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A HYBRIDIZABLE WEAK GALERKIN METHOD FOR THE HELMHOLTZ EQUATION WITH LARGE WAVE NUMBER: hp ANALYSIS

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Abstract. In this paper, an hp hybridizable weak Galerkin (hp-HWG) method is introduced to solve the Helmholtz equation with large wave number in two and three dimensions. By choosing a specific parameter and using the duality argument, we prove that the proposed method is stable under certain mesh constraint. Error estimate is obtained by using the stability analysis and the duality argument. Several numerical results are provided to confirm our theoretical results.

Key words. Weak Galerkin method, hybridizable method, Helmholtz equation, large wave number, error estimates.

1. Introduction

In this paper, we develop an hp-version hybridizable weak Galerkin (hp-HWG) method to solve the Helmholtz equation with Robin boundary condition:

$$\Delta u + \kappa^2 u = \tilde{f} \quad \text{in } \Omega, \tag{1a}$$

$$u = u_0 \quad \text{on } \Gamma_1, \tag{1b}$$

$$\frac{\partial u}{\partial \mathbf{n}} + i\kappa u = \tilde{g} \quad \text{on } \Gamma_2, \tag{1c}$$

where $\Omega \in \mathbb{R}^d$, $d = 2, 3$, is a bounded convex Lipschitz domain, Γ_1 and Γ_2 form a partition of the boundary $\partial\Omega$, $\kappa > 0$ is the wave number, $i = \sqrt{-1}$ is the imaginary unit, and $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$. The condition (1c) is the first order approximation of the radiation condition for Helmholtz scattering problem.

The Helmholtz equation has important applications in electrodynamics, especially in optics and acoustics involving time harmonic wave propagation. The Helmholtz system is not positive definite. When the wave number $\kappa \gg 1$, the solution is highly oscillatory. It is very challenging to design an efficient numerical method to solve the Helmholtz equation with high wave number.

In the literature, there have been extensive investigations devoted to numerical approximations for Helmholtz equations with various boundary conditions. In particular, the finite element method (FEM) has been widely used [3, 7, 17, 18, 21, 22, 35]. It has been shown that the H^1-errors of pth order FEM solutions to the Helmholtz equation have accuracy order $O(\kappa^{p+1}h^p)$ [21, 22, 35, 36]. In [7], Wu et al. analyzed the preasymptotic error of high order FEM and continuous interior penalty FEM (CIP-FEM) for Helmholtz equation with large wave number. They proved that, when $\kappa^{2p+1}h^{2p}$ is sufficiently small, the pollution errors are of order $k^{2p+1}h^{2p}$. Discontinuous Galerkin methods have also been used to solve Helmholtz equations [8, 11, 12, 13, 22, 30]. Detailed analyses have been carried out in [1, 2] on the discrete dispersive relation by hp-FEM and high-order discontinuous Galerkin methods. In [28, 29], Shen and Wang used the spectral method to solve

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the Helmholtz equation in both interior and exterior domains. Their results indicate that high-order methods are preferable, if not necessary, for highly oscillatory problems. In [4, 5, 14], hybridizable discontinuous Galerkin methods were used to solve the Helmholtz equation.

The weak Galerkin (WG) method was first introduced by Wang and Ye [32] for second-order elliptic equations. It can be derived from the variational form of the continuous problem by replacing derivatives involved by weak derivatives with some stabilizers. WG methods have been applied to solve many problem [20, 23, 24, 25, 26, 31, 32, 33, 34]. The HWG method [27] was introduced by Mu et al., which applies Lagrange multiplier so that the computational complexity can be significantly reduced.

In this paper, we will develop an *hp*-HWG method to solve the Helmholtz equation with high wave number. The main difficulty in the analysis of the numerical method is due to the strong indefiniteness of the Helmholtz equation. As a consequence, the stability of the numerical approximation is hard to establish. In this work, we use the duality argument to show that the proposed *hp*-HWG method is stable under proper mesh condition. This stability result not only guarantees the existence of the HWG method but also plays an important role in the error analysis. In particular, we first construct an auxiliary problem and establish its *hp*-HWG error estimates; then we combined the estimates with the stability result to derive the error estimates of the *hp*-HWG scheme for the original Helmholtz problem.

Notation. In this paper, standard notations for Sobolev spaces (e.g., $L^2(\Omega)$, $H^k(\Omega)$ for $k \in \mathbb{N}$, etc.) and the associated norms and seminorms will be adopted. Plain and bold fonts are used for scalars and vectors, respectively.

The rest of this paper is organized as follows. The *hp*-HWG scheme for the Helmholtz equation is developed in Section 2. Section 3 is devoted to show the stability result of the numerical scheme. In Section 4, we derive the error estimate of the numerical scheme. Numerical results are given in Section 5 to confirm the theoretical results.

2. Weak Divergence and the *hp*-HWG Scheme

2.1. Weak divergence. Let K be a subdomain in Ω. A *weak vector-valued function* on K refers to a vector field $\mathbf{v} = \{\mathbf{v}_0,\mathbf{v}_b\}$, where $\mathbf{v}_0 \in [L^2(K)]^d$ carries the information of $\mathbf{v}$ in K, and $\mathbf{v}_b \in [L^2(\partial K)]^d$ represents partial or full information of $\mathbf{v}$ on ∂K. It is important to point out that $\mathbf{v}_b$ may not necessarily be related to the trace of $\mathbf{v}_0$ on ∂K, but shall be well-defined. Denote by $\mathbf{V}(K)$ the space of all weak vector-valued functions on K; that is

$$\mathbf{V}(K) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [L^2(K)]^d, \mathbf{v}_b \in [L^2(\partial K)]^d\}.$$

A weak divergence can be taken for any vector field in $\mathbf{V}(K)$ by following the definition [27].

Definition 2.1. *For any $\mathbf{v} \in \mathbf{V}(K)$, the weak divergence of $\mathbf{v}$, denoted by $\nabla_w \cdot \mathbf{v}$, is defined as a linear functional on $H^1(K)$, whose action on each $\phi \in H^1(K)$ is given by*

$$(\nabla_w \cdot \mathbf{v}, \phi)_K = -(\mathbf{v}_0, \nabla\phi)_K + < \mathbf{v}_b \cdot \mathbf{n}, \phi >_{\partial K}, \tag{2}$$

where $(\cdot,\cdot)_K$ and $< \cdot,\cdot >_{\partial K}$ stand for the inner products in $L^2(K)$ and $L^2(\partial K)$, respectively.

Next, we introduce a discrete weak divergence operator $(\nabla_{w,k}\cdot)$ in a polynomial subspace of the dual of $H^1(K)$ [27].

Definition 2.2. *For any* $\mathbf{v} \in \mathbf{V}(K)$*, the discrete weak divergence of* $\mathbf{v}$*, denoted by* $(\nabla_{w,k} \cdot \mathbf{v})$*, is defined as the unique polynomial in* $P^k(K)$ *satisfying*

$$(\nabla_{w,k} \cdot \mathbf{v}, \psi)_K = -(\mathbf{v}_0, \nabla\psi)_K + < \mathbf{v}_b \cdot \mathbf{n}, \psi >_{\partial K}, \quad \forall \psi \in P^k(K). \tag{3}$$

Here, for $k \geq 0$*,* $P^k(K)$ *is the set of polynomials on* K *with total degree up to* k*.*

2.2. Numerical scheme. Let $\mathcal{T}_h$ be a partition of Ω, with possible hanging nodes. For an element $K \in \mathcal{T}_h$, denote by $h_K = \mathrm{diam}(K)$ the diameter of K. Let $h = \max_{K \in \mathcal{T}_h} h_K$ be the mesh size of $\mathcal{T}_h$. Denote by $\mathcal{E}_h = \cup_{K \in \mathcal{T}_h} \partial K$ the skeleton of the mesh, and set $\mathcal{E}_h^B = \mathcal{E}_h \cap \partial\Omega$ and $\mathcal{E}_h^I = \mathcal{E}_h \setminus \mathcal{E}_h^B$.

To introduce the *hp*-HWG method, we rewrite (1a)-(1c) as a first order system

$$\mathrm{i}\kappa \mathbf{q} + \nabla u = 0 \quad \text{in } \Omega, \tag{4a}$$

$$\mathrm{i}\kappa u + \mathrm{div}\mathbf{q} = f \quad \text{in } \Omega, \tag{4b}$$

$$u = u_0 \quad \text{on } \Gamma_1, \tag{4c}$$

$$-\mathbf{q} \cdot \mathbf{n} + u = g \quad \text{on } \Gamma_2. \tag{4d}$$

Multiplying $\mathbf{v} \in [L^2(\Omega)]^d$ and $w \in L^2(\Omega)$ to (4a) and (4b), respectively, and using integration by parts, we get the weak formulation

$$\mathrm{i}\kappa(\mathbf{q}, \overline{\mathbf{v}})_K - (\overline{\nabla \cdot \mathbf{v}}, u)_K + < u, \overline{\mathbf{v} \cdot \mathbf{n}} >_{\partial K} = 0, \tag{5a}$$

$$\mathrm{i}\kappa(u, \overline{w})_K + (\mathrm{div}\mathbf{q}, \overline{w})_K = (f, \overline{w})_K. \tag{5b}$$

For each element $K \in \mathcal{T}_h$, let $\mathbf{n}$ be the outward normal direction to the boundary ∂K. Let $W^k(K) = P^k(K)$ and

$$\mathbf{V}^k(K) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [P^k(K)]^d, \mathbf{v}_b\big|_e = v_b \mathbf{n} \text{ for } v_b \in P^k(e), e \in \partial K\}.$$

On the wired-basket $\mathcal{E}_h$, define a finite element space

$$\Lambda_h = \{\lambda : \lambda|_e \in P^k(e), \forall e \in \mathcal{E}_h\}.$$

Let Λ_h^0 be a subset of Λ such that

$$\Lambda_h^0 = \{\lambda \in \Lambda_h : \lambda|_e = 0, \forall e \in \mathcal{E}_h \cap \Gamma_1\}.$$

Define discontinuous finite element spaces

$$\mathbf{V}_h = \{\mathbf{v} \in [L^2(\Omega)]^d : \mathbf{v}|_K \in \mathbf{V}^k(K), \forall K \in \mathcal{T}_h\},$$
$$W_h = \{w \in L^2(\Omega) : w|_K \in W^k(K), \forall K \in \mathcal{T}_h\}.$$

Further, for the mesh $\mathcal{T}_h$, we define

$$(u, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (u, v)_K, \qquad (\mathbf{u}, \mathbf{v})_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (\mathbf{u}, \mathbf{v})_K,$$
$$< u, v >_{\partial\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} < u, v >_{\partial K}, < u, v >_{\Gamma_2} = \sum_{e \in \mathcal{E}_h^B \cap \Gamma_2} < u, v >_e .$$

Our hybridizable weak Galerkin method is: find $\mathbf{q}_h = \{\mathbf{q}_0, \mathbf{q}_b\} \in \mathbf{V}_h$, $u_h \in W_h$, $\lambda \in \Lambda_h$, such that $\lambda = \Pi_b u_0$ on Γ_1 and

$$S_t(\mathbf{q}_h, \mathbf{v}) + \mathrm{i}\kappa(\mathbf{q}_0, \overline{\mathbf{v}_0})_{\mathcal{T}_h} - (\overline{\nabla_{w,k} \cdot \mathbf{v}}, u_h)_{\mathcal{T}_h} + < \lambda, \overline{\mathbf{v}_b \cdot \mathbf{n}} >_{\partial\mathcal{T}_h} = 0, \tag{6a}$$

$$\mathrm{i}\kappa(u_h, \overline{w})_{\mathcal{T}_h} + (\nabla_{w,k} \cdot \mathbf{q}_h, \overline{w})_{\mathcal{T}_h} = (f, \overline{w})_{\mathcal{T}_h}, \tag{6b}$$

$$< \mathbf{q}_b \cdot \mathbf{n}, \overline{\phi} >_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{6c}$$

$$< -\mathbf{q}_b \cdot \mathbf{n} + \lambda, \overline{\phi} >_{\Gamma_2} = < g, \overline{\phi} >_{\Gamma_2}, \tag{6d}$$

for all $\mathbf{v} \in \mathbf{V}_h$, $w \in W_h$ and $\phi \in \Lambda_h^0$, where

$$S_t(\mathbf{q}_h, \mathbf{v}) = \sum_{K \in \mathcal{T}_h} S_K(\mathbf{q}_h, \mathbf{v}),$$

$$S_K(\mathbf{q}_h, \mathbf{v}) = \mathrm{i}\tau < (\mathbf{q}_0 - \mathbf{q}_b) \cdot \mathbf{n}, \overline{(\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}} >_{\partial K},$$

is the element-wise stabilizer, τ is a parameter and Π_b is the L^2 projection operator which will be defined later.

Notice that u_h and $\mathbf{q}_0$ are respectively the approximations of u and $\mathbf{q}$ in element K, λ and $\mathbf{q}_b$ are respectively the approximations of u and $\mathbf{q}$ on ∂K.

3. Stability Analysis

Lemma 3.1. *Let $(\mathbf{q}_h, u_h, \lambda) \in \mathbf{V}_h \times W_h \times \Lambda_h$ be the solution of (6a)–(6d). If $u_0 = 0$, then*

$$\|\lambda\|_{0,\Gamma_2}^2 \le \|f\|_{0,\Omega} \|u_h\|_{0,\Omega} + \|g\|_{0,\Gamma_2}^2, \tag{7}$$

$$\tau \|(\mathbf{q}_0 - \mathbf{q}_b) \cdot \mathbf{n}\|_{0,\partial\mathcal{T}_h}^2 + \kappa \|\mathbf{q}_0\|_{0,\Omega}^2 \le \kappa \|u_h\|_{0,\Omega}^2 + \|f\|_{0,\Omega} \|u_h\|_{0,\Omega} + \|g\|_{0,\Gamma_2}^2. \tag{8}$$

Proof. Choosing $\mathbf{v} = \mathbf{q}_h, w = u_h, \phi = \lambda$ in (6a)–(6d), we get

$$S_t(\mathbf{q}_h, \mathbf{q}_h) + \mathrm{i}\kappa(\mathbf{q}_0, \overline{\mathbf{q}_0})_{\mathcal{T}_h} - (\overline{\nabla_{w,k} \cdot \mathbf{q}_h}, u_h)_{\mathcal{T}_h} + < \lambda, \overline{\mathbf{q}_b \cdot \mathbf{n}} >_{\partial\mathcal{T}_h} = 0, \tag{9a}$$

$$\mathrm{i}\kappa(u_h, \overline{u_h})_{\mathcal{T}_h} + (\nabla_{w,k} \cdot \mathbf{q}_h, \overline{u_h})_{\mathcal{T}_h} = (f, \overline{u_h})_{\mathcal{T}_h}, \tag{9b}$$

$$< \mathbf{q}_b \cdot \mathbf{n}, \overline{\lambda} >_{\partial\mathcal{T}_h \backslash \partial\Omega} = 0, \tag{9c}$$

$$< -\mathbf{q}_b \cdot \mathbf{n} + \lambda, \overline{\lambda} >_{\Gamma_2} = < g, \overline{\lambda} >_{\Gamma_2}. \tag{9d}$$

Summing (9a)–(9d) up, and using the fact that $a\overline{b} = \overline{\overline{a}b}$, we get

$$S_t(\mathbf{q}_h, \mathbf{q}_h) + \mathrm{i}\kappa(\mathbf{q}_0, \overline{\mathbf{q}_0})_{\mathcal{T}_h} - \mathrm{i}\kappa(u_h, \overline{u_h})_{\mathcal{T}_h} + < \lambda, \overline{\lambda} >_{\Gamma_2} = < g, \overline{\lambda} >_{\Gamma_2} + (\overline{f}, u_h)_{\mathcal{T}_h}.$$

Taking the real part and the imaginary part of the above equality, we arrive at

$$\|\lambda\|_{0,\Gamma_2}^2 \le \|f\|_{0,\Omega} \|u_h\|_{0,\Omega} + \|g\|_{0,\Gamma_2}^2,$$

$$\tau \|(\mathbf{q}_0 - \mathbf{q}_b) \cdot \mathbf{n}\|_{0,\partial\mathcal{T}_h}^2 + \kappa \|\mathbf{q}_0\|_{0,\Omega}^2 \le \kappa \|u_h\|_{0,\Omega}^2 + \|f\|_{0,\Omega} \|u_h\|_{0,\Omega} + \|g\|_{0,\Gamma_2}^2.$$

It finishes the proof of this lemma. □

To estimate $\|u_h\|_{0,\Omega}$, we will use a duality argument. Given $u_h \in L^2(\Omega)$, we define

$$-\mathrm{i}\kappa\Phi + \nabla\Psi = 0 \quad \text{in } \Omega, \tag{10a}$$

$$\nabla \cdot \Phi - \mathrm{i}\kappa\Psi = u_h \quad \text{in } \Omega, \tag{10b}$$

$$\Psi = 0 \quad \text{on } \Gamma_1, \tag{10c}$$

$$\Phi \cdot \mathbf{n} = \Psi \quad \text{on } \Gamma_2. \tag{10d}$$

The following result is found in [4].

Lemma 3.2. *For Φ and Ψ defined in* (10)*, they admit the following estimate:*

$$\|\Psi\|_{0,\Omega} + \kappa^{-2}\|\Psi\|_{2,\Omega} + \kappa^{-1}\|\Psi\|_{1,\Omega} + \|\Psi\|_{0,\Gamma_2} + \kappa^{-1}\|\Phi\|_{1,\Omega} \le C\|u_h\|_{0,\Omega}. \tag{11}$$

Let Π_0 be the standard L^2 projection form $L^2(K)$ onto $P^k(K)$, $\mathbf{\Pi}_0 = (\Pi_0)^d$ and Π_b be the standard L^2 projection form $L^2(e)$ onto $P^k(e)$, where $K \in \mathcal{T}_h$, $e \in \mathcal{E}_h$. Let $\Pi_h : \mathbf{V}(K) \to \mathbf{V}^k(K)$ be defined as

$$\Pi_h \mathbf{v} = \{\mathbf{\Pi}_0 \mathbf{v}_0,\ (\Pi_b v_b)\mathbf{n}\}, \quad \forall \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{V}(K).$$

The following estimates have been established in [15, 29].

Lemma 3.3. *For any $u \in H^{s+1}(K)$, $s \geq 0, s \in \mathbb{N}$,*

$$\|u - \Pi_0 u\|_{0,K} + \frac{h}{p}\|\nabla(u - \Pi_0 u)\|_{0,K} \leq C(h/p)^{s+1}|u|_{s+1,K},$$

$$\|u - \Pi_0 u\|_{0,\partial K} \leq C h_K^{s+\frac{1}{2}}(p+1)^{-(s+\frac{1}{2})}|u|_{s+1,K}.$$

Moreover, we have the following results.

Lemma 3.4. (1) *For all $w \in P^k(K)$, $\mathbf{q} \in [H^1(\Omega)]^d$, we have*

$$(\nabla_{w,k} \cdot (\Pi_h \mathbf{q}), \overline{w})_K = (\nabla \cdot \mathbf{q}, \overline{w})_K - < \mathbf{q} \cdot \mathbf{n} - \Pi_b(\mathbf{q} \cdot \mathbf{n}), \overline{w} >_{\partial K} . \tag{12}$$

(2) *For all $v = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{V}^k(K)$, $w \in H^1(K)$, we have*

$$\begin{aligned}(\nabla_{w,k} \cdot \mathbf{v}, \overline{\Pi_0 w})_K &= -(\mathbf{v}_0, \overline{\nabla w})_K + < (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}, \overline{w - \Pi_0 w} >_{\partial K} \\ &\quad + < \mathbf{v}_b \cdot \mathbf{n}, \overline{w} >_{\partial K} .\end{aligned} \tag{13}$$

Proof. The results follow the same line as in [33]. □

Now we are ready to derive the stability estimate of u_h, which plays an important role in the error analysis of the Helmholtz equation.

Theorem 3.1. *If $\kappa^2 h/p \leq 1$ and $u_0 = 0$, then there exists a constant C independent of κ, p and h, such that*

$$\|u_h\|_{0,\Omega} \leq C(\|f\|_{0,\Omega} + \|g\|_{0,\Gamma_2}).$$

Proof. Using (10b) and Green's lemma, we get

$$\begin{aligned}(u_h, \overline{u_h})_{\mathcal{T}_h} &= (u_h, \overline{\nabla \cdot \Phi - \mathrm{i}\kappa\Psi})_{\mathcal{T}_h} \\ &= -(\nabla u_h, \overline{\Phi})_{\mathcal{T}_h} + < u_h, \overline{\Phi \cdot \mathbf{n}} >_{\partial \mathcal{T}_h} + \mathrm{i}\kappa(u_h, \overline{\Psi})_{\mathcal{T}_h}.\end{aligned}$$

Together with (12), we have

$$\begin{aligned}(u_h, \overline{u_h})_{\mathcal{T}_h} &= (\overline{\nabla_{w,k} \cdot (\Pi_h \Phi)}, u_h)_{\mathcal{T}_h} \\ &\quad + \sum_{K \in \mathcal{T}_h} < u_h, \overline{\Phi \cdot \mathbf{n} - \Pi_b(\Phi \cdot \mathbf{n})} >_{\partial K} + \mathrm{i}\kappa(u_h, \overline{\Psi})_{\mathcal{T}_h}.\end{aligned} \tag{14}$$

The combination of (6a), (6b) and (14) leads to

$$\begin{aligned}(u_h, \overline{u_h})_{\mathcal{T}_h} &= \sum_{K \in \mathcal{T}_h} < u_h, \overline{\Phi \cdot \mathbf{n} - \Pi_b(\Phi \cdot \mathbf{n})} >_{\partial K} + (f, \overline{\Pi_0 \Psi})_{\mathcal{T}_h} - (\nabla_{w,k} \cdot \mathbf{q}_h, \overline{\Pi_0 \Psi})_{\mathcal{T}_h} \\ &\quad + S_t(\mathbf{q}_h, \Pi_h \Phi) + \mathrm{i}\kappa(\mathbf{q}_0, \overline{\mathbf{\Pi}_0 \Phi})_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} < \lambda, \overline{\Pi_b(\Phi \cdot \mathbf{n})} >_{\partial K} .\end{aligned} \tag{15}$$

Using (10a) and (13), we have

$$\mathrm{i}\kappa(\mathbf{q}_0, \overline{\mathbf{\Pi}_0 \Phi})_{\mathcal{T}_h} = \mathrm{i}\kappa(\mathbf{q}_0, \overline{\Phi})_{\mathcal{T}_h} = \mathrm{i}\kappa(\mathbf{q}_0, \overline{\frac{-\mathrm{i}}{\kappa}\nabla\Psi})_{\mathcal{T}_h} = -(\mathbf{q}_0, \overline{\nabla\Psi})_{\mathcal{T}_h}, \tag{16}$$

$$\begin{aligned}(\nabla_{w,k} \cdot \mathbf{q}_h, \overline{\Pi_0 \Psi})_{\mathcal{T}_h} &= -(\mathbf{q}_0, \overline{\nabla\Psi})_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} < (\mathbf{q}_0 - \mathbf{q}_b) \cdot \mathbf{n}, \overline{\Psi - \Pi_0 \Psi} >_{\partial K} \\ &\quad + < \mathbf{q}_b \cdot \mathbf{n}, \overline{\Psi} >_{\partial\Omega} .\end{aligned}$$

By the definition of $S_t(\mathbf{w}, \mathbf{v})$, we have, for any $\mathbf{w} \in [H^1(\Omega)]^d$,

$$S_t(\mathbf{w}, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h.$$

Hence,

$$S_t(\Pi_h \Phi, \mathbf{v}) = S_t(\Pi_h \Phi - \Phi, \mathbf{v}). \tag{17}$$

The combination of (15)–(17) and the definition of Π_b lead to

$$\begin{aligned}(u_h,\overline{u_h})_{\mathcal{T}_h} &= (f,\overline{\Pi_0\Psi})_{\mathcal{T}_h}+S_t(\mathbf{q}_h,\Pi_h\Phi-\Phi)+<\Pi_b u_0,\overline{\Pi_b(\Phi\cdot\mathbf{n})}>_{\Gamma_1}\\ &\quad -\sum_{K\in\mathcal{T}_h}<(\mathbf{q}_0-\mathbf{q}_b)\cdot\mathbf{n},\overline{\Psi-\Pi_0\Psi}>_{\partial K}+<g,\overline{\Pi_0\Psi}>_{\Gamma_2}.\end{aligned}\tag{18}$$

Hence, if $u_0=0$, we get the following estimate

$$\begin{aligned}\|u_h\|_{0,\Omega}^2\leq&\|f\|_{0,\Omega}\|\Psi\|_{0,\Omega}+\|g\|_{0,\Gamma_2}\|\Psi\|_{0,\Gamma_2}\\ &+\|(\mathbf{q}_0-\mathbf{q}_b)\cdot\mathbf{n}\|_{0,\partial\mathcal{T}_h}(h/p)^{\frac{3}{2}}\|\Psi\|_{2,\Omega}+\tau\|(\mathbf{q}_0-\mathbf{q}_b)\cdot\mathbf{n}\|_{0,\partial\mathcal{T}_h}(h/p)^{\frac{1}{2}}\|\Phi\|_{1,\Omega}.\end{aligned}$$

Using Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}\|u_h\|_{0,\Omega}^2\leq&\|f\|_{0,\Omega}\|u_h\|_{0,\Omega}+\|g\|_{0,\Gamma_2}\|u_h\|_{0,\Omega}\\ &+(\tau^{-\frac{1}{2}}\kappa^2(h/p)^{\frac{3}{2}}+\tau^{\frac{1}{2}}\kappa(h/p)^{\frac{1}{2}})\\ &\times\Big(\kappa^{\frac{1}{2}}\|u_h\|_{0,\Omega}+\|f\|_{0,\Omega}^{\frac{1}{2}}\|u_h\|_{0,\Omega}^{\frac{1}{2}}+\|g\|_{0,\Gamma_2}\Big)\|u_h\|_{0,\Omega}.\end{aligned}$$

We choose $\tau=\dfrac{\kappa h}{p}$ to get the minimum of term $\tau^{-\frac{1}{2}}\kappa^2(h/p)^{\frac{3}{2}}+\tau^{\frac{1}{2}}\kappa(h/p)^{\frac{1}{2}}$. Finally, we obtain

$$\|u_h\|_{0,\Omega}\leq C(\|f\|_{0,\Omega}+\|g\|_{0,\Gamma_2})+\kappa^2\frac{h}{p}\|u_h\|_{0,\Omega}.$$

It follows that, if $\kappa^2h/p<1$, then

$$\|u_h\|_{0,\Omega}\leq C(\|f\|_{0,\Omega}+\|g\|_{0,\Gamma_2}).$$

□

4. Error Analysis

In this section, we will carry out the error analysis of the *hp*-HWG method. For this purpose, we first consider the following auxiliary problem:

$$\mathrm{i}\kappa\mathbf{Q}+\nabla U=0 \quad \text{in }\Omega,\tag{19a}$$
$$\nabla\cdot\mathbf{Q}-\mathrm{i}\kappa U=f-2\mathrm{i}\kappa u \quad \text{in }\Omega,\tag{19b}$$
$$U=u_0 \quad \text{on }\Gamma_1,\tag{19c}$$
$$-\mathbf{Q}\cdot\mathbf{n}+U=g \quad \text{on }\Gamma_2,\tag{19d}$$

where u is the solution of (4a)–(4d), and u_0, f, and g are the same as in (4a)–(4d).

The *hp*-HWG scheme for (19a)–(19d) is: find $(\mathbf{Q}_h,U_h,\lambda_h)\in\mathbf{V}_h\times W_h\times\Lambda_h$ such that $\lambda_h=\Pi_b u_0$ on Γ_1 and

$$S_t(\mathbf{Q}_h,\mathbf{v})+\mathrm{i}\kappa(\mathbf{Q}_0,\overline{\mathbf{v}}_0)_{\mathcal{T}_h}-(\overline{\nabla_{w,k}\cdot\mathbf{v}},U_h)_{\mathcal{T}_h}+<\lambda_h,\overline{\mathbf{v}_b\cdot\mathbf{n}}>_{\partial\mathcal{T}_h}=0,\tag{20a}$$
$$-\mathrm{i}\kappa(U_h,\overline{w})_{\mathcal{T}_h}+(\nabla_{w,k}\cdot\mathbf{Q}_h,\overline{w})_{\mathcal{T}_h}=(f-2\mathrm{i}\kappa u,\overline{w})_{\mathcal{T}_h},\tag{20b}$$
$$<\mathbf{Q}_b\cdot\mathbf{n},\overline{\phi}>_{\partial\mathcal{T}_h\backslash\partial\Omega}=0,\tag{20c}$$
$$<-\mathbf{Q}_b\cdot\mathbf{n}+\lambda,\overline{\phi}>_{\Gamma_2}=<g,\overline{\phi}>_{\Gamma_2},\tag{20d}$$

for all $\mathbf{v}\in\mathbf{V}_h, w\in W_h, \phi\in\Lambda_h^0$.

Denote $\theta_{\mathbf{q}} = \mathbf{q}_h - \mathbf{Q}_h, \theta_u = u_h - U_h$ and $\theta_\lambda = \lambda - \lambda_h$. According to (6a)–(6d) and (20a)–(20d), we have $\theta_\lambda = 0$ on Γ_1 and

$$S_t(\theta_{\mathbf{q}}, \mathbf{v}) + \mathrm{i}\kappa(\theta_{\mathbf{q}}, \overline{\mathbf{v}}_0)_{\mathcal{T}_h} - (\overline{\nabla_{w,k} \cdot \mathbf{v}}, \theta_u)_{\mathcal{T}_h} + < \theta_\lambda, \overline{\mathbf{v}_b \cdot \mathbf{n}} >_{\partial\mathcal{T}_h} = 0, \tag{21a}$$

$$\mathrm{i}\kappa(\theta_u, \overline{w})_{\mathcal{T}_h} + (\nabla_{w,k} \cdot \theta_{\mathbf{q}}, \overline{w})_{\mathcal{T}_h} = 2\mathrm{i}\kappa(U_h - u, \overline{w})_{\mathcal{T}_h}, \tag{21b}$$

$$< \theta_{\mathbf{q}}^b \cdot \mathbf{n}, \overline{\phi} >_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{21c}$$

$$< -\theta_{\mathbf{q}}^b \cdot \mathbf{n} + \theta_\lambda, \overline{\phi} >_{\Gamma_2} = 0. \tag{21d}$$

By Theorem 3.1, in order to get the estimates of $\theta_{\mathbf{q}}, \theta_u$ and θ_λ, we need to estimate $u - U_h$.

4.1. Error estimates of the auxiliary problem. Direct calculation leads to the following *error equations* of the *hp*-HWG method for (19a)–(19d):

$$\begin{aligned} &S_t(\mathbf{R}_{\mathbf{q}}, \mathbf{v}) + \mathrm{i}\kappa(\mathbf{R}_{\mathbf{q}}, \overline{\mathbf{v}_0})_{\mathcal{T}_h} - (\overline{\nabla_{w,k} \cdot \mathbf{v}}, \Pi_0 u - U_h)_{\mathcal{T}_h} \\ &+ \sum_{K\in\mathcal{T}_h} < \overline{(\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}}, u - \Pi_0 u >_{\partial K} + \sum_{K\in\mathcal{T}_h} < \overline{\mathbf{v}_b \cdot \mathbf{n}}, u - \lambda_h >_{\partial K} = 0, \end{aligned} \tag{22a}$$

$$-\mathrm{i}\kappa(R_u, \overline{w})_{\mathcal{T}_h} + (\nabla_{w,k} \cdot (\Pi_h \mathbf{q} - \mathbf{Q}_h), \overline{w})_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h} < \mathbf{q} \cdot \mathbf{n} - \Pi_b(\mathbf{q} \cdot \mathbf{n}), \overline{w} >_{\partial K} = 0, \tag{22b}$$

$$< \mathbf{q} \cdot \mathbf{n} - \mathbf{Q}_b \cdot \mathbf{n}, \overline{\phi} >_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{22c}$$

$$- < \mathbf{q} \cdot \mathbf{n} - \mathbf{Q}_b \cdot \mathbf{n}, \overline{\phi} >_{\Gamma_2} + < u - \lambda_h, \overline{\phi} >_{\Gamma_2} = 0. \tag{22d}$$

In the current application, we will employ the following decomposition:

$$\begin{aligned} &\mathbf{R}_{\mathbf{q}} = \mathbf{q} - \mathbf{Q}_h = \mathbf{q} - \Pi_h \mathbf{q} - (\mathbf{Q}_h - \Pi_h \mathbf{q}), \\ &R_u = u - U_h = u - \Pi_0 u - (U_h - \Pi_0 u), \\ &\zeta_{\mathbf{q}} = \mathbf{Q}_h - \Pi_h \mathbf{q} = \{\zeta_0, \zeta_b\},\ \xi_u = u - \Pi_0 u,\ \eta_u = U_h - \Pi_0 u. \end{aligned}$$

Lemma 4.1. *Suppose that $u \in H^{s+1}(\Omega)$ and $\mathbf{q} \in [H^{s+1}(\Omega)]^d$ are smooth functions on Ω. Then, the following estimates hold true:*

$$\sum_{K\in\mathcal{T}_h} |<(\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}, \overline{\Pi_0 u - u}>_{\partial K}| \le \left(\frac{h}{p}\right)^{s+\frac{1}{2}} \|u\|_{s+1,\Omega} \Big(\sum_{K\in\mathcal{T}_h} \|(\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}\|_{0,\partial K}^2\Big)^{\frac{1}{2}}, \tag{23a}$$

$$S_t(\mathbf{q} - \Pi_h \mathbf{q}, \mathbf{Q}_h - \Pi_h \mathbf{q}) \le C\tau^{\frac{1}{2}} \left(\frac{h}{p}\right)^{s+\frac{1}{2}} \|\mathbf{q}\|_{s+1,\Omega} \Big(\sum_{K\in\mathcal{T}_h} \tau \|(\zeta_0 - \zeta_b) \cdot \mathbf{n}\|_{0,\partial K}^2\Big)^{\frac{1}{2}}, \tag{23b}$$

for all $\mathbf{v} \in \mathbf{V}_h$.

Proof. By Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} |<(\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}, \overline{\Pi_0 u - u}>_{\partial K}|\\
&\le \Big(\sum_{K\in\mathcal{T}_h}\|(\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\|_{0,\partial K}^2\Big)^{\frac12}\Big(\sum_{K\in\mathcal{T}_h}\|\Pi_0 u-u\|_{0,\partial K}^2\Big)^{\frac12}\\
&\le C(h/p)^{s+\frac12}\|u\|_{s+1,\Omega}\Big(\sum_{K\in\mathcal{T}_h}\|(\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\|_{0,\partial K}^2\Big)^{\frac12}.\end{aligned}$$

Using the definition of $S_t(\mathbf{v},\mathbf{r})$, we have

$$\begin{aligned}&S_t(\mathbf{q}-\Pi_h\mathbf{q}, \mathbf{Q}_h-\Pi_h\mathbf{q})\\
\le\ &(\tau\sum_{K\in\mathbb{T}_h}\|\mathbf{q}-\Pi_h\mathbf{q}\|_{0,\partial K}^2)^{\frac12}\cdot(\tau\sum_{K\in\mathcal{T}_h}\|(\zeta_0-\zeta_b)\cdot\mathbf{n}\|_{0,\partial K}^2)^{\frac12}\\
\le\ &C\tau^{\frac12}(h/p)^{s+\frac12}\|\mathbf{q}\|_{s+1,\Omega}\Big(\sum_{K\in\mathcal{T}_h}\tau\|(\zeta_0-\zeta_b)\cdot\mathbf{n}\|_{0,\partial K}^2\Big)^{\frac12}.\end{aligned}$$

It completes the proof of this lemma. □

Theorem 4.1. *Let $u\in H^{s+1}(\Omega)$, $\mathbf{q}\in[H^{s+1}(\Omega)]^d$ be the solution of (4a)–(4d). Let $\mathbf{Q}_h$, U_h be the solution of (20a)–(20d). If $\kappa^2h/p<1$, then there exists a constant C independent of κ, h and p such that*

$$\begin{aligned}&\kappa^{\frac12}\|\zeta_0\|_{0,\Omega}+\kappa^{\frac12}\|\eta_u\|_{0,\Omega}+\tau^{\frac12}\|(\zeta_0-\zeta_b)\cdot\mathbf{n}\|_{0,\partial\mathcal{T}_h}\\
\le&C(1+\tau^{\frac12})\Big(\frac{h}{p}\Big)^{s+\frac12}\Big(\|u\|_{s+1,\Omega}+\|\mathbf{q}\|_{s+1,\Omega}\Big).\end{aligned}$$

Proof. Taking $\mathbf{v}=\mathbf{Q}_h-\Pi_h\mathbf{q}, w=U_h-\Pi_0u$ and $\phi=\Pi_bu-\lambda_h$ in (22a)–(22d), respectively, we get

$$\begin{aligned}&S_t(\mathbf{R}_\mathbf{q},\mathbf{Q}_h-\Pi_h\mathbf{q})+\mathrm{i}\kappa(\mathbf{R}_\mathbf{q},\overline{\zeta_0})_{\mathcal{T}_h}-(\overline{\nabla_{w,k}\cdot(\mathbf{Q}_h-\Pi_h\mathbf{q})},\Pi_0u-U_h)_{\mathcal{T}_h}\\
&+\sum_{K\in\mathcal{T}_h}<\overline{(\zeta_0-\zeta_b)\cdot\mathbf{n}},u-\Pi_0u>_{\partial K}+\sum_{K\in\mathcal{T}_h}<\overline{\zeta_b\cdot\mathbf{n}},u-\lambda_h>_{\partial K}=0, \qquad (24a)\\
&-\mathrm{i}\kappa(R_u,\overline{U_h-\Pi_0u})_{\mathcal{T}_h}+(\nabla_{w,k}\cdot(\Pi_h\mathbf{q}-\mathbf{Q}_h),\overline{U_h-\Pi_0u})_{\mathcal{T}_h}\\
&+\sum_{K\in\mathcal{T}_h}<\mathbf{q}\cdot\mathbf{n}-\Pi_b(\mathbf{q}\cdot\mathbf{n}),\overline{U_h-\Pi_0u}>_{\partial K}=0, \qquad (24b)\\
&<\mathbf{q}\cdot\mathbf{n}-\mathbf{Q}_b\cdot\mathbf{n},\overline{\Pi_bu-\lambda_h}>_{\partial\mathcal{T}_h\backslash\partial\Omega}=0, \qquad (24c)\\
&-<\mathbf{q}\cdot\mathbf{n}-\mathbf{Q}_b\cdot\mathbf{n},\overline{\Pi_bu-\lambda_h}>_{\Gamma_2}+<u-\lambda_h,\overline{\Pi_bu-\lambda_h}>_{\Gamma_2}=0. \qquad (24d)\end{aligned}$$

Summing the conjugate of (24b) and (24a) together, we get

$$\begin{aligned}&S_t(\mathbf{R}_\mathbf{q},\mathbf{Q}_h-\Pi_h\mathbf{q})+\mathrm{i}\kappa(\mathbf{R}_\mathbf{q},\overline{\zeta_0})_{\mathcal{T}_h}+<\overline{(\zeta_0-\zeta_b)\cdot\mathbf{n}},u-\Pi_0u>_{\partial\mathcal{T}_h}\\
&+\mathrm{i}\kappa(\overline{R_u},U_h-\Pi_0u)_{\mathcal{T}_h}+\sum_{K\in\mathcal{T}_h}<\overline{\zeta_b\cdot\mathbf{n}},u-\lambda_h>_{\partial K}\\
&+\sum_{K\in\mathcal{T}_h}<\overline{\mathbf{q}\cdot\mathbf{n}-\Pi_b(\mathbf{q}\cdot\mathbf{n})},U_h-\Pi_0u>_{\partial K}=0.\end{aligned}$$

Meanwhile, summing (24c) and (24d) together, we have

$$\sum_{K\in\mathcal{T}_h}<\mathbf{q}\cdot\mathbf{n}-\mathbf{Q}_b\cdot\mathbf{n},\overline{\Pi_bu-\lambda_h}>_{\partial K}=<u-\lambda_h,\overline{\Pi_bu-\lambda_h}>_{\Gamma_2},$$

where we have used the fact that $u=u_0,\lambda_h=\Pi_bu_0$ on Γ_1.

By using the orthogonality properties of $\Pi_0, \mathbf{\Pi}_0$ and Π_b, we conclude from the operator decompositions of $\mathbf{R_q}$ and R_u that the two equations above can be written as

$$\begin{aligned} &S_t(\mathbf{R_q}, \mathbf{Q}_h - \Pi_h \mathbf{q}) - \mathrm{i}\kappa(\zeta_0, \overline{\zeta_0})_{\mathcal{T}_h} \\ &+ < \overline{(\zeta_0 - \zeta_b)\cdot \mathbf{n}}, u - \Pi_0 u >_{\partial\mathcal{T}_h} - \mathrm{i}\kappa(\overline{\eta_u}, U_h - \Pi_0 u)_{\mathcal{T}_h} \\ &+ < \overline{\zeta_b \cdot \mathbf{n}}, \lambda_h - \Pi_b u >_{\partial\mathcal{T}_h} + < \overline{\mathbf{q}\cdot\mathbf{n} - \Pi_b(\mathbf{q}\cdot\mathbf{n})}, U_h - \Pi_0 u >_{\partial\mathcal{T}_h} = 0, \end{aligned} \tag{25}$$

$$< \Pi_b(\mathbf{q}\cdot\mathbf{n}) - \mathbf{Q}_b\cdot\mathbf{n}, \overline{\Pi_b u - \lambda_h} >_{\partial\mathcal{T}_h} = < \Pi_b u - \lambda_h, \overline{\Pi_b u - \lambda_h} >_{\Gamma_2} . \tag{26}$$

Substituting (26) into (25), one has

$$\begin{aligned} &\mathrm{i}\kappa(\zeta_0, \overline{\zeta_0})_{\mathcal{T}_h} + \mathrm{i}\kappa(\overline{\eta_u}, U_h - \Pi_0 u)_{\mathcal{T}_h} + < \Pi_b u - \lambda_h, \overline{\Pi_b u - \lambda_h} >_{\Gamma_2} \\ = \; & S_t(\mathbf{R_q}, \mathbf{Q}_h - \Pi_h \mathbf{q}) + < \overline{(\zeta_0 - \zeta_b)\cdot\mathbf{n}}, u - \Pi_0 u >_{\partial\mathcal{T}_h} . \end{aligned} \tag{27}$$

Taking the imaginary part of (27), we get

$$\begin{aligned} &\kappa||\zeta_0||_{0,\Omega}^2 + \kappa||\eta_u||_{0,\Omega}^2 + \tau\|(\zeta_0 - \zeta_b)\cdot\mathbf{n}\|_{0,\partial\mathcal{T}_h}^2 \\ \leq \; & |S_t(\mathbf{q} - \Pi_h\mathbf{q}, \mathbf{Q}_h - \Pi_h\mathbf{q})| + |< \overline{(\zeta_0 - \zeta_b)\cdot\mathbf{n}}, u - \Pi_0 u >_{\partial\mathcal{T}_h} |. \end{aligned} \tag{28}$$

Using lemma 4.1 and (28), we arrive at

$$\begin{aligned} &\kappa^{\frac{1}{2}}\|\zeta_0\|_{0,\Omega} + \kappa^{\frac{1}{2}}\|\eta_u\|_{0,\Omega} + \tau^{\frac{1}{2}}\|(\zeta_0 - \zeta_b)\cdot\mathbf{n}\|_{0,\partial\mathcal{T}_h} \\ \leq & C(1 + \tau^{\frac{1}{2}})\Big(\frac{h}{p}\Big)^{s+\frac{1}{2}}\Big(\|u\|_{s+1,\Omega} + \|\mathbf{q}\|_{s+1,\Omega}\Big). \end{aligned}$$

It completes the proof of this Theorem. □

4.2. The estimate of $\|\eta_u\|$. In this subsection, we will use the Aubin-Nitsche duality argument to get an error estimate of $u - U_h$. Consider the following dual problem,

$$-\mathrm{i}\kappa\Phi + \nabla\Psi = 0 \quad \text{in } \Omega, \tag{29a}$$

$$\nabla\cdot\Phi + \mathrm{i}\kappa\Psi = \eta_u \quad \text{in } \Omega, \tag{29b}$$

$$\Psi = 0 \quad \text{on } \Gamma_1, \tag{29c}$$

$$\Phi\cdot\mathbf{n} = \Psi \quad \text{on } \Gamma_2. \tag{29d}$$

The following result is found in [4].

Lemma 4.2. *Let Ψ, Φ be the solution of (29a)–(29d). Then,*

$$\|\Phi\|_{1,\Omega} + \|\Psi\|_{1,\Omega} + \kappa^{-1}\|\Psi\|_{2,\Omega} \leq C\|\eta_u\|_{0,\Omega}.$$

Lemma 4.3. *Let $u \in H^{s+1}(\Omega)$ and $\mathbf{q} \in [H^{s+1}(\Omega)]^d$ be the solution of (4a)–(4d). Then, we have the following estimates:*

(30a)

$$|S_t(\mathbf{q} - \Pi_h \mathbf{q}, \overline{\Pi_h \Phi})| \leq C\tau(\frac{h}{p})^{s+\frac{1}{2}} \|\mathbf{q}\|_{s+1,\Omega} \|\Phi\|_{1,\Omega},$$

(30b)

$$|S_t(\Pi_h \mathbf{q} - \mathbf{Q}_h, \overline{\Pi_h \Phi})| \leq C\tau^{\frac{1}{2}}(1+\tau^{\frac{1}{2}})\Big(\frac{h}{p}\Big)^{s+\frac{1}{2}} \Big(\|u\|_{s+1,\Omega} + \|\mathbf{q}\|_{s+1,\Omega}\Big) \|\Phi\|_{1,\Omega},$$

(30c)

$$\sum_{K \in \mathcal{T}_h} | < \overline{\Pi_0(\Phi \cdot \mathbf{n}) - \Pi_b(\Phi \cdot \mathbf{n})}, \Pi_0 u - u >_{\partial K} | \leq C(\frac{h}{p})^{s+1} \|u\|_{s+1,\Omega} \|\Phi\|_{1,\Omega},$$

(30d)

$$\sum_{K \in \mathcal{T}_h} | < \mathbf{q} \cdot \mathbf{n} - \Pi_b(\mathbf{q} \cdot \mathbf{n}), \overline{\Pi_0 \Psi - \Psi} >_{\partial K} | \leq C(\frac{h}{p})^{s+2} \|\mathbf{q}\|_{s+1,\Omega} \|\Psi\|_{2,\Omega}.$$

Proof. By the definition of $S_t(\cdot,\cdot)$,

$$\begin{aligned} |S_t(\mathbf{q} - \Pi_h \mathbf{q}, \overline{\Pi_h \Phi})| &= |S_t(\mathbf{q} - \Pi_h \mathbf{q}, \overline{\Pi_h \Phi})| \\ &\leq C\tau\Big(\sum_{e \in \mathcal{E}_h} \|\mathbf{q} - \Pi_h \mathbf{q}\|_{0,e}^2\Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{E}_h} \|\Pi_h \Phi\|_{0,e}^2\Big)^{\frac{1}{2}} \\ &\leq C\tau(\frac{h}{p})^{s+\frac{1}{2}} \|\mathbf{q}\|_{s+1,\Omega} \|\Phi\|_{1,\Omega}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |S_t(\Pi_h \mathbf{q} - \mathbf{Q}_h, \overline{\Pi_h \Phi})| &= |S_t(\Pi_h \mathbf{q} - \mathbf{Q}_h, \overline{\Pi_h \Phi - \Phi})| \\ &\leq C\tau\Big(\sum_{e \in \mathcal{E}_h} \|(\zeta_0 - \zeta_b) \cdot \mathbf{n}\|_{0,e}^2\Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{E}_h} \|\Pi_h \Phi - \Phi\|_{0,e}^2\Big)^{\frac{1}{2}} \\ &\leq C\tau^{\frac{1}{2}}(1+\tau^{\frac{1}{2}})\Big(\frac{h}{p}\Big)^{s+1} \Big(\|u\|_{s+1,\Omega} + \|\mathbf{q}\|_{s+1,\Omega}\Big) \|\Phi\|_{1,\Omega}. \end{aligned}$$

Meanwhile,

$$\begin{aligned} &\sum_{K \in \mathcal{T}_h} | < \overline{\Pi_0(\Phi \cdot \mathbf{n}) - \Pi_b(\Phi \cdot \mathbf{n})}, \Pi_0 u - u >_{\partial K} | \\ = &\sum_{K \in \mathcal{T}_h} | < \overline{\Pi_0(\Phi \cdot \mathbf{n}) - \Phi \cdot \mathbf{n} + \Phi \cdot \mathbf{n} - \Pi_b(\Phi \cdot \mathbf{n})}, \Pi_0 u - u >_{\partial K} | \\ \leq &\ C(\frac{h}{p})^{s+1} \|u\|_{s+1,\Omega} \|\Phi\|_{1,\Omega}. \end{aligned}$$

Then, we turn to the estimate of (30d),

$$\begin{aligned} &\sum_{K \in \mathcal{T}_h} | < \mathbf{q} \cdot \mathbf{n} - \Pi_b(\mathbf{q} \cdot \mathbf{n}), \overline{\Pi_0 \Psi - \Psi} >_{\partial K} | \\ \leq &\Big(\sum_{K \in \mathcal{T}_h} \|\mathbf{q} \cdot \mathbf{n} - \Pi_b(\mathbf{q} \cdot \mathbf{n})\|_{0,\partial K}^2\Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\Pi_0 \Psi - \Psi\|_{0,\partial K}^2\Big)^{\frac{1}{2}} \\ \leq &\ C(\frac{h}{p})^{s+2} \|\mathbf{q}\|_{s+1,\Omega} \|\Psi\|_{2,\Omega}. \end{aligned}$$

Hence, we finish the proof of this lemma. □

Theorem 4.2. *Let $\mathbf{Q}_h, U_h$ be the solution of (20a)–(20d). Let $u \in H^{s+1}(\Omega), \mathbf{q} \in [H^{s+1}(\Omega)]^d$ be the exact solution of (4a)–(4d). If $\kappa^2 h/p < 1$, then there exists a constant C independent of κ, h and p, such that*

$$\|u - U_h\|_{0,\Omega} \le C \left(\frac{h}{p}\right)^{s+1} (\|u\|_{s+1,\Omega} + \|\mathbf{q}\|_{s+1,\Omega}).$$

Proof. Using (29b) and (12), we get

$$\begin{aligned}(\eta_u, \overline{\eta_u})_{\mathcal{T}_h} &= (\eta_u, \overline{\nabla \cdot \Phi + \mathrm{i}\kappa\Psi})_{\mathcal{T}_h} \\ &= (\overline{\nabla_{w,k} \cdot (\Pi_h \Phi)}, \eta_u)_{\mathcal{T}_h} + < \eta_u, \overline{\Phi \cdot \mathbf{n} - \Pi_b(\Phi \cdot \mathbf{n})} >_{\partial\mathcal{T}_h} - \mathrm{i}\kappa(\eta_u, \overline{\Psi})_{\mathcal{T}_h}.\end{aligned}$$

Using (22a), we have

$$\begin{aligned}&(\overline{\nabla_{w,k} \cdot (\Pi_h \Phi)}, \eta_u)_{\mathcal{T}_h} \\ =& S_t(\mathbf{R_q}, \Pi_h \Phi) + \mathrm{i}\kappa(\mathbf{R_q}, \overline{\Pi_0 \Phi})_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} < \overline{\Pi_b(\Phi \cdot \mathbf{n})}, u - \lambda_h >_{\partial K} \\ &+ \sum_{K \in \mathcal{T}_h} < \overline{\Pi_0(\Phi \cdot \mathbf{n}) - \Pi_b(\Phi \cdot \mathbf{n})}, u - \Pi_0 u >_{\partial K} .\end{aligned}$$

By the orthogonal property of $\mathbf{\Pi}_0$ and (29a), one has

$$\mathrm{i}\kappa(\mathbf{R_q}, \overline{\mathbf{\Pi}_0 \Phi})_{\mathcal{T}_h} = \mathrm{i}\kappa(\zeta_0, \overline{\Phi})_{\mathcal{T}_h} = \mathrm{i}\kappa(\zeta_0, \overline{\frac{-\mathrm{i}}{\kappa}\nabla\Psi})_{\mathcal{T}_h} = -(\zeta_0, \overline{\nabla\Psi})_{\mathcal{T}_h}.$$

Using (13), (22b) and the orthogonal properties of Π_0, we obtain

$$\begin{aligned}\mathrm{i}(\eta_u, \overline{\Psi})_{\mathcal{T}_h} =& \mathrm{i}(\eta_u, \overline{\Pi_0\Psi})_{\mathcal{T}_h} \\ =& -(\nabla_{w,k} \cdot (\Pi_h \mathbf{q} - \mathbf{Q}_h), \overline{\Pi_0\Psi})_{\mathcal{T}_h} - \sum_{K \in \mathcal{T}_h} < \mathbf{q} \cdot \mathbf{n} - \Pi_b(\mathbf{q} \cdot \mathbf{n}), \overline{\Pi_0 \Psi} >_{\partial K},\end{aligned}$$

and

$$\begin{aligned}(\nabla_{w,k} \cdot (\Pi_h \mathbf{q} - \mathbf{Q}_h), \overline{\Pi_0\Psi})_{\mathcal{T}_h} &= -(\zeta_0, \overline{\nabla\Psi})_{\mathcal{T}_h} + < \zeta_b \cdot \mathbf{n}, \overline{\Psi} >_{\partial\Omega} \\ &\quad + \sum_{K \in \mathcal{T}_h} < (\zeta_0 - \zeta_b) \cdot \mathbf{n}, \overline{\Psi - \Pi_0 \Psi} >_{\partial K} .\end{aligned}$$

The combination of the above equations leads to

$$\begin{aligned}(\eta_u, \overline{\eta_u})_{\mathcal{T}_h} =& S_t(\mathbf{R_q}, \Pi_h \Phi) + \mathrm{i}\kappa(\mathbf{R_q}, \overline{\mathbf{\Pi}_0 \Phi})_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} < \overline{\Pi_b(\Phi \cdot \mathbf{n})}, u - \lambda_h >_{\partial K} \\ &+ \sum_{K \in \mathcal{T}_h} < \overline{\Pi_0(\Phi \cdot \mathbf{n}) - \Pi_b(\Phi \cdot \mathbf{n})}, u - \Pi_0 u >_{\partial K} \\ &+ < \eta_u, \overline{\Phi \cdot \mathbf{n} - \Pi_b(\Phi \cdot \mathbf{n})} >_{\partial\mathcal{T}_h} - \mathrm{i}\kappa(\eta_u, \overline{\Psi})_{\mathcal{T}_h} \\ =& S_t(\mathbf{R_q}, \Pi_h \Phi) - (\zeta_0, \overline{\nabla\Psi})_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} < \overline{\Pi_b(\Phi \cdot \mathbf{n})}, u - \lambda_h >_{\partial K} \\ &+ \sum_{K \in \mathcal{T}_h} < \overline{\Pi_0(\Phi \cdot \mathbf{n}) - \Pi_b(\Phi \cdot \mathbf{n})}, u - \Pi_0 u >_{\partial K} \\ &+ < \eta_u, \overline{\Phi \cdot \mathbf{n} - \Pi_b(\Phi \cdot \mathbf{n})} >_{\partial\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} < \mathbf{q} \cdot \mathbf{n} - \Pi_b(\mathbf{q} \cdot \mathbf{n}), \overline{\Pi_0 \Psi} >_{\partial K} \\ &+ (\zeta_0, \overline{\nabla\Psi})_{\mathcal{T}_h} - < \zeta_b \cdot \mathbf{n}, \overline{\Psi} >_{\partial\Omega} - \sum_{K \in \mathcal{T}_h} < (\zeta_0 - \zeta_b) \cdot \mathbf{n}, \overline{\Psi - \Pi_0 \Psi} >_{\partial K} .\end{aligned} \tag{31}$$

By definitions of Π_b and λ_h, we have

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} < \overline{\Pi_b(\Phi\cdot\mathbf{n})}, u-\lambda_h >_{\partial K} &= < \overline{\Pi_b(\Phi\cdot\mathbf{n})}, u-\lambda_h >_{\Gamma_2}\\ &= < \overline{\Pi_b(\Phi\cdot\mathbf{n})}, \Pi_b u-\lambda_h >_{\Gamma_2} .\end{aligned} \tag{32}$$

Using (22d) and (29c), it follows

$$\begin{aligned} < \zeta_b\cdot\mathbf{n}, \overline{\Psi} >_{\partial\Omega} &= < \zeta_b\cdot\mathbf{n}, \overline{\Psi} >_{\Gamma_2},\\ < \overline{\Pi_b(\Phi\cdot\mathbf{n})}, \Pi_b u-\lambda_h >_{\Gamma_2} &= < \overline{\Pi_b(\Phi\cdot\mathbf{n})}, \mathbf{q}\cdot n-\mathbf{Q}_b\cdot\mathbf{n} >_{\Gamma_2}\\ &= < \overline{\Pi_b(\Phi\cdot\mathbf{n})}, \zeta_b\cdot\mathbf{n} >_{\Gamma_2} .\end{aligned} \tag{33}$$

The combination of (32) and (33) leads to

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} < \overline{\Pi_b(\Phi\cdot\mathbf{n})}, u-\lambda_h >_{\partial K} - < \zeta_b\cdot\mathbf{n}, \overline{\Psi} >_{\partial\Omega}\\ &= < \overline{\Pi_b\Psi-\Psi}, \zeta_b\cdot\mathbf{n} >_{\Gamma_2} .\end{aligned} \tag{34}$$

Substituting (34) into (31), we get from the property of Π_b that

$$\begin{aligned}(\eta_u,\overline{\eta_u})_{\mathcal{T}_h} &= S_t(\mathbf{R_q},\Pi_h\Phi)+\sum_{K\in\mathcal{T}_h} < \overline{\Pi_0(\Phi\cdot\mathbf{n})-\Pi_b(\Phi\cdot\mathbf{n})}, u-\Pi_0 u >_{\partial K}\\ &\quad -\sum_{K\in\mathcal{T}_h} < (\zeta_0-\zeta_b)\cdot\mathbf{n}, \overline{\Psi-\Pi_0\Psi} >_{\partial K} .\end{aligned}$$

Together with lemma 4.3, we get

$$\begin{aligned}\|\eta_u\|^2_{0,\Omega} &\le C\tau\left(\frac{h}{p}\right)^{s+\frac12}\|\mathbf{q}\|_{s+1,\Omega}\|\Phi\|_{1,\Omega}\\ &\quad +C\tau^{\frac12}(1+\tau^{\frac12})\left(\frac{h}{p}\right)^{s+\frac12}(\|u\|_{s+1,\Omega}+\|\mathbf{q}\|_{s+1,\Omega})\|\Phi\|_{1,\Omega}\\ &\quad +C\left(\frac{h}{p}\right)^{s+1}\|u\|_{s+1,\Omega}\|\Phi\|_{1,\Omega}+C\left(\frac{h}{p}\right)^{s+2}\|\mathbf{q}\|_{s+1,\Omega}\|\Psi\|_{2,\Omega}.\end{aligned}$$

Recall that $\tau=\kappa h/p$ and $\kappa^2 h/p<1$. Using lemma 4.2, we obtain

$$\|\eta_u\|_{0,\Omega} \le C\left(\frac{h}{p}\right)^{s+1}(\|u\|_{s+1,\Omega}+\|\mathbf{q}\|_{s+1,\Omega}).$$

The desired result follows from an application of the triangle inequality. □

Theorem 4.3. *Let $\mathbf{q}\in[H^{s+1}(\Omega)]^d$, $u\in H^{s+1}(\Omega)$ be the solution of (4a)–(4d), and $\mathbf{q}_h$, u_h be the solution of (6a)–(6d). If $\kappa^2h/p<1$, then there exists a constant C independent of κ, h and p, such that*

$$\begin{aligned}&\|u-u_h\|_{0,\Omega}\le C(1+\kappa)\left(\frac{h}{p}\right)^{s+1}(\|u\|_{s+1,\Omega}+\|\mathbf{q}\|_{s+1,\Omega}),\\ &\|\mathbf{q}-\mathbf{q}_h\|_{0,\Omega}\le C\left(\frac{1}{\kappa}+\frac{h}{p}+\frac{\kappa h}{p}\right)\left(\frac{h}{p}\right)^{s}\Big(\|u\|_{s+1,\Omega}+\|\mathbf{q}\|_{s+1,\Omega}\Big).\end{aligned}$$

Proof. Direct calculation leads to

$$\begin{aligned}&u-u_h=u-\Pi_0u+(\Pi_0u-U_h)+(U_h-u_h),\\ &\mathbf{q}-\mathbf{q}_h=\mathbf{q}-\boldsymbol{\Pi}_0\mathbf{q}+(\boldsymbol{\Pi}_0\mathbf{q}-\mathbf{Q}_h)+(\mathbf{Q}_h-\mathbf{q}_h).\end{aligned}$$

Then, the combination of Theorem 3.1, Theorem 4.1 and the triangle inequality completes the proof of theorem. □

TABLE 1. Example 1 – Numerical errors and their convergence rates when $\kappa = 20$ and $\tau = 1$.

p	N	$Error_{ur}$	$order$	$Error_{ui}$	$order$	$Error_{qr}$	$order$	$Error_{qi}$	$order$
1	16	7.2759e-2		7.4392e-2		7.5660e-2		7.1359e-2	
	32	1.7038e-2	2.0944	1.7139e-2	2.1179	1.7446e-2	2.1166	1.4450e-2	2.3040
	64	4.2055e-3	2.0184	4.2255e-3	2.0201	4.4354e-3	1.9758	4.4391e-3	1.7027
	128	1.0472e-3	2.0057	1.0521e-3	2.0059	1.1416e-3	1.9580	1.1423e-3	1.9583
	256	2.1650e-4	2.2741	2.6271e-4	2.0017	2.9581e-4	1.9483	2.9567e-4	1.9499
2	4	5.3263e-1		5.0299e-1		4.8872e-1		4.5507e-1	
	8	6.7472e-2	2.9808	6.3474e-2	2.9863	5.9255e-2	3.0440	6.1081e-2	2.8973
	16	6.8141e-3	3.3077	6.7084e-3	3.2421	6.8501e-3	3.1127	6.8423e-3	3.1582
	32	8.4300e-4	3.0149	8.4020e-4	2.9972	8.7973e-4	2.9610	8.7920e-4	2.9602
	64	1.0516e-4	3.0029	1.0484e-4	3.0025	1.1300e-4	2.9607	1.1301e-4	2.9597
3	4	1.6288e-1		1.4473e-1		1.2886e-1		1.3923e-1	
	8	8.1095e-3	4.3281	8.2020e-3	4.1412	8.0787e-3	3.9955	8.0980e-3	4.1038
	16	5.1154e-4	3.9867	5.1311e-4	3.9986	5.2742e-4	3.9371	5.2765e-4	3.9399
	32	3.2136e-5	3.9926	3.2213e-5	3.9936	3.4070e-5	3.9524	3.4062e-5	3.9533
	64	2.0121e-6	3.9974	2.0165e-6	3.9977	2.1946e-6	3.9565	2.1924e-6	3.9576
4	4	2.2940e-2		2.2989e-2		2.8650e-2		2.8591e-2	
	8	9.8821e-4	4.5369	9.8018e-4	4.5518	9.9811e-4	4.8432	9.9685e-4	4.8420
	16	3.1637e-5	4.9651	3.1565e-5	4.9566	3.2955e-5	4.9206	3.2962e-5	4.9185
	32	9.9616e-7	4.9891	9.9433e-7	4.9885	1.0637e-6	4.9533	1.0646e-6	4.9524
	64	3.1220e-8	4.9958	3.1155e-8	4.9962	3.4267e-8	4.9561	3.4326e-8	4.9549

TABLE 2. Example 1 – Numerical errors and their convergence rates when $\kappa = 100$ and $\tau = 1$.

p	N	$Error_{ur}$	$order$	$Error_{ui}$	$order$	$Error_{qr}$	$order$	$Error_{qi}$	$order$
5	16	5.9324e-2		1.6886e-1		1.6667e-1		5.9069e-2	
	32	4.0396e-4	7.1983	3.8333e-4	8.7830	4.0162e-4	8.6969	3.7255e-4	7.3088
	64	6.2622e-6	6.0114	6.2583e-6	5.9367	6.2545e-6	6.0048	6.2522e-6	5.8969
	128	9.1352e-8	6.0991	9.1402e-8	6.0974	9.0489e-8	6.0920	9.2463e-8	6.0793
6	16	5.8303e-3		8.8604e-3		8.8508e-3		5.5560e-3	
	32	4.1480e-5	7.1350	4.4167e-5	7.6483	4.1876e-5	7.7235	4.2935e-5	7.0157
	64	3.4917e-7	6.8923	3.5206e-7	6.9710	3.5291e-7	6.8907	3.5020e-7	6.9378
	128	2.7492e-9	6.9888	2.8055e-9	6.9714	2.7976e-9	6.9790	2.7866e-9	6.9735

TABLE 3. Example 1 – Numerical errors and their convergence rates when $\frac{\kappa h}{p} = 1.1$ and $\tau = 1$.

κ	p	N	$Error_{ur}$	$Error_{ui}$	$Error_{qr}$	$Error_{qi}$
22	5	4	1.0280e-2	1.0298e-2	1.0033e-2	1.0010e-2
44		8	1.0479e-2	1.1833e-2	1.1136e-2	1.0048e-2
88		16	1.0594e-2	1.2286e-2	1.1329e-2	1.0085e-2
176		32	1.0731e-2	1.2389e-2	1.1597e-2	1.0085e-2
356		64	1.0946e-2	1.2477e-2	1.2189e-2	1.0155e-2
712		128	1.3098e-2	1.5654e-2	1.6432e-2	1.2870e-2
1424		256	6.0231e-2	6.9078e-2	7.0456e-2	6.1176e-2

5. Numerical Results

We shall provide several numerical examples to support the theoretical analysis of the HWG scheme (6a)–(6d). The exact solution can be written as $u = ur + \mathrm{i}\, ui$.

TABLE 4. Example 1 – Numerical errors and their convergence rates when $\frac{\kappa h}{p} = 1.125$ and $\tau = 1$.

κ	p	N	$Error_{ur}$	$Error_{ui}$	$Error_{qr}$	$Error_{qi}$
18	4	4	1.7612e-2	1.7283e-2	1.7029e-2	1.7037e-2
36		8	1.7956e-2	1.8068e-2	1.7346e-2	1.7272e-2
72		16	1.8035e-2	2.0094e-2	1.8906e-2	1.7100e-2
144		32	1.8795e-2	2.0975e-2	1.9701e-2	1.7810e-2
288		64	1.8946e-2	2.1377e-2	2.0189e-2	1.8155e-2
576		128	2.0754e-2	2.3007e-2	2.3368e-2	2.0009e-2
1158		256	5.5679e-2	5.7053e-2	5.8876e-2	5.6851e-2

In this section, the L^2-errors are computed as follows:

$$Error_{ur} = \Big(\sum_{K\in\mathcal{T}_h} \int_K |ur - ur_h|^2 d\mathbf{x}\Big)^{\frac{1}{2}},$$

$$Error_{ui} = \Big(\sum_{K\in\mathcal{T}_h} \int_K |ui - ui_h|^2 d\mathbf{x}\Big)^{\frac{1}{2}}.$$

Example 1. Let $\Omega = [0,1]^2$. We firstly consider a pure Dirichlet boundary value problem (i.e. $\Gamma_2 = \emptyset$). The exact solution is taken as $u = \exp(-\mathrm{i}\kappa x)$. We choose the parameter $\tau = O(1)$. Tables 1 and 2 show the errors and their convergent rates of the numerical solutions by the HWG scheme for $\kappa = 20$ and $\kappa = 100$, respectively. It is noticed that some numerical convergence rates are superior than the theoretical rates.

On the other hand, in Tables 3 and 4, we provide the data when $\kappa h/p$ is fixed as 1.1 and 1.125, respectively. it is observed that the errors $u - u_h$ and $\mathbf{q} - \mathbf{q}_h$ are not improved when $\kappa h/p$ is fixed.

TABLE 5. Example 2 – Numerical errors for different κ values.

κ	p	$Error_{ur}$	$Error_{ui}$	$Error_{qr}$	$Error_{qi}$
1	10	7.5376e-6	1.1257e-4	9.2169e-4	1.2222e-3
	20	8.8877e-9	3.0055e-8	2.5482e-7	3.0337e-7
	30	2.2117e-12	6.3917e-12	5.4866e-11	6.6376e-11
20	20	1.5129e-1	2.5313e-1	1.7234e-1	2.6452e-1
	30	1.4897e-4	1.9865e-4	1.4906e-4	1.8976e-4
	50	1.3124e-7	2.0058e-7	1.4036e-7	1.9218e-7
50	50	5.6034e-2	6.7556e-2	6.9807e-2	7.0098e-2
	70	1.5440e-3	2.0078e-3	1.6871e-3	1.9899e-3
	100	1.9800e-6	1.8764e-6	1.9549e-6	2.0276e-6

Example 2. In this example, we choose $\Omega = B_1 \setminus S_1$, where $B_1 = \{(x,y) : x^2+y^2 \leq 1\}$ and $S_1 = [-0.25, 0.25]^2$. The exact solution is taken as $u = \exp(-\mathrm{i}\kappa x)$. For simplicity, only Dirichlet boundary condition is applied. Table 5 presents the errors $u_h - u$ and $\mathbf{q}_h - \mathbf{q}$ for various κ values. Figures 1 and 2 are the surface plots of the HWG solutions u_h with $p = 30$, where $\kappa = 20$ for the exact solution u, and four elements of Ω partitioned by $y = \pm x$ are used. It is evident that high order method is very effective for solving Helmholtz equations.

Example 3. Let $\Omega = [0,1]^2$, which is partitioned by a uniform mesh. We choose the inhomogeneous boundary conditions in such a way that the analytical solutions are the circular waves given, in polar coordinates $x = (r\cos\theta, r\sin\theta)$, by

$$u(x) = J_\xi(\kappa r)\cos(\xi\theta), \quad \xi \geq 0,$$

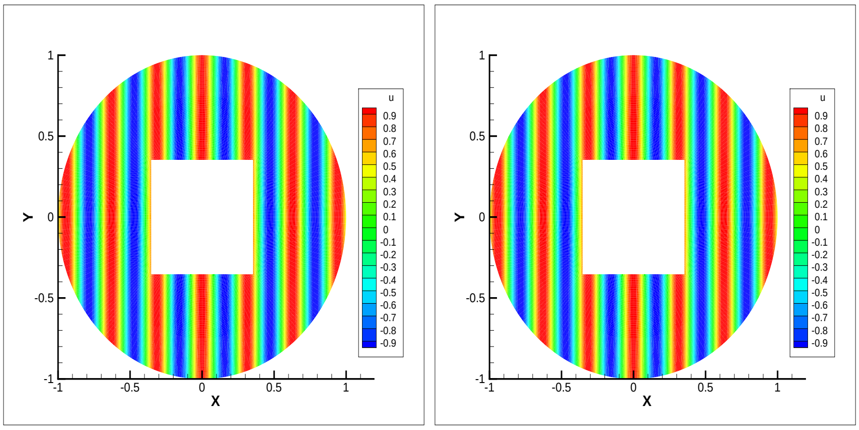

FIGURE 1. Example 2 – The numerical real part and exact real part for $\kappa = 20, p = 30$.

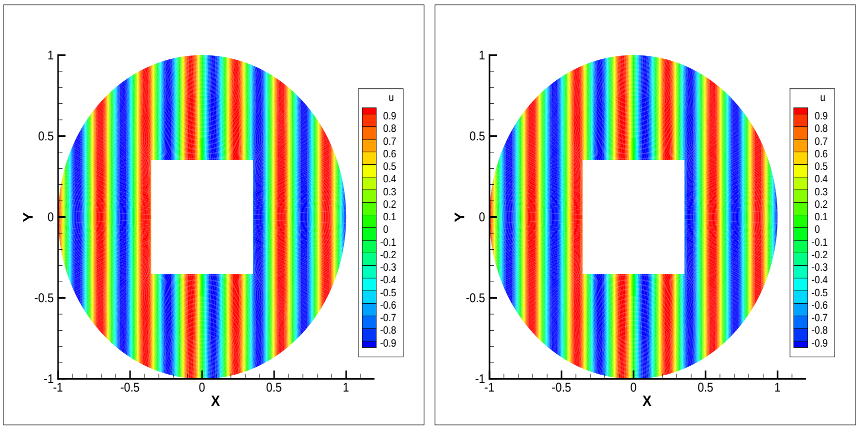

FIGURE 2. Example 2 – The numerical imaginary part and exact imaginary part for $\kappa = 20, p = 30$.

TABLE 6. Example 3 – Numerical errors and their convergence rates when $\kappa = 100$ and $\xi = \frac{3}{2}$.

p	N	$Error_{ur}$	$order$	$Error_{qi}$	$order$
3	16	1.1633e-1		1.0804e-1	
	32	5.6997e-3	4.3512	5.7057e-3	4.2430
	64	6.9273e-5	6.3624	8.0156e-5	6.1534
	128	4.3604e-6	3.9879	1.3374e-5	2.5833
	256	3.9708e-7	3.4570	4.6952e-6	1.5102
	512	7.1748e-8	2.4684	1.7398e-6	1.4323
7	16	2.1940e-5		4.7681e-5	
	32	5.4183e-7	5.3396	7.1597e-6	2.7354
	64	1.3008e-7	2.0584	2.6734e-6	1.4212
	128	3.1574e-8	2.0426	9.8352e-7	1.4426

where, J_ξ denotes the Bessel function of the first kind of order ξ. If $\xi \in \mathbb{N}$, u can be analytically extended to a Helmholtz solution in $\mathbb{R}^2$. If $\xi \notin \mathbb{N}$, its derivatives have a singularity at the origin. Then $u \in H^{\xi+1-\epsilon}(\Omega)$ can be extended to a Helmholtz solution in $\mathbb{R}^2$ provided $\epsilon > 0$, but $u \notin H^{\xi+1}(\Omega)$ [16].

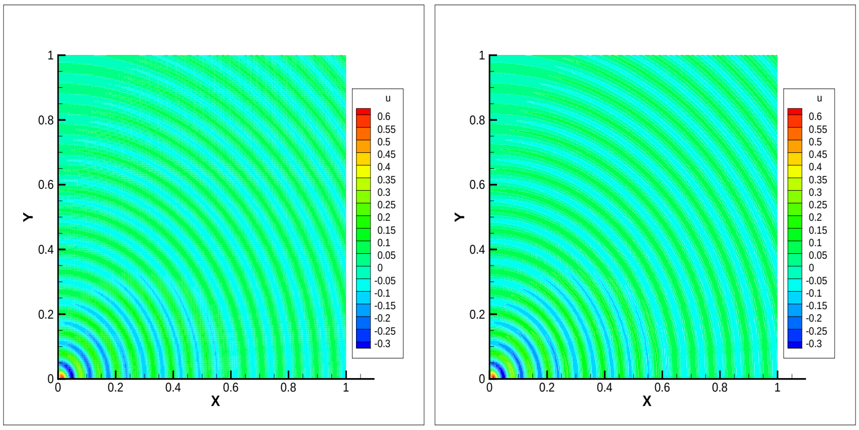

FIGURE 3. Example 3 – The numerical solution (Left) and the exact solution (Right) of u for $\kappa = 100, \xi = \frac{2}{3}, p = 4, N = 32(error = 2.0077 \times 10^{-4})$.

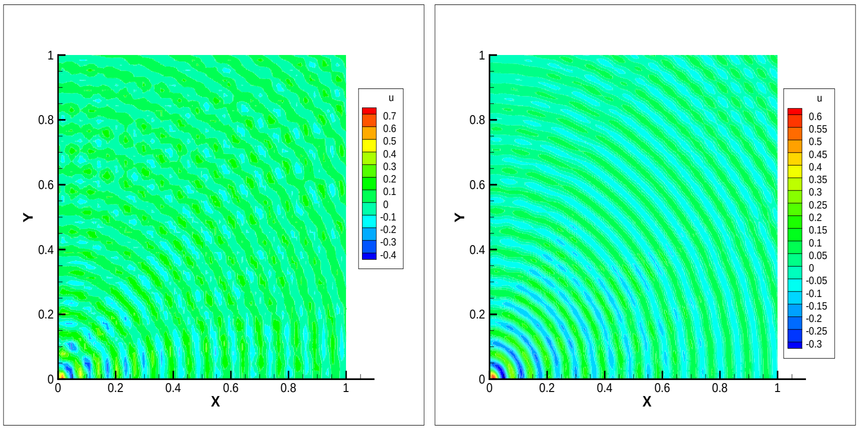

FIGURE 4. Example 3 – The numerical solution of u for $\kappa = 100, \xi = \frac{2}{3}, p = 1, N = 64$(Left, $error = 3.5489 \times 10^{-2}$), $p = 2, N = 64$(Right,$error = 1.2259 \times 10^{-2}$).

We compare the HWG solutions with the exact solutions for various κ and ξ on uniform meshes. Table 6 documents the errors $u - u_h$ and $\mathbf{q} - \mathbf{q}_h$ for $\kappa = 100$ and $\xi = \frac{3}{2}$. Figures 3 and 4 present the surface plots of the HWG solutions u_h and the exact solutions u for various κ and ξ. It is observed that high order methods are very effective.

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