\Big) + \frac{\delta}{2}\,\|\eta_w\|^2. \qquad (75)
\end{aligned}$$

For the estimation of τ_4, only term involving boundary terms needs to be evaluated as the other terms have exactly same estimates as in the proof of Theorem 4.5 of [12]. Therefore, to estimate the boundary term, a use approximation property (49) with the Cauchy-Schwarz inequality, Lemma 4.6 and the Young's inequality yields

$$\begin{aligned}
&|\sum_{j=1}^{N}\Big(\Big(\{\eta_u\} - \frac{1}{2}\{\zeta_u\}\Big)\,\{\zeta_u\}\,[\eta_u]\Big)_{j+1/2}|\\
&\leq C\sum_{j=1}^{N}\Big(|\eta_u^-| + |\eta_u^+| + |\zeta^+|\Big)_{j+1/2}|\zeta_u^+|_{j+1/2}|\,[\eta_u]_{j+1/2}\\
&\leq Ch^{k+1/2}\,\sum_{j=1}^{N}\Big(|\eta_u^-| + |\eta_u^+| + h^{k+1/2}\Big)_{j+1/2}\,[\eta_u]_{j+1/2}\\
&\leq Ch^{k+1/2}\,\sum_{j=1}^{N}h^{1/2}\Big(|\eta_u^-| + |\eta_u^+| + h^{k+1/2}\Big)_{j+1/2}\,h^{-1/2}[\eta_u]_{j+1/2}\\
&\leq Ch^{k+1/2}\,h^{1/2}\,\|\eta_u\|_{\Gamma_h}\,\Big(\sum_{j=1}^{N}h^{-1}[\eta_u]^2_{j+1/2}\Big)^{1/2}\\
&\leq C\,\epsilon^{-1/2}\,h^{k+1/2}\,\|\eta_u\|\,\Big(h^{k+1} + \|\eta_w\|\Big)\\
&\leq C\,\Big(h^{2(k+1)} + \epsilon^{-1}\,h^{2k+1}\,\|\eta_u\|^2\Big) + \frac{\delta}{2}\|\eta_w\|^2. \qquad (76)
\end{aligned}$$

Following the estimates of the rest of the terms in τ_4 in the proof of Theorem 4.5 of [12], we arrive with (76) at

$$|\tau_4| \leq C\,\Big(h^{2(k+1)} + \epsilon^{-1}\,h^{2k+1}\,\|\eta_u\|^2\Big) + \frac{\delta}{2}\|\eta_w\|^2. \qquad (77)$$

On substitution of estimates (73), (75) and (84) in (63), we use $\delta = 1/2$ to find as in the proof of Lemma 4.2 a counter part of inequality (64) as

$$\begin{aligned}\frac{d}{dt}\|\eta_u\|^2 + \|\eta_w\|^2 &\leq C\Big((\epsilon^{-1}+1)h^{2(k+1)} + (1+\epsilon^{-1}h^{2k+1})\|\eta_u\|^2 + h^{-3/2}\|\eta_u\|^3\Big)\\ &\leq C\Big((\epsilon^{-1}+1)h^{2(k+1)} + (1+\epsilon^{-1}h^{2k+1})\|\eta_u\|^2 + h^{-3}\|\eta_u\|^4\Big). \qquad (78)\end{aligned}$$

The rest of the proof of Lemma 4.2 follows with modification of Φ as

$$\Phi(t) = (1+\epsilon^{-1})h^{2k+2} + \int_0^t \Big((1+\epsilon^{-1}\; h^{2k+1})\; |||(\eta_u,\eta_w)(\tau)|||^2 + h^{-3}|||(\eta_u,\eta_w)(\tau)|||^4\Big)\, d\tau \qquad (79)$$

Note that the result holds for fixed $\epsilon > 0$. Proceed similarly as in the proof of the Lemma 4.2, we arrive at for small h

$$\Phi(t) \leq (1+\epsilon^{-1})e^{C_*(1+\epsilon^{-1}h^{2k+1})T}\; h^{2(k+1)}.$$

Therefore,

$$|||(\eta_u,\eta_w)(t)|||^2 \leq C\Phi(t) \leq C\,\tilde{C}\,(1+\epsilon^{-1})\,h^{2(k+1)} \leq C\,\frac{1}{\epsilon}\,h^{2(k+1)} \qquad (80)$$

This concludes the rest of the proof. □

Remark 4.1. Note that the estimates in Theorem 4.5 are optimal for fixed ϵ and are not valid uniformly with respect to ϵ as $\epsilon \mapsto 0$ as against the results of the Theorem 4.3. This is mainly due to the use of Lemma 4.6 for taking care of the nonlinearity specially in the estimates of (74) and (76). But a more careful observation of the estimate in (74) reveals with out using Lemma 4.6 and applying trace inequality (10) and the global projection property like (49) that

$$\begin{aligned}\Big|-\sum_{j=1}^{N} u\,\{\zeta_u\}\,[\eta_u]_{j+1/2}\Big| &\leq C\sum_{j=1}^{N}\left(|\zeta_u^-|_{j+1/2}| + |\zeta_u^+|_{j+1/2}|\right)|[\eta_u]_{j+1/2}|\\ &\leq C\,\|\zeta_u\|_{\Gamma_h}\,\|\eta_u\|_{\Gamma_h}\\ &\leq C\,h^k\,\|\eta_u\|. \qquad (81)\end{aligned}$$

Hence, the estimate of τ_2 term in Theorem 4.5 now becomes

$$|\tau_2| \leq C\left(h^{2k} + \|\eta_u\|^2\right). \qquad (82)$$

More over, for the estimate (76) in the term τ_4 of the proof of the Theorem 4.5, we now apply(10) and the global projection property like (49) to obtain

$$
\begin{aligned}
&|\sum_{j=1}^{N}\Big(\Big(\{\eta_u\}-\frac{1}{2}\{\zeta_u\}\Big)\ \{\zeta_u\}\ [\eta_u]\Big)_{j+1/2}| \\
\leq\ & C\sum_{j=1}^{N}\Big(|\eta_u^-|+|\eta_u^+|+|\zeta^+|\Big)_{j+1/2}|\zeta_u^+|_{j+1/2}|\ [\eta_u]_{j+1/2} \\
\leq\ & Ch^{k+1/2}\ \sum_{j=1}^{N}\Big(|\eta_u^-|+|\eta_u^+|+h^{k+1/2}\Big)_{j+1/2}\ [\eta_u]_{j+1/2} \\
\leq\ & Ch^{k+1/2}\ \sum_{j=1}^{N}\Big(|\eta_u^-|+|\eta_u^+|+h^{k+1/2}\Big)_{j+1/2}\ [\eta_u]_{j+1/2} \\
\leq\ & Ch^{k+1/2}\ \|\eta_u\|^2_{\Gamma_h} \\
\leq\ & C\ h^{k-1/2}\ \|\eta_u\|^2 \\
\leq\ & C\ \Big(h^{2(k+1)}+h^{-3}\ \|\eta_u\|^4\Big).
\end{aligned}
\tag{83}
$$

Following the estimates of the rest of the terms in τ_4 in the proof of Theorem 4.5 of [12], we arrive with (83) at

$$
|\tau_4|\leq C\ \Big(h^{2(k+1)}+h^{-3}\ \|\eta_u\|^4\Big). \tag{84}
$$

Similar to the proof of the Theorem 4.5, substitute the estimates to arrive at

$$
\frac{d}{dt}\|\eta_u\|^2+\|\eta_w\|^2\quad\leq\quad C\Big(h^{2k}+h^{-3}\|\eta_u\|^4\Big). \tag{85}
$$

The rest of the analysis follows as in the proof of Theorem 4.3 to derive sub-optimal estimate

$$
\|u-u_h\|_{L^\infty(0,T;L^2(I))}+\|w-w_h\|_{L^2(0,T;L^2(I))}=O(h^k), \tag{86}
$$

which does not depend explicitly on $\epsilon^{-1/2}$.

5. Numerical results

In this section, we perform the numerical simulations on general viscous BP system

$$
u_t+(\frac{u^2}{2}-\phi)_x-\epsilon u_{xx}=f(x,t),\qquad x\in[0,L]=I,\quad t>0 \tag{87}
$$

$$
\phi_{xx}-\phi=u, \tag{88}
$$

with the same boundary and initial conditions as (1)-(2). Our proposed scheme reduces the problem (87)-(88) into the system of ODEs

$$
\frac{d}{dt}\vec{a}=\mathcal{L}(\vec{a},t), \tag{89}
$$

where $\vec{a}=\vec{a}(t)$ is the coefficient vector of u_h. To further approximate the solution of the system (89), we use the third order TVD RK scheme [8]

TABLE 1

k	N	$\theta = 0, \epsilon = 1/10$					
		$\|u - u_h\|$	order	$\|u - u_h\|_\infty$	order	$\|w - w_h\|$	order
1	5	2.0388e-01		2.7734e-01		2.2951e-01	
	10	4.9798e-02	2.0335	8.2017e-02	1.7577	8.3916e-02	1.4516
	20	1.1284e-02	2.1418	1.9282e-02	2.0886	2.2834e-02	1.8777
	40	2.7039e-03	2.0611	4.4963e-03	2.1005	5.7866e-03	1.9804
	80	6.6806e-04	2.0170	1.0674e-03	2.0747	1.4518e-03	1.9948
2	5	1.6241e-02		2.5013e-02		1.6845e-02	
	10	2.0839e-03	2.9623	3.9864e-03	2.6495	2.4213e-03	2.7984
	20	2.6629e-04	2.9682	5.1223e-04	2.9602	3.2222e-04	2.9097
	40	3.3466e-05	2.9922	6.2396e-05	3.0373	4.1259e-05	2.9653
	80	4.1884e-06	2.9982	7.7201e-06	3.0147	5.2006e-06	2.9880
3	5	1.3740e-03		3.2986e-03		2.0556e-03	
	10	8.2364e-05	4.0602	2.1133e-04	3.9643	1.5526e-04	3.7268
	20	5.1575e-06	3.9973	1.2728e-05	4.0534	1.0401e-05	3.9000
	40	3.2328e-07	3.9958	7.3910e-07	4.1061	6.6583e-07	3.9654
	80	2.0224e-08	3.9986	4.3919e-08	4.0728	4.1911e-08	3.9898
4	5	8.0351e-05		1.7147e-04		9.7122e-05	
	10	2.5626e-06	4.9706	5.9801e-06	4.8417	3.1020e-06	4.9685
	20	8.0488e-08	4.9927	1.8076e-07	5.0480	9.7244e-08	4.9955
	40	2.5183e-09	4.9982	5.5438e-09	5.0270	3.0471e-09	4.9961
	80	7.8731e-11	4.9994	1.7246e-10	5.0066	9.5402e-11	4.9973

$$\begin{aligned}
\vec{a}_1 &= \vec{a}(t) + \Delta t \mathcal{L}(\vec{a}(t), t) \\
\vec{a}_2 &= \frac{3}{4}\vec{a}(t) + \frac{1}{4}\vec{a}_1 + \frac{1}{4}\Delta t \mathcal{L}(\vec{a}_1, t + \Delta t) \\
\vec{a}(t + \Delta t) &= \frac{1}{3}\vec{a}(t) + \frac{2}{3}\vec{a}_2 + \frac{2}{3}\Delta t \mathcal{L}(\vec{a}_2, t + \frac{\Delta t}{2})
\end{aligned}$$

Below, we discuss two examples: one with periodic boundary conditions and other one with Dirichlet boundary conditions.

5.1. Example 1. We test the proposed scheme on the non-homogeneous problem (87)-(88) with $f(x,t) = -\frac{1}{2}\cos(x-t) + \epsilon\sin(x-t) + \cos(x-t)\sin(x-t)$ and $u(x,0) = \sin(x)$. The exact solution of this problem is given by

$$\begin{aligned}
u(x,t) &= \sin(x-t) \\
\phi(x,t) &= -\frac{1}{2}\sin(x-t).
\end{aligned}$$

5.1.1. Accuracy test. We run the simulation on the domain $[0, 2\pi]$ at $t = 1$ using $\Delta t = 0.0001$. The value of ϵ is fixed at $1/10$. For $\theta = 1/2$, we use $\epsilon = 1/10$. For $\theta = 0$, we use $\epsilon = 1/10, 1/100, 1/1000, 1/10000$.

The results in the first four tables show that we can achieve $(k+1)$-order of accuracy if $\theta = 0$, which confirms our theoretic findings given in Theorem 4.5. However, as ϵ becomes smaller and smaller, we start to lose the superconvergence for k odd. Finally, when $\epsilon = 0$, it is observed in [12] that the $(k+1)$-order of accuracy can be achieved only for k even. This result is consistent with the inviscous Burgers-Poisson equation [12]. The results in tables 5 corresponds to the case $\theta = 1/4$.

Table 2

k	N	$\theta = 0, \epsilon = 1/100$					
		$\|u-u_h\|$	order	$\|u-u_h\|_\infty$	order	$\|w-w_h\|$	order
1	5	3.2328e-01		3.6130e-01		1.5506e-01	
	10	1.2984e-01	1.3161	1.7635e-01	1.0348	1.2534e-01	0.3070
	20	3.1738e-02	2.0324	3.9362e-02	2.1635	5.9163e-02	1.0831
	40	5.3134e-03	2.5785	8.5844e-03	2.1970	1.7721e-02	1.7393
	80	8.9217e-04	2.5742	1.6385e-03	2.3893	4.5447e-03	1.9632
2	5	1.9219e-02		4.0640e-02		9.4163e-03	
	10	1.7010e-03	3.4981	2.5775e-03	3.9789	1.8927e-03	2.3147
	20	2.2288e-04	2.9321	4.7986e-04	2.4253	4.0077e-04	2.2396
	40	3.0563e-05	2.8664	7.1167e-05	2.7534	7.8836e-05	2.3458
	80	4.0637e-06	2.9109	8.6607e-06	3.0387	1.3215e-05	2.5766
3	5	2.7282e-03		3.9711e-03		2.7922e-03	
	10	1.6787e-04	4.0225	3.8538e-04	3.3652	3.3356e-04	3.0654
	20	7.1141e-06	4.5606	1.9341e-05	4.3166	2.7439e-05	3.6036
	40	3.4985e-07	4.3459	1.0059e-06	4.2651	1.8187e-06	3.9153
	80	2.0310e-08	4.1065	5.6500e-08	4.1541	1.2020e-07	3.9194
4	5	9.2490e-05		2.2458e-04		1.0465e-04	
	10	2.3969e-06	5.2701	5.8055e-06	5.2737	4.5874e-06	4.5117
	20	7.7214e-08	4.9562	2.1897e-07	4.7286	2.0415e-07	4.4900
	40	2.4919e-09	4.9536	6.5668e-09	5.0594	8.0486e-09	4.6647
	80	7.8521e-11	4.9880	1.8774e-10	5.1284	2.7818e-10	4.8547

Table 3

k	N	$\theta = 0, \epsilon = 1/1000$					
		$\|u-u_h\|$	order	$\|u-u_h\|_\infty$	order	$\|w-w_h\|$	order
1	5	3.5181e-01		3.9157e-01		5.4044e-02	
	10	1.8182e-01	0.9523	2.4786e-01	0.6597	5.6150e-02	-0.0551
	20	8.2851e-02	1.1339	1.3673e-01	0.8582	5.0705e-02	0.1471
	40	2.7697e-02	1.5808	3.6894e-02	1.8898	3.3757e-02	0.5870
	80	5.3698e-03	2.3668	6.1613e-03	2.5821	1.2970e-02	1.3800
2	5	1.9584e-02		4.2807e-02		3.1992e-03	
	10	1.5649e-03	3.6455	2.2156e-03	4.2720	6.5149e-04	2.2959
	20	1.8270e-04	3.0985	3.1696e-04	2.8054	1.5950e-04	2.0302
	40	2.3350e-05	2.9680	4.8526e-05	2.7075	3.7702e-05	2.0808
	80	3.0681e-06	2.9280	7.6729e-06	2.6609	8.9789e-06	2.0701
3	5	3.9413e-03		4.3871e-03		1.2223e-03	
	10	4.4060e-04	3.1611	9.5367e-04	2.2017	3.2856e-04	1.8954
	20	3.3511e-05	3.7167	6.2666e-05	3.9277	4.9405e-05	2.7334
	40	1.5009e-06	4.4808	2.8277e-06	4.4700	4.9075e-06	3.3316
	80	5.4373e-08	4.7868	1.2395e-07	4.5117	3.4383e-07	3.8352
4	5	1.0052e-04		2.3377e-04		4.8462e-05	
	10	2.0176e-06	5.6386	3.8261e-06	5.9331	1.8901e-06	4.6803
	20	5.9925e-08	5.0733	1.3550e-07	4.8195	1.1244e-07	4.0713
	40	1.9704e-09	4.9266	5.9395e-09	4.5118	6.2640e-09	4.1659
	80	6.7755e-11	4.8620	2.3095e-10	4.6847	3.4924e-10	4.1648

Table 4

k	N	$\theta = 0, \epsilon = 1/10000$					
		$\|u - u_h\|$	order	$\|u - u_h\|_\infty$	order	$\|w - w_h\|$	order
1	5	3.5496e-01		3.9478e-01		1.7261e-02	
	10	1.8890e-01	0.9100	2.5724e-01	0.6179	1.8452e-02	-0.0962
	20	9.6043e-02	0.9759	1.6220e-01	0.6653	1.8572e-02	-0.0093
	40	4.6361e-02	1.0508	7.9448e-02	1.0297	1.7825e-02	0.0592
	80	1.9337e-02	1.2616	2.8940e-02	1.4569	1.4841e-02	0.2643
2	5	1.9647e-02		4.2946e-02		1.0233e-03	
	10	1.5621e-03	3.6527	2.1982e-03	4.2881	2.1129e-04	2.2760
	20	1.7878e-04	3.1272	2.7232e-04	3.0129	5.2349e-05	2.0130
	40	2.1894e-05	3.0295	3.3385e-05	3.0280	1.2601e-05	2.0546
	80	2.7510e-06	2.9925	5.1780e-06	2.6887	3.1059e-06	2.0205
3	5	4.1383e-03		4.6185e-03		4.0243e-04	
	10	5.5488e-04	2.8988	1.4944e-03	1.6279	1.3873e-04	1.5365
	20	6.4602e-05	3.1025	1.7191e-04	3.1198	3.1246e-05	2.1506
	40	6.1030e-06	3.4040	1.5968e-05	3.4284	6.1922e-06	2.3351
	80	3.6911e-07	4.0474	6.7325e-07	4.5679	7.7617e-07	2.9960
4	5	1.0255e-04		2.2554e-04		1.6424e-05	
	10	2.0505e-06	5.6442	3.8575e-06	5.8696	6.4710e-07	4.6657
	20	5.6146e-08	5.1907	1.0990e-07	5.1334	3.8065e-08	4.0874
	40	1.7335e-09	5.0175	4.1854e-09	4.7146	2.1697e-09	4.1329
	80	5.4746e-11	4.9848	1.5017e-10	4.8007	1.3589e-10	3.9970

Table 5

k	N	$\theta = 1/4, \epsilon = 1/10$					
		$\|u - u_h\|$	order	$\|u - u_h\|_\infty$	order	$\|w - w_h\|$	order
1	5	2.6814e-01		3.2185e-01		1.8409e-01	
	10	8.9032e-02	1.5906	1.1929e-01	1.4320	1.1095e-01	0.7306
	20	2.1332e-02	2.0613	3.3916e-02	1.8144	4.0375e-02	1.4583
	40	4.8176e-03	2.1467	7.8323e-03	2.1145	1.1209e-02	1.8488
	80	1.1492e-03	2.0677	1.7592e-03	2.1545	2.8724e-03	1.9643
2	5	1.6221e-02		3.2720e-02		1.2031e-02	
	10	1.6509e-03	3.2965	3.1440e-03	3.3795	1.4365e-03	3.0662
	20	2.0201e-04	3.0308	3.7388e-04	3.0720	1.7460e-04	3.0404
	40	2.5167e-05	3.0048	4.6143e-05	3.0184	2.1722e-05	3.0068
	80	3.1434e-06	3.0011	5.7479e-06	3.0050	2.7147e-06	3.0003
3	5	1.9972e-03		4.6369e-03		2.3566e-03	
	10	1.3250e-04	3.9139	3.3880e-04	3.7747	2.4305e-04	3.2774
	20	8.4209e-06	3.9759	2.1092e-05	4.0057	1.9115e-05	3.6685
	40	5.2986e-07	3.9903	1.2048e-06	4.1298	1.2965e-06	3.8820
	80	3.3155e-08	3.9983	6.9668e-08	4.1122	8.2972e-08	3.9659
4	5	7.7137e-05		1.9411e-04		8.1797e-05	
	10	2.0459e-06	5.2366	4.5215e-06	5.4239	1.8708e-06	5.4503
	20	6.2286e-08	5.0377	1.3263e-07	5.0913	5.2795e-08	5.1472
	40	1.9349e-09	5.0086	4.1126e-09	5.0112	1.6093e-09	5.0358
	80	6.0385e-11	5.0019	1.2811e-10	5.0046	5.0008e-11	5.0082

TABLE 6

k	N	$\theta = 1/2, \epsilon = 1/10$					
		$\|u - u_h\|$	order	$\|u - u_h\|_\infty$	order	$\|w - w_h\|$	order
0	5	3.6079e-01		4.1791e-01		5.7311e-02	
	15	1.2005e-01	1.0017	1.3769e-01	1.0106	1.4206e-02	1.2696
	45	4.0026e-02	0.9998	4.5541e-02	1.0071	4.5310e-03	1.0402
	135	1.3342e-02	1.0000	1.5120e-02	1.0036	1.5036e-03	1.0041
2	5	1.5333e-02		2.6582e-02		7.5385e-03	
	15	4.1908e-04	3.2766	5.9120e-04	3.4642	1.3368e-04	3.6704
	45	1.5231e-05	3.0172	2.1322e-05	3.0242	4.8172e-06	3.0250
	135	5.6298e-07	3.0018	7.8770e-07	3.0023	1.7803e-07	3.0020
3	5	3.7800e-03		5.8459e-03		1.7401e-03	
	15	1.4584e-04	2.9628	2.4965e-04	2.8704	7.5612e-05	2.8546
	45	5.4232e-06	2.9963	9.3695e-06	2.9880	2.8253e-06	2.9920
	135	2.0094e-07	2.9996	3.4788e-07	2.9977	1.0472e-07	2.9993
4	5	7.6576e-05		1.3690e-04		5.0985e-05	
	15	2.3275e-07	5.2758	3.7040e-07	5.3817	7.4292e-08	5.9450
	45	9.4001e-10	5.0171	1.4902e-09	5.0206	2.9729e-10	5.0255
	135	3.9985e-12	4.9699	6.8782e-12	4.8956	1.3249e-12	4.9274

Moreover in table 6, convergence rates for $\theta = 1/2$ are considered and it is shown that optimal rates of convergence are achieved only for k even as predicted by the Theorem 4.3.

5.1.2. Energy-preserving test. We run the same example on a longer period of time to test the Energy-preserving property of the proposed scheme against the the scheme (26)-(30) with the Lax-Friedrich flux

$$\widehat{u^2} = \frac{1}{2}\left((u_h^-)^2 + (u_h^+)^2 - \sigma(u_h^+ - u_h^+)\right), \sigma = 2 \max_{u \in [u_h^-, u_h^+]} |u|. \tag{90}$$

Using $k = 2$, $N = 80$, and $\Delta t = 0.001$, we plot the decaying of energy $\|u(\cdot,t)\| - \|u(\cdot,0)\|$ from the initial time to the time $t = 100$ using $\theta = 1/4$ and $\theta = 1/2$ in Figures 1 and 2, respectively.

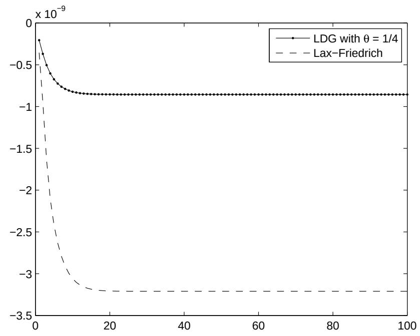

FIGURE 1. The loss of energy from $t = 0$ to $t = 100$.

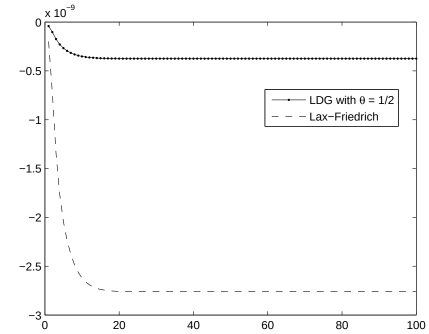

FIGURE 2. The loss of energy from $t = 0$ to $t = 100$.

TABLE 7

k	N	$\theta = 0, \epsilon = 1/10$					
		$\|u - u_h\|$	order	$\|u - u_h\|_\infty$	order	$\|w - w_h\|$	order
1	40	1.2231e-01		1.6244e-01		2.6912e-01	
	80	4.9729e-02	1.2984	1.0810e-01	0.5875	2.0980e-01	0.3592
	160	1.3170e-02	1.9168	3.3023e-02	1.7108	7.3256e-02	1.5180
	320	3.8009e-03	1.7929	1.1698e-02	1.4972	1.9559e-02	1.9052
	640	1.0471e-03	1.8600	3.4005e-03	1.7825	4.9544e-03	1.9810
2	40	5.6573e-02		8.4813e-02		2.1547e-01	
	80	6.2863e-03	3.1698	1.3734e-02	2.6265	2.6513e-02	3.0227
	160	8.5312e-04	2.8814	2.1423e-03	2.6806	3.7012e-03	2.8406
	320	1.1718e-04	2.8641	3.5382e-04	2.5981	4.9873e-04	2.8917
	640	1.5550e-05	2.9137	5.0821e-05	2.7995	6.3964e-05	2.9629
3	40	6.0195e-03		1.7721e-02		1.8538e-02	
	80	6.7208e-04	3.1630	1.9713e-03	3.1683	2.6617e-03	2.8001
	160	4.8549e-05	3.7911	1.9500e-04	3.3376	2.2289e-04	3.5779
	320	3.2886e-06	3.8839	1.4183e-05	3.7813	1.4688e-05	3.9236
	640	2.1486e-07	3.9360	9.9745e-07	3.8297	9.3265e-07	3.9772
4	40	2.1454e-03		4.2698e-03		8.0431e-03	
	80	7.8971e-05	4.7638	2.4591e-04	4.1180	4.0057e-04	4.3276
	160	2.6014e-06	4.9240	1.0234e-05	4.5867	1.1966e-05	5.0651
	320	8.7346e-08	4.8964	3.8211e-07	4.7432	3.8816e-07	4.9461
	640	2.8526e-09	4.9364	1.2742e-08	4.9063	1.2258e-08	4.9848

5.2. Example 2. We test the proposed scheme on the non-homogeneous problem with Dirichlet boundary conditions. The exact solutions are given by

$$
\begin{aligned}
u(x,t) &= -\operatorname{sech}(t-x) - \operatorname{sech}^3(t-x) + \operatorname{sech}(t-x)\tanh^2(t-x)\\
\phi(x,t) &= \operatorname{sech}(x-t).
\end{aligned}
$$

Here, zero boundary conditions are used in place of (1).

We run the simulation on the domain $[-20, 20]$ at $t = 0.1$ using $\Delta t = 0.00001$. The values of ϵ and θ are fixed at $1/10$ and 0 as shown in the table below. It is observed that the order of convergence is $k + 1$.

6. Conclusion

In this article, the LDG method is applied to the viscous Burgers-Poisson system and optimal convergence rates are proved only for even polynomial degrees k in Theorem 4.3, when $\theta = 1/2$. It is further observed that the bounds in the error analysis are valid uniformly with respect to ϵ. Subsequently in Theorem 4.5, optimal error estimates are shown for both even and odd polynomial degrees, but the constants in the error estimates depend on $\epsilon^{-1/2}$ for $\theta \in [0, 1/2)$. With appropriate changes in our error analysis, it is possible to prove similar convergence rate for the problem (1) -(2) with either Dirichlet or Neumann boundary conditions.

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