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STOCHASTIC SPLINE-COLLOCATION METHOD FOR CONSTRAINED OPTIMAL CONTROL PROBLEM GOVERNED BY RANDOM ELLIPTIC PDE

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Abstract. In this paper, we investigate a stochastic spline-collocation approximation scheme for an optimal control problem governed by an elliptic PDE with random field coefficients. We obtain the necessary and sufficient optimality conditions for the optimal control problem and establish a scheme to approximate the optimality system through the discretization with respect to the spatial space by finite elements method and the probability space by stochastic spline-collocation method. We further investigate Smolyak approximation schemes, which are effective collocation strategies for smooth problems that depend on a moderately large number of random variables. For more general control problems where the state may be non-smooth with respect to the random variables in some areas, we adopt a domain decomposition strategy to partition the random space into smooth and non-smooth parts and then apply Smolyak scheme and spline approximation respectively. A priori error estimates are derived for the state, the co-state and the control variables. Numerical examples are presented to illustrate our theoretical results.

Key words. Random elliptic PDE, priori error estimates, stochastic spline-collocation method, Smolyak approximation, optimal control problem, deterministic constrained control.

1. Introduction

In recent years, there are increasing interests in modeling uncertainty in many complex physical and engineering systems, such as uncertain parameters, coefficients, forcing term, and boundary conditions. It is well known that these systems can be described by stochastic partial differential equations(SPDEs). Since stochastic PDEs are conveniently used in many areas, such as fluid flows in porous media,chemistry, transport of pollutants in groundwater and oil recovery processes, the numerical solutions for Stochastic PDEs have been a main subject of growing interest in the scientific community([4]-[22]).

The well-known Monte Carlo (MC) method is the most commonly used method for simulating stochastic PDEs and for dealing with the statistic characteristics of the solution [4, 5]. Although MC method only needs to do repetitive deterministic simulations, it is a rather computationally expensive method for the reason that the statistic convergence rate is relatively slow, especially when there are large amounts of computations in the deterministic systems. Another alternative to the Monte Carlo method is the so-called stochastic Galerkin method[9, 15] for solving stochastic PDEs with random fields input data. This method allows us to utilize standard approximations in space(finite elements, finite volumes, spectral or h-p finite elements, etc.) and polynomial approximation in the probability domain, either on full polynomial spaces [16, 20, 21], tensor product polynomial spaces [17, 18, 19], or on piecewise polynomial spaces [6, 7, 8, 17]. By applying stochastic Galerkin method, we can utilize the regularity of the solution and acquire faster convergence rates. However, in general, this technique requires to solve a system

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of equations that couples all degrees of freedom when approximating the stochastic systems.

Due to this issue, the stochastic collocation method has gained much attention recently in the computational community [10, 11, 12, 13, 20], which was originally introduced in [10,20]. In principle, stochastic collocation method consists of a Galerkin approximation in physical space and a collocation in the zeros of suitable tensor product orthogonal polynomials (Gauss points) in the probability space[10]. Compared with stochastic Galerkin methods, this method solves uncoupled deterministic PDEs at the collocation points that are trivially parallelizable, as in the Monte Carlo method. This method also can treat efficiently the case of dependent random variables by introducing an auxiliary density $\hat{\rho}$. And this method deals easily with unbounded random variables such as Gaussian or exponential variables. Hence, stochastic collocation is an attractive method for computing solutions of stochastic PDEs with random field input data.

In many applications, optimization of physical and engineering systems can be formulated as optimal control problems that are constrained by PDEs. Computational methods for deterministic optimal control problems constrained by PDEs have been well developed and investigated for several decades([1]-[3],[23]-[29]). Recently efficient numerical methods for optimal control problem governed by stochastic PDEs are becoming a new hot topic. Comparison with the deterministic optimal control, efficient computation of stochastic optimal control problems constrained by stochastic PDEs is still in its infancy, see the very recent work([30]-[37]). Based on the work([6]-[22]), [30] dealed with optimal control governed by random steady PDEs with deterministic Neumann boundary control, and the existence of an optimal solution and of a Lagrange multiplier were demonstrated. The authors also proposed the stochastic finite element solution of the optimality system and estimated its error through the discretizations with respect to both spatial and random parameter spaces. In [31], one-shot stochastic finite element methods were used to find approximate solutions with 'pure' stochastic control function as well as 'semi' stochastic control function for an optimal control problem constrained by stochastic steady diffusion problems. In [32] and [33], stochastic optimal control governed by stochastic elliptic PDEs with deterministic distributed control function were introduced, and the authors proved the existence of the optimal solution, established the validity of the Lagrange multiplier rule and obtained stochastic optimality system. Computationally, the numerical solutions of the optimality system were given by the stochastic finite element method. In [34], the author proposed framework combines space-time multigrid methods with sparse-grid collocation techniques to solve nonlinear parabolic optimal control problems with random coefficients for unconstrained control. In [35], we studied an optimal control problem governed by an elliptic PDE with random field in coefficients and constrained control, and obtained the necessary and sufficient optimality conditions by applying the well-known Lions' lemma. Then a stochastic finite element approximation scheme is applied and the a priori error estimate for the state, the co-state and the control variables is derived. In [36] a stochastic finite element approximation scheme and the a priori error estimate for the state, the co-state and the control variables were developed for an optimal control problem governed by an elliptic integro-differential equation with random coefficients. Furthermore in [37], stochastic finite element is applied to an optimal control problem governed by a parabolic PDE with random field in its coefficients, and a priori error estimates for the state, the co-state and the control variables have been given. However, to our best knowledge, there has been a lack of results about stochastic collocation approximation for optimal control problem governed by random elliptic PDE with constrained control and possible non-smoothness in the probability space.

In this paper, we develop a stochastic spline-collocation method for an optimal control problem governed by an elliptic PDE with random field in its coefficients. We divide the computational area of probability space into two parts and apply Smolyak and lower order spline approximations in different areas. The outline of the following paper is as follows: in Section 2, we introduce some function spaces and the stochastic optimal control problem. By applying the well-known Lions' lemma to the optimal control problem, we obtain the necessary and sufficient optimality conditions. In Section 3, we introduce the stochastic collocation method and Smolyak approximation schemes for the optimal control problem. We give a priori error estimates for the state, the co-state and the control variables. Numerical examples are presented to illustrate our theoretical results in Section 4.

2. Model problem

2.1 Function spaces and notations. Let $D \subset \mathbb{R}^d$ be a convex bounded polygonal spatial domain with boundary ∂D and $1 \leq d \leq 3$. Denote by $B(D)$ the Borel σ-algebra generated by the open subset of D. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space, where Ω is a set of outcomes, $\mathcal{F}$ is a σ-algebra of events and $\mathcal{P} : \mathcal{F} \to [0, 1]$ is a probability measure. Let $\xi = \xi(\omega) = (\xi_1(\omega), \cdots, \xi_N(\omega))$ with independent components $\xi_i(\omega)$, $i = 1, \cdots, N \in \mathbb{N}$. Let $\Gamma_i = [a_i, b_i] = \xi_i(\Omega) \subset \mathbb{R}$ be a bounded interval for $i = 1, \cdots, N$ and $\rho_i : \Gamma_i \to \mathbb{R}_+$ be the probability density functions of the random variables $\xi_i(\omega)$, $\omega \in \Omega$. Then we can use the joint probability density function $\rho(\xi) = \prod_{i=1}^N \rho_i(\xi_i)$ for random vector ξ with the support $\Gamma = \prod_{i=1}^N \Gamma_i \subset \mathbb{R}^N$. On Γ, we have the probability measure $\rho(\xi)d\xi$.

Remark (unbounded random variables). By using a similar approach in [10], we can deal with unbounded random variables, such as Gaussian or exponential ones. For simplicity, here we only focus our study on bounded random variables.

Let $L^2_\rho(\Gamma)$ denote the probabilistic Hilbert space[38], in which the random processes based upon the random variables ξ have finite second moments. The inner product of this Hilbert space is given by

$$(X, Y)_{L^2_\rho(\Gamma)} = \int_\Gamma X(\xi)Y(\xi)\rho(\xi)d\xi, \ \forall \ X, \ Y \in L^2_\rho(\Gamma),$$

where we have used independence of the random variables to allow us to write the measure as product of measures in each random direction. We similarly define the expectation of a random process $X \in L^2_\rho(\Gamma)$ as

$$\mathbb{E}[X(\xi)] = \int_\Gamma X(\xi)\rho(\xi)d\xi,$$

and we refer to the expectation of the powers $\mathbb{E}[X^i(\xi)]$ as the ith moment of the random process.

Additionally, we define the mapping $f : (x, \xi) \in D \times \Gamma \to \mathbb{R}$ to be a set of random processes, which are indexed by the spatial position $x \in D$. Such a set of processes is referred to as a random field [39] and can also be interpreted as a function-valued random variable, because for every $\xi \in \Gamma$ the realization $f(\cdot, \xi) : D \to \mathbb{R}$ is a real valued function on D.

For a vector-space W on D, let the class $L^2_\rho(\Gamma; W)$ denote the space of random fields whose realizations lie in W for a.e (almost every) $\xi \in \Gamma$. If W is a Banach space, a norm on $L^2_\rho(\Gamma;W)$ is induced by $||f(x,\xi)||^2_{L^2_\rho(\Gamma;W))} = \mathbb{E}[\|f(x,\xi)\|^2_W]$; for example, on $L^2_\rho(\Gamma;L^2(D))$ we have

$$||f(x,\xi)||^2_{L^2_\rho(\Gamma;L^2(D))} = \mathbb{E}[\|f(x,\xi)\|^2_{L^2(D)}] = \int_\Gamma\int_D (f(x,\xi))^2\rho(\xi)dxd\xi,$$

which denotes the expected value of the $L^2(D)$-norm of the function $f(x,\xi)$. Similarly, we have the norm

$$||f(x,\xi)||^2_{L^2_\rho(\Gamma;H^1(D))} = \mathbb{E}[\|f(x,\xi)\|^2_{H^1(D)}] = \int_\Gamma\int_D \{(f(x,\xi))^2 + |\nabla f(x,\xi)|^2\}\rho(\xi)dxd\xi.$$

2.2 Model problem. In this paper, we consider the following control problem governed by a random elliptic equation with a constrained control:

$$\min_{u\in K}\mathcal{J}(u) = \min_{u\in K}\mathbb{E}\Big[\frac{1}{2}\int_D |y-y_d|^2dx + \frac{\alpha}{2}\int_D |u|^2dx\Big] \tag{1}$$

subject to

$$\begin{cases} -\nabla\cdot[a(x,\xi)\nabla y(x,\xi)] = u(x), & x\in D,\ \xi\in\Gamma,\\ y(x,\xi)=0, & x\in\partial D,\ \xi\in\Gamma. \end{cases} \tag{2}$$

where $\mathcal{J}$ is a cost functional, $y:\bar{D}\times\Gamma\to\mathbb{R}$ is the state variable, $y_d:\ \bar{D}\times\Gamma\to\mathbb{R}$ is a given target solution, $a: D\times\Gamma\to\mathbb{R}$ is a random function that will be determined below, $u:\ D\to\mathbb{R}$ is a deterministic control, α is a positive constant measuring the importance between two terms in $\mathcal{J}$. The operator ∇ means derivatives with respect to the spatial variable $x\in D$ only. Here, K is a closed convex subset in the control space $L^2(D)$. In the following context, we will discuss some different cases on the choice of K.

Although the objective functional $\mathcal{J}$ in (1) contains stochastic function y subject to (2), its outcome is deterministic by using the expectation $\mathbb{E}$. If we denote by $B(D)$ the Borel σ-algebra generated by the open subsets of D, then a is assumed measurable with respect to the σ-algebras $B(\Gamma\otimes D)$. To ensure regularity of the solution y, we assume that there are positive constants a_{min} and a_{max} such that

$$a_{min}\le a(x,\xi)\le a_{max},\qquad \text{a. e. } (x,\ \xi)\in D\times\Gamma. \tag{3}$$

Then, under the assumption (3), we know that there exists a unique weak solution y for (2)[17].

In the following, we set the state space $Y_\rho = L^2_\rho(\Gamma;H^1_0(D))$, the control space $U = L^2(D)$. To present the weak formulation of equation (2), we introduce the following bilinear forms:

$$A[y,v] = \int_\Gamma\int_D a\nabla y\cdot\nabla v\rho dxd\xi,\quad \forall\ y,\ v\in Y_\rho, \tag{4}$$

and

$$[u,v] = \int_\Gamma\int_D uv\rho dxd\xi,\quad \forall\ u\in U,\ \ v\in Y_\rho, \tag{5}$$

$$(u,w) = \int_D uwdx,\quad \forall\ u,\ w\in U. \tag{6}$$

Then, we can easily obtain the weak formulation of (1)-(2) as follows:

$$\min_{u\in K}\mathcal{J}(u) = \min_{u\in K}\mathbb{E}\Big[\frac{1}{2}\int_D |y-y_d|^2dx + \frac{\alpha}{2}\int_D |u|^2dx\Big] \tag{7}$$

subject to

$$A[y, v] = [u, v], \quad \forall\, v \in Y_\rho. \tag{8}$$

Under the assumption (3) and from [1, 32], we know that there is at least one solution of (7)-(8).

2.3 Stochastic optimality system. In order to set up suitable finite element approximation of (7)-(8) and obtain the error estimates, it is essential to derive the optimality conditions for the above constrained optimal control problem. In [32, 33], the authors used the infinite dimensional Lagrange multiplier theory which is quite complex to apply to study the stochastic control problems with un-constrained control. Furthermore, it is not trivial to extend the infinite dimensional Lagrange multiplier theory into our cases. In this paper, we use a different approach which is much simpler and widely used in the literature [25, 29] to study the optimal control problem (7)-(8), and we will explain in the following.

It is well-known that according to Lions' theorem [1], the PDE-constrained optimal control problem (7)-(8) has a unique minimizer , which satisfies the following variational inequality:

$$\mathcal{J}'(u)(w - u) \geq 0, \quad \forall\, w \in K. \tag{9}$$

Here, the directional derivative of functional $\mathcal{J}$ at $u \in K$ along the direction $w \in K$ is defined by

$$\mathcal{J}'(u)(w) = \lim_{t \to 0^+} \frac{\mathcal{J}'(u + tw) - \mathcal{J}'(u)}{t}. \tag{10}$$

Applying the above theory to our control problem, we have the following theorem:

Theorem 2.1.[35] The optimal control problem (7)-(8) has a unique solution $(y, u) \in Y_\rho \times K$. Furthermore, a pair (y, u) is the solution of (7)-(8) iff there is a co-state variable $p \in Y_\rho$, such that the triplet (y, p, u) satisfies the following optimality system:

$$\begin{cases} A[y, v] = [u, v], \quad \forall\, v \in Y_\rho, \\ A[p, q] = [y - y_d, q], \quad \forall\, q \in Y_\rho, \\ [p + \alpha u, w - u] \geq 0, \quad \forall\, w \in K \subset U. \end{cases} \tag{11}$$

It is known that the inequality in (11) is just the necessary and sufficient optimality condition.

The explicit solution of the variational inequality in (11) depends heavily on the choice of the joint probability density ρ and the convex set K. In the simple case, if the joint probability density ρ is uniform on Γ, we can have the following explicit solutions for some cases[1, 25]. For example,

Case I: Let K be given by

$$K = \{u \in L^2(D) : \ u(x) \geq 0, \ \text{a.e. } x \in D\}. \tag{12}$$

Then, the solution is

$$u(x) = max\big\{0, -\frac{1}{\alpha}\mathbb{E}[p(x, \xi)]\big\}, \quad \text{a.e. } \ x \in D. \tag{13}$$

Case II: Let K be given by

$$K = \{u \in L^2(D) : \ \int_D u(x) \geq 0 \ \ \}. \tag{14}$$

Then, the solution is

$$u(x) = -\frac{1}{\alpha}\mathbb{E}[p(x,\xi)] + max\{0, \frac{1}{\alpha}\overline{\mathbb{E}p}\}, \quad \text{a.e. } \ x \in D, \tag{15}$$

where $\overline{\mathbb{E}p} = \dfrac{\int_D \mathbb{E}[p(x,\xi)]dx}{\int_D dx}$.

Case III: Let K be given by

$$K = \{u \in L^2(D): \ c \le u(x) \le d, \ \text{a.e. } x \in D\}, \tag{16}$$

where constants c, $d \in \mathbb{R}$ and $c < d$. Then, the solution is

$$u(x) = \begin{cases} c, & \text{if } \ \mathbb{E}[p(x,\xi)] + \alpha u(x) > 0, \\ -\frac{1}{\alpha}\mathbb{E}[p(x,\xi)], & \text{if } \ \mathbb{E}[p(x,\xi)] + \alpha u(x) = 0, \\ d, & \text{if } \ \mathbb{E}[p(x,\xi)] + \alpha u(x) < 0, \end{cases} \quad \text{a.e. } x \in D. \tag{17}$$

Also, we can rewrite the solution as

$$u(x) = Proj_{[c,d]}\{-\frac{1}{\alpha}\mathbb{E}[p(x,\xi)]\}, \ \text{a.e. } x \in D, \tag{18}$$

where $Proj_{[c,d]}$ denotes the projection mapping from $\mathbb{R}$ onto $[c,d]$.

3. Stochastic spline-collocation method and Smolyak approximation

To present the discretization of the optimality system (11), a stochastic collocation scheme and the Smolyak approximation scheme will be formulated in this section. The reason for considering a spline-collocation scheme is that in general $a(x,\cdot)$ is not globally smooth, and thus we need to refine some non-smooth areas in the sample space while applying the Smolyak approximation in the smooth areas.

3.1 Semi-discrete approximation scheme. First of all, we consider finite element spaces defined on spatial domain $D \subset \mathbb{R}^d$[32]. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of regular triangulation of D such that $\bar{D} = \cup_{\tau\in\mathcal{T}_h}\bar{\tau}$. Let $h_s = \max_{\tau\in\mathcal{T}_h} h_\tau$, where h_τ denotes the diameter of the element τ and N_h is the number of all the nodes. Consider two finite element spaces $V_{h_s} \subset H_0^1(D)$ and $W_{h_s} \subset L^2(D)$, consisting of piecewise linear continuous functions and piecewise constants on $\{\mathcal{T}_h\}$, respectively. Let $K_h = W_{h_s} \cap K$, then the semi-discrete scheme of optimal control problem (7)-(8) can be formulated as follows:

$$\min_{u_h\in K_h} \mathcal{J}_h(u_h) = \min_{u_h\in K_h} \mathbb{E}\big[\frac{1}{2}\int_D |y_h - y_d|^2 dx + \frac{\alpha}{2}\int_D |u_h|^2 dx\big] \tag{19}$$

subject to

$$A[y_h, v_h] = [u_h, v_h], \ \ \forall \ v_h \in V_{h_s} \times L^2_\rho(\Gamma). \tag{20}$$

Similarly, following from [1] that the optimal control problem (19)-(20) has a unique solution $(y_h, u_h) \in (V_{h_s} \times L^2_\rho(\Gamma)) \times K_h$. Furthermore, a pair (y_h, u_h) is the solution of (19)-(20) iff there is a co-state variable $p_h \in V_{h_s} \times L^2_\rho(\Gamma)$, such that the triplet (y_h, p_h, u_h) satisfies the following optimality system:

$$\begin{cases} A[y_h, v_h] = [u_h, v_h], \ \ \forall \ v_h \in V_{h_s} \times L^2_\rho(\Gamma), \\ A[p_h, \hat{v}_h] = [y_h - y_d, \hat{v}_h], \ \ \forall \ \hat{v}_h \in V_{h_s} \times L^2_\rho(\Gamma), \\ [p_h + \alpha u_h, w_h - u_h] \ge 0, \ \ \forall \ w_h \in K_h. \end{cases} \tag{21}$$

It is known that the inequality in (21) is just the necessary and sufficient optimality condition. According to the result of finite element approximation[35], we have

$$
\begin{aligned}
&\|y-y_h\|_{L^2_\rho(\Gamma;H^1_0(D))}+\|p-p_h\|_{L^2_\rho(\Gamma;H^1_0(D))}+\|u-u_h\|_{L^2(D)}\\
\leq{}& \frac{1}{\sqrt{a_{min}}}\inf_{v\in V_{h_s}\times L^2_\rho(\Gamma)}\Big(\int_{\Gamma\times D}\rho a|\nabla(y-v)|^2\Big)^{\frac{1}{2}}\\
&+\frac{1}{\sqrt{a_{min}}}\inf_{v\in V_{h_s}\times L^2_\rho(\Gamma)}\Big(\int_{\Gamma\times D}\rho a|\nabla(p-v)|^2\Big)^{\frac{1}{2}}+C\|u-u_h\|_{L^2(D)}\\
\leq{}& Ch_s\big\{\|y\|_{L^2_\rho(\Gamma;H^2(D)\cap H^1_0(D))}+\|p\|_{L^2_\rho(\Gamma;H^2(D)\cap H^1_0(D))}+\|u\|_{H^1(D)}\big\}.
\end{aligned}
\tag{22}
$$

3.2 Spline-collocation approximate scheme and convergence analysis. In this section, we investigate the piecewise Lagrange interpolation approximation in consideration of the oscillatory nature of high-degree interpolation polynomials. For $\Gamma_i=[a_i,b_i], i=1,2,\cdots,N$, let $a_i=\xi_{i,0}<\xi_{i,1}<\xi_{i,2}<\cdots<\xi_{i,j_i}=b_i$, be a partition of interval $[a_i,b_i]$, let $h_{i,j}=\xi_{i,j}-\xi_{i,j-1}, 1\leq j\leq j_i, h_i=\max_{1\leq j\leq j_i}h_{i,j}$, for convenience, suppose $h_{i,j}<1$. Let $I_{i,j}=[\xi_{i,j-1},\xi_{i,j}], 1\leq j\leq j_i$, then $\Gamma_i=\bigcup_{j=1}^{j_i}I_{i,j}$. Thus, if $f\in C^\nu(\Gamma_i;H^1_0(D))$ and ν is big enough to keep the regularity of f with respect to ξ_i, in every subinterval $I_{i,j}$, we approximate f by using the $k_{i,j}-$order Lagrange polynomials on $k_{i,j}+1$ nodes $\xi_{i,j-1}=\xi^0_{i,j}<\xi^1_{i,j}<\xi^2_{i,j}<\cdots<\xi^{k_{i,j}}_{i,j}=\xi_{i,j}$, as follows:

$$
\mathcal{I}^i(f)(\xi)=\sum_{s=0}^{k_{i,j}}f(\xi^s_{i,j})l^s_{i,j}(\xi),\qquad \xi\in I_{i,j}.
$$

Where

$$
l^s_{i,j}(\xi)=\prod_{r=0,r\neq s}^{k_{i,j}}\frac{\xi-\xi^r_{i,j}}{\xi^s_{i,j}-\xi^r_{i,j}}.
$$

Let $M_{i,j}=\max_{\xi\in I_{i,j}}f^{(k_{i,j}+1)}(\xi), M_i=\max_{1\leq j\leq j_i}M_{i,j}$ and $k_i=\min_{1\leq j\leq j_i}k_{i,j}$, then, by interpolation error formula, we get

$$
\Big|f(\xi)-\mathcal{I}^i(f)(\xi)\Big|\leq\frac{M_i}{(k_i+1)!}(h_i)^{k_i+1},
$$

Now, in the multivariate case $N>1$, for $f\in C^\nu(\Gamma;H^1_0(D))$, we define the full tensor product interpolation formulas

$$
\begin{aligned}
\mathcal{I}(f)(\xi_1,\xi_2,...,\xi_N)&=(\mathcal{I}^1\otimes\mathcal{I}^2\otimes\cdots\otimes\mathcal{I}^N)(f)(\xi_1,\xi_2,...,\xi_N)\\
&=\sum_{s_1=0}^{k_{1,r_1}}\cdots\sum_{s_N=0}^{k_{N,r_N}}f(\xi^{s_1}_{1,r_1},\xi^{s_2}_{2,r_2},...,\xi^{s_N}_{N,r_N})\prod_{i=1}^{N}l^{s_i}_{i,r_i}(\xi_i),\\
&\qquad(\xi_1,\xi_2,...,\xi_N)\in I_{1,r_1}\times I_{2,r_2}\times\cdots\times I_{N,r_N}.
\end{aligned}
$$

We then have the interpolation error estimate

$$
\Big|f(\xi_1,\xi_2,...,\xi_N)-\mathcal{I}(f)(\xi_1,\xi_2,...,\xi_N)\Big|\leq C\sum_{i=1}^{N}\frac{M_i}{(k_i+1)!}(h_i)^{k_i+1},
$$

where C is a constant independent of k_i,h_i.

Next, we consider a finite dimensional space defined on $\Gamma\subset\mathbb{R}^N$. Let $\mathcal{P}(\Gamma_i)$ be the set of all the functions which is linear spanning in the subinterval $I_{i,j}$ of $l^s_{i,j}, 0\leq s\leq k_{i,j}, 1\leq j\leq j_i$, and let $\mathcal{P}(\Gamma)=\bigotimes_{i=1}^N\mathcal{P}(\Gamma_i)$. Combining spaces V_{h_s}, W_{h_s} and $\mathcal{P}(\Gamma)$ together, we define tensor product finite element space on $D\times\Gamma$. We will use $Y_{h,q}=V_{h_s}\bigotimes\mathcal{P}(\Gamma)$ for the finite element space of the state variable y

and co-state variable p, $U_h = W_{h_s}$ for the control variable u and let $K_h = U_h \cap K$ be the finite element space of the convex set K. Then, the optimal control problem (7)-(8) can be formulated as follows:

$$\min_{u_h \in K_h} \mathcal{J}_h(u_h) = \min_{u_h \in K_h} \mathbb{E}\Big[\frac{1}{2}\int_D |y_{h,q} - y_d|^2 dx + \frac{\alpha}{2}\int_D |u_h|^2 dx\Big] \tag{23}$$

subject to

$$A[y_{h,q}, v_{h,q}] = [u_h, v_{h,q}], \quad \forall\, v_{h,q} \in Y_{h,q}. \tag{24}$$

In the following, for convenience, we set $j_i = 1, k_{i,j} = k, i = 1, 2, \cdots, N$, to any vector of indexes $[s_1, \cdots, s_N]$ we associate the global index

$$n = s_1 + (k+1)(s_2 - 1) + (k+1)^2(s_3 - 1) + \cdots + (k+1)^{N-1}(s_N - 1),$$

here we also denote the point $[\xi_{1,1}^{s_1}, \xi_{2,1}^{s_2}, \cdots, \xi_{N,1}^{s_N}] \in \Gamma$ by ξ_n, and set $l_n(\xi) = \prod_{i=1}^N l_{i,1}^{s_i}$, let $N_q = (k+1)^N$.

Let $\hat{\rho} : \Gamma \to \mathbb{R}^+$ be an auxiliary probability density function which can be seen as the joint probability of N independent random variables; i.e., it factorizes as

$$\hat{\rho}(\xi) = \prod_{n=1}^{N} \hat{\rho}_n(\xi_n), \;\; \forall\, \xi \in \Gamma, \qquad \text{and such that} \quad \|\frac{\rho}{\hat{\rho}}\|_{L^\infty(\Gamma)} \le +\infty. \tag{25}$$

For any continuous function $g : \Gamma \to \mathbb{R}$, we introduce the quadrature formula $E_{\hat{\rho}}^q[g]$ approximating the integral $\int_\Gamma g(\xi)\hat{\rho}(\xi)d\xi$ as follows:

$$E_{\hat{\rho}}^q[g] = \sum_{k=1}^{N_q} \omega_k g(\xi_k), \qquad \omega_k = \prod_{n=1}^{N} \omega_{k_n}, \qquad \omega_{k_n} = \int_{\Gamma_n} l_{k_n}^2(\xi)\hat{\rho}_n(\xi)\, d\xi. \tag{26}$$

For simplicity, we denote $\widetilde{\mathbb{E}}[g] = E_{\hat{\rho}}^q[\frac{\rho}{\hat{\rho}} g]$ for a given continuous function $g(\xi), \;\; \forall\, \xi \in \Gamma$.

Replacing the integrals over Γ in (24) by the quadrature formula (26), the collocation method for the optimal control problem (23)-(24) is:

$$\min_{u_h \in K_h} \mathcal{J}_h(u_h) = \min_{u_h \in K_h} \widetilde{\mathbb{E}}\Big[\frac{1}{2}\int_D |y_{h,q} - y_d|^2 dx + \frac{\alpha}{2}\int_D |u_h|^2 dx\Big] \tag{27}$$

subject to

$$\widetilde{\mathbb{E}}\big[(a\nabla y_{h,q}, \nabla v_{h,q})_{L^2(D)}\big] = \widetilde{\mathbb{E}}\big[(u_h, v_{h,q})_{L^2(D)}\big], \;\; \forall\, v_{h,q} \in Y_{h,q}. \tag{28}$$

Then, the discrete solution $y_{h,q}$ in $D \times I_{1,r_1} \times I_{2,r_2} \times \cdots \times I_{N,r_N}$ has the form

$$y_{h,q} = \sum_{j=1}^{N_h} \sum_{s_1=0}^{k_{1,r_1}} \cdots \sum_{s_N=0}^{k_{N,r_N}} y_{j,r_1,r_2,\ldots,r_N}^{s_1,s_2,\ldots,s_N} \phi_j(x) \prod_{i=1}^{N} l_{i,r_i}^{s_i}(\xi_i).$$

Letting $y_{h,q} = \sum_{i=1}^{N_h} \sum_{n=1}^{N_q} y_{in}\phi_i(x) l_n(\xi) \in Y_{h,q}$, $u_h \in K_h$, we choose $v_{h,q}(x, \xi) = \phi_j(x) l_m(\xi) (j = 1, 2, ..., N_h; m = 1, 2, ..., N_q)$ as the test functions in (28), where $\phi_i(x), \phi_j(x) \in V_{h_s}$ and $l_n(\xi), l_m(\xi)$ is the Lagrange basis function. Then, from (28) we have

$$\begin{aligned} &\widetilde{\mathbb{E}}\big[(a(x,\xi)\nabla\Big(\sum_{i=1}^{N_h}\sum_{n=1}^{N_q} y_{in}\phi_i(x)l_n(\xi)\Big), \nabla(\phi_j(x)l_m(\xi)))_{L^2(D)}\big] \\ =&\widetilde{\mathbb{E}}\big[(u_h, \phi_j(x)l_m(\xi))_{L^2(D)}\big]. \end{aligned} \tag{29}$$

Here, the left of (29) is

$$\begin{aligned}&\int_D \sum_{s=1}^{N_q} a(x,\xi_s)\nabla\Big(\sum_{i=1}^{N_h}\sum_{n=1}^{N_q} y_{in}\phi_i(x)l_n(\xi_s)\Big)\nabla\phi_j(x)l_m(\xi_s)\rho(\xi_s)\omega_s dx\\ =&\int_D a(x,\xi_m)\nabla\Big(\sum_{i=1}^{N_h} y_{im}\phi_i(x)\Big)\nabla\phi_j(x)\rho(\xi_m)\omega_m dx.\end{aligned}$$

The right of (29) is

$$\int_D \sum_{s=1}^{N_q} u_h\phi_j(x)l_m(\xi_s)\rho(\xi_s)\omega_s dx = \int_D u_h\phi_j(x)\rho(\xi_m)\omega_m dx.$$

From the above two equations, (29) is equivalent to

$$\int_D a(x,\xi_m)\nabla\Big(\sum_{i=1}^{N_h} y_{im}\phi_i(x)\Big)\nabla\phi_j(x) = \int_D u_h\phi_j(x)dx. \tag{30}$$

This leads to solve a sequence of uncoupled problems of the form

$$\int_D a(x,\xi)\nabla y_h(\xi)\cdot\nabla v_h dx = \int_D u_h v_h dx, \quad \forall\ v_h\in V_{h_s}, \tag{31}$$

collocated in the points ξ_m.

Similarly, if we introduce the Lagrange interpolant operator $\mathcal{I}_q:\ C^0(\Gamma;\ H_0^1(D)) \to \mathcal{P}(\Gamma)\bigotimes H_0^1(D)$, such that

$$\mathcal{I}_q v(\xi) = \sum_{s=1}^{N_q} v(\xi_s)l_s(\xi), \qquad \forall\ v\in C^0(\Gamma;\ H_0^1(D)),$$

then we have simply $y_{h,q} = \mathcal{I}_q y_h$.

According to Lions' lemma[1], we have the following theorem similar to theorem 2.1.

Theorem 3.1. The control problem (27)-(28) has a unique pair solution $(y_{h,q}, u_h) \in Y_{h,q}\times K_h$. Furthermore, a pair $(y_{h,q}, u_h)$ is the solution if and only if there is a co-state variable $p_{h,q}\in Y_{h,q}$, such that $\{y_{h,q},\ p_{h,q},\ u_h\}\in Y_{h,q}\times Y_{h,q}\times K_h$ satisfies the following system

$$\left\{\begin{aligned}&\widetilde{\mathbb{E}}\big[(a\nabla y_{h,q},\nabla v_{h,q})_{L^2(D)}\big] = \widetilde{\mathbb{E}}\big[(u_h, v_{h,q})_{L^2(D)}\big],\ \ \forall\ v_{h,q}\in Y_{h,q},\\ &\widetilde{\mathbb{E}}\big[(a\nabla p_{h,q},\nabla \hat v_{h,q})_{L^2(D)}\big] = \widetilde{\mathbb{E}}\big[(y_{h,q}-y_d,\ \hat v_{h,q})_{L^2(D)}\big],\ \ \forall\ \hat v_{h,q}\in Y_{h,q},\\ &\widetilde{\mathbb{E}}\big[(p_{h,q}+\alpha u_h, w_h-u_h)_{L^2(D)}\big]\geq 0,\ \ \forall\ w_h\in K_h\subset U_h.\end{aligned}\right. \tag{32}$$

Proof. Let $\mathcal{J}_h(u_h) = g(y_{h,q}(u_h)) + j(u_h)$, where

$$g(y_{h,q}(u_h)) = \widetilde{\mathbb{E}}\big[\frac{1}{2}\int_D |y_{h,q}(u_h)-y_d|^2dx\big], \quad \text{and}\quad j(u_h) = \widetilde{\mathbb{E}}\big[\frac{\alpha}{2}\int_D |u_h|^2dx\big].$$

Applying the Lions' lemma, the optimal condition reads

$$j'(u_h)(w_h-u_h) + (g(y_{h,q}(u_h)))'(w_h-u_h)\geq 0,\ \ \forall\ w_h\in K_h. \tag{33}$$

It is clear that

$$\begin{aligned} j'(u_h)(w_h-u_h) &= \lim_{t\to 0^+}\frac{1}{t}\widetilde{\mathbb{E}}\big[\frac{\alpha}{2}\int_D \big[|u_h+t(w_h-u_h)|^2-|u_h|^2\big]dx\big]\\ &= \widetilde{\mathbb{E}}\big[\int_D \alpha u_h(w_h-u_h)dx\big] = \widetilde{\mathbb{E}}\big[\big(\alpha u_h, w_h-u_h\big)_{L^2(D)}\big]\end{aligned} \tag{34}$$

and
(35)

$$\begin{aligned}&\big(g(y_{h,q}(u_h))\big)'(w_h-u_h)=\lim_{t\to 0^+}\frac{1}{t}\big(g\big(y_{h,q}(u_h+t(w_h-u_h))\big)-g(y_{h,q}(u_h))\big)\\ =&\lim_{t\to 0^+}\frac{1}{2t}\widetilde{\mathbb{E}}\Big[\int_D\big[|y_{h,q}(u_h+t(w_h-u_h))-y_{h,q}(u_h)|^2\\ &\qquad+2(y_{h,q}(u_h+t(w_h-u_h))-y_{h,q}(u_h),y_{h,q}-y_d)\big]dx\Big]\\ =&\widetilde{\mathbb{E}}\Big[\int_D y'_{h,q}(u_h)(w_h-u_h)\cdot(y_{h,q}-y_d)dx\Big]\\ =&\widetilde{\mathbb{E}}\big[\big(y'_{h,q}(u_h)(w_h-u_h),y_{h,q}-y_d\big)_{L^2(D)}\big].\end{aligned}$$

Next, let us differentiate the state equation (28) at u_h in the direction w_h-u_h. By (28), we have

$$\begin{aligned}&\frac{1}{t}\widetilde{\mathbb{E}}\Big[\big(a(\nabla y_{h,q}(u_h+t(w_h-u_h))-\nabla y_{h,q}(u_h)),\nabla v_{h,q}\big)_{L^2(D)}\Big]\\ =&\widetilde{\mathbb{E}}\big[(w_h-u_h,v_{h,q})_{L^2(D)}\big],\quad\forall\, v_{h,q}\in Y_{h,q}.\end{aligned}\tag{36}$$

Taking limit in (36) as $t\to 0$, we obtain

$$\begin{aligned}&\widetilde{\mathbb{E}}\big[\big(a\nabla y'_{h,q}(u_h)(w_h-u_h),\nabla v_{h,q}\big)_{L^2(D)}\big]\\ =&\widetilde{\mathbb{E}}\big[(w_h-u_h,v_{h,q})_{L^2(D)}\big],\quad\forall\, w_h\in K_h,\quad v_{h,q}\in Y_{h,q}.\end{aligned}\tag{37}$$

Define the co-state $p_{h,q}\in Y_{h,q}$ satisfying

$$\widetilde{\mathbb{E}}\big[(a\nabla p_{h,q},\nabla\hat{v}_{h,q})_{L^2(D)}\big]=\widetilde{\mathbb{E}}\big[(y_{h,q}-y_d,\,\hat{v}_{h,q})_{L^2(D)}\big],\quad\forall\,\hat{v}_{h,q}\in Y_{h,q}.\tag{38}$$

Letting $v_{h,q}=p_{h,q}$ in (37) and $\hat{v}_{h,q}=y'_{h,q}(u_h)(w_h-u_h)$, we have

$$\begin{aligned}\widetilde{\mathbb{E}}\big[(w_h-u_h,p_{h,q})_{L^2(D)}\big]&=\widetilde{\mathbb{E}}\big[\big(a\nabla y'_{h,q}(u_h)(w_h-u_h),\nabla p_{h,q}\big)_{L^2(D)}\big]\\ &=\widetilde{\mathbb{E}}\big[\big(y'_{h,q}(u_h)(w_h-u_h),y_{h,q}-y_d\big)_{L^2(D)}\big]\\ &=\big(g(y_{h,q}(u_h))\big)'(w_h-u_h).\end{aligned}\tag{39}$$

By (33)-(34) and (39), the optimality condition reads

$$\mathcal{J}'_h(u_h)(w_h-u_h)=\widetilde{\mathbb{E}}\big[(p_{h,q}+\alpha u_h,w_h-u_h)_{L^2(D)}\big]\geq 0,\qquad\forall\, w_h\in K_h,\tag{40}$$

where $p_{h,q}$ is defined in (38).This completes the proof. □

Theorem 3.2. Let (y,p,u) be the solution of the optimal control problem (11) and $(y_{h,q},p_{h,q},u_h)$ be the solution of the discretized problem (32), respectively. Then the following error estimate holds:

$$\begin{aligned}&\|y-y_{h,q}\|_{L^2_\rho(\Gamma;H^1(D))}+\|p-p_{h,q}\|_{L^2_\rho(\Gamma;H^1(D))}+\|u-u_h\|_{L^2(D)}\\ \leq& Ch_s\big\{\|y\|_{L^2_\rho(\Gamma;H^2(D)\cap H^1_0(D))}+\|p\|_{L^2_\rho(\Gamma;H^2(D)\cap H^1_0(D))}+\|u\|_{H^1(D)}\big\}\\ &+\sum_{i=1}^{N}\frac{Q_i+R_i}{(k+1)!}(h_i)^{k+1}.\end{aligned}\tag{41}$$

where Q_i, and R_i are derived similar to M_i, by using y_h,p_h to substitute f.

Proof. According to the deduction of (3.11)-(3.13), we have $y_{h,q}=\mathcal{I}_p y_h$, $p_{h,q}=\mathcal{I}_p p_h$. By interpolation error formula, and notice that y_h and p_h have the same regularity as the exact solution y and p with respect to ξ_i, we have

(42)
$$\|y_h - y_{h,q}\|_{L^2_\rho(\Gamma;H^1(D))} + \|p_h - p_{h,q}\|_{L^2_\rho(\Gamma;H^1(D))} \leq C\sum_{i=1}^{N}\frac{Q_i+R_i}{(k_i+1)!}(h_i)^{k+1},$$
then combining (22) and (42), we can get theorem 3.2. This completes the proof. □

A disadvantage of piecewise Lagrange interpolation approximation is that there have no differentiability at the endpoints of the subintervals, which means that the interpolating function is not smooth. In this case, often Q_i, and R_i will in fact depend on the size k_i. In fact they often become explosive when k_i are getting smaller, and this makes the error estimates useless. An alternative procedure is to use spline interpolation approximation, in particular, when the function f has a certain degree of smoothness, we will obtain a better approximation effect by using spline function approximation. The most common spline interpolation is called cubic spline interpolation.

If function S satisfying: (1) $S \in C^{k-1}[a_i, b_i]$; (2) S is polynomial which degree is no more than k on subinterval $I_{i,j} = [\xi_{i,j-1}, \xi_{i,j}], 1 \leq j \leq j_i$, then, S is called a $k-$degree spline function with $\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,j_i}$ as the node. Given a function f defined on $[a_i, b_i]$, if S satisfying:
$$S(\xi_{i,j}) = f(\xi_{i,j}), \quad j = 1, 2, \cdots, j_i,$$
then S is called $k-$degree spline function of f.

If $f \in C^{k+1}[a_i, b_i]$, and S is $k-$degree spline function of f, then, we have the following interpolation error estimator:

(43)
$$\| f^{(r)} - S^{(r)} \|_\infty \leq C \| f^{(k+1)} \|_\infty h_i^{k+1-r}, \qquad r = 0, 1, \cdots k,$$
where C is a constant independent of k, h_i.

If $a(x, \xi_1, \xi_2, \cdots, \xi_N)$ is $k_i + 1-$ order continuously differentiable about $\xi_i, i = 1, 2, \cdots, N$, then, we can approximate y and p by using k_i- degree spline function on i-th direction of probability space Γ. The approximation solution obtained by spline function have better smoothness than that of the approximate solution obtained by piecewise Lagrange interpolation approximation, and we have the following error estimate:

(44)
$$\begin{aligned}&\|y - y_{h,q}\|_{L^2_\rho(\Gamma;H^1(D))} + \|p - p_{h,q}\|_{L^2_\rho(\Gamma;H^1(D))} + \|u - u_h\|_{L^2(D)}\\ &\leq Ch_s\big\{\|y\|_{L^2_\rho(\Gamma;H^2(D)\cap H^1_0(D))} + \|p\|_{L^2_\rho(\Gamma;H^2(D)\cap H^1_0(D))} + \|u\|_{H^1(D)}\big\}.\\ &\quad +C\sum_{i=1}^{N} h_i^{k_i+1}\Big(\| y_{\xi_i}^{(k_i+1)} \|_\infty + \| p_{\xi_i}^{(k_i+1)} \|_\infty \Big).\end{aligned}$$

3.3 Smolyak approximation. In this subsection, we consider the case where f is analytic. We will develop the convergence properties of the collocation techniques by Smolyak approximation which depends on the regularity of the solution $y(x, \xi)$ with respect to ξ [12]. Denote $\Gamma_n^* = \prod_{j=1,j\neq n}^{N}\Gamma_j$, and let ξ_n^* be an arbitrary element of Γ_n^*. Here we require the solution of (2.12) to satisfy the following assumption.

Regularity assumption 3.3. ([12]) For each $\xi_n \in \Gamma_n$, there exists $\tau_n > 0$ such that the function $y(x, \xi_n, \xi_n^*)$ as a function of $\xi_n, y : \Gamma_n \longrightarrow C^0(\Gamma_n^*; H_0^1(D))$ admits an analytic extension $y(x, z, y_n^*), z \in \mathbb{C}$, in the region of the complex plane
$$\sum(\Gamma_n; \tau_n) \equiv \{z \in \mathbb{C}, \mathbf{dist}(z, \Gamma_n) \leq \tau_n\}.$$

Moreover, $\forall z \in \sum(\Gamma_n;\tau_n)$,

$$\|y(z)\|_{C^0(\Gamma_n^*;H_0^1(D))} \leq \lambda,$$

with λ a constant independent of n.

For convenience, we briefly redefine the interpolation operator based on Lagrange polynomials. We first introduce an index $i \in \mathbb{N}$, $i \geq 1$. Then, for each value of i, let $\{\xi_1^i, ..., \xi_{m_i}^i\} \subset \Gamma_n, n = 1, 2, ...N$ be a sequence of abscissas for Lagrange interpolation on Γ_n.

For $v \in C^0(\Gamma_n; H_0^1(D))$, we introduce a sequence of one-dimensional Lagrange interpolation operators $\mathcal{I}_{m_i}^i : C^0(\Gamma_n; H_0^1(D)) \to V_{m_i}(\Gamma_n; H_0^1(D))$,

$$\mathcal{I}_{m_i}^i(v)(\xi) = \sum_{j=1}^{m_i} v(\xi_j^i) l_j^i(\xi), \forall v \in C^0(\Gamma_n; H_0^1(D)), \tag{45}$$

where $l_j^i \in \mathcal{P}_{m_i-1}(\Gamma_n)$ are the Lagrange polynomials of degree $m_i - 1$, i.e., $l_j^i(\xi) = \Pi_{k=1,k\neq j}^{m_i} \frac{\xi-\xi_k^i}{\xi_j^i-\xi_k^i}$, and

$$\begin{aligned} &V_{m_i}(\Gamma_n; H_0^1(D)) \\ =&\Big\{v \in C^0(\Gamma_n; H_0^1(D)) : v(x,\xi) = \sum_{k=1}^{m_i} \widetilde{v}_k(x) l_k(\xi), \{\widetilde{v}_k\}_{k=1}^{m_i} \in H_0^1(D)\Big\}. \end{aligned}$$

Now, in the multivariate case, for each $v \in C^0(\Gamma; H_0^1(D))$ and the multi-index $\mathbf{i} = (i_1, ..., i_N), \mathbf{m} = (m_{i_1}, ..., m_{i_N}) \in \mathbb{N}_+^N$ we define the full tensor product interpolation formulas

$$\mathcal{I}_{\mathbf{m}}^{\mathbf{i}} v(\xi) = (\mathcal{I}_{m_{i_1}}^{i_1} \otimes \cdots \otimes \mathcal{I}_{m_{i_N}}^{i_N})(v)(\xi) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_N=1}^{m_{i_N}} v(\xi_{j_1}^{i_1}, \cdots, \xi_{j_N}^{i_N})(l_{j_1}^{i_1} \otimes \cdots \otimes l_{j_N}^{i_N}). \tag{46}$$

Here we follow closely the work [40] to describe the Smolyak isotropic formulas $\mathcal{A}(\mathrm{w}, N)$. The Smolyak formulas are just linear combinations of product formulas (2.5) with the following key properties: only products with a relatively small number of points are used. Let $\mathcal{I}_{m_0}^0 = 0$, and for $i \in \mathbb{N}_+$ define

$$\Delta^i := \mathcal{I}_{m_i}^i - \mathcal{I}_{m_{i-1}}^{i-1}. \tag{47}$$

Moreover, given an integer $\mathrm{w} \in \mathbb{N}_+$, hereafter called the level, we define the sets

$$X(\mathrm{w}, N) := \Big\{\mathbf{i} \in \mathbb{N}_+^N, \mathbf{i} \geq \mathbf{1} : \sum_{n=1}^{N}(i_n - 1) \leq \mathrm{w}\Big\}, \tag{48}$$

$$\widetilde{X}(\mathrm{w}, N) := \Big\{\mathbf{i} \in \mathbb{N}_+^N, \mathbf{i} \geq \mathbf{1} : \sum_{n=1}^{N}(i_n - 1) = \mathrm{w}\Big\}, \tag{49}$$

$$Y(\mathrm{w}, N) := \Big\{\mathbf{i} \in \mathbb{N}_+^N, \mathbf{i} \geq \mathbf{1} : \mathrm{w} - N + 1 \leq \sum_{n=1}^{N}(i_n - 1) \leq \mathrm{w}\Big\}, \tag{50}$$

and for $\mathbf{i} \in \mathbb{N}_+$ we set $|\mathbf{i}| = i_1 + \cdots + i_N$. Then the isotropic Smolyak formula is given by

$$\mathcal{A}(\mathrm{w}, N) = \sum_{\mathbf{i}\in X(\mathrm{w},N)} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_N}). \tag{51}$$

Equivalently, the above equation can be written as [41]

$$\mathcal{A}(\mathrm{w},N)=\sum_{\mathbf{i}\in Y(\mathrm{w},N)}(-1)^{\mathrm{w}+N-|\mathbf{i}|}\begin{pmatrix}N-1\\ \mathrm{w}+N-|\mathbf{i}|\end{pmatrix}\cdot\mathcal{I}^{i_1}_{m_{i_1}}\otimes\cdots\otimes\mathcal{I}^{i_N}_{m_{i_N}}. \tag{52}$$

To compute $\mathcal{A}(\mathrm{w},N)(v)$, one only needs to know function values on the "sparse grid"

$$\mathcal{H}(\mathrm{w},N)=\bigcup_{\mathbf{i}\in Y(\mathrm{w},N)}(\vartheta^{i_1}\times\cdots\times\vartheta^{i_N})\subset\Gamma, \tag{53}$$

where $\vartheta^i=\{\xi^i_1,...,\xi^i_{m_i}\}$ denotes the set of abscissas used by $\mathcal{I}^i_{m_i}$.If the sets are nested, i.e.,$\vartheta^i\subset\vartheta^{i+1}$, then $\mathcal{H}(\mathrm{w},N)\subset\mathcal{H}(\mathrm{w}{+}1,N)$ and

$$\mathcal{H}(\mathrm{w},N)=\bigcup_{\mathbf{i}\in\widetilde{X}(\mathrm{w},N)}(\vartheta^{i_1}\times\cdots\times\vartheta^{i_N}). \tag{54}$$

The Smolyak formula is actually interpolatory whenever nested points are used. This result has been proved in [40].

3.4 Clenshaw-Curtis abscissas. In this subsection, we use Clenshaw-Curtis abscissas[42] for the construction of the Smolyak formula. These abscissas are the extrema of Chebyshev polynomials and, for any choice of $m_i>1$, are given by

$$\xi^i_j=-\cos\Big(\frac{\pi(j-1)}{m_i-1}\Big),\qquad j=1,2,...,m_i.$$

In addition, one sets $\xi^i_1=0$ if $m_i=1$ and lets the number of abscissas m_i in each level grow according to the following formula:

$$m_1=1,\qquad m_i=2^{i-1}+1,\quad i>1. \tag{55}$$

With this particular choice, one obtains nested sets of abscissas, i.e., $\vartheta^i\subset\vartheta^{i+1}$, and thereby $\mathcal{H}(\mathrm{w},N)\subset\mathcal{H}(\mathrm{w}{+}1,N)$. It is important to choose $m_1=1$ if we are interested in optimal approximation in relatively large N, because in all other cases the number of points used by $\mathcal{A}(\mathrm{w},N)$ increases too fast with N.

3.5 Analysis of the approximation error. In this subsection we present the error estimates for the isotropic Smolyak approximation based on Clenshaw-Curtis abscissas.

Let $\widehat{\sigma}_n=\log\Big(\frac{2\tau_n}{|\Gamma_n|}+\sqrt{1+\frac{4\tau_n^2}{|\Gamma_n|^2}}\Big)$, $\sigma=\frac{1}{2}\min_{1\le n\le N}\min_{\xi_n^*\in\Gamma_n^*}\widehat{\sigma}_n, C=\frac{4}{e^{2\sigma}-1}$, $\mu_1=\frac{\sigma}{1+\log(2N)},\delta^*=(e\log(2)-1)/\widetilde{C}_2(\sigma)$ and $\eta=\eta(\mathrm{w},N)=\#\mathcal{H}(\mathrm{w},N)$ be the total number of the collocation points used in the Smolyak formula (described by (3.33)) with Clenshaw-Curtis abscissas, which is

$$\eta=\sum_{\mathbf{i}\in X(\mathrm{w},N)}\prod_{n=1}^{N}r(i_n),\qquad r(i):=\begin{cases}1, & i=1,\\ 2, & i=2,\\ 2^{i-2}, & i>2.\end{cases}$$

$$\widetilde{C}_2(\sigma):=1+\frac{1}{\log(2)}\sqrt{\frac{\pi}{2\sigma}},$$

$$C_1(\sigma,\delta):=\frac{4C}{e\delta\sigma}\exp\Big(\delta\sigma\Big\{\frac{1}{\sigma\log^2(2)}+\frac{1}{\log(2)\sqrt{2\sigma}}+2\Big(1+\frac{1}{\log(2)}\sqrt{\frac{\pi}{2\sigma}}\Big)\Big\}\Big).$$

Theorem 3.4. (algebraic convergence)[13]. For functions $y_h, p_h \in C^0(\Gamma; H_0^1(D))$ satisfying the regular assumption 3.3, the isotropic Smolyak formula (3.32) based on Clenshaw-Curtis abscissas satisfies

$$
\begin{aligned}
&\|y_h - \mathcal{A}(\mathrm{w}, N)(y_h)\|_{L^\infty_\rho(\Gamma; H_0^1(D))} + \|p_h - \mathcal{A}(\mathrm{w}, N)(p_h)\|_{L^\infty_\rho(\Gamma; H_0^1(D))} \\
&\le \tfrac{C_1(\sigma,\delta^*)e^\sigma}{|1-C_1(\sigma,\delta^*)|} \max\{1, C_1(\sigma,\delta^*)\}^N \eta^{-\mu_1}.
\end{aligned} \tag{56}
$$

Theorem 3.4 indicates at least algebraic convergence with respect to the number of collocation points η.

Theorem 3.5. (subexponential convergence)[13]. Under the same assumptions of theorem 3.4 and for $\mathrm{w} > \frac{N}{\log(2)}$, we have

$$
\begin{aligned}
&\|y_h - \mathcal{A}(\mathrm{w}, N)(y_h)\|_{L^\infty_\rho(\Gamma; H_0^1(D))} + \|p_h - \mathcal{A}(\mathrm{w}, N)(p_h)\|_{L^\infty_\rho(\Gamma; H_0^1(D))} \\
&\le \tfrac{C_1(\sigma,\delta^*)}{e^{\sigma\delta^*\widetilde{C}_2(\sigma)}} \tfrac{\max\{1, C_1(\sigma,\delta^*)\}^N}{|1-C_1(\sigma,\delta^*)|} \eta^{\mu_3} e^{-\frac{N\sigma}{2^{1/N}}\eta^{\mu_2}},
\end{aligned} \tag{57}
$$

where

$$
\mu_2 = \frac{\log(2)}{N(1+\log(2N))}, \qquad \mu_3 = \frac{\sigma\delta^*\widetilde{C}_2(\sigma)}{1+\log(2N)}.
$$

Theorem 3.6. Under regular assumption 3.3, let (y, p, u) be the solution of the optimal control problem (11). $(y_{h,q}, p_{h,q}, u_h)$ is the solution of the discretized problem (32) by Smolyak approximation based on Clenshaw-Curtis abscissas . Then the following error estimate holds:

$$
\begin{aligned}
&\|y - y_{h,q}\|_{L^2_\rho(\Gamma; H_0^1(D))} + \|p - p_{h,q}\|_{L^2_\rho(\Gamma; H_0^1(D))} + \|u - u_h\|_{L^2(D)} \\
&\le Ch_s\big\{\|y\|_{L^2_\rho(\Gamma; H^2(D)\cap H_0^1(D))} + \|p\|_{L^2_\rho(\Gamma; H^2(D)\cap H_0^1(D))} + \|u\|_{H^1(D)}\big\} \\
&\quad + \frac{C_1(\sigma,\delta^*)e^\sigma}{|1-C_1(\sigma,\delta^*)|} \max\{1, C_1(\sigma,\delta^*)\}^N \eta^{-\mu_1},
\end{aligned}
$$

and if $\mathrm{w} > \frac{N}{\log(2)}$, we have

$$
\begin{aligned}
&\|y - y_{h,q}\|_{L^2_\rho(\Gamma; H_0^1(D))} + \|p - p_{h,q}\|_{L^2_\rho(\Gamma; H_0^1(D))} + \|u - u_h\|_{L^2(D)} \\
&\le Ch_s\big\{\|y\|_{L^2_\rho(\Gamma; H^2(D)\cap H_0^1(D))} + \|p\|_{L^2_\rho(\Gamma; H^2(D)\cap H_0^1(D))} + \|u\|_{H^1(D)}\big\} \\
&\quad + \frac{C_1(\sigma,\delta^*)}{e^{\sigma\delta^*\widetilde{C}_2(\sigma)}} \frac{\max\{1, C_1(\sigma,\delta^*)\}^N}{|1-C_1(\sigma,\delta^*)|} \eta^{\mu_3} e^{-\frac{N\sigma}{2^{1/N}}\eta^{\mu_2}}.
\end{aligned}
$$

4. Domain decomposition and numerical examples

As discussed above in real applications, $a(x,\cdot)$ often has irregular points, and we can make a small sub-domain including these points. Then, we can use lower regularity bases with compact supports to approximate the solution in this sub-domain, while approximating it via the Smolyak schemes as shown in Example 2. We first discuss a smooth case where $a(x,\cdot)$ is analytic.

Example 1 We take space domain $D = [0,1]\times[0,1]$, each stochastic domain Γ_i is $[-1,1]$, $a(x,\xi_1,\xi_2) = 3+\xi_1+\xi_2$, where ξ_1 and ξ_2 are uniform distributions on $[-1,1]$. We consider the following model problem:

$$
\min_u \mathcal{J}(u) = \min_u\Big(\frac{1}{2}\int_{-1}^1\int_{-1}^1\int_D (y-y_d)^2 dx d\xi_1 d\xi_2 + \frac{\alpha}{2}\int_{-1}^1\int_{-1}^1\int_D u^2 dx d\xi_1 d\xi_2\Big) \tag{58}
$$

subject to

$$
\begin{aligned}
-\nabla\cdot(a(x,\xi_1,\xi_2)\nabla y(x,\xi_1,\xi_2)) &= f+u, \quad x\in D, \;\; \xi_1\in[-1,1], \;\; \xi_2\in[-1,1], \\
y(x,\xi_1,\xi_1) &= 0, \quad x\in\partial D, \;\; \xi_1\in[-1,1], \;\; \xi_2\in[-1,1].
\end{aligned}
$$

The target solution $y_d = \sin(2\pi x_1)\sin(2\pi x_2)$. The objective is to minimize the expectation of a cost functional, and the deterministic control is constrained by the condition $u(x) \geq 0, \forall x \in D$. The solutions for this problem are

$$\begin{aligned} p &= \sin(2\pi x_1)\sin(2\pi x_2), \\ y &= (1+8\pi^2(3+\xi_1+\xi_2))\sin(2\pi x_1)\sin(2\pi x_2), \\ u &= \max\{0, -\frac{p}{\alpha}\}, \\ f &= 8\pi^2(3+\xi_1+\xi_2)(1+8\pi^2(3+\xi_1+\xi_2))\sin(2\pi x_1)\sin(2\pi x_2) - u. \end{aligned}$$

In this example, because y, p have enough regularity with respect to ξ_1 and ξ_2, we take Clenshaw-Curtis abscissas in every direction of the random space, and take Lagrange interpolation function as the basis functions. On the space D, for the control u, we use the discontinuous piecewise constant finite element; for the state y and co-state p, we use the piecewise linear finite element. In Table 1, we present the numerical result using Smolyak approximation scheme for the above problem. In Figure 1, we present the numerical control. Table 1 illustrates that the numerical results are consistent with our theoretical results.

TABLE 1. The relative error for $\alpha = 1$ and w=2.

node/side/element	$\frac{\|u_h-u\|_{L^2(D)}}{\|u\|_{L^2(D)}}$	$\frac{\|y_{h,q}-y\|_{L^2_\rho(\Gamma,H^1(D))}}{\|y\|_{L^2_\rho(\Gamma,H^1(D))}}$	$\frac{\|p_{h,q}-p\|_{L^2_\rho(\Gamma,H^1(D))}}{\|p\|_{L^2_\rho(\Gamma,H^1(D))}}$
16/33/18	1.2822	0.8754	0.9079
49/120/72	0.6591	0.4506	0.4822
169/456/288	0.2685	0.2294	0.2353
625/1776/1152	0.1184	0.1155	0.1163
2401/7008/4608	0.0567	0.0578	0.0579

Where, the node, side and element in Table 1 are nodes and sides of triangles (element) generated by triangulation of the space D.

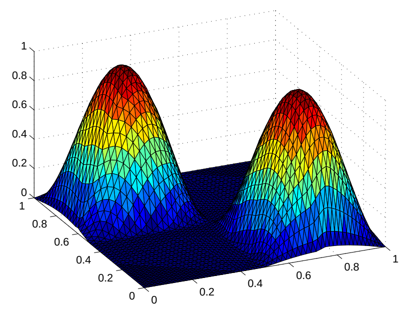


FIGURE 1. the solutions of control u for Example 1.

Example 2 We take space domain $D = [0,1]\times[0,1]$, and each stochastic domain Γ_i is $[-1,1]$,

$$a(x,\xi_1,\xi_2) = \begin{cases} 0.3-|\xi_1|+0.3-|\xi_2|+0.5, & if|\xi_1| < 0.3 \text{ and } |\xi_2| < 0.3 \\ 0.5, & else \end{cases}$$

We consider the following model problem:

$$\min_u \mathcal{J}(u) = \min_u\left(\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\int_D (y-y_d)^2 dx d\xi_1 d\xi_2 + \frac{\alpha}{2}\int_{-1}^{1}\int_{-1}^{1}\int_D u^2 dx d\xi_1 d\xi_2\right) \tag{59}$$

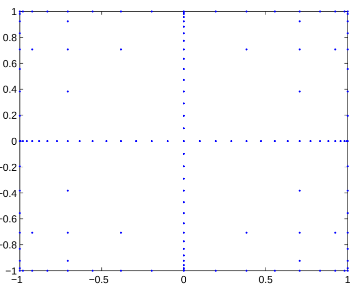
subject to

$$\begin{aligned} -\nabla\cdot(a(x,\xi_1,\xi_2)\nabla y(x,\xi_1,\xi_2)) &= f+u, \quad x\in D, \;\; \xi_1\in[-1,1], \;\; \xi_2\in[-1,1], \\ y(x,\xi_1,\xi_1) &= 0, \quad x\in\partial D, \;\; \xi_1\in[-1,1], \;\; \xi_2\in[-1,1]. \end{aligned}$$

The target solution $y_d = \sin(2\pi x_1)\sin(\pi x_2)$. The objective is to minimize the expectation of a cost functional, and the deterministic control is constrained by the condition $u(x)\geq 0, \forall x\in D$. The solutions for this problem are

$$\begin{aligned} p &= \sin(2\pi x_1)\sin(\pi x_2), \\ y &= (1+5\pi^2 a(x,\xi_1,\xi_2))\sin(2\pi x_1)\sin(\pi x_2), \\ u &= \max\{0, -\frac{p}{\alpha}\}, \\ f &= 5\pi^2 a(x,\xi_1,\xi_2)(1+5\pi^2 a(x,\xi_1,\xi_2))\sin(2\pi x_1)\sin(\pi x_2) - u. \end{aligned}$$

In this example, since y has underivable points with respect to ξ_1 and ξ_2, then, we can make a partition to Γ so that $\Gamma = \Gamma_c\cup\Gamma_s$, where $\Gamma_c = [-0.3, 0.3]\times[-0.3, 0.3]$. In Γ_c, we choose piecewise spline function as bases functions in every direction. Otherwise, we take Clenshaw-Curtis abscissas in every direction of Γ_s, and take Lagrange interpolation function as the basis functions. In our computation, we use 25 spline points inside Γ_c, while use 28 Smolyak approximation collocation for Γ_s. The partition is shown in Figure 2(right). On the space D, for the control u, we use the discontinuous piecewise constant finite element; for the state y and co-state p, we use the piecewise linear finite element. For the comparison, we also give a computation with no partition. We use 145 points with no partition shown in Figure 2(left) and the result by Smolyak approximation is given in Table 2, with the result by spline in $D\times\Gamma_c$ combined with Smolyak approximation in $D\times\Gamma_s$.



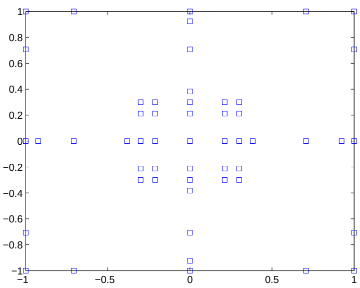


FIGURE 2. Colllocation points with no partition(left) and with partition (right).

TABLE 2. The absolute error for $\alpha = 1$ and nodes = 625, sides = 1776, elements = 1152.

	$\|u_h-u\|_{L^2(D)}$	$\|y_{h,q}-y\|_{L^2_\rho(\Gamma,H^1(D))}$	$\|p_{h,q}-p\|_{L^2_\rho(\Gamma,H^1(D))}$
145 points with no partition	4.7854e-02	1.8002e+01	6.5894e-01
53 points with partition	2.9456e-02	1.4508e+01	4.0560e-01

It is clear that with fewer collocation points, the spline-collocation can give better atropinization. Then, further numerical results using stochastic spline-collocation combined with Smolyak approximation scheme is shown in Table 3 and the numerical control is given in Figure 3. It is clear that Table 3 illustrates that the numerical results are consistent with our theoretical results.

TABLE 3. The relative error for $\alpha = 1$ and 53 points shown in Figure 2 (right).

node/side/element	$\frac{\|u_h-u\|_{L^2(D)}}{\|u\|_{L^2(D)}}$	$\frac{\|y_{h,q}-y\|_{L^2_\rho(\Gamma,H^1(D))}}{\|y\|_{L^2_\rho(\Gamma,H^1(D))}}$	$\frac{\|p_{h,q}-p\|_{L^2_\rho(\Gamma,H^1(D))}}{\|p\|_{L^2_\rho(\Gamma,H^1(D))}}$
16/33/18	0.4783	0.8741	0.4790
49/120/72	0.2175	0.4471	0.2378
169/456/288	0.0906	0.2348	0.1155
625/1776/1152	0.0417	0.1345	0.0572

Where, the node, side and element in Table 3 are nodes and sides of triangles (element) generated by triangulation of the space D.

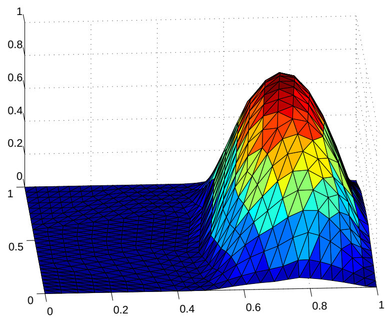


FIGURE 3. the solutions of control u for Example 2.

5. Conclusions

In this paper, we establish stochastic spline-collocation approximation scheme, combined with Smolyak approximation scheme according to the regularity of state variable and costate variable with respect to random variable for an optimal control problem governed by an elliptic PDE with random field coefficients. Although the Smolyak approximation schemes are effective collocation strategies for problems when the solutions are smooth enough with respect to the random variables, in the actual calculation, the state or the costate often has a discontinuity point or non-smooth points. In such cases, we can make a small sub-domain including these point, and choose piecewise low order interpolation functions as bases functions by the stochastic collocation approximation. In the rest domain, since the state and costate are smooth enough with respect to random variable, we can take Clenshaw-Curtis abscissas in every direction of the random space and take Lagrange interpolation function as the basis functions and Smolyak approximation scheme. By numerical tests, we conclude that such a domain decomposition strategy works well for the general non-smooth case.

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References

[1] J.L. Lions, Optimal control of systems governed by partial differential equations, Springer-Verlag, Berlin, 1971.
[2] R. Glowinski, J.L. Lions, Exact and approximate controllability for distributed parameter systems, Cambridge University Press, 1996.
[3] F. Tröltzsch, Optimal control of partial differential equations: theory, methods, and applications, volume 112, American Mathematical Society, 2010.
[4] M. Papadrakakis, V. Papadopoulos, Robust and efficient methods for stochastic finite element analysis using Monte Carlo simulation, Comput. Methods Appl. Mech. Engrg. 134(1996) 325-340.
[5] I. Doltsinis, Inelastic deformation processes with random parametersmethods of analysis and design, Comput. Methods Appl. Mech. Engrg. 192(2003) 2405-2423.
[6] I. Babuska, P. Chatzipantelidis, On solving elliptic stochastic partial differential equations, Comput. Methods Appl. Mech. Engrg. 191(2002) 4093-4122.
[7] I. Babuska, K. Liu, R. Tempone, Solving stochastic partial differential equations based on the experimental data, Math. Models Methods Appl. Sci. 13(3)(2003) 415-444.
[8] M.K. Deb, I. Babuska, J.T. Oden, Solution of stochastic partial differential equations using Galerkin finite element techniques, Comput. Methods Appl. Mech. Engrg. 190(2001) 6359-6372.
[9] M. Kleiber, T.D. Hien, The stochastic finite element method, John Wiley, Chichester, 1992.
[10] I. Babuska, F. Nobile, R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data. SIAM J. Numer. Anal. 45(3)(2007) 1005-1034.
[11] I. Babuska, F. Nobile, R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data. SIAM Rev. 52(2)(2010) 317-355.
[12] F. Nobile, R. Tempone, C.G. Webster, An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. SIAM J. Numer. Anal. 46(5)(2008) 2411-2442.
[13] F. Nobile, R. Tempone, C.G. Webster, A sparse grid stochastic collocationmethod for partial differential equations with random input data. SIAM J. Numer. Anal. 46(5)(2008) 2309-2345.
[14] J. Beck, F. Nobile, L. Tamellini, R. Tempone, Stochastic spectral Galerkin and collocation methods for PDEs with random coefficients: A numerical comparison. In: Spectral and High Order Methods for Partial Differential Equations, volume 76, Springer, Berlin, 2011, 43-62.
[15] R. Ghanem, P.D. Spanos, Stochastic finite elements: a spectral approach, Springer-Verlag, Berlin, 1991.
[16] R. Ghanem, Ingredients for a general purpose stochastic finite elements implementation, Comput. Methods Appl. Mech. Engrg. 168(1999) 19-34.
[17] I. Babuska, R. Tempone, G. E. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations, SIAM J. Numer. Anal. 42(2)(2004) 800-825.
[18] L. J. Roman, M. Sarkis, Stochastic Galerkin method for elliptic SPDEs: A white noise approach, Discrete Contin. Dyn. Syst. Ser. B, 6 (2006), 941-955.
[19] P. Frauenfelder, C. Schwab, R. A. Todor, Finite elements for elliptic problems with stochastic coefficients, Comput. Methods Appl. Mech. Engrg. 194 (2005), 205-228.
[20] D. Xiu, J.S. Hesthaven, High-order collocation methods for differential equations with random inputs. SIAM J. Sci. Comput. 27(3)(2005) 1118-1139.
[21] H.G. Matthies, C. Bucher, Finite elements for stochastic media problems, Comput. Methods Appl. Mech. Engrg. 168(1999) 3-17.
[22] C. Schwab, R.A. Todor, Sparse finite elements for elliptic problems with stochastic loading, Numer. Math. 95(2003) 707-734.
[23] W.B. Liu, D. Tiba, Error estimates for the finite element approximation of a class of nonlinear optimal control problems, J. Numer. Func. Optim., 22(2001), 953-972.
[24] W.B. Liu, N. N. Yan, A posteriori error estimates for convex boundary control problems, SIAM Numer. Anal. 39(2001), 73-99.

[25] W.B. Liu, N.N. Yan, Adaptive finite element methods for optimal control governed by PDEs, Series in Information and Computational Science 41, Science Press, Beijing, 2008.
[26] T.J. Sun, Discontinuous Galerkin finite element method with interior penalties for convection diffusion optimal control problem, Int. J. Numer. Anal. Mod. 7(1)(2010) 87-107.
[27] T.J. Sun, L. Ge, W.B. Liu, Equivalent a posteriori error estimates for a constrained optimal control problem governed by parabolic equations, Int. J. Numer. Anal. Mod. 10(1)(2013) 1-23.
[28] W.B. Liu, D.P. Yang, L. Yuan, C.Q. Ma, Finite elemnet approximation of an optimal control problem with integral state constraint, SIAM J. Numer. Anal. 48(3)(2010), 1163-1185.
[29] W. F. Shen, L. Ge, D. P. Yang, Finite element methods for optimal control problems governed by linear quasi-parabolic integer-differential equations, Int. J. Numer. Anal. Mod. 10(3)(2013) 536-550.
[30] M. D. Gunzburger, H.C. Lee, J. Lee, Error estimates of stochastic optimal Neumann boundary control problems, SIAM J. Numer. Anal. 49(4)(2011) 1532-1552.
[31] E. Rosseel, G.N. Wells, Optimal control with stochastic PDE constrains and uncertain controls, Comput. Methods Appl. Mech. Engrg. 213-216(2012) 152-167.
[32] L. S. Hou, J. Lee, H. Manouzi, Finite element approximations of stochastic optimal control problems constrained by stochastic elliptic PDEs, J. Math. Anal. Appl. 384(2011) 87-103.
[33] H. C. Lee, J. Lee, A stochastic Galerkin method for stochastic control problems, Commum. Comput. Phys. 14(1)(2013) 77-106.
[34] A. Borzi, R. Griesse, Multigrid methods and sparse-grid collocation techniques for parabolic optimal control problems with random coefficients, SIAM J. Sci. Comput. 31(3), 2172-2192.
[35] T.J. Sun, W.F. Shen, B.X. Gong, W.B. Liu, A priori error estimate of stochastic Galerkin method for optimal control problem governed by stochastic elliptic PDE with constrained control, accepted by J.S.C.,67(2016) 405-431.
[36] W.F. Shen, T.J. Sun, B.X. Gong, W.B. Liu, Stochastic Galerkin method for constrained optimal control problem governed by an elliptic integro-differential PDE with stochastic coefficients, International Journal of Numerical Analysis & Modeling, 12(4)(2015) 593-616.
[37] B.X. Gong, T.J. Sun, W.F. Shen, W.B. Liu, A Priori Error Estimate of Stochastic Galerkin Method for Optimal Control Problem Governed by Random Parabolic PDE with Constrained Control, International Journal of Computational Methods, 13(5)(2016).
[38] P. Malliavin, Stochastic analysis, Springer, Berlin, 1997.
[39] A. Keese, Numerical solution of systems with stochastic uncertainties: a general purpose framework for stochastic Finite elements, Ph.D. thesis, Technical University Braunschweig, Braunschweig, Germany, 2004.
[40] V.Barthelmann, E.Novak, K.Ritter, High dimensional polynomial interpolation on sparse grids, Adv. Comput. Math., 12 (2000) 273-288.
[41] G. W. Wasilkowski, H. Wozniakowski, Explicit cost bounds of algorithms for multivariate tensor product problems, J. Complexity, 11 (1995) 1-56.
[42] C. W. Clenshaw, A. R. Curtis, A method for numerical integration on an automatic computer, Numer. Math., 2 (1960) 197-205.

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