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A FINITE ELEMENT METHOD FOR THE ONE-DIMENSIONAL PRESCRIBED CURVATURE PROBLEM

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Abstract. We develop a finite element method for solving the Dirichlet problem of the one-dimensional prescribed curvature equation due to its irreplaceable role in applications. Specifically, we first analyze the existence and uniqueness of the solution of the problem and then develop a finite element method to solve it. The well-posedness of the finite element method is shown by employing the Banach fixed-point theorem. The optimal error estimates of the proposed method in both the H^1 norm and the L^2 norm are established. We also design a Newton type iteration scheme to solve the resulting discrete nonlinear system. Numerical experiments are presented to confirm the order of convergence of the proposed method.

Key words. Prescribed curvature equation, finite element method, Newton iteration, Banach fixed-point theorem.

1. Introduction

The purpose of this paper is to develop a finite element method for solving the Dirichlet problem of the one-dimensional prescribed curvature equation. The study of the prescribed curvature equation originates from Thomas Young's [35] and Pierre-Simon Laplace's [24] independent research about the properties of capillary surfaces which date back to 1805. Its mathematical theory was built by Gauss [16] in 1830, and was enriched by numerous researchers [17, 28]. At the present time, there is still great interest in the study of this equation including the one dimensional case [19, 23, 29] and high dimensional cases [17, 18, 32]. The prescribed curvature equation appears in many important fields including classical problems in differential geometry (e.g. minimal surfaces [26, 28]; constant curvature surfaces [28]) and the static fluid problem in fluid mechanics such as the Young-Laplace equation [15, 35]. In particular, the one-dimensional equation plays an irreplaceable role in applications such as modeling corneal shape [10, 27, 33] and modeling electrostatic micro-electro mechanical systems [6, 7, 13].

The interest in the one-dimensional prescribed curvature equation has led to much progress in analyzing the existence, non-existence and multiplicity of its solutions. Studying the equation was inspired by an open problem proposed by Haim Brezis et al. in [1] for investigating the multiplicity and structure of the solution of a specific semilinear elliptic problem related to a simplified version of the equation. The equation under study has more severe nonlinearity in its operator and more complexity of the multiplicity of its solution. There exist a large number of papers which focused on the existence of solutions of the equation by using the barrier method [23], the time map method [29] and the sub-super solution method [25]. Especially, the equation with a general forcing term that depends on the unknown solutions and their gradient was considered in [3, 29]. Moreover, some fascinating aspects of the Dirichlet problem of the one-dimensional prescribed curvature equation were obtained in [30, 31], including the disappearing solution behavior and the bifurcation property of the solution. For the computational issue of this problem,

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the shooting method was studied in [2], the finite difference method was investigated in [8] and the conjugate gradient method was considered in [20]. These methods are difficult to be extended to a higher dimensional case. Specifically, the shooting method converts the original problem to an equivalent initial value problem involved in many parameters. The finite difference method for solving this problem in higher dimensions suffers from the difficulty in handling the curve boundaries imposed the Dirichlet boundary condition and as a result it is difficult to extend it to a higher dimensional case. The conjugate gradient method treats the original problem as an equivalent minimization problem, and it is difficult to extend it reliably and easily to high dimensions for more complicated and important case.

At present, the finite element method are used for special two dimensional cases of the prescribed curvature problem. In [21], Johnson and Thomée developed the finite element error analysis to obtain the optimal H^1 and $L^p, 1 \leq p < 2$ estimates for the minimal surface problem by using the piecewise linear approximate functions. A posteriori error analysis for the Dirichlet problem of the prescribed mean curvature equation with homogeneous boundary conditions was developed in [14] and the finite element method for the discrete Plateau's problem and the corresponding H^1, L^2 error estimates were obtained in [11, 12] (see also the references cited therein) by dealing with the equivalent energy functional. In the aforementioned finite element methods, the strict convexity of the corresponding energy functional provides conveniences for the study of the existence and uniqueness of the approximate solution of the specific two dimensional cases. However, for the one-dimensional prescribed curvature equation, when the forcing term depends on the unknown solution, the finite element method may result in a nonconvexity of the corresponding energy functional. This requires a new approach to deal with the existence and uniqueness of the approximate solution of the one-dimensional case. We shall accomplish this by applying the Banach fixed point theorem. Therefore, the finite element method for one-dimensional problem deserves further investigation.

Our goal is to develop a finite element method for solving the Dirichlet boundary value problem of the one-dimensional prescribed curvature equation, with potential of easy extension to handle cases where the forcing term depends on both the unknown solution and its gradient and handle the high dimensional case. We shall adopt the standard Lagrange finite element for this purpose. As explained in [4], advantages of using the standard Lagrange finite element include the simplicity of its implementation and the ability of handling the general case in which the forcing term may depend on the unknown solution and its gradient. For simplicity of presentation, we consider in this paper the equation with the forcing term independent of the unknown solution. The method developed in this paper can be easily extended to the general case. Specifically, we establish the existence and uniqueness of the solution for this problem by using the shooting method. We study the regularity of the solution, which lays a foundation for the convergence analysis of the proposed method. We construct the finite element scheme by using the simple and practical Lagrange finite element so that its discrete linearization is consistent with the linearization of the original nonlinear equation. We identify a fixed point of the constructive nonlinear operator in a small ball by using the Banach fixed-point theorem to simultaneously show the well-posedness of the finite element method and derive an optimal H^1 error estimate. Furthermore, we obtain the optimal L^2 error estimate by extending the Nitsche strategy naturally within our framework.

A critical issue in analyzing the proposed method for this problem is the nonlinearity of the differential operator involved in the equation. To overcome this challenging issue, we split the nonlinear operator derived by the variational form of the problem into a standard linear elliptic operator plus a nonlinear operator which inherits the nonlinearity of the original operator. We make use the stability analysis of the standard linear elliptic operator with a contraction property of the derived nonlinear operator to complete the convergence analysis of the proposed method.

This paper is organized in seven sections. In section 2, we establish the existence and uniqueness of the solution for the Dirichlet boundary value problem of the one-dimensional prescribed curvature equation. We develop in section 3 the finite element method for solving this problem. Section 4 is devoted to a complete error analysis of the proposed finite element method. By using the Banach fixed-point theorem, we establish the well-posedness of the method and the optimal error estimates, in both the H^1 morn and the L^2 morn, of the approximate solution. In section 5, we describe the Newton iteration for solving the discrete nonlinear system that results from the finite element method and prove its convergence order. In section 6, we present numerical results to verify the convergence rate of the approximate solution. Finally, we draw a conclusion in section 7.

2. The Dirichlet Boundary Value Problem

We investigate in this section the existence, uniqueness and regularity of the solution of the Dirichlet boundary value problem of the one-dimensional prescribed curvature equation. We accomplish this goal by converting the boundary value problem to a related initial value problem and using the shooting method.

We first describe the Dirichlet boundary value problem that we consider in this paper. Set $\Omega := (-1, 1)$ and $\overline{\Omega} := [-1, 1]$. Let f be a continuous function defined on $\overline{\Omega}$ and $\ell \in \mathbb{R}$. We consider the following Dirichlet boundary value problem of the one-dimensional prescribed curvature equation:

$$-\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right)' = f(x), \quad x \in \Omega \tag{1a}$$

$$u(-1) = 0 \tag{1b}$$

$$u(1) = \ell. \tag{1c}$$

In order to apply the shooting method to the above problem, we first analyze the existence of a solution of equation (1a) on $\overline{\Omega}$ with the left boundary condition (1b). That is, we first investigate whether there exists a shooting curve starting from the point $(-1, 0)$ determined by equation (1a) on $\overline{\Omega}$. To this end, we let

$$F(x) := \int_{-1}^{x} f(t)dt, \quad x \in \overline{\Omega}.$$

and

$$M := \max_{x\in\overline{\Omega}} F(x), \quad m := \min_{x\in\overline{\Omega}} F(x).$$

A comment on the numbers M and m is in order.

Remark 2.1. If f vanishes at points $x_1 < \cdots < x_k$ inside Ω, then

$$M = \max_{j=0,1,\ldots,k+1} F(x_j) \ \text{ and } \ m = \min_{j=0,1,\ldots,k+1} F(x_j)$$

where $x_0 = -1$ and $x_{k+1} = 1$.

For the sake of conciseness and clarity, we introduce the notation

$$G(a) := \frac{a}{\sqrt{1+a^2}}, \tag{2}$$

and

$$H(a) := \frac{a}{\sqrt{1-a^2}}, \tag{3}$$

where a may be a function or a function value.

Lemma 2.1. *If problem* (1a) *and* (1b) *has a solution on* $\overline{\Omega}$, *then*

$$M - m < 2. \tag{4}$$

Proof. We use the idea of the shooting method to obtain this result. Instead of problem (1a) and (1b), we consider its equivalent first order system

$$u' = y \qquad u(-1) = 0, \tag{5a}$$

$$-\left(\frac{y}{\sqrt{1+y^2}}\right)' = f \qquad y(-1) = u'(-1). \tag{5b}$$

If problem (1a) and (1b) has a solution u^* on $\overline{\Omega}$, then u^* satisfies equations (5). For simplicity, we let $\alpha^* := (u^*)'(-1)$.

The existence of a solution of problem (5a) clearly requires that y exists on $\overline{\Omega}$. To this end, using definition (2) of notation $G(a)$, we turn to equation (5b) and integrate both sides of the equation to obtain that

$$-G(y(x)) = F(x) - \bar{\alpha}, \quad x \in \overline{\Omega},$$

where $\bar{\alpha} := G(\alpha^*)$. By using definition (3) of notation $H(a)$, we conclude that

$$y(x) = -H(F(x) - \bar{\alpha}), \quad x \in \overline{\Omega}.$$

Hence, y exists on $\overline{\Omega}$ ensures that the following condition is satisfied

$$1 - (F(x) - \bar{\alpha})^2 > 0, \quad x \in \overline{\Omega}.$$

Furthermore, we have that

$$-1 + \bar{\alpha} < F(x) < 1 + \bar{\alpha}, \quad x \in \overline{\Omega}.$$

This implies that

$$m > -1 + \bar{\alpha} \text{ and } M < 1 + \bar{\alpha}.$$

Consequently, if initial value problem (5) has a solution on $\overline{\Omega}$, then $M - m < 2$, which is the necessary condition for a solution of initial value problem (1a) and (1b) to exist. □

Condition (4) in Lemma 2.1 is only necessary. However, it involves only the forcing term so that it is easy to verify. Below, we shall provide a necessary and sufficient condition which ensures initial value problem (1a), (1b) has a solution on $\overline{\Omega}$. To this end, by using notation (2), we define

$$\alpha := G(u'(-1)). \tag{6}$$

Lemma 2.2. *Suppose that the parameter* α *is well defined by* (6). *The solution* $u(\cdot, \alpha)$ *of the initial value problem defined by* (1a), (1b) *and* (6) *exists on* $\overline{\Omega}$ *if and only if*

$$-1 + M < \alpha < 1 + m. \tag{7}$$

Proof. For $x \in \overline{\Omega}$, by using notation (2), we integrate both sides of equation (1a) on $[-1, x]$ and obtain

$$-G(u'(x)) = F(x) - \alpha, \quad x \in \overline{\Omega}. \tag{8}$$

Using (3), we solve u' from the above equation and get that

$$u'(x) = -H(F(x) - \alpha), \quad x \in \overline{\Omega}. \tag{9}$$

From the above equation, we observe immediately that the solution of the initial value problem defined by (1a), (1b) and (6) exists on $\overline{\Omega}$ if and only if u' is continuous on $\overline{\Omega}$, which is equivalent to

$$1 - (F(x) - \alpha)^2 > 0, \quad x \in \overline{\Omega}.$$

Moreover, we have that

$$m > -1 + \alpha \text{ and } M < 1 + \alpha.$$

Consequently, we conclude that the solution of the initial value problem defined by (1a), (1b) and (6) exists on $\overline{\Omega}$ if and only if $-1 + M < \alpha < 1 + m$. □

Initial value problem (1a), (1b) has a family of solutions depending on the parameter α that satisfies (6). We denote by $u(\cdot, \alpha)$ the solution of initial value problem (1a), (1b) determined by α. Next, with the condition on α as stated in Lemma 2.2, we use the function

$$\Phi(\alpha) := u(1, \alpha) \tag{10}$$

to describe the value of the point where the shooting curve intersects with the line $x = 1$. We shall characterize the shooting curve $u(x, \alpha)$ which satisfies boundary condition (1b). We first establish a property of the mapping $\Phi(\alpha)$.

Lemma 2.3. *The function $\Phi(\alpha)$ is a strictly increasing function in α on the open interval $(-1 + M, 1 + m)$.*

Proof. For two different values $\alpha_1, \alpha_2 \in (-1 + M, 1 + m)$ with $\alpha_1 < \alpha_2$, we wish to prove $\Phi(\alpha_1) < \Phi(\alpha_2)$. We consider two shooting curves determined by the initial value problem defined by (1a), (1b) and (6) with two different values α_1 and α_2. By using notation (2), we integrate both sides of equation (1a) in terms of α_1 and α_2 respectively and obtain for $x \in \overline{\Omega}$ that

$$-G(u_1'(x)) = F(x) - \alpha_1 \tag{11a}$$

$$-G(u_2'(x)) = F(x) - \alpha_2. \tag{11b}$$

Noticing that $\alpha_1 < \alpha_2$ and $G(x)$ is a strictly increasing function, we conclude from above equations (11) that $u_1'(x) < u_2'(x)$ on Ω. Evidently, we have that $u(1, \alpha_1) < u(1, \alpha_2)$, which proves the lemma. □

From the above lemma, we see that the nonlinear boundary value problem (1) can have at most one solution. With the preceding lemmas established, we are now ready to present the following result regarding the existence and uniqueness of the solution of problem (1). Let I be the open interval defined by

$$I := \{\Phi(\alpha) : -1 + M < \alpha < 1 + m\}. \tag{12}$$

Theorem 2.4. *The Dirichlet boundary value problem (1) is solvable if and only if $\ell \in I$.*

Proof. The basic idea of the shooting method is to replace the Dirichlet boundary value problem by the initial value problem. Hence, in order to prove the existence of a solution of the Dirichlet boundary value problem (1), we first take account of the initial value problem (1a) and (1b). We then investigate whether the shooting curve, that is, the solution of the initial value problem (1a) and (1b), intersects with the line $x = 1$ at the point $(1, \ell)$ (which is the same as boundary condition (1c)).

Let u be a solution of Dirichlet boundary value problem (1). We shall follow the proof of Lemma 2.2 to conclude that u is the solution of initial value problem (1a), (1b). By using the left boundary condition (1b) with definition (6) of the parameter α, for $x \in \overline{\Omega}$, we integrate both sides of equation (1a) on $[-1, x]$, and obtain equation (8). Then solving u' from (8) we have (9). Next, for $s \in \overline{\Omega}$, we integrate both sides of equation (9) on $[-1, s]$, and obtain that

$$u(s) = -\int_{-1}^{s} H(F(x) - \alpha)dx.$$

We can clearly see that the function u determined by the above formula involved α is a solution of initial value problem (1a), (1b). Conversely, we assume that $u(\cdot, \alpha)$ with α satisfying (6) is a solution of initial value problem (1a), (1b). If $u(\cdot, \alpha)$ satisfies boundary condition (1c), namely $u(1, \alpha) = \ell$, then we obtain that $u(\cdot, \alpha)$ is a solution of Dirichlet boundary value problem (1). Consequently, the existence of a solution of Dirichlet boundary value problem (1) is converted equivalently to the existence of a solution $u(\cdot, \alpha)$ with α satisfying (6) which satisfies initial value problem (1a), (1b) and boundary condition (1c).

Next, we turn to proving that the solution $u(\cdot, \alpha)$ of the initial value problem defined by (1a), (1b), (6) satisfying boundary condition (1c) exists on $\overline{\Omega}$ if and only if $\ell \in I$. If $\ell \in I$, by definition (6) of α, we prove that there exists a solution $u(\cdot, \alpha)$ of initial value problem (1a), (1b) satisfying boundary condition (1c). By employing Lemma 2.3 with definition (6) and (10), the mapping $\Phi(\alpha) : \alpha \to u(1, \alpha)$ is continuous and strictly increasing in α on the open interval $(-1 + M, 1 + m)$, by applying the intermediate value theorem we conclude that there exists a unique $\hat{\alpha}$ in $(-1 + M, 1 + m)$ such that $u(1, \hat{\alpha}) = \ell \in I$, and by Lemma 2.2, $u(\cdot, \hat{\alpha})$ is a solution of initial value problem (1a), (1b). Conversely, let $u(\cdot, \tilde{\alpha})$ with $\tilde{\alpha}$ be a solution of initial value problem (1a), (1b) satisfying boundary condition (1c), namely $u(1, \tilde{\alpha}) = \ell$. By employing Lemma 2.2, we have that $\tilde{\alpha}$ satisfies (7). By using definition (10) and (12), we obtain $\ell \in I$.

In conclusion, boundary value problem (1) is solvable if and only if $\ell \in I$. □

With Theorem 2.4 proved above, we establish the following regularity result of the solution of the one-dimensional prescribed curvature problem (1).

Proposition 2.5. *If u is a solution of the Dirichlet boundary value problem* (1) *and $f \in C^k(\overline{\Omega})$ for an integer $k \geq 0$, then $u \in C^{k+2}(\overline{\Omega})$.*

Proof. Let u be a solution of Dirichlet boundary value problem (1). By employing Theorem 2.4 with definition (10) of Φ, there exists a unique $\alpha^\star$ satisfying (6) such that $\ell = u(1, \alpha^\star)$, and $u(\cdot, \alpha^\star)$ is the solution of initial value problem (1a), (1b). Integrating both sides of equation (1a), by using notation (2), we obtain that

$$-G(u'(x)) = F(x) - \alpha^\star, \quad x \in \overline{\Omega}. \tag{13}$$

Using notation (3), we solve u' from above equation (13), and obtain

$$u'(x) = -H(F(x) - \alpha^\star), \quad x \in \overline{\Omega}. \tag{14}$$

Next, for $s \in \overline{\Omega}$, we integrate both sides of above equation (14) on $[-1, s]$, and obtain

$$u(s) = -\int_{-1}^{s} H(F(x) - \alpha^\star)dx. \tag{15}$$

Consequently, we can readily verify from above equation (15) that if $f \in C^k(\overline{\Omega})$, then u belongs to $C^{k+2}(\overline{\Omega})$. □

3. Finite Element Method

In this section, we develop the finite element method to solve the nonlinear prescribed curvature problem (1).

We begin with setting the notation used throughout the paper. For $k \geq 0$ and $0 \leq p \leq \infty$, we define the Sobolev space $W_p^k(\Omega)$ to be the set of all $L^p(\Omega)$ functions whose distributional derivatives of order up to k are in $L^p(\Omega)$ with the (semi) norm:

$$\|v\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p}, \quad |v|_{W_p^k(\Omega)} := \left(\sum_{|\alpha| = k} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p},$$

where $D^\alpha v$ denotes the weak derivative of function v, and in the case $p = \infty$:

$$\|v\|_{W_\infty^k(\Omega)} := \max_{|\alpha| \leq k} \|D^\alpha v\|_{L^\infty(\Omega)}.$$

Moreover, let $W_p^{k,0}(\Omega)$ denote the set of W_p^k functions whose traces vanish up to order $k-1$ on $\partial\Omega$. Especially, in the case $p = 2$, we set $H^k(\Omega) := W_2^k(\Omega)$ and $H_0^k(\Omega) := W_2^{k,0}(\Omega)$. For simple notation, we let $\|\cdot\|_0$ denote the L^2 norm $\|\cdot\|_{L^2(\Omega)}$, and $\|\cdot\|_\infty$ denote the L^∞ norm $\|\cdot\|_{L^\infty(\Omega)}$, for an integer $m \geq 1$, we let $\|\cdot\|_m$ (or $|\cdot|_m$) denote the H^m norm $\|\cdot\|_{H^m(\Omega)}$ (or H^m semi-norm $|\cdot|_{H^m(\Omega)}$) and $\|\cdot\|_\infty^m$ denote the W_∞^m norm $\|\cdot\|_{W_\infty^m(\Omega)}$. We also denote by X' the dual space of a normed linear space X, and $\langle\cdot,\cdot\rangle$ the pairing between X' and X. For our purpose, we take

$$H_0^1(\Omega) := \{u : u \in H^1(\Omega) : u(-1) = u(1) = 0\},$$

where $H^1(\Omega)$ is defined by

$$H^1(\Omega) := \{u : u \in L^2(\Omega),\, Du \in L^2(\Omega)\}.$$

Let $f \in L^2(\Omega)$. For convenience, we rewrite problem (1) with inhomogenous boundary condition (1a), (1b) as the following equivalent equation with the homogenous boundary condition

$$-\left(\frac{u' + \frac{\ell}{2}}{\sqrt{1 + (u' + \frac{\ell}{2})^2}} \right)' = f \tag{16a}$$

$$u(-1) = 0 \tag{16b}$$

$$u(1) = 0. \tag{16c}$$

Clearly, the relationship between the solution u of the problem (1) and the solution v of the above problem (16) may be expressed as

$$u(x) = v(x) + \frac{\ell}{2}(x+1), \quad x \in \overline{\Omega}.$$

We now describe the construction of the finite element method for the problem (16). To this end, recalling definition (2) of notation $G(a)$, we firstly derive the following variational formulation of problem (16): Find $u \in H_0^1(\Omega)$ such that

$$(G(u' + \frac{\ell}{2}), v') = (f, v), \quad \text{for all} \quad v \in H_0^1(\Omega), \tag{17}$$

where $(\cdot, \cdot)$ denotes the inner product on $L^2(\Omega)$. Furthermore, we introduce the nonlinear mapping $\mathcal{N} : H_0^1(\Omega) \to [H_0^1(\Omega)]'$ by

$$\langle \mathcal{N} w, v \rangle := \int_\Omega G(w' + \frac{\ell}{2}) v' dx - \int_\Omega f v dx, \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{18}$$

Employing this notation, we then rewrite equation (17) in a compact form:

$$\mathcal{N} u = 0. \tag{19}$$

We next derive the finite element scheme for (19). For this purpose, we construct a finite element space. Let $\overline{\Omega}_h := \{I_j\}$ be a partition of $\overline{\Omega}$ as

$$-1 = x_0 < x_1 < \cdots < x_{j-1} < x_j < x_{j+1} < \cdots < x_n < x_{n+1} = 1$$

where $I_j := [x_{j-1}, x_j]$. Set $h_j := x_j - x_{j-1}$ and $h := \max_{1 \le j \le n+1} (h_j)$. We define the finite element space $V_h \subset H_0^1(\Omega)$ as follows:

$$V_h := \{v \in C^0(\overline{\Omega}) : v|_{I_j} \in \mathcal{P}_k(I_j), \text{ for } j = 1, 2, \ldots, n+1, v(-1) = v(1) = 0\}$$

where $\mathcal{P}_k(I_j)$ is the set of polynomials of degree $\le k$ on I_j.

We recall that V_h' denotes the dual space of V_h. We denote by $\mathcal{N}_h : V_h \to V_h'$ the restriction of $\mathcal{N}$ on V_h. That is, $\mathcal{N}_h$ satisfies the following finite dimensional variational equation

$$\langle \mathcal{N}_h u_h, v \rangle = \int_\Omega G(u_h' + \frac{\ell}{2}) v' dx - \int_\Omega f v dx, \quad \text{for all} \quad v \in V_h. \tag{20}$$

Hence, the finite element method for (16) is to find $u_h \in V_h$ such that

$$\mathcal{N}_h u_h = 0. \tag{21}$$

or equivalently,

$$\int_\Omega G(u_h' + \frac{\ell}{2}) v' dx - \int_\Omega f v dx = 0, \quad \text{for all} \quad v \in V_h. \tag{22}$$

Equation (21) (or (22)) is a numerical method to find an approximate solution of the Dirichlet boundary value problem of the one-dimensional prescribed curvature equation. Upon choosing a basis of V_h, equation (21) is in fact a nonlinear algebraic equation. We shall solve the resulting nonlinear system by employing the Newton iteration.

4. Convergence Analysis

In this section, we establish the stability and the optimal convergence rate of the proposed finite element method. Specifically, we rewrite nonlinear operator (18) in an equivalent split form of a linear part and a nonlinear part to analyze the convergence of finite element method (21). We use the Banach fixed point theorem to show the existence of a solution to resulting nonlinear equation (21) and develop the H^1 error estimate of the method. Sequentially, we obtain the L^2 error estimate of the method by employing the Nitsche strategy.

We first consider a split form of the nonlinear operator defined by (18). In order to ensure finite element scheme (21) consistent, a key idea is to split nonlinear operator (18) into two components: a linear component and a nonlinear component such that the proposed finite element scheme inherits a stable discrete linearization.

Namely, the linear component is a standard one which can be discretized by known methods. For our purpose, we used the simple and practical Lagrange finite element method. Following [4], we conduct linearization of the nonlinear operator $\mathcal{N}$ (18) at $u \in H_0^1(\Omega)$. For all $w \in H_0^1(\Omega)$, we apply $\mathcal{N}$ to $u + w$, and write this as a sum of a linear functional and a nonlinear functional in w. That is,

$$\mathcal{N}(u + w) = \mathcal{L}w + \mathcal{R}w, \quad \text{for all} \quad w \in H_0^1(\Omega), \tag{23}$$

where $\mathcal{L}$ is a linear mapping and $\mathcal{R}$ is a nonlinear mapping to be derived below.

With this idea in mind, we consider the linearization of the nonlinear operator $\mathcal{N}$ defined in (18). For the sake of simplicity, we introduce the notation

$$K(a) := 1/(1 + a^2)^{\frac{3}{2}}, \tag{24}$$

where a may be a function or a function value. The linearization of $\mathcal{N}$ results in the linear mapping $\mathcal{L} : H_0^1(\Omega) \to [H_0^1(\Omega)]'$ defined by

$$\langle \mathcal{L}w, v\rangle := \int_\Omega K(u' + \frac{\ell}{2})w'v'dx, \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{25}$$

The nonlinear mapping $\mathcal{R} : H_0^1(\Omega) \to [H_0^1(\Omega)]'$ is then defined by

$$\mathcal{R}w := \mathcal{N}(u + w) - \mathcal{L}w. \tag{26}$$

Remark 4.1. Noting that $\mathcal{L}$ is linear and $\mathcal{R}$ is nonlinear, it follows from identity (23) that $\mathcal{L}$ is the linearization of $\mathcal{N}$ at u. In fact, $\mathcal{L}$ is the linear operator associated with the variational form for the differential operator

$$L_u w := -\frac{d}{dx}[K(u'(x) + \frac{\ell}{2})\frac{d}{dx}w(x)], \quad \text{for } w \in C^2(\Omega),$$

with the homogeneous Dirichlet boundary condition.

We introduce a linearization $\mathcal{L}_h : V_h \to V_h'$ of $\mathcal{N}_h$ defined by (20) at u by

$$\langle \mathcal{L}_h w, v\rangle := \int_\Omega K(u' + \frac{\ell}{2})w'v'dx, \quad \text{for all} \quad v \in V_h. \tag{27}$$

We remark that $\mathcal{L}_h$ is the discretization of the linear operator $\mathcal{L}$ by applying the standard Lagrange finite elements. This important fact will lead to the stability analysis of proposed finite element method (21).

Based on the above fundamental split form of operator (18), we conduct the convergence analysis for proposed finite element method (21) by using the Banach fixed point theorem. To this end, we introduce some notation. Since $\mathcal{L}_h$ defined by (27) is the restriction of $\mathcal{L}$ to V_h, we let $\mathcal{L}_h^{-1} : V_h' \to V_h$ denote its inverse. We define the mapping $\mathcal{M} : H_0^1(\Omega) \to V_h$ as

$$\mathcal{M} := \mathcal{L}_h^{-1}(\mathcal{L} - \mathcal{N}), \tag{28}$$

and let $\mathcal{M}_h : V_h \to V_h$ be the restriction of $\mathcal{M}$ to V_h. That is,

$$\mathcal{M}_h := \mathcal{I}d_h - \mathcal{L}_h^{-1}\mathcal{N}_h, \tag{29}$$

where $\mathcal{I}d_h$ is the identity map on V_h.

We turn to the investigation of equation (21). The existence of a solution of equation (21) near u will be proved by establishing a fixed point for M_h in a small ball centered at $u_{c,h}$, where $u_{c,h} \in V_h$ is an elliptic projection of u defined by

$$u_{c,h} := \mathcal{L}_h^{-1}\mathcal{L}u, \tag{30}$$

or equivalently,

$$\langle \mathcal{L}u_{c,h}, v\rangle = \langle \mathcal{L}u, v\rangle, \quad \text{for all} \quad v \in V_h.$$

We define the corresponding discrete negative norm as

$$\|q\|_{-1,h} := \sup_{0\neq v\in V_h} \frac{\langle q, v\rangle}{\|v\|_1}, \quad \text{for } q \in V_h'. \tag{31}$$

Remark 4.2. In order to avoid the proliferation of constants, we adopt the notation $A \lesssim B$ to represent the relation $A \leq$ constant $\times B$, where the constant is independent of the mesh parameter h.

Remark 4.3. Based on the regularity of the solution u of prescribed curvature problem (1) in Proposition 2.5, for an integer $s > 2$, the regularity condition $u \in H^s(\Omega)$ is assumed throughout this paper.

Suppose that k is the polynomial degree of the finite element space V_h. Let $t := \min\{k+1, s\}$. The requirement $k \geq 2$ is assumed throughout the paper.

Remark 4.4. By approximation properties of V_h [4, 5], if $u \in H^s(\Omega)$ then there exists $v \in V_h$ such that

$$\|u - v\|_1 \lesssim h^{t-1}\|u\|_t. \tag{32}$$

We next establish that finite element scheme (21) inherits a stability from its linearization. We begin with an analysis of the stability of the linear operator $\mathcal{L}_h$.

Lemma 4.1. *If $u \in H^s(\Omega)$ for some $s > 2$ and the linear mapping $\mathcal{L}_h : V_h \to V_h'$ is defined as in* (27)*, then*

$$\|\mathcal{L}_h v\|_{-1,h} \lesssim \|v\|_1, \quad \textit{for all} \;\; v \in V_h. \tag{33}$$

Moreover, the map $\mathcal{L}_h$ is invertible and

$$\|\mathcal{L}_h^{-1} q\|_1 \lesssim \|q\|_{-1,h}, \quad \textit{for all} \;\; q \in V_h'. \tag{34}$$

Proof. We prove these results by showing that the linear mapping $\mathcal{L}_h$ is continuous and coercive on V_h. If $u \in H^s(\Omega)$ for some $s > 2$, then $u \in W^1_\infty(\Omega)$ by a Sobolev inequality. According to definition (27) of linear mapping $\mathcal{L}_h$ and the boundedness of $K(u' + \frac{\ell}{2})$, by the Cauchy-Schwarz inequality, there exists a positive constant η such that

$$\langle \mathcal{L}_h w, v\rangle \leq \eta \|w\|_1 \|v\|_1, \quad \text{for all} \quad w, v \in V_h,$$

which implies the continuity of the operator $\mathcal{L}_h$. In addition, by using the definition of discrete negative norm (31) directly, we may obtain the desired result (33).

It remains to prove (34). We recall the Poincaré inequality [5, 9]

$$\|v\|_1 \lesssim |v|_1, \quad \text{for all } v \in H_0^1(\Omega). \tag{35}$$

By definition (25) of linear operator $\mathcal{L}$ and applying the Poincaré inequality (35), there exists a positive constant λ such that

$$\langle \mathcal{L} w, w\rangle \geq \lambda \|w\|_1^2, \quad \text{for all} \;\; w \in H_0^1(\Omega). \tag{36}$$

This is the coercivity of the operator $\mathcal{L}$ and thus, it implies the coercivity of the operator $\mathcal{L}_h$. Consequently, $\mathcal{L}_h$ is invertible, and for any $q \in V_h'$, there exists a v such that $\mathcal{L}_h v = q$. Therefore, combining estimate (36) with the definition of discrete negative norm (31), we have that

$$\|\mathcal{L}_h^{-1} q\|_1 = \|\mathcal{L}_h^{-1}(\mathcal{L}_h v)\|_1 \lesssim \|q\|_{-1,h},$$

which completes the proof of (34). □

Remark 4.5. By Lemma 4.1, $u_{c,h}$ and the operators $\mathcal{M}, \mathcal{M}_h$ are both well-defined due to the stability of $\mathcal{L}_h$.

We next estimate the error between u and its elliptic projection $u_{c,h}$. This estimate plays an important role in the error estimate of u_h.

Lemma 4.2. *If $u \in H^s(\Omega)$ for $s > 2$ and $u_{c,h} \in V_h$ is the elliptic projection of u defined by* (30)*, then*

$$\|u - u_{c,h}\|_1 \lesssim h^{t-1}\|u\|_t. \tag{37}$$

Proof. We prove estimate (37) by applying Lemma 4.1. By using stability estimate (34) of operator $\mathcal{L}_h$, we have for any $v \in V_h$ that

$$\begin{aligned}\|u - u_{c,h}\|_1 &\le \|u - v\|_1 + \|\mathcal{L}_h^{-1}\mathcal{L}_h(v - u_{c,h})\|_1 \\ &\lesssim \|u - v\|_1 + \|\mathcal{L}_h(v - u_{c,h})\|_{-1,h}.\end{aligned}$$

According to definition (30) of $u_{c,h}$ and employing stability estimate (33), we obtain that

$$\|u - u_{c,h}\|_1 \lesssim \|u - v\|_1, \quad \text{for all} \quad v \in V_h.$$

As v is arbitrary and the assumption about u, the conclusion follows immediately from (32). □

We next apply the Banach fixed-point theorem to study nonlinear equation (21). To this end, we reexpress the mapping $\mathcal{M}$ defined in (28). By the guiding principle (23), we have for all $w \in H_0^1(\Omega)$ that

$$\mathcal{M}w = \mathcal{L}_h^{-1}(\mathcal{L}w - \mathcal{N}w) = \mathcal{L}_h^{-1}(\mathcal{L}u - \mathcal{R}(w - u)).$$

This together with the definition of $u_{c,h}$ gives rise to the formula

$$\mathcal{M}w = u_{c,h} - \mathcal{L}_h^{-1}\mathcal{R}(w - u), \quad \text{for all} \quad w \in H_0^1(\Omega), \tag{38}$$

which further implies that

$$\mathcal{M}w_1 - \mathcal{M}w_2 = \mathcal{L}_h^{-1}(\mathcal{R}(w_2 - u) - \mathcal{R}(w_1 - u)), \quad \text{for all} \quad w_1, w_2 \in H_0^1(\Omega). \tag{39}$$

We observe from formulas (38) and (39) that the nonlinear operator $\mathcal{R}$ is of importance in the analysis of the mapping $\mathcal{M}$, so a contraction property of $\mathcal{R}$ is established in the next lemma. We rewrite the operator $\mathcal{R}$ in a convenient form for our analysis. With the help of notation (2) and (24), we have for all $v \in H_0^1(\Omega)$ that

$$\langle \mathcal{R}w, v\rangle = \int_\Omega G(u' + w' + \frac{\ell}{2})v'dx - \int_\Omega K(u' + \frac{\ell}{2})w'v'dx - \int_\Omega fvdx. \tag{40}$$

We now establish a contraction property of the nonlinear operator $\mathcal{R}$.

Lemma 4.3. *If $u \in H^s(\Omega)$ for $s > 2$, then for all $w_1, w_2 \in H_0^1(\Omega)$,*

$$\|\mathcal{R}w_1 - \mathcal{R}w_2\|_{-1,h} \lesssim h^{-1}(\|w_1\|_1 + \|w_2\|_1)\|w_1 - w_2\|_1. \tag{41}$$

Proof. We use definition (40) of operator $\mathcal{R}$ to prove this result. For all $w_1, w_2, v \in H_0^1(\Omega)$, we consider

$$\begin{aligned}&\langle \mathcal{R}w_1 - \mathcal{R}w_2, v\rangle \\ &= \int_\Omega \Bigg[\Big(G(u' + w_1' + \frac{\ell}{2}) - K(u' + \frac{\ell}{2})w_1'\Big) \\ &\quad - \Big(G(u' + w_2' + \frac{\ell}{2}) - K(u' + \frac{\ell}{2})w_2'\Big)\Bigg] v'dx.\end{aligned} \tag{42}$$

For simplicity of the presentation, we define the notation

$$A := u' + w_1' + \frac{\ell}{2}, \quad B := u' + w_2' + \frac{\ell}{2}, \quad C := u' + \frac{\ell}{2}.$$

With the above notation, we rewrite equation (42) as

$$\begin{aligned}&\langle \mathcal{R}w_1 - \mathcal{R}w_2, v\rangle \\ =&\int_\Omega \left\{\left(\frac{A-B}{\sqrt{1+A^2}} - \frac{A-B}{\sqrt{1+C^2}}\right) - \left[\left(\frac{B}{\sqrt{1+B^2}} - \frac{B}{\sqrt{1+A^2}}\right) - \frac{(A-B)C^2}{(\sqrt{1+C^2})^3}\right]\right\} v'dx.\end{aligned}$$

Applying the triangle inequality to the above equation leads to

$$\begin{aligned}&\langle \mathcal{R}w_1 - \mathcal{R}w_2, v\rangle \le \int_\Omega |A-B| \\ &\times \left\{|A-C| + \left|\frac{B(A+B)}{\sqrt{1+A^2}\sqrt{1+B^2}(\sqrt{1+A^2}+\sqrt{1+B^2})} - \frac{C^2}{(\sqrt{1+C^2})^3}\right|\right\} |v'|dx.\end{aligned} \tag{43}$$

Next, we estimate the second term in the curly brace of the right hand side of equation (43). To this end, we define

$$\Delta_1 := \frac{B(A+B)}{\sqrt{1+A^2}\sqrt{1+B^2}(\sqrt{1+A^2}+\sqrt{1+B^2})} - \frac{C^2}{(\sqrt{1+C^2})^3}. \tag{44}$$

Using a splitting strategy, we rewrite Δ_1 as

$$\begin{aligned}\Delta_1 =& \frac{A+B}{\sqrt{1+A^2}+\sqrt{1+B^2}}\left(\frac{B}{\sqrt{1+A^2}\sqrt{1+B^2}} - \frac{C}{(\sqrt{1+C^2})^2}\right) \\ &+ \frac{C}{(\sqrt{1+C^2})^2}\left(\frac{A+B}{\sqrt{1+A^2}+\sqrt{1+B^2}} - \frac{2C}{2\sqrt{1+C^2}}\right).\end{aligned}$$

Applying the triangle inequality to the above equation leads to

$$|\Delta_1| \le \left|\frac{B}{\sqrt{1+A^2}\sqrt{1+B^2}} - \frac{C}{(\sqrt{1+C^2})^2}\right| + \frac{1}{2}\left|\frac{A+B}{\sqrt{1+A^2}+\sqrt{1+B^2}} - \frac{2C}{2\sqrt{1+C^2}}\right|. \tag{45}$$

We now estimate the right hand side of equation (45). For the sake of convenience, we define

$$\Delta_2 := \frac{B}{\sqrt{1+A^2}\sqrt{1+B^2}} - \frac{C}{(\sqrt{1+C^2})^2}, \quad \Delta_3 := \frac{A+B}{\sqrt{1+A^2}+\sqrt{1+B^2}} - \frac{2C}{2\sqrt{1+C^2}}.$$

Using the splitting strategy and the triangle inequality, we have that

$$\begin{aligned}|\Delta_2| \le& \left|\frac{B-C}{\sqrt{1+A^2}\sqrt{1+B^2}} + \left(\frac{C}{\sqrt{1+A^2}\sqrt{1+B^2}} - \frac{C}{(\sqrt{1+C^2})^2}\right)\right| \\ \le& |B-C| + |C|\left|\frac{1}{\sqrt{1+A^2}}\left(\frac{1}{\sqrt{1+B^2}} - \frac{1}{\sqrt{1+C^2}}\right)\right. \\ &\left.+\frac{1}{\sqrt{1+C^2}}\left(\frac{1}{\sqrt{1+A^2}} - \frac{1}{\sqrt{1+C^2}}\right)\right| \\ \le& |C||A-C| + (|C|+1)|B-C|.\end{aligned}$$

Likewise, we have that

$$
\begin{aligned}
|\Delta_3| &= \left| \frac{A+B-2C}{\sqrt{1+A^2}+\sqrt{1+B^2}} + \left(\frac{2C}{\sqrt{1+A^2}+\sqrt{1+B^2}} - \frac{2C}{2\sqrt{1+C^2}} \right) \right| \\
&\le \frac{1}{2}(|A-C|+|B-C|) + 2\left| \frac{C(2\sqrt{1+C^2}-\sqrt{1+A^2}-\sqrt{1+B^2})}{(\sqrt{1+A^2}+\sqrt{1+B^2})\sqrt{1+C^2}} \right| \\
&\le \frac{3}{2}|A-C| + \frac{3}{2}|B-C|.
\end{aligned}
$$

Substituting the above two estimates into the right hand side of (45) yields that

$$
|\Delta_1| \le \left(|C| + \frac{3}{4}\right)|A-C| + \left(|C| + \frac{7}{4}\right)|B-C|.
$$

Sequentially, substituting the estimate above into the right hand side of (43), we obtain that

$$
\langle \mathcal{R}w_1 - \mathcal{R}w_2, v\rangle \le \left(|C| + \frac{7}{4}\right)\int_\Omega |A-B|(|A-C|+|B-C|)|v'|dx.
$$

By the assumption on u, we have that $u \in W^1_\infty(\Omega)$. It implies that C is bounded. Recall the Sobolev inequality [5]

$$
\|w\|^1_\infty \lesssim \|w\|_2, \quad \text{for all } w \in H^2(\Omega), \tag{46}
$$

and the inverse estimate [5]

$$
\|v\|_2 \lesssim h^{-1}\|v\|_1, \quad \text{for all } v \in V_h. \tag{47}
$$

By employing the triangle inequality and the Cauchy-Schwarz inequality, we obtain that

$$
\langle \mathcal{R}w_1 - \mathcal{R}w_2, v\rangle \lesssim (\|w_1\|_1 + \|w_2\|_1)\|w_1 - w_2\|_1\|v'\|_\infty.
$$

Using estimates (46) and (47), we have that

$$
\langle \mathcal{R}w_1 - \mathcal{R}w_2, v\rangle \lesssim h^{-1}(\|w_1\|_1 + \|w_2\|_1)\|w_1 - w_2\|_1\|v\|_1.
$$

Using definition (31) of the discrete negative norm, we obtain the desired estimate (41). □

We next translate the contraction property of $\mathcal{R}$ established in Lemma 4.3 to that of $\mathcal{M}$.

Lemma 4.4. *If $u \in H^s(\Omega)$ for some $s > 2$, then for all $w_1, w_2 \in H^1_0(\Omega)$,*

$$
\|\mathcal{M}w_1 - \mathcal{M}w_2\|_1 \lesssim h^{-1}(\|u - w_1\|_1 + \|u - w_2\|_1)\|w_1 - w_2\|_1. \tag{48}
$$

Proof. Estimate (48) is immediately obtained by combining contraction property (41) of $\mathcal{R}$ proved in Lemma 4.3, formula (39) of the mapping $\mathcal{M}$ and stability estimate (34) of the operator $\mathcal{L}_h$. □

We now establish the contraction property of $\mathcal{M}_h$ on the discrete closed ball of V_h with center $u_{c,h}$ and radius ρ, defined by

$$
\mathbb{B}_\rho(u_{c,h}) := \{v \in V_h; \|u_{c,h} - v\|_1 \le \rho\}. \tag{49}
$$

Lemma 4.5. *If $u \in H^s(\Omega)$ for some $s > 2$, then there exists a constant $C_1 > 0$ such that for all $v_1, v_2 \in \mathbb{B}_\rho(u_{c,h})$,*

$$
\|\mathcal{M}_h v_1 - \mathcal{M}_h v_2\|_1 \le C_1 h^{-1}(\rho + h^{t-1}\|u\|_t)\|v_1 - v_2\|_1. \tag{50}
$$

Proof. We establish (50) by employing Lemma 4.4. Since $\mathcal{M}_h$ is the restriction of $\mathcal{M}$ to V_h, we have for all $v_1, v_2 \in \mathbb{B}_\rho(u_{c,h})$ that

$$\|\mathcal{M}_h v_1 - \mathcal{M}_h v_2\|_1 = \|\mathcal{M} v_1 - \mathcal{M} v_2\|_1.$$

This combined with Lemma 4.4 yields for all $v_1, v_2 \in \mathbb{B}_\rho(u_{c,h})$ that

$$\|\mathcal{M}_h v_1 - \mathcal{M}_h v_2\|_1 \lesssim h^{-1}(\|u - v_1\|_1 + \|u - v_2\|_1)\|v_1 - v_2\|_1.$$

Therefore, by the triangle inequality and definition (49) of the discrete closed ball $\mathbb{B}_\rho(u_{c,h})$, we observe for all $v_1, v_2 \in \mathbb{B}_\rho(u_{c,h})$ that

$$\begin{aligned}\|\mathcal{M}_h v_1 - \mathcal{M}_h v_2\|_1 &\lesssim h^{-1}(\|u_{c,h} - v_1\|_1 + \|u_{c,h} - v_2\|_1 + 2\|u - u_{c,h}\|_1)\|v_1 - v_2\|_1 \\ &\lesssim h^{-1}(\rho + \|u - u_{c,h}\|_1)\|v_1 - v_2\|_1.\end{aligned}$$

We now obtain the desired result by combining the above estimate and error estimate (37) of Lemma 4.2 between u and its elliptic projection $u_{c,h}$. □

Next, we need to estimate the error between $u_{c,h}$ and the image of the ball $\mathbb{B}_\rho(u_{c,h})$ under the mapping $\mathcal{M}_h$.

Lemma 4.6. *If $u \in H^s(\Omega)$ for some $s > 2$, then there exists a constant $C_2 > 0$ such that for all $v \in \mathbb{B}_\rho(u_{c,h})$*

$$\|u_{c,h} - \mathcal{M}_h v\|_1 \le C_2 h^{-1}(\rho^2 + h^{2(t-1)}\|u\|_t^2). \tag{51}$$

Proof. Because $\mathcal{M}_h$ is the restriction of the mapping $\mathcal{M}$ on V_h, using formula (38) of the mapping $\mathcal{M}$, we observe for all $v \in \mathbb{B}_\rho(u_{c,h})$ that

$$u_{c,h} - \mathcal{M}_h v = u_{c,h} - \mathcal{M} v = \mathcal{L}_h^{-1}\mathcal{R}(v - u).$$

Using stability result (34) of $\mathcal{L}_h$ and using definition (26) of $\mathcal{R}$ with the fact that $\mathcal{R}(0) = 0$, by applying Lemma 4.3, we have for all $v \in \mathbb{B}_\rho(u_{c,h})$ that

$$\|u_{c,h} - \mathcal{M}_h v\|_1 \lesssim \|\mathcal{R}(v - u) - \mathcal{R}(0)\| \lesssim h^{-1}\|u - v\|_1^2.$$

Therefore, by using the definition of H^1 norm and employing Lemma 4.2, by definition (49) of the ball $\mathbb{B}_\rho(u_{c,h})$, we have for all $v \in \mathbb{B}_\rho(u_{c,h})$ that

$$\|u_{c,h} - \mathcal{M}_h v\|_1 \lesssim h^{-1}(\|u - u_{c,h}\|_1^2 + \|u_{c,h} - v\|_1^2) \lesssim h^{-1}(h^{2(t-1)}\|u\|_t^2 + \rho^2),$$

yielding the desired result. □

With the technical lemmas established above, we next prove our main result about the well-posedness of finite element method (21) and the H^1 error estimate of the approximate solution. This is done by establishing the fixed point of the mapping $\mathcal{M}_h$ in a small ball centered at $u_{c,h}$. We shall use the following Banach fixed-point theorem [22].

Lemma 4.7. *Let (X, d) be a non-empty complete metric space. If $T : X \to X$ is a contraction mapping on X, that is, there is a nonnegative real number $q < 1$ such that for all $x, y \in X$*

$$d(T(x), T(y)) \le q d(x, y),$$

then T admits a unique fixed-point in X.

We are now ready to prove our main theorem of this paper.

Theorem 4.8. *If $u \in H^s(\Omega)$ for some $s > 2$, then there exists an $h_0 > 0$ such that for each $h \le h_0$ the finite element method (21) has a unique solution u_h, and*

$$\|u - u_h\|_1 \lesssim h^{t-1}\|u\|_t. \tag{52}$$

Proof. We shall use the Banach fixed point theorem (Lemma 4.7) to show that there exists a unique solution u_h for finite element method (21). For this purpose, we shall identify a fixed-point of $\mathcal{M}_h$ in a small ball centered at $u_{c,h}$. This requires to prove that $\mathcal{M}_h$ is indeed a contraction mapping.

Noticing that $t = \min\{k+1, s\} > 2$, we choose $h_0 > 0$ such that for all $h \le h_0$

$$\delta := 2h^{-1} \max\{C_1, C_2\} h^{t-1} \|u\|_t < 1, \tag{53}$$

where C_1, C_2 are constants that appear in Lemmas 4.5 and 4.6, respectively. Fix $h \le h_0$ and set

$$\rho_0 := h^{t-1} \|u\|_t. \tag{54}$$

According to Lemma 4.6, we have for all $v \in \mathbb{B}_{\rho_0}(u_{c,h})$ that

$$\|u_{c,h} - \mathcal{M}_h v\|_1 \le C_2 h^{-1}(\rho_0^2 + h^{2(t-1)} \|u\|_t^2) = (2h^{-1} C_2 h^{t-1} \|u\|_t) \rho_0 \le \rho_0.$$

This yields that $\mathcal{M}_h$ maps $\mathbb{B}_{\rho_0}(u_{c,h})$ into $\mathbb{B}_{\rho_0}(u_{c,h})$. Furthermore, we shall prove $\mathcal{M}_h$ defined in a small ball centered at $u_{c,h}$ with radius ρ_0 is actually a contraction mapping. Applying Lemma 4.5 with the fact (53) and (54) for all $v_1, v_2 \in \mathbb{B}_{\rho_0}(u_{c,h})$, we obtain that

$$\begin{aligned} \|\mathcal{M}_h v_1 - \mathcal{M}_h v_2\|_1 &\le C_1 h^{-1}(\rho_0 + h^{t-1}\|u\|_t)\|v_1 - v_2\|_1 \\ &= (2h^{-1} C_1 h^{t-1} \|u\|_t) \|v_1 - v_2\|_1 \\ &\le \delta \|v_1 - v_2\|_1. \end{aligned}$$

Hence, by employing Lemma 4.7, $\mathcal{M}_h$ has a unique fixed point $u_h \in \mathbb{B}_{\rho_0}(u_{c,h})$, which is a unique solution of (21).

It remains to establish error estimate (52). By applying Lemma 4.2 with $u_h \in \mathbb{B}_{\rho_0}(u_{c,h})$ and the definition (54) of ρ_0, we have that

$$\|u - u_h\|_1 \le \|u - u_{c,h}\|_1 + \|u_{c,h} - u_h\|_1 \lesssim h^{t-1} \|u\|_t.$$

This gives the optimal H^1 error estimate. □

Next, we establish the L^2 error estimate of the proposed finite element method.

Theorem 4.9. *If $u \in W_\infty^2(\Omega)$ and u_h is the solution of the finite element method* (21)*, then*

$$\|u - u_h\|_0 \lesssim h^t \|u\|_t. \tag{55}$$

Proof. In order to derive the optimal L^2 error estimate for finite element method (21), we use the Nitsche technique. Let $\varphi \in H_0^1(\Omega)$ be the solution to the following auxiliary problem

$$-(K(u' + \frac{\ell}{2})\varphi')' = u - u_h, \qquad \text{in} \quad \Omega \tag{56a}$$

$$\varphi = 0, \qquad \text{on} \quad \partial\Omega. \tag{56b}$$

Following above auxiliary equation (56) and definition (25) of the linear operator $\mathcal{L}$, we obtain that

$$\|u - u_h\|_0^2 = -(u - u_h, (K(u' + \frac{\ell}{2})\varphi')') = \langle \mathcal{L}(u - u_h), \varphi \rangle . \tag{57}$$

We further rewrite (57) as

$$\|u - u_h\|_0^2 = \langle \mathcal{L}(u - u_h), \varphi - \varphi_h \rangle + \langle \mathcal{L}(u - u_h), \varphi_h \rangle . \tag{58}$$

We next prove error estimate (55) by estimating the two terms in the right hand side of above equation (58).

We first consider the first term of equation (58). By definition (31) of the discrete negative norm, we obtain that

$$\langle \mathcal{L}(u - u_h), \varphi - \varphi_h \rangle \leq \|\mathcal{L}(u - u_h)\|_{-1,h} \|\varphi - \varphi_h\|_1. \tag{59}$$

According to the hypothesis of u, we observe that $u' \in W^1_\infty(\Omega)$. Hence, by the elliptic regularity theory [17], we find that

$$\|\varphi\|_2 \lesssim \|u - u_h\|_0. \tag{60}$$

Let $\varphi_h \in V_h$ be chosen so that

$$\|\varphi - \varphi_h\|_1 \lesssim h\|\varphi\|_2 \lesssim h\|u - u_h\|_0. \tag{61}$$

By applying stability estimate (33) of $\mathcal{L}_h$ and above estimate (61) to (59), we have that

$$\langle \mathcal{L}(u - u_h), \varphi - \varphi_h \rangle \lesssim h\|u - u_h\|_1 \|u - u_h\|_0. \tag{62}$$

We now consider the second term of equation (58). We need a sharper estimate about it, by using definition (30) of the elliptic projection $u_{c,h}$, we have that

$$\langle \mathcal{L}(u - u_h), \varphi_h \rangle = \langle \mathcal{L}_h(u_{c,h} - u_h), \varphi_h \rangle. \tag{63}$$

Recalling formula (38) of the mapping $\mathcal{M}$, letting $w := u_h$ in (38) and noticing that $\mathcal{M}_h$ is a restriction of $\mathcal{M}$ on V_h, we obtain that

$$\mathcal{M}_h u_h = u_{c,h} - \mathcal{L}_h^{-1} \mathcal{R}(u_h - u).$$

This yields that

$$\mathcal{R}(u_h - u) = \mathcal{L}_h(u_{c,h} - \mathcal{M}_h u_h). \tag{64}$$

With above equation (64), using the fact in the proof of Theorem 4.8 that u_h is the fixed point of $\mathcal{M}_h$, we have that

$$\mathcal{R}(u_h - u) = \mathcal{L}_h(u_{c,h} - u_h). \tag{65}$$

Applying above equation (65) to equation (63) yields that

$$\langle \mathcal{L}(u - u_h), \varphi_h \rangle = \langle \mathcal{R}(u_h - u), \varphi_h \rangle. \tag{66}$$

Definition (26) of $\mathcal{R}$ with the fact that $\mathcal{R}(0) = 0$ ensures that

$$\langle \mathcal{L}(u - u_h), \varphi_h \rangle \leq \|\mathcal{R}(u_h - u) - \mathcal{R}(0)\|_{-1,h} \|\varphi_h\|_1.$$

By estimate (41) in Lemma 4.3, we obtain that

$$\langle \mathcal{L}(u - u_h), \varphi_h \rangle \lesssim h^{-1} \|u - u_h\|_1^2 \|\varphi_h\|_1. \tag{67}$$

It suffices to estimate $\|\varphi_h\|_1$. Based on the definition of the Sobolev norm in the last section, we have that

$$\|\varphi\|_1 \lesssim \|\varphi\|_2.$$

Together with estimates (60) and (61), we obtain that

$$\|\varphi_h\|_1 \lesssim \|\varphi\|_1 + \|\varphi - \varphi_h\|_1 \lesssim \|u - u_h\|_0. \tag{68}$$

Combining estimates (68) and (67), we obtain that

$$\langle \mathcal{L}(u - u_h), \varphi_h \rangle \lesssim h^{-1} \|u - u_h\|_1^2 \|u - u_h\|_0. \tag{69}$$

Consequently, substituting estimates (62) and (69) into the right hand side of equation (58) and employing H^1 error estimate (52), we conclude the following estimate

$$\|u - u_h\|_0 \lesssim h^t \|u\|_t + h^{2t-3} \|u\|_t^2. \tag{70}$$

Under the assumption of u, with the fact that $t = \min\{k+1, s\} \geq 3$, we have the optimal order of L^2 error estimate that

$$\|u - u_h\|_0 \lesssim h^t \|u\|_t.$$

□

5. Newton Iteration

In this section, we describe the Newton iteration algorithm for solving nonlinear equation (21) and present its convergence analysis.

We now describe the Newton iteration for solving finite element method (21). Given an initial guess $u_0 \in V_h$, the Newton approximation to u_h forms a sequence $\{u_k\}_{k=0}^{\infty} \subset V_h$ that satisfies

$$D\mathcal{N}_h[u_k](u_{k+1} - u_k) = -\mathcal{N}_h u_k, \tag{71}$$

where $D\mathcal{N}_h : V_h \to L(V_h; V_h')$ denotes the Gâteaux derivative of $\mathcal{N}_h$, that is,

$$\langle D\mathcal{N}_h[q](w), v\rangle := \int_\Omega K(q' + \frac{\ell}{2}) w' v' dx, \quad \text{for all } q, w, v \in V_h. \tag{72}$$

Here, $L(V_h; V_h')$ denotes the space of linear mapping from V_h to V_h'. Together with equation (20), by using notation (2), equation (71) can be written in a concrete form

$$\int_\Omega K(u_k' + \frac{\ell}{2}) w' v' dx = -\int_\Omega G(u_k' + \frac{\ell}{2}) v' dx + \int_\Omega f v dx, \quad \text{for all } v \in V_h,$$

where $w := u_{k+1} - u_k \in V_h$.

Next, we present finite element method (21) based on Newton iteration (71) described above.

Algorithm 5.1. *The Discrete Form*

- Step 1: Set $k = 0$ and choose $u_0 \in V_h$.
- Step 2: Compute $w_k \in V_h$ from the equation

$$\int_\Omega K(u_k' + \frac{\ell}{2}) w_k' v' dx = -\int_\Omega G(u_k' + \frac{\ell}{2}) v' dx + \int_\Omega f v dx, \quad \text{for all } \ v \in V_h. \tag{73}$$

- Step 3: Let $u_{k+1} = u_k + w_k$. Set $k \leftarrow k+1$ and go back to Step 2 until a tolerance condition is satisfied.

Step 2 in Algorithm 5.1 requires solving $w_k \in V_h$ from integral differential equation (73). We shall use the Galerkin principle with the Lagrange finite elements to discretize the integral differential equation.

Next, we establish the convergence result concerning the Newton iteration algorithm. To this end, we investigate

$$\mathcal{G}[q_1; q_2, w] := D\mathcal{N}_h[q_1](w) - D\mathcal{N}_h[q_2](w), \tag{74}$$

where $q_1, q_2, w \in V_h$.

Proposition 5.1. *If $D\mathcal{N}_h$ is defined as* (72), *then for all $q_1, q_2, w \in V_h$*

$$\|\mathcal{G}[q_1; q_2, w]\|_0 \lesssim h^{-1} \|q_1 - q_2\|_1 \|w\|_1. \tag{75}$$

Proof. By using the Cauchy-Schwarz inequality, for all $q_1, q_2, w \in V_h$, we obtain that

$$\|\mathcal{G}[q_1; q_2, w]\|_0 \lesssim \int_\Omega |K(q_1' + \frac{\ell}{2}) - K(q_2' + \frac{\ell}{2})| |w'| |v'| dx \lesssim \|q_1' - q_2'\|_\infty \|w\|_1.$$

With above estimate, using estimates (46) and (47), we have that

$$\|\mathcal{G}[q_1;q_2,w]\|_0 \lesssim \|q_1-q_2\|_\infty^1\|w\|_1 \lesssim \|q_1-q_2\|_2\|w\|_1 \lesssim h^{-1}\|q_1-q_2\|_1\|w\|_1,$$

which yields the desired result. □

We now present the main result regarding the convergence of the Newton iteration. We define the discrete closed ball of V_h at center u_h with radius ρ by

$$\mathbb{B}_\rho(u_h) := \{v\in V_h; \|u_h - v\|_1 \le \rho\}. \tag{76}$$

Theorem 5.2. *If u_h is the solution of the finite element method* (21)*, then there exists an $h_1>0$ and $\rho_h>0$ such that for $h\le h_1$ and $u_0\in\mathbb{B}_{\rho_h}(u_h)$, the sequence generated by* (71) *is well-defined and*

$$\|u_{k+1}-u_h\|_1 \lesssim h^{-1}\|u_k-u_h\|_1^2. \tag{77}$$

Proof. By using Theorem 4.8, we choose $h_1 := h_0$ such that for each $h\le h_1$ the finite element method (21) has a unique solution. For some $\epsilon_0>0$, we set

$$\rho_h := h^{1+\epsilon_0}. \tag{78}$$

Suppose that $u_k\in\mathbb{B}_{\rho_h}(u_h)$, we first show that u_{k+1} determined by (71) is well defined. Using definition (72) of the operator $D\mathcal{N}_h$ and employing the Poincaré inequality (35), we have that

$$\langle D\mathcal{N}_h[q](w),w\rangle \gtrsim \|w\|_1^2, \quad \text{for all} \quad q,w\in V_h. \tag{79}$$

This is the coercivity of $D\mathcal{N}_h$ and consequently, $D\mathcal{N}_h$ is invertible. According to Newton iteration (71), u_{k+1} is well defined.

Next, we prove estimate (77). For this purpose, we rewrite equation (71). According to the fact that u_h is the solution of finite element method (21), we obtain that

$$\begin{aligned} D\mathcal{N}_h[u_k](u_{k+1}-u_h) &= D\mathcal{N}_h[u_k](u_k-u_h) - \mathcal{N}_h u_k \\ &= D\mathcal{N}_h[u_k](u_k-u_h) - (\mathcal{N}_h u_k - \mathcal{N}_h u_h). \end{aligned}$$

With above equation, by using (74) and employing the mean value theorem, there exists a $\theta\in(0,1)$ such that

$$D\mathcal{N}_h[u_k](u_{k+1}-u_h) = \mathcal{G}[u_k;\theta(u_k-u_h)+u_h, u_k-u_h]. \tag{80}$$

Following estimate (79), we obtain that

$$\|w\|_1 \lesssim \|D\mathcal{N}_h[q](w)\|_0, \quad \text{for all} \quad q,w\in V_h. \tag{81}$$

Below, we use estimate (81) and equation (80) to prove (77). By using estimate (81), we obtain that

$$\|u_{k+1}-u_h\|_1 \lesssim \|D\mathcal{N}_h[u_k](u_{k+1}-u_h)\|_0. \tag{82}$$

Substituting equation (80) into the right hand side of inequality (82) yields the estimate

$$\|u_{k+1}-u_h\|_1 \lesssim \|\mathcal{G}[u_k;\theta(u_k-u_h)+u_h,u_k-u_h]\|_0.$$

By employing Proposition 5.1, we have for the $\theta\in(0,1)$ that

$$\|u_{k+1}-u_h\|_1 \lesssim h^{-1}\|(1-\theta)(u_k-u_h)\|_1\|u_k-u_h\|_1 \lesssim h^{-1}\|u_k-u_h\|_1^2,$$

yields the desired result (77). □

As a consequence of estimate (77) in Theorem 5.2, $u_k\in\mathbb{B}_{\rho_h}(u_h)$ for all $k\ge 1$ provided $u_0\in\mathbb{B}_{\rho_h}(u_h)$.

6. Numerical Experiments

In this section, we perform numerical experiments to verify the approximation accuracy of proposed finite element method (21).

In the following examples, we choose a positive integer N and use Algorithm 5.1 with a uniform partition of the Ω with $h := \frac{2}{N+1}$. We apply the shooting method [8, 34] to prescribed curvature problem (1) and use the resulting solution as an initial guess. For the choice of bases for the proposed finite element method, we choose the Lagrange linear and quadratic polynomials bases. Notice from Theorems 4.8 and 4.9, to guarantee convergence, we require the degree of polynomials used in the proposed methods to be at least 2. In the numerical experiments presented in this section, we also include the linear elements even though they do not meet the theoretical requirement.

Specifically, the piecewise linear basis functions $\varphi_j \in V_h, j = 1, 2, \ldots, N$ are defined by

$$\varphi_j(x) := \begin{cases} 1 + h^{-1}(x - x_j), & x_{j-1} \le x \le x_j \\ 1 - h^{-1}(x - x_j), & x_j \le x \le x_{j+1} \\ 0, & \text{else} \end{cases}$$

and the piecewise quadratic basis functions φ_j, $\varphi_{j+\frac{1}{2}} \in V_h, j = 1, 2, \ldots, N$ are defied by

$$\varphi_j(x) := \begin{cases} (2h^{-1}(x - x_j) + 1)(h^{-1}(x - x_j) + 1), & x_{j-1} \le x \le x_j \\ (2h^{-1}(x - x_j) - 1)(h^{-1}(x - x_j) - 1), & x_j \le x \le x_{j+1} \\ 0, & \text{else} \end{cases}$$

$$\varphi_{j+\frac{1}{2}}(x) := \begin{cases} 4h^{-1}(x - x_j)(1 - h^{-1}(x - x_j)), & x_j \le x \le x_{j+1} \\ 0, & \text{else} \end{cases}$$

where $x_{j+\frac{1}{2}} := x_j + \frac{1}{2}h$.

Below, we consider three numerical examples and present their numerical results in six tables. The convergence order of the approximate solution with respect to the L^2, H^1 norms reported in the tables is computed by

$$\log_2 \frac{\|u - u_h\|}{\|u - u_{h/2}\|},$$

where $\|\cdot\|$ may be the L^2 norm or the H^1 norm.

Table 1. Numerical results of Example 1 by using the linear basis.

N	$\|u - u_h\|_0$	rate	$\|u - u_h\|_1$	rate
7	8.529E-03		8.756E-02	
15	2.335E-03	1.869	4.596E-02	0.930
31	6.042E-04	1.950	2.340E-02	0.974
63	1.526E-04	1.985	1.176E-02	0.992
127	3.826E-05	1.996	5.891E-03	0.998
255	9.573E-06	1.999	2.946E-03	0.999
511	2.394E-06	2.000	1.473E-03	1.000
1023	5.987E-07	2.000	7.367E-04	1.000
2047	1.496E-07	2.001	3.683E-04	1.000
4095	3.737E-08	2.001	1.842E-04	1.000

TABLE 2. Numerical results of Example 1 by using the quadratic basis.

N	$\|u-u_h\|_0$	rate	$\|u-u_h\|_1$	rate
7	4.152E-04		1.155E-02	
15	6.360E-05	2.707	3.757E-03	1.620
31	8.585E-06	2.889	1.048E-03	1.843
63	1.099E-06	2.966	2.713E-04	1.949
127	1.383E-07	2.991	6.848E-05	1.986
255	1.731E-08	2.998	1.716E-05	1.996
511	2.165E-09	2.999	4.293E-06	1.999
1023	2.707E-10	3.000	1.073E-06	2.000
2047	3.384E-11	3.000	2.684E-07	2.000
4095	4.229E-12	3.000	6.709E-08	2.000

Example 1.

We solve problem (1) with $\ell := 0$ and data

$$f(x) := \frac{3}{8}(1-\frac{39}{64}x^2)^{-3/2}, \qquad x \in \Omega.$$

We can verify by Theorem 2.4 that this boundary value problem is solvable. Its exact solution is given by

$$u(x) := \frac{1}{2}(1-\frac{3}{4}x^2)^{1/2} - \frac{1}{4}.$$

We apply Algorithm 5.1 based on the Galerkin method via the Lagrange linear and quadratic polynomials basis to solve the above problem. We report numerical results of this example in Tables 1 and 2, respectively, for the cases using the linear and quadratic bases.

TABLE 3. Numerical results of Example 2 by using the linear basis.

N	$\|u-u_h\|_0$	rate	$\|u-u_h\|_1$	rate
7	1.211E-02		1.263E-01	
15	3.060E-03	1.985	6.318E-02	0.999
31	7.669E-04	1.997	3.159E-02	1.000
63	1.918E-04	1.999	1.580E-02	1.000
127	4.797E-05	2.000	7.899E-03	1.000
255	1.199E-05	2.000	3.949E-03	1.000
511	2.998E-06	2.000	1.975E-03	1.000
1023	7.495E-07	2.000	9.874E-04	1.000
2047	1.874E-07	2.000	4.937E-04	1.000
4095	4.684E-08	2.000	2.468E-04	1.000

Example 2.

We solve problem (1) with $\ell := \frac{2}{3}$ and data

$$f(x) := -K(\frac{1}{3}x^3 + x^2)x(x+2), \qquad x \in \Omega,$$

where K is defined by (24). The exact solution of this problem is given by

$$u(x) := \frac{1}{12}x^4 + \frac{1}{3}x^3 + \frac{1}{4}.$$

TABLE 4. Numerical results of Example 2 by using the quadratic basis.

N	$\|u-u_h\|_0$	rate	$\|u-u_h\|_1$	rate
7	2.731E-04		7.683E-03	
15	3.163E-05	3.110	1.907E-03	2.010
31	3.866E-06	3.032	4.758E-04	2.003
63	4.805E-07	3.008	1.189E-04	2.001
127	5.997E-08	3.002	2.972E-05	2.000
255	7.494E-09	3.000	7.429E-06	2.000
511	9.366E-10	3.000	1.857E-06	2.000
1023	1.171E-10	3.000	4.643E-07	2.000
2047	1.463E-11	3.001	1.161E-07	2.000
4095	1.829E-12	3.000	2.902E-08	2.000

We solve this problem by using Algorithm 5.1 with the Lagrange linear and quadratic polynomials bases. We report numerical results for this example in Tables 3 and 4, respectively, for the cases using the linear and quadratic bases.

Example 3.

We solve problem (1) with $\ell := \frac{1}{8}$ and data

$$f(x) := -K(\frac{11}{8}x^3 - \frac{9}{16}x + \frac{1}{16})(\frac{33}{8}x^2 - \frac{9}{16}), \qquad x \in \Omega.$$

The exact solution of this problem is given by

$$u(x) = \frac{11}{32}x^4 - \frac{9}{32}x^2 + \frac{1}{16}x.$$

Again, we solve this problem by using Algorithm 5.1 with the Lagrange linear and quadratic polynomials bases. We report numerical results for this example in Tables 5 and 6, respectively, for the cases using the linear and quadratic bases.

TABLE 5. Numerical results of Example 3 by using the linear basis.

N	$\|u-u_h\|_0$	rate	$\|u-u_h\|_1$	rate
7	1.473E-02		1.480E-01	
15	3.925E-03	1.908	7.497E-02	0.981
31	9.965E-04	1.978	3.758E-02	0.996
63	2.501E-04	1.994	1.880E-02	0.999
127	6.258E-05	1.999	9.402E-03	1.000
255	1.565E-05	2.000	4.701E-03	1.000
511	3.913E-06	2.000	2.351E-03	1.000
1023	9.781E-07	2.000	1.175E-03	1.000
2047	2.445E-07	2.000	5.877E-04	1.000
4095	6.113E-08	2.000	2.938E-04	1.000

We observe that computed convergence rates reported in the above three examples conform the theoretical estimates presented in Theorems 4.8-4.9 for the quadratic case. For the linear case, even though it does not meet the hypothesis of Theorems 4.8-4.9, it seems from the numerical results that its convergent rates enjoy $O(h)$ and $O(h^2)$ for the H^1 norm and the L^2 norm, respectively.

TABLE 6. Numerical results of Example 3 by using the quadratic basis.

N	$\|u - u_h\|_0$	rate	$\|u - u_h\|_1$	rate
7	5.909E-04		1.573E-02	
15	6.726E-05	3.110	3.928E-03	2.010
31	8.046E-06	3.032	9.810E-04	2.003
63	9.933E-07	3.008	2.452E-04	2.001
127	1.238E-07	3.002	6.129E-05	2.000
255	1.546E-08	3.000	1.532E-05	2.000
511	1.932E-09	3.000	3.831E-06	2.000
1023	2.415E-10	3.000	9.576E-07	2.000
2047	3.018E-11	3.001	2.394E-07	2.000
4095	3.773E-12	3.000	5.985E-08	2.000

7. Conclusion

In this paper, we have developed the finite element method for solving the Dirichlet boundary value problem of the one-dimensional prescribed curvature equation. We have established the optimal order of convergence for the proposed method in both the H^1 morn and the L^2 morn. The convergence order is verified by numerical examples, where the resulting nonlinear system is solved by the Newton iteration.

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