INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 4-5, Pages 591–603

A SIMPLE FINITE ELEMENT METHOD OF THE CAUCHY PROBLEM FOR POISSON EQUATION

XIAOZHE HU, LIN MU, AND XIU YE

Abstract. In this paper, we introduce a simple method for the Cauchy problem. This new finite element method is based on least squares methodology with discontinuous approximations which can be implemented and analyzed easily. This discontinuous Galerkin finite element method is flexible to work with general unstructured meshes. Error estimates of the finite element solution are derived. The numerical examples are presented to demonstrate the robustness and flexibility of the proposed method.

Key words. Finite element methods, Cauchy problem, polyhedral meshes.

1. Introduction

We consider the Cauchy problem for Poisson equation

$$\begin{aligned} \Delta u &= f, && \text{in} \quad \Omega, \\ u &= 0, && \text{on} \quad \Gamma_1, \\ \nabla u \cdot \boldsymbol{n} &= g, && \text{on} \quad \Gamma_1, \end{aligned} \tag{1}$$

where Ω is a bounded convex polytopal domain in $\mathbb{R}^d$ with $d = 2, 3$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Assume that Γ_1 is simply connected.

The Cauchy problem (1) is well-known to be ill-posed [1, 4]. It has applications in many different areas such as plasma physic, electrocardiography, and corrosion non-destructive evaluation (e.g., [5, 12, 16, 21]). Due to the ill-posedness, the numerical approximation of the Cauchy problem is very difficult and challenging. Traditionally, regularization techniques, such as Tikhonov regularization [26] and the quasi-reversibility approach [23], were used to provide robust numerical schemes. Many different finite element methods have also been developed for solving the Cauchy problem (1). In [15, 24, 25], Galerkin type approaches are proposed based on structured grids or special formulation of the continuous problem. The regularization techniques are also used in finite element settings, e.g., [6, 3, 7, 13]. In [2, 11, 19, 18], the Cauchy problem (1) is reformulated as minimization problems and then solved numerically with possible regularizations. More recently, primal-dual formulation is proposed and solved by discontinuous Galerkin (DG) finite element methods with suitable stabilization/regularization, see [9, 10].

The purpose of this paper is to develop a simple finite element method to approximate the solution of the Cauchy problem (1) when it exists and is unique. This method is designed aiming on easy implementation and easy error analysis. The methodology of the scheme is combining the least squares technique with discontinuous approximations. Suitable stabilization terms are added to ensure the

Received by the editors on December 31, 2016, accepted on April 21, 2017.

2000 *Mathematics Subject Classification.* Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50.

stability of the discretization. As a result, our method leads to a symmetric and positive definite linear system of equations and is flexible to use on general polygonal meshes with hanging nodes. We prove that our discontinues finite element solution approaches to the solution of the model problem (1) when the mesh size approaches to zero. Convergence rate are studied in both energy norm and L^2-norm based on the conditional stability of the continues Cauchy problem. Comparing with existing methods, our approach is attractive due to its simplicity. The numerical results also show the efficiency of the proposed approach which confirms our theoretical results.

The rest of the paper is organized as follows. In Section 2, we recall Cauchy problem and its conditional stability results based on a traditional weak formulation. Our new simple discretization is given in Section 3. We study its stability and error estimates in Section 4 and 5, respectively. Finally, we present some numerical experiments to demonstrate the stability of the WG formulation in Section 6 .

2. Cauchy Problem

We denote the standard Lebesgue sapces by $L^2(\mathcal{D})$ and $\mathcal{D} \in \mathbb{R}^d$, $d = 2, 3$, with corresponding norms $\|\cdot\|_{L^2(\mathcal{D})}$ (or $\|\cdot\|_{\mathcal{D}}$). $H^s(\mathcal{D})$ denote the standard Sobolev space of index $s \geq 0$ along with the corresponding norm and semi-norm $\|\cdot\|_{H^s(\mathcal{D})}$ (or $\|\cdot\|_{s,\mathcal{D}}$) and $|\cdot|_{H^s(\mathcal{D})}$ (or $|\cdot|_{s,\mathcal{D}}$), respectively.

For the Cauchy problem (1), if the $(d-1)$-measure of Γ_2 is nonempty, it is an ill-conditioned problem. In practice, as shown in [4], such Cauchy problem is not well-posed due to measurement errors. However, following the traditional arguments, if the underlying physical process is stable, i.e., if the boundary data are known on the whole boundary, then the problem is well-posed, it is natural to assume that the Cauchy problem (1) has a unique solution in the idealized case with unperturbed data. Therefore, we assume that $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\Gamma_1)$, and that there is a unique solution $u \in H^2(\Omega)$ satisfies (1). Our analysis will be based on this assumption and the so-called conditional stability described later in Section 2.2.

2.1. A Traditional Weak Formulation. In order to introduce the conditional stability of the Cauchy problem (1), we need to first look at the weak formulation of the Cauchy problem (1). Following [1], we introduce two Sobolev spaces

$$H^1_{\Gamma_1}(\Omega) := \{v \in H^1(\Omega) : \ v|_{\Gamma_1} = 0\},$$

and

$$H^1_{\Gamma_2}(\Omega) := \{v \in H^1(\Omega) : \ v|_{\partial\Omega\setminus\Gamma_1} = 0\}.$$

The weak formulation for (1) is: find $u \in H^1_{\Gamma_1}(\Omega)$ such that

$$a_0(u, v) = l(v), \quad \forall\, v \in H^1_{\Gamma_2}(\Omega), \tag{2}$$

where

$$a_0(u, v) := \int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x,$$

and

$$l(v) := -\int_\Omega f v \mathrm{d}x + \int_{\Gamma_1} g v \mathrm{d}s.$$

Again, we are not assuming this weak formulation of Cauchy problem is well-posed since inf-sup stability does not hold in general [4].

Remark 2.1. Although the weak formulation (2) can be naturally derived from the Cauchy problem (1), which makes it a usual choice when developing numerical schemes (see, e.g., [9, 10]). Our numerical scheme is not designed based on this weak formulation.

2.2. Conditional Stability. It is well-known that the Cauchy problem (1) and the weak formulation (2) is ill-conditioned in general. Therefore, we use a conditional stability result [1, 10] in our error estimates, which can be stated as the following theorem as presented in [10].

Theorem 2.1 ([10], Section 2.1, equation (2.5))**.** *Let Ω be a connected open set of Lipschitz class. Let $u \in H^1_{\Gamma_1}(\Omega)$ be a weak solution to the Cauchy problem* (2)*, where $f \in L^2(\Omega)$, $g \in H^{-\frac{1}{2}}(\Gamma_1)$, and $\|l\|_{(H^1_{\Gamma_2})'} \leq \epsilon$. If u satisfies $\|u\|_{H^1(\Omega)} \leq E$, then*

$$\|u\|_{L^2(\Omega)} \leq \omega(\epsilon)$$

where $\omega(t) = C_1(E)\left(|\log(t)| + C_2(E)\right)^{-\mu}$, for $C_1(E), C_2(E) > 0$, $0 < t < 1$ and $0 < \mu < 1$.

As we shall see later, this conditional stability of the Cauchy problem (2) helps us to derive a L^2 error estimate for our proposed scheme.

3. A Discontinuous Finite Element Methods

In this section, we propose our new discontinuous Galerkin finite element method for solving the Cauchy problem (1). In order to do that, we first introduce some notations that are needed for the discretizations. Let $\mathcal{T}_h$ be a shape regular partition of the domain Ω consisting of polygons in two dimension or polyhedrons in three dimension satisfying a set of conditions specified in [27]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$ and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and the mesh size of the whole partition $\mathcal{T}_h$ is defined as $h = \max_{T \in \mathcal{T}_h} h_T$. Moreover, we use h_e to denote the diameter of an edge or a flat face $e \in E_h$.

Next, based on the partition $\mathcal{T}_h$, we can define a discontinuous finite element space V_h as follows for $k \geq 2$,

$$V_h = \{v \in L^2(\Omega) : \ v|_T \in P_k(T), \ \forall\, T \in \mathcal{T}\}. \tag{3}$$

Since we use discontinuous functions, we need define jumps as usual. Let two neighboring elements T_1 and T_2 have e as a common edge/face and unit normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ on e pointing exterior to T_1 and T_2, respectively. We then define jumps $[\phi]$ and $[\nabla\phi \cdot \boldsymbol{n}]$ on $e \in \mathcal{E}_h^0$ as following

$$[\phi] := \phi|_{\partial T_1}\boldsymbol{n}_1 + \phi|_{\partial T_2}\boldsymbol{n}_2, \quad [\nabla\phi \cdot \boldsymbol{n}] := \nabla\phi|_{\partial T_1} \cdot \boldsymbol{n}_1 + \nabla\phi|_{\partial T_2} \cdot \boldsymbol{n}_2,$$

and for $e \in \Gamma_1$, we define

$$[\phi] := \phi, \quad [\nabla\phi \cdot \boldsymbol{n}] := \nabla\phi \cdot \boldsymbol{n}.$$

Based on those jumps, now we introduce two bilinear forms as following

$$
\begin{aligned}
s(v,w) &:= \sum_{e\in\mathcal{E}_h\backslash\Gamma_2}\int_e (h_e^{-3}[v][w] + h_e^{-1}[\nabla v\cdot \boldsymbol{n}][\nabla w\cdot\boldsymbol{n}])ds,\\
a(v,w) &:= \sum_{T\in\mathcal{T}_h}(\Delta v,\Delta w)_T + s(v,w),
\end{aligned}
$$

where $(\cdot,\cdot)_T$ is the usual L^2-inner product on a element $T \in \mathcal{T}_h$.

Now we are in the position to propose our discontinuous finite element discretization for the Cauchy problem (1) as follows,

Algorithm 1. *A numerical approximation for (1) can be obtained by seeking $u_h \in V_h$ satisfying the following equation:*

$$
a(u_h, v) = (f,\ \Delta v) + \sum_{e\in\Gamma_1}\int_e h_e^{-1} g\nabla v\cdot\boldsymbol{n}ds,\quad \forall v\in V_h. \tag{4}
$$

Remark 3.1. Here, for the sake of the simplicity, we only consider the case that both the Dirichlet and Neumann boundary conditions are defined on the same subset Γ_1 of the boundary $\partial\Omega$. However, the discontinuous finite element method and the results we will present in the following sections can be simply adjusted so that they still hold for the general case when Dirichlet and Neumann boundary condition parts are different.

4. Stability Estimates

Because the continuous problem (1) is ill-posed, it is important to study the stability of the discretization (4), i.e., the well-posedness of the proposed discontinuous Galerkin finite element scheme. This is the main focus of this section.

For studying the stability, we need to choose a suitable norm and show that the DG discretization (4) is well-posed with respect to this norm. Let us first define a semi-norm induced by the bilinear form $a(\cdot,\cdot)$ as following

$$
\begin{aligned}
|||v|||^2 &:= a(v,v) = \sum_{T\in\mathcal{T}_h}\|\Delta v\|_T^2 + s(v,v)\\
&= \sum_{T\in\mathcal{T}_h}\|\Delta v\|_T^2 + \sum_{e\in\mathcal{E}_h\backslash\Gamma_2}\int_e h_e^{-3}[v]^2 \mathrm{d}s + \sum_{e\in\mathcal{E}_h\backslash\Gamma_2}\int_e h_e^{-1}[\nabla v\cdot\boldsymbol{n}]^2\mathrm{d}s.
\end{aligned} \tag{5}
$$

and another semi-norm induced by $s(\cdot,\cdot)$ as $|v|_s^2 := s(v,v)$.

Next Lemma shows that the semi-norm $|||\cdot|||$ actually is a norm.

Lemma 4.1. *Under the assumption that the Cauchy problem* (1) *has a unique solution, the semi-norm $|||\cdot|||$ defines a norm.*

Proof. It is sufficient to show that $|||v||| = 0$ implies $v = 0$. Note that when $|||v||| = 0$, we have $\Delta v = 0$ on each element T and v and $\nabla v\cdot\boldsymbol{n}$ are continuous across the edges. Moreover, $v = 0$ and $\nabla v\cdot\boldsymbol{n} = 0$ on Γ_1. Therefore, v is a solution of the Cauchy problem (1) with $f = g = 0$. By the assumption of uniqueness of the solution, we have $v = 0$ which shows that $|||\cdot|||$ is in fact a norm. $\square$

Based on the definition of the norm, we can show the following fundamental stability result for the DG discretization (4), i.e., the coercivity and continuity of the bilinear form $a(\cdot,\cdot)$.

Lemma 4.2. *The bilinear form $a(\cdot,\cdot)$ satisfies the following continuity property,*

$$a(v,w) \le C\|\!|v|\!\|\|\!|w|\!\|, \tag{6}$$

and the coercivity property

$$a(v,v) = \|\!|v|\!\|^2. \tag{7}$$

Proof. (7) follows from the definition and (6) follows from the Cauchy-Schwarz inequality,

$$\begin{aligned} a(v,w) &= \sum_{T\in\mathcal{T}_h}(\Delta v,\Delta w)_T + \sum_{e\in\mathcal{E}_h\setminus\Gamma_2}\int_e (h_e^{-3}[v][w]ds + h_e^{-1}[\nabla v\cdot\boldsymbol{n}][\nabla w\cdot\boldsymbol{n}])ds \\ &\le (\sum_{T\in\mathcal{T}_h}\|\Delta v\|_T^2)^{\frac12}(\sum_{T\in\mathcal{T}_h}\|\Delta w\|_T^2)^{\frac12} \\ &+ (\sum_{e\in\mathcal{E}_h\setminus\Gamma_2}\int_e h_e^{-3}[v]^2ds)^{\frac12}(\sum_{e\in\mathcal{E}_h\setminus\Gamma_2}\int_e h_e^{-3}[w]^2ds)^{\frac12} \\ &+ (\sum_{e\in\mathcal{E}_h\setminus\Gamma_2}\int_e h_e^{-1}[\nabla v\cdot\boldsymbol{n}]^2ds)^{\frac12}(\sum_{e\in\mathcal{E}_h\setminus\Gamma_2}\int_e h_e^{-1}[\nabla w\cdot\boldsymbol{n}]^2)^{\frac12} \\ &\le C\|\!|v|\!\|\|\!|w|\!\|. \end{aligned}$$

We have proved the lemma. □

Now we naturally have the well-posedness result of the DG discretization (4).

Theorem 4.1. *Under the assumption that the Cauchy problem* (1) *has a unique solution, the discontinuous Galerkin finite element scheme (4) has a unique solution.*

Proof. This is a direct result of Lemma 4.2 and Lax-Milgram Theorem. □

Remark 4.1. Due to the simplicity of our DG scheme, and the naturally induced norm, the stability result can be obtained easily following the standard arguments. This is different from the primal dual formulation proposed recently in [10], where a inf-sup condition is needed.

5. Error Estimate

In the previous section, we have shown that the proposed DG scheme (4) is well-posed. In this section, we focus on the error estimates and derive the error analysis for the DG solution u_h obtained from (4). Error estimates in both energy norm $\|\!|\cdot|\!\|$ and L^2-norm are considered.

First, we recall the standard trace inequality which is used later in our analysis. For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [27] for details):

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{8}$$

Next we define the L^2-projection as usual. For any $v \in L^2(\Omega)$, the L^2-projection $Q_h v \in V_h$ is defined as following

$$(Q_h v, w) = (v, w), \quad \forall\, w \in V_h. \tag{9}$$

The following lemma shows the approximation property of the L^2-projection under certain regularity assumption of the functions.

Lemma 5.1. *Let $v \in H^{k+1}(\Omega)$, $k \geq 2$, and $Q_h v \in V_h$ be the L^2 projection of v. Then there exist a constant C independent of the mesh size h, such that,*

$$|||v - Q_h v||| \leq C h^{k-1} \|v\|_{k+1}. \tag{10}$$

Proof. Using the definition of Q_h (9), and the trace inequality (8), we have

$$\begin{aligned}
|||v - Q_h v|||^2 &= \sum_{T \in \mathcal{T}_h} \|\Delta(v - Q_h v)\|_T^2 \\
&\quad + \sum_{e \in \mathcal{E}_h \setminus \Gamma_2} (h_e^{-3} \|[v - Q_h v]\|_e^2 + h_e^{-1} \|[\nabla(v - Q_h v) \cdot \boldsymbol{n}]\|_e^2) \\
&\leq C \sum_{T \in \mathcal{T}_h} (|v - Q_h v|_{2,T}^2 + h^{-4} \|v - Q_h v\|_T^2 + h^{-2} \|\nabla(v - Q_h v)\|_T^2) \\
&\leq C h^{2k-2} \|v\|_{k+1}^2,
\end{aligned}$$

which finishes the proof of the lemma. □

Now we are ready to derive the error estimate in the energy norm $|||\cdot|||$. The analysis follows directly from the coercivity (7) and continuity (6) of the bilinear from $a(\cdot,\cdot)$ and the approximation property of the L^2-projection (10).

Theorem 5.1. *Let $u_h \in V_h$ be the finite element solution of the problem (1) obtained by the DG discretization (4). Under the assumption that the Cauchy problem (1) has a unique solution u and further assume that $u \in H^{k+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$, $k \geq 2$, then there exists a constant C independent of the mesh size h, such that,*

$$|||u - u_h||| \leq C h^{k-1} \|u\|_{k+1}. \tag{11}$$

Proof. Obviously, the true solution u of (1) satisfies

$$a(u, v) = (f, \Delta v) + \sum_{e \in \Gamma_1} \int_e h_e^{-1} g \nabla v \cdot \boldsymbol{n} ds.$$

Subtracting (4) from the above equation implies

$$a(u - u_h, v) = 0, \quad \forall v \in V_h.$$

Then we have, for any $v \in V_h$,

$$\begin{aligned}
|||u - u_h|||^2 &= a(u - u_h, u - u_h) \\
&= a(u - u_h, u - v) + a(u - u_h, v - u_h) \\
&= a(u - u_h, u - v),
\end{aligned}$$

which, by the continuity of the bilinear form $a(\cdot,\cdot)$, implies

$$|||u - u_h||| \leq C \inf_{v \in V_h} |||u - v|||.$$

Now, by choosing $v = Q_h u$ and using the approximation results (10), we obtain (11), which completes the proof. □

Next, we consider the L^2 error estimate. Due to the lack of regularity of the Cauchy problem, standard duality argument does not work. Here, we follow the idea from [10] and decompose the error $u-u_h = (u-\bar{u}_h)+(\bar{u}_h-u_h)$, where $\bar{u}_h$ is a C^1-conforming approximation of u constructed from u_h. For the sake of simplicity, we restrict our discussion on triangular or rectangular meshes in two dimension or tetrahedral or hexahedral meshes in three dimension. On those meshes, such construction can be done on C^1-conforming space consisting of macro-elements of degree $k+2$. As pointed out in [17], this is done by a recovery operator which is constructed via averages of the nodal basis functions (cf., [22, 8, 20]) and is a high-order version of the classical Heish-Clough-Tocher element [14]. Since we only need the existence of such a $\bar{u}_h$ and the construction is not essential to our analysis, we do not give the complete construction here and refer interested readers to [17] for a detailed description. Note that, in our case, we only need $\bar{u}_h \in H^1_{\Gamma_1}(\Omega)$. Therefore, we can use a slightly modification of the construction given in [17] so that only the nodal values of $\bar{u}_h$ on Γ_1 vanish but other degrees of freedom are defined via averaging. Following the similar argument in [17], we have the following lemma about the approximation property of $\bar{u}_h$.

Lemma 5.2 (Lemma 3.1 [17]). *Let $\bar{u}_h$ be constructed as shown in Definition 3.1* [17], *we have*

$$\sum_{T\in\mathcal{T}_h}\left(\|u_h-\bar{u}_h\|_T^2 + h^2|u_h-\bar{u}_h|_{1,T}^2 + h^4|u_h-\bar{u}_h|_{2,T}^2\right) \le Ch^4|u_h|_{\mathcal{E}_h}^2. \tag{12}$$

where $|v|^2_{\mathcal{E}_h} := \sum_{e\in\mathcal{E}_h\setminus\Gamma_2}\int_e h_e^{-3}[v]^2 \mathrm{d}s + \sum_{e\in\mathcal{E}_h^0}\int_e h_e^{-1}[\nabla v\cdot\boldsymbol{n}]^2 ds$ *and C is a constant independent of h and u_h.*

Now we have $(u-\bar{u}_h) \in H^1_{\Gamma_1}(\Omega)$ and we can think it satisfies the Cauchy problem in the weak sense, i.e. the weak formulation (2) with a different right hand side, i.e.,

$$a_0(u-\bar{u}_h, v) = \bar{l}(v), \forall\ v \in H^1_{\Gamma_2}(\Omega). \tag{13}$$

For the L^2 error estimates, we need to estimate $\|\bar{l}\|_{(H^1_{\Gamma_2}(\Omega))'}$.

$$\begin{aligned}
\|\bar{l}\|_{(H^1_{\Gamma_2}(\Omega))'} &= \sup_{0\neq v\in H^1_{\Gamma_2}(\Omega)} \frac{\bar{l}(v)}{\|v\|_{H^1}} = \sup_{0\neq v\in H^1_{\Gamma_2}(\Omega)} \frac{a_0(u-\bar{u}_h, v)}{\|v\|_{H^1(\Omega)}} \\
&= \sup_{0\neq v\in H^1_{\Gamma_2}(\Omega)} \frac{-\int_\Omega \Delta(u-\bar{u}_h)v\mathrm{d}x + \sum_{e\in\Gamma_1}\int_e (g-\nabla\bar{u}_h\cdot\boldsymbol{n})v\mathrm{d}s}{\|v\|_{H^1(\Omega)}} \\
&\le \frac{\|\Delta(u-\bar{u}_h)\|\|v\| + \sum_{e\in\Gamma_1} h_e^{-1/2}\|g-\nabla\bar{u}_h\cdot\boldsymbol{n}\|_e\, h_e^{1/2}\|v\|_e}{\|v\|_{H^1(\Omega)}} \\
&\le C\left(\|\Delta(u-\bar{u}_h)\|^2 + \sum_{e\in\Gamma_1} h_e^{-1}\|g-\nabla\bar{u}_h\cdot\boldsymbol{n}\|_e^2\right)^{1/2} \\
&= C|||u-\bar{u}_h|||. \qquad (14)
\end{aligned}$$

Note that, since $a(u-u_h, v_h) = 0$, $\forall\, v_h \in V_h$, and $u_h - \bar{u}_h \in V_h$, we have

$$
\begin{aligned}
|||u-\bar{u}_h|||^2 &= |||u-u_h|||^2 + |||u_h-\bar{u}_h|||^2 \\
&= |||u-u_h|||^2 + \sum_{T\in\mathcal{T}_h} \|\Delta(u_h-\bar{u}_h)\|_T^2 \\
&\quad + \sum_{e\in\mathcal{E}_h\setminus\Gamma_2} (h_e^{-3}\|[u_h-\bar{u}_h]\|_e^2 + h_e^{-1}\|[\nabla(u_h-\bar{u}_h)\cdot\boldsymbol{n}]\|_e^2) \\
&\le |||u-u_h|||^2 + C\sum_{T\in\mathcal{T}_h} (\|\Delta(u_h-\bar{u}_h)\|_T^2 \\
&\quad + h^{-4}\|u_h-\bar{u}_h\|_T^2 + h^{-2}\|\nabla(u_h-\bar{u}_h)\|_T^2) \\
&\le |||u-u_h|||^2 + C|u_h|_{\mathcal{E}_h}^2. \qquad (15)
\end{aligned}
$$

Based on the definition of $|\cdot|_{\mathcal{E}_h}$ and $u \in H^2(\Omega)\cap H^1_{\Gamma_1}(\Omega)$, we have

$$
|u_h|_{\mathcal{E}_h}^2 = |u-u_h|_{\mathcal{E}_h}^2 \le |u-u_h|_s^2 \le |||u-u_h|||^2. \qquad (16)
$$

Therefore, substitute (15) and (16) back into (14) and apply Theorem 5.1, we have the following estimate

$$
\|\bar{l}\|_{(H^1_{\Gamma_2}(\Omega))'} \le C|||u-u_h||| \le Ch^{k-1}\|u\|_{k+1},
$$

and, if h is small enough, we have $\|\bar{l}\|_{(H^1_{\Gamma_2}(\Omega))'} \le \epsilon$, which is needed for the L^2 error estimate.

In order to apply Theorem 2.1, we also need to show that $\|u-\bar{u}_h\|_{H^1(\Omega)}$ is bounded. To this end, since $(u-\bar{u}_h) \in H^2(\Omega)\cap H^1_{\Gamma_1}(\Omega)$, then apply the classical Poincaré-Friedrichs inequality, we have

$$
\|u-\bar{u}_h\|_{H^1(\Omega)} \le C(|u-\bar{u}_h|_{H^2(\Omega)} + \|\nabla(u-\bar{u}_h)\cdot\boldsymbol{n}\|_{\Gamma_1}) \le C|||u-\bar{u}_h|||.
$$

Then use (15) and (16) and apply Theorem (5.1), we have

$$
\|u-\bar{u}_h\|_{H^1(\Omega)} \le C|||u-u_h||| \le Ch^{k-1}\|u\|_{k+1} := E_h. \qquad (17)
$$

This means the H^1-norm conforming part of the error is bounded.

Based on the above estimate and applying Theorem 2.1, we have the following error estimate in the L^2-norm.

Theorem 5.2. *Let $u \in H^{k+1}(\Omega)\cap H^1_{\Gamma_1}(\Omega)$ be the solution of the Cauchy problem* (1) *and $u_h \in V_h$ be the finite element solution of the Cauchy problem* (1) *obtained by the DG discretization* (4) *and E_h is defined in* (17)*. Assume that h is small enough so that $\|\bar{l}\|_{(H^1_{\Gamma_2})'} \le \epsilon$ holds. Here $\bar{l}$ is defined in* (13)*, then we have*

$$
\|u-u_h\|_{L^2(\Omega)} \le \omega(\epsilon) + Ch^2|u_h|_{\mathcal{E}_h}. \qquad (18)
$$

where $\omega(t)$ is defined in Theorem 2.1. Furthermore, we have,

$$
\|u-u_h\|_{L^2(\Omega)} \le \omega(\epsilon) + Ch^{k+1}\|u\|_{k+1}. \qquad (19)
$$

Here C is a constant independent of the mesh size h.

Proof. As shown in (17), $\|u-\bar{u}_h\|_{H^1(\Omega)} \le E_h$, then apply Theorem 2.1, we have $\|u-\bar{u}_h\|_{L^2(\Omega)} \le \omega(\epsilon)$, where ω is defined in Theorem 2.1. Then apply the triangular inequality and estimate (12), we obtain (18). (19) can be obtained from (18) and (16) directly, which completes the proof.

□

Remark 5.1. From the error estimate in the L^2 norm, we can see that it contains an optimal part ($h^{k+1}\|u\|_{k+1}$). However, the convergence behavior of the other term $\omega(\epsilon)$ depends on the conditional stability of the original Cauchy problem (1). Therefore, the overall convergence rate also depends on the conditional stability. In the numerical experiments, we observe suboptimal convergence rate when $\Gamma_1 \subset \partial\Omega$. The details will be given in the next section.

6. Numerical Experiments

In this section, we shall apply the proposed numerical schemes (4) to two test problems for validating the theoretical conclusions in the previous sections and demonstrating the effectiveness of the proposed DG method.

6.1. Test 1. Consider the Cauchy problem (1) on $\Omega = (0,1) \times (0,1)$ and let

$$f = -2\pi^2(4\cos(\pi x)^2\cos(\pi y)^2 - 3\cos(\pi x)^2 - 3\cos(\pi y)^2 + 2)$$

such that the exact solution of (1) is given as the following,

$$u = \sin^2(\pi x)\sin^2(\pi y).$$

We shall consider three different choices of the boundary Γ_1, which are shown in the Figure 1. In this test, due to the choice of exact solution, it is easy to see that equation (1) is coupled with homogeneous boundary condition, i.e. $g = 0$.

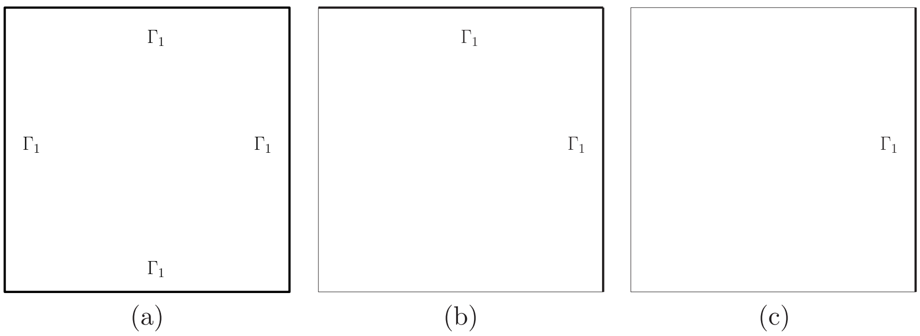

FIGURE 1. Different choices of boundary Γ_1: (a) Case 1: $\Gamma_1 = \{x=0; x=1; y=0; y=1\}$; (b) Case 2: $\Gamma_1 = \{x=1, y=1\}$; (c) Case 3: $\Gamma_1 = \{x=1\}$.

Uniform rectangular mesh is employed in the numerical experiment. Denote the size of mesh as h. We perform the proposed numerical scheme with polynomials of degree two, i.e., $k = 2$. The error profiles and convergence tests are reported in Table 1. From Theorem 5.1, it is expected that the energy norm converges with optimal order $\mathcal{O}(h)$, which is verified in the numerical test. The error measured in the L^2-norm has also been reported in Table 1. It can be seen that the convergence rate for L^2-error are affected by the choices of the boundary Γ_1. When $\Gamma_1 \subset \partial\Omega$ (case 2 and 3), the convergence rate deteriorates as expected according to the conditional stability of the Cauchy problem (1) and error estimates given in Theorem 5.2.

TABLE 1. Test 1: Error Profiles and Convergence Test with Γ_1 for $k = 2$.

$1/h$	$\|u - u_h\|$	order	$\|\Delta u - \Delta u_h\|$	order
Case 1 with $\Gamma_1 = \partial\Omega$				
2	$3.7500E-01$		$1.3958E+01$	
4	$8.6956E-02$	2.11	$7.2021E+00$	0.95
8	$2.5455E-02$	1.77	$3.8049E+00$	0.92
16	$7.3808E-03$	1.79	$1.9292E+00$	0.98
32	$1.9807E-03$	1.90	$9.6787E-01$	1.00
64	$5.0901E-04$	1.96	$4.8434E-01$	1.00
128	$1.2865E-04$	1.98	$2.4222E-01$	1.00
Case 2 with $\Gamma_1 = \{x = 1, y = 1\}$				
2	$3.7500E-01$		$1.3957E+01$	
4	$1.6592E-01$	1.18	$7.2018E+00$	0.95
8	$1.5816E-01$	0.07	$3.8050E+00$	0.92
16	$8.4246E-02$	0.91	$1.9294E+00$	0.98
32	$3.1840E-02$	1.40	$9.6793E-01$	1.00
64	$1.0616E-02$	1.58	$4.8435E-01$	1.00
128	$3.3567E-03$	1.66	$2.4222E-01$	1.00
Case 3 with $\Gamma_1 = \{x = 1\}$				
2	$3.7500E-01$		$1.3957E+01$	
4	$4.8766E-01$	-0.38	$7.2017E+00$	0.95
8	$2.8303E-01$	0.78	$3.8050E+00$	0.92
16	$9.8429E-02$	1.52	$1.9294E+00$	0.98
32	$3.2958E-02$	1.58	$9.6793E-01$	1.00
64	$1.4843E-02$	1.15	$4.8435E-01$	1.00
128	$7.3844E-03$	1.01	$2.4222E-01$	1.00

6.2. Case 2. In this numerical experiment, we shall perform the numerical scheme (4) to the Cauchy problem (1) with non-homogeneous boundary conditions. Let the domain be $\Omega = (0, 1) \times (0, 1)$ and choose the right-hand side function as the following,

$$f = -2\pi^2 \cos(\pi x) \cos(\pi y).$$

The exact solution is

$$u = \cos(\pi x) \cos(\pi y).$$

The different choices of the boundary Γ_1 are the same as the three choices in the first test as shown in Figure 1.

Again, we use DG scheme (4) with polynomials of degree two, i.e., $k = 2$, and report the errors and convergence rate in Table 2 and we can see that the convergence rate for the energy norm error has optimal order $\mathcal{O}(h)$, which confirms the theoretical conclusion given in Theorem 5.1. For the error in the L^2-norm, similar with the first numerical example, we observe degenerated rate when $\Gamma_1 \subset \Omega$, which confirms Theorem 5.2 due to the conditional stability of the Cauchy problem (1).

TABLE 2. Test 2: Error Profiles and Convergence Test with Γ_1 with $k = 2$.

$1/h$	$\|u - u_h\|$	order	$\|\Delta u - \Delta u_h\|$	order
Case 1 with $\Gamma_1 = \partial\Omega$				
2	$8.1104E-02$		$5.7804E+00$	
4	$1.2417E-02$	2.71	$3.0928E+00$	0.90
8	$1.5082E-03$	3.04	$1.5733E+00$	0.98
16	$2.0064E-04$	2.91	$7.9004E-01$	0.99
32	$3.3193E-05$	2.60	$3.9543E-01$	1.00
64	$6.4818E-06$	2.36	$1.9777E-01$	1.00
128	$6.4818E-06$	2.00	$4.9443E-02$	2.00
Case 2 with $\Gamma_1 = \{x = 1, y = 1\}$				
2	$3.1301E-01$		$5.7821E+00$	
4	$1.3439E-01$	1.22	$3.0942E+00$	0.90
8	$3.0415E-02$	2.14	$1.5737E+00$	0.98
16	$5.4631E-03$	2.48	$7.9009E-01$	0.99
32	$3.2055E-03$	0.77	$3.9544E-01$	1.00
64	$1.7224E-03$	0.90	$1.9777E-01$	1.00
128	$7.8712E-04$	1.13	$9.8891E-02$	1.00
Case 3 with $\Gamma_1 = \{x = 1\}$				
2	$6.6573E-01$		$5.7801E+00$	
4	$4.0398E-01$	0.72	$3.0932E+00$	0.90
8	$1.2248E-01$	1.72	$1.5738E+00$	0.97
16	$2.5969E-02$	2.24	$7.9014E-01$	0.99
32	$3.9986E-03$	2.70	$3.9545E-01$	1.00
64	$1.2349E-03$	1.70	$1.9777E-01$	1.00
128	$8.2412E-04$	0.58	$9.8891E-02$	1.00

Acknowledgments

The valuable and careful comments offered by anonymous referees are gratefully acknowledged. This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research.This manuscript has been authored by UT-Battelle, LLC under Contract No. DE-AC05-00OR22725 with the U.S. Department of Energy. The United States Government retains and the publisher, by accepting the article for publication, acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for United States Government purposes. The Department of Energy will provide public access to these results of federally sponsored research in accordance with the DOE Public Access Plan (`http://energy.gov/downloads/doe-public-access-plan`).The work of X. Hu was supported in part by the National Science Foundation under grant DMS-1620063. The work of X. Ye was supported in part by National Science Foundation Grant DMS-1620016.

References

[1] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella, The stability for the Cauchy problem for elliptic equations, Inverse problems, 25 (2009), p. 123004.

[2] S. Andrieux, T. Baranger, and A. B. Abda, Solving Cauchy problems by minimizing an energy-like functional, Inverse problems, 22 (2006), p. 115.

[3] M. Azaïez, F. B. Belgacem, and H. El Fekih, On Cauchy's problem: Ii. completion, regularization and approximation, Inverse problems, 22 (2006), p. 1307.

[4] F. B. Belgacem, Why is the Cauchy problem severely ill-posed?, Inverse Problems, 23 (2007), p. 823.

[5] J. Blum, Numerical simulation and optimal control in plasma physics, New York, NY; John Wiley and Sons Inc., 1989.

[6] L. Bourgeois, A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace's equation, Inverse problems, 21 (2005), p. 1087.

[7] L. Bourgeois, Convergence rates for the quasi-reversibility method to solve the Cauchy problem for Laplace's equation, Inverse problems, 22 (2006), p. 413.

[8] S. C. Brenner and L.-Y. Sung, C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains, Journal of Scientific Computing, 22 (2005), pp. 83–118.

[9] E. Burman, Stabilized finite element methods for nonsymmetric, noncoercive, and ill-posed problems. Part I: Elliptic equations, SIAM Journal on Scientific Computing, 35 (2013), pp. A2752–A2780.

[10] E. Burman, A stabilized nonconforming finite element method for the elliptic Cauchy problem, Mathematics of Computation, (2016).

[11] A. Chakib and A. Nachaoui, Convergence analysis for finite element approximation to an inverse Cauchy problem, Inverse Problems, 22 (2006), p. 1191.

[12] P. Colli-Franzone, L. Guerri, S. Tentoni, C. Viganotti, S. Baruffi, S. Spaggiari, and B. Taccardi, A mathematical procedure for solving the inverse potential problem of electrocardiography. Analysis of the time-space accuracy from in vitro experimental data, Mathematical Biosciences, 77 (1985), pp. 353–396.

[13] J. Dardé, A. Hannukainen, and N. Hyvonen, An $H(\text{div})$-Based Mixed Quasi-reversibility Method for Solving Elliptic Cauchy Problems, SIAM Journal on Numerical Analysis, 51 (2013), pp. 2123–2148.

[14] J. Douglas Jr, T. Dupont, P. Percell, and R. Scott, A family of C^1 finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems, RAIRO-Analyse numérique, 13 (1979), pp. 227–255.

[15] R. Falk and P. Monk, Logarithmic convexity for discrete harmonic functions and the approximation of the Cauchy problem for Poisson's equation, Mathematics of computation, 47 (1986), pp. 135–149.

[16] D. Fasino and G. Inglese, An inverse Robin problem for Laplace's equation: theoretical results and numerical methods, Inverse problems, 15 (1999), p. 41.

[17] E. H. Georgoulis, P. Houston, and J. Virtanen, An a posteriori error indicator for discontinuous Galerkin approximations of fourth-order elliptic problems, IMA journal of numerical analysis, (2009), p. drp023.

[18] H. Han, L. Ling, and T. Takeuchi, An energy regularization for Cauchy problems of Laplace equation in annulus domain, Communications in Computational Physics, 9 (2011), pp. 878–896.

[19] W. Han, J. Huang, K. Kazmi, and Y. Chen, A numerical method for a Cauchy problem for elliptic partial differential equations, Inverse Problems, 23 (2007), p. 2401.

[20] P. Houston, D. Schötzau, and T. P. Wihler, Energy norm a posteriori error estimation of hp-adaptive discontinuous Galerkin methods for elliptic problems, Mathematical Models and Methods in Applied Sciences, 17 (2007), pp. 33–62.

[21] G. Inglese, An inverse problem in corrosion detection, Inverse problems, 13 (1997), p. 977.

[22] O. A. Karakashian and F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems, SIAM Journal on Numerical Analysis, 41 (2003), pp. 2374–2399.

[23] R. Lattés and J. Lions, The method of quasi-reversibility: applications to partial differential equations, no. 18 in Translated from the French edition and edited by Richard Bellman. Modern Analytic and Computational Methods in Science and Mathematics, American Elsevier Publishing Co., Inc., New York,, 1969.

[24] W. Lucht, A finite element method for an ill-posed problem, Applied numerical mathematics, 18 (1995), pp. 253–266.

[25] H.-J. Reinhardt, H. Han, and D. N. Hào, Stability and regularization of a discrete approximation to the Cauchy problem for Laplace's equation, SIAM Journal on Numerical Analysis, 36 (1999), pp. 890–905.

[26] A. N. Tikhonov and V. Y. Arsenin, Solutions of ill-posed problems, Winston, 1977.
[27] J. Wang and X. Ye, A weak Galerkin mixed finite element method for second order elliptic problems, Mathematics of Computation, 83 (2014), pp. 2101–2126.

Department of Mathematics, Tufts University, Medford 02155, United States

E-mail: Xiaozhe.Hu@tufts.edu

Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, United States

E-mail: mul1@ornl.gov

Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, United States

E-mail, Corresponding author: xxye@ualr.edu