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A POSTERIORI ERROR ESTIMATES FOR MIXED FINITE ELEMENT GALERKIN APPROXIMATIONS TO SECOND ORDER LINEAR HYPERBOLIC EQUATIONS

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Abstract. In this article, *a posteriori* error analysis for mixed finite element Galerkin approximations of second order linear hyperbolic equations is discussed. Based on mixed elliptic reconstructions and an integration tool, which is a variation of Baker's technique introduced earlier by G. Baker (SIAM J. Numer. Anal., 13 (1976), 564-576) in the context of *a priori* estimates for a second order wave equation, *a posteriori* error estimates of the displacement in $L^\infty(L^2)$-norm for the semidiscrete scheme are derived. Finally, a first order implicit-in-time discrete scheme is analyzed and *a posteriori* error estimators are established.

Key words. Second order linear wave equation, mixed finite element methods, mixed elliptic reconstructions, semidiscrete method, first order implicit completely discrete scheme, and *a posteriori* error estimates.

1. Introduction

In this paper, we discuss *a posteriori* error estimates for mixed finite element Galerkin approximations to the following class of second order linear hyperbolic problems:

$$u_{tt} - \nabla \cdot (A\nabla u) = f \qquad \text{in } \Omega \times \ (0,T], \tag{1}$$

$$u|_{\partial\Omega} = 0 \qquad u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1. \tag{2}$$

Here, $\Omega \subset I\!\!R^2$ is a bounded polygonal domain with boundary $\partial\Omega$, $0 < T < \infty$, $u_t = \frac{\partial u}{\partial t}$ and $A(x) = (a_{ij}(x))_{1\le i,j\le 2}$ is a symmetric and uniformly positive definite matrix. All the coefficients a_{ij}'s are smooth functions of x with uniformly bounded derivatives in $\bar{\Omega}$. Moreover, the initial functions $u_0 = u_0(x)$, $u_1 = u_1(x)$ and the forcing function $f = f(x,t)$ are assumed to be smooth functions in their respective domains.

In recent years, there has been a growing demand for designing reliable and efficient space-time algorithms for the numerical computation of time dependent partial differential equations. Most of these algorithms are based on *a posteriori* error estimators, which provide appropriate tools for adaptive mesh refinements. For elliptic boundary value problems, *a posteriori* error estimates are well developed (see, [3, 32]). Adaptivity with *a posteriori* error control for parabolic problems has also been an active research area for the last two decades (cf. [18, 33, 25, 30, 8, 9, 5] and references, therein). For the time discretization, some results on *a posteriori* error estimations for abstract first order evolution problems are available in the literature (cf. [4, 21, 26, 28, 30]).

In the context of second order wave equations, only few results are available on *a posteriori* error analysis, see, [24, 1, 14, 13, 7, 31]. Further, it is observed that the design and implementation of adaptive algorithms for these equations based on rigorous *a posteriori* error estimators are less complete compared to elliptic

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and parabolic equations. Based on a space-time finite element discretization with basis functions being continuous in space and discontinuous in time, *a priori* and *a posteriori* error estimates for second order linear wave equations are proved in [24]. Asymptotically exact *a posteriori* estimates for the standard finite element method are proposed and analyzed in [1, 2] by solving a set of local elliptic problems. The recent results in [7, 20] cover only first order time discrete schemes. In [7], the second order wave equation is written as a first order system and a first order implicit backward Euler scheme in time is used with continuous piecewise affine finite elements in space. Further, rigorous *a posteriori* bounds have been established using energy arguments and adaptive algorithms based on the *a posteriori* bounds are discussed. In [20], based on Baker's technique *a posteriori* bounds are derived for the semidiscrete scheme in $L^\infty(L^2)$-norm and for first order implicit-in-time fully discrete schemes in $\ell^\infty(L^2)$-norm. The fully discrete analysis relies crucially on a novel time reconstruction satisfying a local vanishing-moment property, and on a space reconstruction technique used earlier in [28] for parabolic problems. In [14], an adaptive algorithm in space and time which is based on Galerkin space-time discretizations leading to Newmark scheme is analyzed. Further, goal oriented *a posteriori* error estimates are derived and some numerical results are provided to demonstrate the efficiency of error estimators. In [31], the author has studied an anisotropic *a posteriori* error estimate for a finite element discretization of a two dimensional wave equation. The estimate is derived in the $L^2(0, T, H^1(\Omega))$-norm and it turns out to be sharp on anisotropic meshes.

For higher order time reconstruction for abstract second order evolution equations, one may refer to the recent papers [23, 22]. In [23], an adaptive time stepping Galerkin method is analyzed for second order evolution problems. Based on the energy approach and the duality argument, optimal order *a posteriori* error estimates and *a posteriori* nodal superconvergence results have been derived. An adaptive time stepping strategy is discussed and some numerical experiments are conducted to assess the effectiveness of the proposed scheme. In a recent work [22], second order explicit and implicit two-step time discretization schemes such as leap-frog and cosine methods are discussed and *a posteriori* estimates using a novel time reconstruction are derived. Further, some numerical experiments are conducted to confirm their theoretical findings.

For space-time adaptivity, the finite element discretization depends on the space-time variational formulation and its error indicators include both space and time errors. Recently, attempts have been made to exploit elliptic reconstruction to prove optimal *a posteriori* error estimates in finite element methods for parabolic problems [28]. In fact, the role of the elliptic reconstruction operator in *a posteriori* estimates is quite similar to the role played by elliptic projection introduced earlier by Wheeler [34] for recovering optimal *a priori* error estimates of finite element Galerkin approximations to parabolic problems. This analysis is, further, developed for completely discrete scheme based on backward Euler method [26], for maximum norm estimates [17] and for discontinuous Galerkin methods for parabolic problems [21]. In recent works [29] and [27], the analysis is further extended to mixed FE Galerkin methods applied to parabolic problems.

In this article, an *a posteriori* analysis is discussed for mixed finite element Galerkin approximations of a class of second order linear hyperbolic problems. One notable advantage of mixed finite element scheme is that it offers a simultaneous approximations of displacements and stresses, resulting in better convergences rates for the stress variable. This property is important in applications such as in the modeling boundary controllability of the wave equation. In the first part of this article, a semidiscrete scheme is derived using mixed finite element method in spatial direction, while keeping time variable constant. Based on mixed elliptic reconstructions presented in [29], which depend explicitly on residuals and a time integration tool, a variant of Baker's technique, *a posteriori* error estimates in $L^\infty(L^2)$-norm are derived for the displacement u. For the time discretization, the time discrete scheme with the time-reconstruction proposed in [20] is applied and then, using summation tool, *a posteriori* error estimators in $\ell^\infty(L^2)$-norms are developed. Compared to [20], our analysis is not only for mixed finite element method, but also it differs from the analysis of [20] in the sense that a time integration tool is used for deriving $L^\infty(L^2)$ *a posteriori* estimators, as against the time testing procedure of Baker [6] used in [20].

The outline of this article is as follows. In Section 2, we introduce both weak primal and mixed formulations for the hyperbolic problem (1)-(2) and establish their equivalence. Section 3 deals with mixed elliptic reconstruction techniques proposed in [29] and *a posteriori* estimates for the semidiscrete problem for both displacement u and its stress $\boldsymbol{\sigma}$ in $L^\infty(L^2)$-norms are derived. Based on a first order backward differencing implicit method, a completely discrete scheme is proposed and related *a posteriori* error estimators are established in Section 4. Finally, results are summarized in Section 5 with a brief outline on future work.

2. On primal and mixed formulations

We use the usual notations for the L^2, H_0^1 and H^2 spaces and their norms and semi-norms. Let H^{-1} be the dual space of H_0^1 and let $\langle \cdot,\cdot\rangle$ be a duality paring between H^{-1} and H_0^1. Since we shall be dealing with time-space domain, we further introduce for a Banach space X with norm $\|\cdot\|_X$, the space $L^p(0,T;X)$ denoted by $L^p(X)$, for $1\le p\le\infty$ with norm

$$\|v\|_{L^p(X)} = \left(\int_0^T \|v\|_X dt\right)^{1/p}, \quad 1\le p<\infty,$$

and for $p=\infty$,

$$\|v\|_{L^\infty(X)} = \operatorname*{ess\,sup}_{0<t<T} \|v(t)\|_X.$$

Moreover, we denote by $H^m(X)$ the space of vector valued functions $\phi:(0,T)\longrightarrow X$ such that $\frac{d^j}{dt^j}\phi\in L^2(X)$ for $j=0,1,\ldots,m$ with the standard norm $\|\cdot\|_{H^m(X)}$. Further, we define $\mathcal{D}(\Omega)$ as the space of infinitely differentiable functions with compact support in Ω and call its topological dual as $\mathcal{D}'(\Omega)$.

For the weak primal formulation, define a bilinear form for $w,z\in H_0^1$

$$a(w,z) := (A\nabla w,\nabla z).$$

Given $f\in L^2(L^2)$, $u_0\in H_0^1$ and $u_1\in L^2$, the weak formulation of (1)-(2) is to seek a function $u:(0,T]\longrightarrow H_0^1$ with $u(0)=u_0$ and $u_t(0)=u_1$ such that

$$\langle u_{tt},w\rangle + a(u,w) = (f,w)\ \forall\ w\in H_0^1. \tag{3}$$

Note that for $f\in L^2(L^2)$, $u_0\in H_0^1$ and $u_1\in L^2$, there exists a unique weak solution u of (1)-(2) satisfying $u\in L^2(H_0^1), u_t\in L^2(L^2)$ and $u_{tt}\in L^2(H^{-1})$. Moreover, the equation (3) is satisfied for almost all $t\in(0,T]$. For a proof, refer to Evans ([19], pp. 399-408).

For mixed formulation, let

$$\mathbf{H}(div,\Omega) = \{\boldsymbol{\phi}\in(L^2(\Omega))^2 : \nabla\cdot\boldsymbol{\phi}\in L^2(\Omega)\}$$

be a Hilbert space equipped with norm $\|\phi\|_{\mathbf{H}(div,\Omega)} = (\|\phi\|^2 + \|\nabla\cdot\phi\|^2)^{\frac{1}{2}}$.

Now, introduce

$$\boldsymbol{\sigma} = -A\nabla u, \tag{4}$$

and $\alpha = A^{-1}$. Then, the equation (1) is rewritten as

$$\alpha\boldsymbol{\sigma} + \nabla u = 0, \quad u_{tt} + \nabla\cdot\boldsymbol{\sigma} = f. \tag{5}$$

Set $W = L^2(\Omega)$ and $\mathbf{V} = \mathbf{H}(div,\Omega)$. For given $f \in L^2(W)$, $u_0, u_1 \in W$, a weak mixed formulation for (1)-(2) is to find $(u,\boldsymbol{\sigma}) : (0,T] \to W \times \mathbf{V}$ with $u(0) = u_0$ and $u_t(0) = u_1$ such that $(u,\boldsymbol{\sigma})$ satisfies

$$u, u_t \in L^2(W),\ u_{tt} \in L^2(H^{-1}) \ \text{ and } \boldsymbol{\sigma} \in L^2(\mathbf{V})$$

and

$$(\alpha\boldsymbol{\sigma}, \mathbf{v}) - (u, \nabla\cdot\mathbf{v}) = 0\ \forall\ \mathbf{v} \in \mathbf{V}, \tag{6}$$

$$\langle u_{tt}, w\rangle + (\nabla\cdot\boldsymbol{\sigma}, w) = (f, w)\ \forall\ w \in W. \tag{7}$$

Below, we discuss the equivalence of weak primal and weak mixed formulations.

Theorem 2.1. *The pair $(u,\boldsymbol{\sigma}) \in L^2(W)\times L^2(\mathbf{V})$ with $u_{tt} \in L^2(H^{-1})$ and $(u_0,u_1) \in W\times W$ is a solution of the mixed formulation* (6)-(7) *if and only if u is a solution of the weak formulation* (3) *and $\boldsymbol{\sigma} = -A\nabla u$ with $u \in L^2(H_0^1)$ and $u_0 \in H_0^1$.*

Proof. Let $(u,\boldsymbol{\sigma}) \in L^2(W)\times L^2(\mathbf{V})$ with $u_{tt} \in L^2(H^{-1})$ be a solution of (6)-(7) and let $\phi \in \mathcal{D}(\Omega)$. Choose $\mathbf{v} = \text{Curl}\ \phi := (-\partial\phi/\partial x_2, \partial\phi/\partial x_1)$ in (6). Since it is divergence free, we obtain

$$(\alpha\boldsymbol{\sigma}, \text{Curl}\ \phi) = 0 \quad \text{a.e. } t \in (0,T).$$

Using distributional derivative, it follows that

$$\langle \text{Curl}\ (\alpha\boldsymbol{\sigma}), \phi\rangle = 0\ \forall\phi \in \mathcal{D}(\Omega), \quad \text{a.e. } t \in (0,T),$$

and hence,

$$\text{Curl}\ (A^{-1}\boldsymbol{\sigma}) = 0 \ \text{ in } \ \mathcal{D}'(\Omega), \quad \text{a.e. } t \in (0,T).$$

Now, a use of Helmholtz decomposition yields for some $\psi \in H_0^1$

$$A^{-1}\boldsymbol{\sigma} = \nabla\psi \quad \text{a.e. } t \in (0,T).$$

Apply this in (6) to arrive at

$$(\nabla\cdot\mathbf{v}, \psi + u) = 0\ \forall\mathbf{v} \in \mathbf{V}, \quad \text{a.e. } t \in (0,T), \tag{8}$$

which shows $u = -\psi \in H_0^1$. Hence for *a.e.* $t \in (0,T)$,

$$A^{-1}\boldsymbol{\sigma} = -\nabla u$$

and

$$\boldsymbol{\sigma} = -A\nabla u \ \text{ a.e. } t \in (0,T).$$

On substitution in (7) yields (1) and hence, it satisfies (3) for *a.e.* $t \in (0,T)$. Since, $u, u_t \in L^2(W)$, $u \in C^0[0,T]$ and (8) holds for $t = 0$. Thus, $u_0 = -\psi(0) \in H_0^1$.

For the converse, let u be a weak solution of (1) satisfying (3). Now, set $\boldsymbol{\sigma} = -A\nabla u \in L^2(L^2)$. Then, the equation (1) becomes

$$u_{tt} + \nabla\cdot\boldsymbol{\sigma} = f \quad \text{in } \ \mathcal{D}'(\Omega), \quad \text{a.e. } t \in (0,T).$$

Since, $f \in L^2(L^2)$ and $\nabla\cdot\boldsymbol{\sigma} \in L^2(L^2)$, therefore, $u_{tt} + \nabla\cdot\boldsymbol{\sigma} \in L^2(L^2)$ and (7) is satisfied for *a.e.* $t \in (0,T)$. Note that $\alpha\boldsymbol{\sigma} = -A^{-1}\boldsymbol{\sigma} = -\nabla u$. Multiply by $\mathbf{v} \in \mathbf{V}$, and integrate over Ω to show that (6) is satisfied. This concludes the rest of the proof. □

Since the weak primal formulation (3) is well-posed, by the Theorem 2.1 on equivalence, the weak mixed formulation is well-posed. As a byproduct, since $f \in L^2(L^2)$ and $\nabla \cdot \boldsymbol{\sigma} \in L^2(L^2)$, therefore, $u_{tt} \in L^2(L^2)$.

Since, the matrix A is uniformly positive definite, there exist two positive constants a_0 and a_1 such that

$$a_0\|\boldsymbol{\sigma}\| \leq \|\boldsymbol{\sigma}\|_{A^{-1}} \leq a_1\|\boldsymbol{\sigma}\|, \text{ where } \|\boldsymbol{\sigma}\|^2_{A^{-1}} := (\alpha\boldsymbol{\sigma}, \boldsymbol{\sigma}).$$

3. A posteriori error estimates for the semidiscrete scheme

This section focuses on a mixed finite element method for the hyperbolic problem (1)-(2) and *a posteriori* error estimates are derived for the semidiscrete mixed Galerkin approximation to (1)-(2).

For the semidiscrete mixed formulation corresponding to (6)-(7), let $\mathcal{T}_h = \{K\}$ be a shape-regular partition of the domain Ω into triangles of diameter $h_K = \text{diam}(K)$. To each triangulation $\mathcal{T}_h$, we now associate a positive piecewise constant function $h(x)$ defined on $\bar{\Omega}$ by $h|_K = h_K \ \forall K \in \mathcal{T}_h$. Let Γ_h denote the set of all internal edges E of the triangulation $\mathcal{T}_h$. Further, let $\mathbf{V}_h$ and W_h be appropriate finite element subspaces of $\mathbf{V}$ and W satisfying the Ladyzenskaya-Babuska-Brezzi (LBB) condition. More precisely, for the spaces $\mathbf{V}_h$ and W_h, assume that there exists a linear operator $\Pi_h : \mathbf{V} \to \mathbf{V}_h$ such that $\nabla \cdot \Pi_h = P_h(\nabla \cdot\ )$, where $P_h : \ W \to W_h$ is the L^2-projection defined by

$$(\phi - P_h\phi, w_h) = 0 \ \forall \ w_h \in W_h, \ \phi \in W.$$

Further, we assume that the finite element spaces satisfy the following properties:

$$\|\mathbf{v} - \Pi_h\mathbf{v}\| \leq C_I h^r\|\mathbf{v}\|_r, \quad 1 \leq r \leq \ell + 1, \ \|w - P_h w\| \leq C_I h^r\|w\|_r, \quad 0 \leq r \leq \ell + 1.$$

Note that for $\mathbf{v} \in \mathbf{H}(div, \Omega)$ and $w \in L^2(\Omega)$, the following properties hold true:

$$(\nabla \cdot (\mathbf{v} - \Pi_h\mathbf{v}), w_h) = 0, \ w_h \in W_h; \ (w - P_h w, \nabla \cdot \mathbf{v}_h) = 0, \ \mathbf{v}_h \in \mathbf{V}_h.$$

For example, let W_h and $\mathbf{V_h}$ be the Raviart-Thomas spaces of index $l \geq 0$ defined by

$$W_h = \{w \in L^2(\Omega) : \ w|_K \in P_l(K) \ \forall K \in T_h\}$$

and

$$\mathbf{V_h} = \{\mathbf{v} \in \mathbf{H}(div, \Omega) : \ w|_K \in RT_l(K) \ \forall K \in T_h\},$$

where $RT_l(K) = (P_l(K))^2 + xP_l(K)$, $l \geq 0$. For more examples of these spaces including Brezzi-Douglas-Marini spaces and Brezzi-Douglas-Fortin-Marini spaces,etc., see [11].

The corresponding semidiscrete mixed finite element formulation is to seek a pair $(u_h, \boldsymbol{\sigma}_h) : (0, T] \to W_h \times \mathbf{V}_h$ such that

$$(\alpha\boldsymbol{\sigma}_h, \mathbf{v}_h) - (u_h, \nabla \cdot \mathbf{v}_h) = 0 \ \forall \ \mathbf{v}_h \in \mathbf{V}_h, \tag{9}$$

$$(u_{h,tt}, w_h) + (\nabla \cdot \boldsymbol{\sigma}_h, w_h) = (f, w_h) \ \forall \ w_h \in W_h \tag{10}$$

with $u_h(0) \in W_h$ and $u_{h,t}(0) \in W_h$ to be defined later. Since W_h and $\mathbf{V}_h$ are finite dimensional, from (9) we can eliminate $\boldsymbol{\sigma}_h$ in the discrete level by writing it in terms of u_h. Therefore, substituting in (10), we obtain a second order linear system of ODEs and existence follows using ODE linear theory. Then, as a consequence of LBB condition and energy estimates, uniqueness can be proved easily and hence, we skip the proof.

Set $e_u = u_h - u$ and $e_{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_h - \boldsymbol{\sigma}$. From (6)-(7) and (9)-(10), e_u and $e_{\boldsymbol{\sigma}}$ satisfy the following equations

$$(\alpha e_{\boldsymbol{\sigma}}, \mathbf{v}) - (e_u, \nabla\cdot\mathbf{v}) = \mathbf{r}_1(\mathbf{v}) \ \forall\ \mathbf{v}\in\mathbf{V}, \tag{11}$$

$$(e_{u,tt}, w) + (\nabla\cdot e_{\boldsymbol{\sigma}}, w) = \mathbf{r}_2(w) \ \forall\ w\in W, \tag{12}$$

where the residuals $\mathbf{r}_1$ and $\mathbf{r}_2$ are given by

$$\mathbf{r}_1(\mathbf{v}) := (\alpha\boldsymbol{\sigma}_h, \mathbf{v}) - (u_h, \nabla\cdot\mathbf{v}),$$

and

$$\mathbf{r}_2(w) := (u_{h,tt}, w) + (\nabla\cdot\boldsymbol{\sigma}_h, w) - (f, w).$$

Following [29], now introduce mixed elliptic reconstructions $\tilde{u}(t)\in H_0^1(\Omega)$ and $\tilde{\boldsymbol{\sigma}}(t)\in\mathbf{V}$ of $u_h(t)$ and $\boldsymbol{\sigma}_h(t)$ for $t\in(0,T]$, respectively, as follows: for given u_h and $\boldsymbol{\sigma}_h$, let the mixed elliptic reconstructions $\tilde{u}$ and $\tilde{\boldsymbol{\sigma}}$ satisfy

$$(\nabla\cdot(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h), w) = -\mathbf{r}_2(w),\ \ \forall w\in W, \tag{13}$$

$$(\alpha(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h), \mathbf{v}) - (\tilde{u} - u_h, \nabla\cdot\mathbf{v}) = -\mathbf{r}_1(\mathbf{v}),\ \ \forall\mathbf{v}\in\mathbf{V}. \tag{14}$$

Using Theorem 4.3 (pp. 132) of [10], one can verify that for a given $u_h, \boldsymbol{\sigma}_h, \mathbf{r}_1$ and $\mathbf{r}_2$, the system (13)-(14) has a unique pair of solution $\{\tilde{u}(t), \tilde{\boldsymbol{\sigma}}(t)\}\in W\times\mathbf{V}$, for $t\in(0,T]$. Here elliptic reconstructions are assumed to be smooth in time.

Note that $\mathbf{r}_1(\mathbf{v}_h) = 0\ \ \forall\mathbf{v}_h\in\mathbf{V}_h$, and $\mathbf{r}_2(w_h) = 0\ \ \forall w_h\in W_h$. Then, $\boldsymbol{\sigma}_h$ and u_h are indeed mixed elliptic projections of $\tilde{\boldsymbol{\sigma}}$ and $\tilde{u}$, respectively.

Using mixed elliptic reconstructions, we now rewrite

$$e_u := (\tilde{u} - u) - (\tilde{u} - u_h) =: \xi_u - \eta_u,$$

and

$$e_{\boldsymbol{\sigma}} := (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}) - (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h) =: \xi_{\boldsymbol{\sigma}} - \eta_{\boldsymbol{\sigma}}.$$

An application of (13)-(14) in (11)-(12) yields

$$(\alpha\xi_{\boldsymbol{\sigma}}, \mathbf{v}) - (\xi_u, \nabla\cdot\mathbf{v}) \ = \ 0\ \ \forall\mathbf{v}\in\mathbf{V}, \tag{15}$$

$$(\xi_{u,tt}, w) + (\nabla\cdot\xi_{\boldsymbol{\sigma}}, w) \ = \ (\eta_{u,tt}, w)\ \ \forall w\in W. \tag{16}$$

With mixed elliptic reconstructions $\tilde{u}$ and $\tilde{\boldsymbol{\sigma}}$ satisfying (13)-(14), apply (15) to check that

$$\alpha\tilde{\boldsymbol{\sigma}} = -\nabla\tilde{u}. \tag{17}$$

Lemma 3.1. *Let ξ_u and $\xi_{\boldsymbol{\sigma}}$ satisfy* (15)-(16)*. Then, the following estimates hold:*

$$\|\xi_{u,t}(t)\| + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}(t)\| \le \|\xi_{u,t}(0)\| + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}(0)\| + 2\int_0^t \|\eta_{u,tt}(s)\|\,ds. \tag{18}$$

and

$$\|\xi_u(t)\| \le \|\xi_u(0)\| + 2\int_0^t \|\eta_{u,t}(s)\|\,ds.$$

Proof. Differentiate (15) with respect to t and set $\mathbf{v} = \xi_{\boldsymbol{\sigma}}$ in the resulting equation to find that

$$(\alpha\xi_{\boldsymbol{\sigma},t}, \xi_{\boldsymbol{\sigma}}) - (\xi_{u,t}, \nabla\cdot\xi_{\boldsymbol{\sigma}}) = 0. \tag{19}$$

Choose $w = \xi_{u,t}$ in (16). Then, add the resulting equations to (19) to arrive at

$$\frac{1}{2}\frac{d}{dt}(\|\xi_{u,t}\|^2 + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}\|^2) \ = \ (\eta_{u,tt}, \xi_{u,t}).$$

On integrating the above equation from 0 to t, a use of the Cauchy-Schwarz inequality yields

$$\begin{aligned} \|\xi_{u,t}(t)\|^2 + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}\|^2 &\leq \|\xi_{u,t}(0)\|^2 + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}(0)\|^2 \\ &+ 2\int_0^t \|\eta_{u,tt}(s)\|\,\|\xi_{u,t}(s)\|ds. \end{aligned} \tag{20}$$

Setting

$$|||(\xi_{u,t},\xi_{\boldsymbol{\sigma}})(t)||| = (\|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}(t)\|^2 + \|\xi_{u,t}(t)\|^2)^{1/2},$$

let $t^* \in [0,t]$ be such that

$$|||(\xi_{u,t},\xi_{\boldsymbol{\sigma}})(t^*)||| = \max_{0\leq s\leq t} |||(\xi_{u,t},\xi_{\boldsymbol{\sigma}})(s)|||.$$

Then at time $t = t^*$, (20) becomes

$$|||(\xi_{u,t},\xi_{\boldsymbol{\sigma}})(t^*)||| \leq |||(\xi_{u,t},\xi_{\boldsymbol{\sigma}})(0)||| + 2\int_0^{t^*} \|\eta_{u,tt}(s)\|\,ds,$$

and hence,

$$|||(\xi_{u,t},\xi_{\boldsymbol{\sigma}})(t)||| \leq |||(\xi_{u,t},\xi_{\boldsymbol{\sigma}})(0)||| + 2\int_0^{t^*} \|\eta_{u,tt}(s)\|\,ds. \tag{21}$$

This completes the proof of (18). Note that from (21), we obtain $L^\infty(L^2)$-estimate of the displacement using $\xi(t) = \xi(0) + \int_0^t \xi_{u,t}(s)\ ds$. Now in order to reduce the regularity, an integration tool which is a variant of Baker's time testing procedure is used in a crucial way. To motivate our tool, integrate (16) with respect to time to arrive at

$$(\xi_{u,t},w) + (\nabla\cdot\hat{\xi}_{\boldsymbol{\sigma}},w) = (\xi_{u,t}(0),w) + (\eta_{u,t},w) - (\eta_{u,t}(0),w), \tag{22}$$

where $\hat{\xi}_{\boldsymbol{\sigma}} = \int_0^t \xi_{\boldsymbol{\sigma}}(s)\ ds$. Choose $w = \xi_u$ in (22) and $\mathbf{v} = \hat{\xi}_{\boldsymbol{\sigma}}$ in (15) and adding the resulting equations to obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|\xi_u(t)\|^2 + \|\alpha^{1/2}\hat{\xi}_{\boldsymbol{\sigma}}(t)\|^2\Big) = (e_{u,t}(0),\xi_u) + (\eta_{u,t},\xi_u).$$

Then, integrate with respect to time and use kick back arguments to arrive at

$$\|\xi_u(t)\| \leq \|\xi_u(0)\| + 2\int_0^t \|\eta_{u,t}(s)\|\,ds.$$

This completes the rest of the proof. □

To prove the main theorem of this section, we need the following *a posteriori* estimates of $\eta_u, \eta_{u,t}$ and $\eta_{\boldsymbol{\sigma}}$ related to the mixed elliptic reconstructions (13)-(14). For a proof, see [15].

Lemma 3.2. *For Raviart-Thomas elements, there exists a positive constant C which depends only on the coefficient matrix A, the domain Ω, the shape regularity of the elements and polynomial degree ℓ such that for $\ell = 0,1$,*

$$\|\eta_u\| \leq C\Big(\|h^{\ell+1}\mathbf{r}_2\| + \min_{w_h\in W_h} \|h(\alpha\boldsymbol{\sigma}_h - \nabla_h w_h)\|\Big), \tag{23}$$

and for $j = 1,2$,

$$\left\|\frac{\partial^j\eta_u}{\partial t^j}\right\| \leq C\left(\left\|h^{\ell+1}\frac{\partial^j\mathbf{r}_2}{\partial t^j}\right\| + \min_{w_h\in W_h}\left\|h\left(\alpha\frac{\partial^j\boldsymbol{\sigma}_h}{\partial t^j} - \nabla_h w_h\right)\right\|\right), \tag{24}$$

and

$$\|\alpha^{1/2}\eta_{\boldsymbol{\sigma}}\| \leq C\left(\|h\mathbf{r}_2\| + \|h^{1/2}J(\alpha\boldsymbol{\sigma}_h\cdot\mathbf{t})\|_{0,\Gamma_h} + \|h\, curl_h\,(\alpha\boldsymbol{\sigma}_h)\|\right), \tag{25}$$

where $\mathbf{r}_2 = (u_{h,tt} - f + \nabla\cdot\boldsymbol{\sigma}_h)$ is a residual, $J(\alpha\boldsymbol{\sigma}_h\cdot\mathbf{t})$ denotes the jump of $\alpha\boldsymbol{\sigma}_h\cdot\mathbf{t}$ across element edge E with $\mathbf{t}$ being the tangential unit vector along the edge $E\in\Gamma_h$ and ∇_h is piecewise gradient defined by $\nabla_h w_h|_K := \nabla(w_h|_K)$ for all $K\in\mathcal{T}_h$.

Now, let $\mathcal{E}_1(\mathbf{r}_2,\boldsymbol{\sigma}_h;\mathcal{T}_h)$, $\mathcal{E}_1(\frac{\partial^j\mathbf{r}_2}{\partial t^j},\frac{\partial^j\boldsymbol{\sigma}_h}{\partial t^j};\mathcal{T}_h)$ and $\mathcal{E}_2(\mathbf{r}_2,\boldsymbol{\sigma}_h;\mathcal{T}_h)$ denote the terms on the right-hand sides of (23), (24) and (25), respectively. Then, using Lemmas 3.1-3.2, we finally obtain the main theorem of this section as:

Theorem 3.1. *Let* $(u,\boldsymbol{\sigma})$ *be a solution of the mixed formulation* (6)-(7) *and let* $(u_h,\boldsymbol{\sigma}_h)$ *be a solution of the semidiscrete mixed formulation* (9)-(10). *Then, the following a posteriori estimates hold:*

$$\begin{aligned}\|e_{u,t}\|_{L^\infty(0,T;L^2(\Omega))} &+ \|\alpha^{1/2}e_{\boldsymbol{\sigma}}\|_{L^\infty(0,T;L^2(\Omega))}\\ &\lesssim \|e_{u,t}(0)\| + \|\alpha^{1/2}e_{\boldsymbol{\sigma}}(0)\| + \mathcal{E}_1(\mathbf{r}_{2,t}(0),\boldsymbol{\sigma}_{h,t}(0);\mathcal{T}_h)\\ &\quad+\mathcal{E}_2(\mathbf{r}_2(0),\boldsymbol{\sigma}_h(0);\mathcal{T}_h) + \|\mathcal{E}_1(\mathbf{r}_{2,t},\boldsymbol{\sigma}_{h,t};\mathcal{T}_h)\|_{L^\infty(0,T)}\\ &\quad+\|\mathcal{E}_2(\mathbf{r}_2,\boldsymbol{\sigma}_h;\mathcal{T}_h)\|_{L^\infty(0,T)} + \int_0^T \mathcal{E}_1(\mathbf{r}_{2,tt},\boldsymbol{\sigma}_{h,tt};\mathcal{T}_h)\,ds,\end{aligned}$$

and

$$\begin{aligned}\|e_u\|_{L^\infty(0,T;L^2(\Omega))} &\lesssim \|e_u(0)\| + \mathcal{E}_1(\mathbf{r}_2(0),\boldsymbol{\sigma}_h(0);\mathcal{T}_h)\\ &\quad+\|\mathcal{E}_1(\mathbf{r}_2,\boldsymbol{\sigma}_h;\mathcal{T}_h)\|_{L^\infty(0,T)} + \int_0^T \mathcal{E}_1(\mathbf{r}_{2,t},\boldsymbol{\sigma}_{h,t};\mathcal{T}_h)\,ds.\end{aligned}$$

4. Completely discrete scheme

This section deals with *a posteriori* analysis for a completely discrete mixed approximation based on backward differencing.

Let $0 = t_0 < t_1 < \ldots < t_N = T$, $I_n = (t_{n-1}, t_n]$ and $k_n = t_n - t_{n-1}$. For $n\in[0:N]$, let $\mathcal{T}_n$ be a refinement of a macro-triangulation which is a triangulation of the domain Ω that satisfies the same conformity and shape regularity assumptions made on its refinements. Let

$$h_n(x) := \text{diam }(K),\quad \text{where } \ K\in\mathcal{T}_n \text{ and } \ x\in K,$$

for all $x\in\Omega$. Given two compatible triangulations $\mathcal{T}_{n-1}$ and $\mathcal{T}_n$, i.e., they are refinements of the same macro-triangulation, let $\hat{\mathcal{T}}_n$ be the finest common coarsening of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$, whose meshsize is given by $\hat{h}_n := \max(h_n, h_{n-1})$, see ([26], pp. 1655).

We consider $\mathbf{V}_h^n$ and W_h^n defined over the triangulations $\mathcal{T}^n$ as Raviart-Thomas finite element spaces of index $\ell\ge 0$ of $\mathbf{H}(div,\Omega)$ and $L^2(\Omega)$, respectively. Let $P_h^n : L^2(\Omega)\longrightarrow W_h^n$ be the L^2-projection defined by

$$(P_h^n w,\phi^n) = (w,\phi^n)\quad \forall\phi^n\in W_h^n.$$

Given $U^0 = P_h^0 u_0$, find $\{(U^n,\boldsymbol{\Sigma}^n)\}$ with $(U^n,\boldsymbol{\Sigma}^n)\in W_h^n\times\mathbf{V}_h^n$ for $n\in[1:N]$ such that

$$(\partial_t^2 U^n, w) + (\nabla\cdot\boldsymbol{\Sigma}^n, w) = (\bar{f}^n, w)\ \ \forall w\in W_h^n, \tag{26}$$

$$(\alpha\boldsymbol{\Sigma}^n,\mathbf{v}) - (U^n,\nabla\cdot\mathbf{v}) = 0\quad \forall\mathbf{v}\in\mathbf{V}_h^n, \tag{27}$$

where $\bar{f}^n$ is either chosen as point-wise value $\bar{f}^n(\cdot) := f(t_n,\cdot)$ for $n\ge 0$ or the one through average value as in Remark 4.2. Here, the backward second and first finite differences are given, respectively, by

$$\partial_t^2 U^n = \frac{\partial_t U^n - \partial_t U^{n-1}}{k_n},$$

and

$$\partial_t U^n := \begin{cases} \dfrac{U^n - U^{n-1}}{k_n}, & \text{for } n = 1, \cdots, N, \\ P_h^0 u_1, & \text{for } n = 0. \end{cases}$$

Throughout the rest of the paper, we shall use the following notation:

$$P_h^n(\partial_t \phi^n) = \frac{1}{k_n}(\phi^n - P_h^n \phi^{n-1}).$$

Following [20], we define for given a sequence of discrete values $\{V^n\}_{n=0}^N$, the time reconstruction $V : [0, T] \times \Omega \to I\!R$ or $I\!R^2$ as

$$V(t) = V^n + (t - t_n)\partial_t V^n - \frac{(t - t_{n-1})(t_n - t)^2}{k_n} \partial_t^2 V^n, \quad t_{n-1} < t \le t_n, \tag{28}$$

for $n = 1, \cdots, N$. Note that we have used the fact that $\partial_t V^0$ is well defined.

We shall use the above C^1-function $V(t)$ such that for $n = 0, 1, \cdots, N$,

$$V(t_n) = V^n, \qquad V_t(t_n) = \partial_t V^n, \qquad V_{tt}(t) = (1 + \mu^n)\partial_t^2 V^n, \tag{29}$$

for $t \in (t_{n-1}, t_n]$, where

$$\mu^n(t) := -6k_n^{-1}(t - t_{n-1/2}).$$

Similarly, we define C^1-functions $U(t)$ and $\mathbf{\Sigma}(t)$ in time variable using the discrete sequences $\{U^n\}_{n=0}^N$ and $\{\mathbf{\Sigma}^n\}_{n=0}^N$, respectively.

As in Section 3, for given $\{U^n, \mathbf{\Sigma}^n\}_{n=0}^N$, we now define the mixed elliptic reconstructions $\tilde{u}^n \in H_0^1(\Omega)$ and $\tilde{\boldsymbol{\sigma}}^n \in \mathbf{V}$ at $t = t_n$ as:

$$(\nabla \cdot (\tilde{\boldsymbol{\sigma}}^n - \mathbf{\Sigma}^n), w) = -\mathbf{r}_2^n(w), \;\; w \in W, \tag{30}$$

$$(\alpha(\tilde{\boldsymbol{\sigma}}^n - \mathbf{\Sigma}^n), \mathbf{v}) - (\tilde{u}^n - U^n, \nabla \cdot \mathbf{v}) = -\mathbf{r}_1^n(\mathbf{v}), \;\; \mathbf{v} \in \mathbf{V}, \tag{31}$$

where $\mathbf{r}_1^n(\mathbf{v}) := (\alpha \mathbf{\Sigma}^n, \mathbf{v}) - (U^n, \nabla \cdot \mathbf{v})$ and $\mathbf{r}_2^n(w) := (P_h^n(\partial_t^2 U^n), w) + (\nabla \cdot \mathbf{\Sigma}^n, w) - (\bar{f}^n, w)$.

Since $\mathbf{r}_1^n(\mathbf{v}_h) = 0 \;\; \forall \mathbf{v}_h \in \mathbf{V}_h^n$, $n \ge 0$ and $\mathbf{r}_2^n(w_h) = 0 \;\; \forall w_h \in W_h^n$, $n \ge 1$, in fact, $\mathbf{\Sigma}^n$ and U^n are mixed elliptic projections of $\tilde{\boldsymbol{\sigma}}^n$ and $\tilde{u}^n$ at time $t = t_n$, respectively. Now given $\{\tilde{u}^n\}_{n=0}^N$ and $\{\tilde{\boldsymbol{\sigma}}^n\}_{n=0}^N$, we define the C^1-functions $\tilde{u}(t)$ and $\tilde{\boldsymbol{\sigma}}(t)$ in time $t \in (0, T]$, respectively, as

$$\tilde{u}(t) = \tilde{u}^n + (t - t_n)\partial_t \tilde{u}^n - \frac{(t - t_{n-1})(t_n - t)^2}{k_n} \partial_t^2 \tilde{u}^n, \quad t_{n-1} < t \le t_n, \tag{32}$$

and

$$\tilde{\boldsymbol{\sigma}}(t) = \tilde{\boldsymbol{\sigma}}^n + (t - t_n)\partial \tilde{\boldsymbol{\sigma}}^n - \frac{(t - t_{n-1})(t_n - t)^2}{k_n} \partial_t^2 \tilde{\boldsymbol{\sigma}}^n, \quad t_{n-1} < t \le t_n, \tag{33}$$

provided that $\partial_t \tilde{u}^0$ and $\partial_t \tilde{\boldsymbol{\sigma}}^0$ are well defined.

For $t \in (0, T]$, the mixed elliptic reconstruction $\{\tilde{u}, \tilde{\boldsymbol{\sigma}}\}$ satisfies

$$(\nabla \cdot (\tilde{\boldsymbol{\sigma}} - \mathbf{\Sigma}), w) = -\mathbf{r}_2(w), \;\; w \in W, \tag{34}$$

$$(\alpha(\tilde{\boldsymbol{\sigma}} - \mathbf{\Sigma}), \mathbf{v}) - (\tilde{u} - U, \nabla \cdot \mathbf{v}) = -\mathbf{r}_1(\mathbf{v}), \;\; \mathbf{v} \in \mathbf{V}, \tag{35}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are defined as C^1-functions in time using $\{\mathbf{r}_1^n, \mathbf{r}_2^n\}_{n=1}^N$ as in (28).

Again, set

$$e_u = (\tilde{u} - u) - (\tilde{u} - U) =: \xi_u - \eta_u,$$

and

$$e_{\boldsymbol{\sigma}} = (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}) - (\tilde{\boldsymbol{\sigma}} - \mathbf{\Sigma}) =: \xi_{\boldsymbol{\sigma}} - \eta_{\boldsymbol{\sigma}}.$$

Now, the pair $\{e_u, e_{\boldsymbol{\sigma}}\}$ satisfies

$$(e_{u,tt}, w) + (\nabla \cdot e_{\boldsymbol{\sigma}}, w) = (U_{tt}, w) + (\nabla \cdot \mathbf{\Sigma}, w) - (f, w). \tag{36}$$

On splitting e_u and $e_{\boldsymbol{\sigma}}$, we obtain from (36)

$$(37)\quad (\xi_{u,tt}, w) + (\nabla\cdot\xi_{\boldsymbol{\sigma}}, w) = (\eta_{u,tt}, w) + ((I-P_h^n)U_{tt}, w) + \mu^n(t)(\partial_t^2 U^n, P_h^n w) + (\nabla\cdot(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^n), w) + (\bar{f}^n - f, w) \quad \forall w \in W.$$

Similarly, we also arrive at

$$(38)\quad (\alpha\xi_{\boldsymbol{\sigma}}, \mathbf{v}) - (\xi_u, \nabla\cdot\mathbf{v}) = (\alpha(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^n), \mathbf{v}) - (\tilde{u} - \tilde{u}^n, \nabla\cdot\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}.$$

Note that

$$(39)\quad \tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^n = (t-t_n)\partial_t\tilde{\boldsymbol{\sigma}}^n + \left(k_n^{-1}(t_n-t)^3 - (t_n-t)^2\right)\partial_t^2\tilde{\boldsymbol{\sigma}}^n,$$

and

$$(40)\quad \tilde{u} - \tilde{u}^n = (t-t_n)\partial_t\tilde{u}^n + \left(k_n^{-1}(t_n-t)^3 - (t_n-t)^2\right)\partial_t^2\tilde{u}^n.$$

Now, it follows that

$$(41)\quad (\alpha(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^n), \mathbf{v}) - (\tilde{u} - \tilde{u}^n, \nabla\cdot\mathbf{v}) = (t-t_n)\Big\{(\alpha\partial_t\tilde{\boldsymbol{\sigma}}^n, \mathbf{v}) - (\partial_t\tilde{u}^n, \nabla\cdot\mathbf{v})\Big\} + \left(k_n^{-1}(t_n-t) - (t_n-t)^2\right)\Big\{(\alpha\partial_t^2\tilde{\boldsymbol{\sigma}}^n, \mathbf{v}) - (\partial_t^2\tilde{u}^n, \nabla\cdot\mathbf{v})\Big\},$$

and from (30) with definition of $\mathbf{r}_1(\mathbf{v})$, we find that

$$(42)\quad (\alpha\tilde{\boldsymbol{\sigma}}^n, \mathbf{v}) - (\tilde{u}^n, \nabla\cdot\mathbf{v}) = 0 \ \ \forall \mathbf{v} \in \mathbf{V}.$$

From (42), the equation (41) takes the form

$$(\alpha(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^n), \mathbf{v}) - (\tilde{u} - \tilde{u}^n, \nabla\cdot\mathbf{v}) = 0,$$

and thus, (38) becomes

$$(43)\quad (\alpha\xi_{\boldsymbol{\sigma}}, \mathbf{v}) - (\xi_u, \nabla\cdot\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}.$$

Theorem 4.1. *Let $(u, \boldsymbol{\sigma})$ and $(U, \boldsymbol{\Sigma})$ be the solution of (6)-(7) and (26)-(27), respectively. Then, the following estimates hold for $t \in (t_{n-1}, t_n]$*

$$\|\xi_{u,t}(t)\| + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}(t)\| \le \|\xi_{u,t}(0)\| + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}(0)\| + 2\sum_{j=1}^{4}\mathcal{E}_{1,j}(t) + 2\int_0^t \|\eta_{u,tt}(s)\|\,ds,$$

where

$$\mathcal{E}_{1,1}(t) = \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} ||(I-P_h^j)U_{tt}||\,ds + \int_{t_{n-1}}^{t} ||(I-P_h^n)U_{tt}||\,ds,$$

$$\mathcal{E}_{1,2}(t) = \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} ||\mu^j\partial_t^2U^j||\,ds + \int_{t_{n-1}}^{t} ||\mu^n\partial_t^2U^n||\,ds,$$

$$\begin{aligned}\mathcal{E}_{1,3}(t) = {} & \sum_{j=1}^{n-1}\left(\frac{k_j^2}{2}||\partial_t(\mathbf{r}_2^j - \nabla\cdot\boldsymbol{\Sigma}^j)|| + \frac{k_j^3}{12}||\partial_t^2(\mathbf{r}_2^j - \nabla\cdot\boldsymbol{\Sigma}^j)||\right) \\ & + \int_{t_{n-1}}^{t}(t_n-s)||\partial_t(\mathbf{r}_2^n - \nabla\cdot\boldsymbol{\Sigma}^n)||\,ds \\ & + \int_{t_{n-1}}^{t}\left((t_n-s)^2 - \frac{(t_n-s)^3}{k_n}\right)||\partial_t^2(\mathbf{r}_2^n - \nabla\cdot\boldsymbol{\Sigma}^n)||\,ds,\end{aligned}$$

$$\mathcal{E}_{1,4}(t) = \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} ||\bar{f}^j - f||\,ds + \int_{t_{n-1}}^{t} ||\bar{f}^n - f||\,ds.$$

Proof . Differentiate (43) with respect to t. Then, choose $v = \xi_{\boldsymbol{\sigma}}$ in the resulting equation and $w = \xi_{u,t}$ in (37) to obtain for $t \in (t_{n-1}, t_n]$

$$\begin{aligned}\frac{1}{2}\left(\|\xi_{u,t}\|^2 + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}\|^2\right) &= (\eta_{u,tt}, \xi_{u,t}) + ((I - P_h^n)U_{tt}, \xi_{u,t}) \\ &\quad + \mu^n(t)(\partial_t^2 U^n, P_h^n \xi_{u,t}) \\ &\quad + (\nabla\cdot(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^n), \xi_{u,t}) + (\bar{f}^n - f, \xi_{u,t}).\end{aligned}$$

On integrating from 0 to t with $t \in (t_{n-1}, t_n]$, we find that

$$\begin{aligned}&\frac{1}{2}\left(\|\xi_{u,t}\|^2 + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}\|^2\right) \\ &= \frac{1}{2}\left(\|\xi_{u,t}(0)\|^2 + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}(0)\|^2\right) + \int_0^t (\eta_{u,tt}, \xi_{u,t})\, ds \\ &\quad + J_{1,1}^n(\xi_{u,t}) + J_{1,2}^n(\xi_{u,t}) + J_{1,3}^n(\xi_{u,t}) + J_{1,4}^n(\xi_{u,t}),\end{aligned} \tag{44}$$

where

$$\begin{aligned}J_{1,1}^n(\xi_{u,t}) &:= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\left((I - P_h^j)U_{tt}, \xi_{u,t}\right) ds + \int_{t_{n-1}}^{t}((I - P_h^n)U_{tt}, \xi_{u,t})\, ds, \\ J_{1,2}^n(\xi_{u,t}) &:= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\mu^j(\partial_t^2 U^j, P_h^j\xi_{u,t})\, ds + \int_{t_{n-1}}^{t}\mu^n(\partial_t^2 U^n, P_h^n\xi_{u,t})\, ds, \\ J_{1,3}^n(\xi_{u,t}) &:= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}(\nabla\cdot(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^j), \xi_{u,t})\, ds + \int_{t_{n-1}}^{t}(\nabla\cdot(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^n), \xi_{u,t})\, ds,\end{aligned}$$

and

$$J_{1,4}^n(\xi_{u,t}) := \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}(\bar{f}^j - f, \xi_{u,t})\, ds + \int_{t_{n-1}}^{t}(\bar{f}^n - f, \xi_{u,t})\, ds.$$

Set

$$E_1^2(t) := \|\xi_{u,t}(t)\|^2 + \|\alpha^{1/2}\xi_{\boldsymbol{\sigma}}(t)\|^2,$$

and let at $t = t^* \in (0, t]$ be such that

$$E_1(t^*) = \max_{0\le s\le t} E_1(s).$$

Now, a use of the Cauchy-Schwarz inequality yields

$$|J_{1,1}^n(\xi_{u,t})| \le \left(\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\|(I - P_h^j)U_{tt}\|\, ds + \int_{t_{n-1}}^{t}\|(I - P_h^n)U_{tt}\|\, ds\right) E_1(t^*).$$

and similarly,

$$|J_{1,2}^n(\xi_{u,t})| \le \left(\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}||\mu^j\partial_t^2 U^j||\, ds + \int_{t_{n-1}}^{t}||\mu^n\partial_t^2 U^n||\, ds\right) E_1(t^*).$$

For $J_{1,3}^n$, we rewrite using (39) as

$$\begin{aligned}(\nabla\cdot(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^n), w) &= (t - t_n)\,(\nabla\cdot\partial_t\tilde{\boldsymbol{\sigma}}^n, w) \\ &\quad + \left(k_n^{-1}(t_n - t)^3 - (t_n - t)^2\right)\left(\nabla\cdot\partial_t^2\tilde{\boldsymbol{\sigma}}^n, w\right).\end{aligned} \tag{45}$$

From (3.6), we obtain

$$(\nabla\cdot\tilde{\boldsymbol{\sigma}}^n, w) = -\mathbf{r}_2^n(w) + (\nabla\cdot\boldsymbol{\Sigma}^n, w)\,, \tag{46}$$

and therefore, for $j = 1, 2$

$$\left(\nabla \cdot \partial_t^j \tilde{\boldsymbol{\sigma}}^n, w\right) = -(\partial_t^j \mathbf{r}_2^n)(w) + \left(\nabla \cdot \partial_t^j \boldsymbol{\Sigma}^n, w\right). \tag{47}$$

On substituting (47) for $j = 1, 2$ in $J_{1,3}^n$, we arrive at

$$\begin{aligned} J_{1,3}^n(\xi_{u,t}) = & \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \Big((t_j - s) \Big\{ (\partial_t \mathbf{r}_2^j)(\xi_{u,t}) - \big(\nabla \cdot \partial_t \boldsymbol{\Sigma}^j, \xi_{u,t}\big) \Big\} \\ & - \big(k_j^{-1}(t_j - s)^3 - (t_j - s)^2\big) \Big\{ (\partial_t^2 \mathbf{r}_2^j)(\xi_{u,t}) - \big(\nabla \cdot \partial_t^2 \boldsymbol{\Sigma}^j, \xi_{u,t}\big) \Big\} \Big) \\ & + \int_{t_{n-1}}^{t} \Big((t_n - s) \left\{ (\partial_t \mathbf{r}_2^n)(\xi_{u,t}) - (\nabla \cdot \partial_t \boldsymbol{\Sigma}^n, \xi_{u,t}) \right\} \\ & - \big(k_n^{-1}(t_n - s)^3 - (t_n - s)^2\big) \left\{ (\partial_t^2 \mathbf{r}_2^n)(\xi_{u,t}) - \big(\nabla \cdot \partial_t^2 \boldsymbol{\Sigma}^n, \xi_{u,t}\big) \right\} \Big). \end{aligned}$$

Using the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \left|J_{1,3}^n(\xi_{u,t})\right| \leq & \sum_{j=1}^{n-1} \left(\frac{k_j^2}{2} \left\| \partial_t(\mathbf{r}_2^j - \nabla \cdot \boldsymbol{\Sigma}^j) \right\| + \frac{k_j^3}{12} \left\| \partial_t^2(\mathbf{r}_2^j - \nabla \cdot \boldsymbol{\Sigma}^j) \right\| \right) E_1(t^*) \\ & + \Big(\int_{t_{n-1}}^{t} \big\{ (t_n - s) \|\partial_t(\mathbf{r}_2^n - \nabla \cdot \boldsymbol{\Sigma}^n)\| \\ & + \big((t_n - s)^2 - k_n^{-1}(t_n - s)^3\big) \|\partial_t^2(\mathbf{r}_2^n - \nabla \cdot \boldsymbol{\Sigma}^n)\| \big\} \Big) E_1(t^*). \end{aligned}$$

For $J_{1,4}^n$, we note that

$$\begin{aligned} \left|J_{1,4}^n(\xi_{u,t})\right| = & \left| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (\bar{f}^j - f, \xi_{u,t})\, ds + \int_{t_{n-1}}^{t} (\bar{f}^n - f, \xi_{u,t})\, ds \right| \\ \leq & \left(\sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \|\bar{f}^j - f\|\, ds + \int_{t_{n-1}}^{t} \|\bar{f}^n - f\|\, ds \right) E_1(t^*). \end{aligned}$$

On substituting the estimates of $J_{1,j}^n(\xi_{u,t})$, $j = 1, \cdots, 4$, in (44), we arrive at

$$E_1(t) \leq E_1(t^*) \leq E_1(0) + 2\sum_{j=1}^{4} \mathcal{E}_{1,j}(t) + 2\int_0^t \|\eta_{u,tt}(s)\|\, ds.$$

This completes the rest of the proof. □

Remark 4.1. *The term* $(\mathbf{r}_2^j - \nabla \cdot \boldsymbol{\Sigma}^j)$ *can be replaced by* $(\partial_t^2 U^j - \bar{f}^j)$.

For obtaining $L^\infty(L^2)$ estimate for e_u, we now integrate (37) with respect to time from 0 to t with $t \in (t_{n-1}, t_n]$, to arrive at

$$\begin{aligned} (\xi_{u,t}, w) + \left(\nabla \cdot \hat{\xi}_{\boldsymbol{\sigma}}, w\right) = & (e_{u,t}(0), w) + (\eta_{u,t}, w) \\ & + J_{2,1}^n(w) + J_{2,2}^n(w) + J_{2,3}^n(w) + J_{2,4}^n(w), \end{aligned} \tag{48}$$

where

$$
\begin{aligned}
J_{2,1}^n(w) &:= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\Big((I-P_h^j)U_{tt},w\Big)\,ds+\int_{t_{n-1}}^{t}((I-P_h^n)U_{tt},w)\,ds,\\
J_{2,2}^n(w) &:= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\mu^j(\partial_t^2U^j,w)\,ds+\int_{t_{n-1}}^{t}\mu^n(\partial_t^2U^n,w)\,ds,\\
J_{2,3}^n(w) &:= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}(\nabla\cdot(\tilde{\boldsymbol\sigma}-\tilde{\boldsymbol\sigma}^j),w)\,ds+\int_{t_{n-1}}^{t}(\nabla\cdot(\tilde{\boldsymbol\sigma}-\tilde{\boldsymbol\sigma}^n),w)\,ds,\\
J_{2,4}^n(w) &:= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}(\bar f^j-f,w)\,ds+\int_{t_{n-1}}^{t}(\bar f^n-f,w)\,ds.
\end{aligned}
$$

Note that

$$J_{2,4}=\int_{t_{n-1}}^{t}(\bar f^n-f,w)\,ds \quad\text{ and }\quad J_{2,2}^n(w)=\int_{t_{n-1}}^{t}\mu^n(\partial_t^2U^n,w)\,ds$$

as $\int_{t_{j-1}}^{t_j}\mu^j=0$. Further, since P_h^j commutes with time derivative, $J_{2,1}^n(w)$ can be written as

$$
\begin{aligned}
J_{2,1}^n(w) =&\sum_{j=1}^{n-1}\Big((I-P_h^j)U_t(t_j)-(I-P_h^j)U_t(t_{j-1}),w\Big)\\
&+((I-P_h^n)U_t(t),w)-((I-P_h^n)U_t(t_{n-1}),w)\\
=&\sum_{j=1}^{n-1}\Big((I-P_h^j)U_t(t_j)-(I-P_h^{j-1})U_t(t_{j-1}),w\Big)+\sum_{j=1}^{n-1}(P_h^j-P_h^{j-1})U_t(t_{j-1})\\
&+((I-P_h^n)U_t(t),w)-((I-P_h^n)U_t(t_{n-1}),w)\\
=&\sum_{j=0}^{n-1}\Big((P_h^{j+1}-P_h^j)U_t(t_j),w\Big)-\Big((I-P_h^0)U_t(0),w\Big)\\
&+((I-P_h^n)U_t(t),w)\,. \qquad (49)
\end{aligned}
$$

Below, we prove one of the main results of this section.

Theorem 4.2. *Let $(u,\boldsymbol\sigma)$ and $(U,\boldsymbol\Sigma)$ be the solution of (6)-(7) and (26)-(27), respectively. Then, for $t\in(t_{n-1},t_n]$, the following estimate holds*

$$\|\xi_u(t)\|\le\|\xi_u(0)\|+2\sum_{j=1}^{4}\mathcal{E}_{2,j}(t)+2\int_0^t\|\eta_{u,t}(s)\|\,ds,$$

where $\mathcal{E}_{2,1},\cdots,\mathcal{E}_{2,4}$ will be given in the following proof.

Proof . Choose $w=\xi_u$ in (48) and $v=\hat\xi_{\boldsymbol\sigma}$ in (43). Then, add the resulting equations to obtain

$$\frac12\frac{d}{dt}\Big(\|\xi_u\|^2+\|\alpha^{1/2}\hat\xi_{\boldsymbol\sigma}\|^2\Big)=(e_{u,t}(0),\xi_u)+(\eta_{u,t},\xi_u)+\sum_{j=1}^{4}J_{2,j}(\xi_u).$$

On integrating from 0 to t with $t\in(t_{n-1},t_n]$, it follows that

$$\|\xi_u\|^2+\|\alpha^{1/2}\hat\xi_{\boldsymbol\sigma}\|^2=\|\xi_u(0)\|^2+2\,(e_{u,t}(0),\xi_u)+2\int_0^t(\eta_{u,t},\xi_u)\,ds+2\sum_{j=1}^{4}K_j(t), \qquad (50)$$

where $K_j(t) = \int_0^t J_{2,j}(\xi_u)\,ds$. Set

$$E_2^2(t) := \|\xi_u(t)\|^2 + \|\alpha^{1/2}\hat{\xi}_{\boldsymbol{\sigma}}(t)\|^2,$$

and let $t = t^{**} \in (0, t]$ be such that

$$E_2(t^{**}) = \max_{0\le s\le t} E_2(s).$$

Then, for $t_{n-1} < t \le t_n$, we obtain from (49)

$$\begin{aligned} K_1(t) \quad = \quad & \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\sum_{l=0}^{j-1}\big((P_h^{l+1}-P_h^l)U_t(t_l),\xi_u(s)\big)\,ds \\ & + \int_{t_{n-1}}^{t}\sum_{l=0}^{n-1}\big((P_h^{l+1}-P_h^l)U_t(t_l),\xi_u(s)\big)\,ds - \int_0^t \big((I-P_h^0)U_t(0),\xi_u(s)\big)\,ds \\ & + \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\Big((I-P_h^j)U_t(s),\xi_u(s)\Big)\,ds + \int_{t_{n-1}}^{t}((I-P_h^n)U_t(s),\xi_u(s))\,ds. \end{aligned}$$

Hence,

$$\begin{aligned} |K_1(t)| \quad \le \quad & \Bigg[\sum_{j=1}^{n-1}\Bigg(k_j\sum_{l=0}^{j-1}\|(P_h^{l+1}-P_h^l)\partial_t U^l\| + (t-t_{n-1})\|(P_h^{j+1}-P_h^j)\partial_t U^j\| \\ & + \int_{t_{j-1}}^{t_j}\|(I-P_h^j)U_t(s)\|ds\Bigg) + (t-t_{n-1})\|(P_h^1-P_h^0)\partial_t U^0\| \\ & + t\|(I-P_h^0)\partial_t U^0\| + \int_{t_{n-1}}^{t}\|(I-P_h^n)U_t(s)\|ds\Bigg]\,E_2(t^{**}) \\ =: \quad & \mathcal{E}_{2,1}(t)E_2(t^{**}). \end{aligned}$$

The second term can be written as

$$\begin{aligned} K_2(t) \quad = \quad & \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\left(\int_{t_{j-1}}^{s}\mu^j(\tau)(\partial_t^2 U^j,\xi_u(s))d\tau\right)ds \\ & + \int_{t_{n-1}}^{t}\left(\int_{t_{n-1}}^{s}\mu^n(\tau)(\partial_t^2 U^n,\xi_u(s))d\tau\right)ds \end{aligned}$$

for $t_{n-1} < t \le t_n$. Since

$$\int_{t_{j-1}}^{s}\mu^j(\tau)d\tau = -3k_j^{-1}\left[(s-t_{j-1/2})^2 - \frac{k_j^2}{4}\right],$$

we obtain

$$
\begin{aligned}
&|K_2(t)| \\
=&\left|-3\sum_{j=1}^{n-1} k_j^{-1}\int_{t_{j-1}}^{t_j}(s-t_{j-1/2})^2(\partial_t^2U^j,\xi_u(s))ds+\frac{3}{4}\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}k_j(\partial_t^2U^j,\xi_u(s))ds\right.\\
&\left.-3k_n^{-1}\int_{t_{n-1}}^{t}(s-t_{n-1/2})^2(\partial_t^2U^n,\xi_u(s))ds+\frac{3}{4}k_n\int_{t_{n-1}}^{t}(\partial_t^2U^n,\xi_u(s))ds\right|\\
\le&\left[\sum_{j=1}^{n-1}\|k_j^2\partial_t^2U^j\|+\left\|k_n^{-1}\left((t-t_{n-1/2})^3+\frac{3k_n^2}{4}(t-t_{n-1})+\frac{k_n^3}{8}\right)\partial_t^2U^n\right\|\right]E_2(t^{**})\\
=:&\mathcal{E}_{2,2}(t)E_2(t^{**}).
\end{aligned}
$$

For the third term $K_3(t)$, we obtain for $t_{n-1}<t\le t_n$ and $t_{j-1}<s\le t_j$,

$$
\begin{aligned}
K_3(t) &= \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\sum_{l=1}^{j-1}\int_{t_{l-1}}^{t_l}\Big(\nabla\cdot(\tilde{\boldsymbol\sigma}(\tau)-\tilde{\boldsymbol\sigma}^l),\xi_u(s)\Big)\,d\tau ds\\
&\quad+\int_{t_{n-1}}^{t}\sum_{l=1}^{n-1}\int_{t_{l-1}}^{t_l}\Big(\nabla\cdot(\tilde{\boldsymbol\sigma}(\tau)-\tilde{\boldsymbol\sigma}^l),\xi_u(s)\Big)\,d\tau ds\\
&\quad+\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{s}\big(\nabla\cdot(\tilde{\boldsymbol\sigma}(\tau)-\tilde{\boldsymbol\sigma}^j),\xi_u(s)\big)\,d\tau ds\\
&\quad+\int_{t_{n-1}}^{t}\int_{t_{n-1}}^{s}(\nabla\cdot(\tilde{\boldsymbol\sigma}(\tau)-\tilde{\boldsymbol\sigma}^n),\xi_u(s))\,d\tau ds\\
&= I_1+I_2+I_3+I_4.
\end{aligned}
$$

Using (45) and the fact that $\int_{t_{l-1}}^{t_l}\big(k_l^{-1}(t_l-\tau)^3-(t_l-\tau)^2\big)\,d\tau=-\frac{k_l^3}{12}$, we find that

$$
\begin{aligned}
I_1 =&\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\sum_{l=1}^{j-1}\frac{k_l^2}{2}\Big(\nabla\cdot\partial_t\tilde{\boldsymbol\sigma}^l,\xi_u(s)\Big)\,ds+\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\sum_{l=1}^{j-1}\frac{k_l^3}{12}\Big(\nabla\cdot\partial_t^2\tilde{\boldsymbol\sigma}^l,\xi_u(s)\Big)\,ds,\\
|I_1|\le&\left[\sum_{j=1}^{n-1}k_j\sum_{l=1}^{j-1}\left(\frac{k_l^2}{2}\|\nabla\cdot\partial_t\tilde{\boldsymbol\sigma}^l\|+\frac{k_l^3}{12}\|\nabla\cdot\partial_t^2\tilde{\boldsymbol\sigma}^l\|\right)\right]E_2(t^{**}).
\end{aligned}
$$

Similarly,

$$
|I_2| \quad\le\quad \left[(t-t_{n-1})\sum_{j=1}^{n-1}\left(\frac{k_j^2}{2}\|\nabla\cdot\partial_t\tilde{\boldsymbol\sigma}^j\|+\frac{k_j^3}{12}\|\nabla\cdot\partial_t^2\tilde{\boldsymbol\sigma}^j\|\right)\right]E_2(t^{**}).
$$

For the I_3 and I_4 terms, we easily obtain

$$\begin{aligned}
|I_3| &\leq \left[\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\left(\frac{(t_j-s)^2}{2}-\frac{k_j^2}{2}\right)\|\nabla\cdot\partial_t\tilde{\boldsymbol{\sigma}}^j\|\,ds\right. \\
&\quad \left.+\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\left(\frac{(t_j-s)^3}{3}-k_j^{-1}\frac{(t_j-s)^4}{4}\right)\|\nabla\cdot\partial_t^2\tilde{\boldsymbol{\sigma}}^j\|\,ds\right]E_2(t^{**}) \\
&\leq \left[\sum_{j=1}^{n-1}\left(\frac{k_j^3}{3}\|\nabla\cdot\partial_t\tilde{\boldsymbol{\sigma}}^j\|+\frac{k_j^4}{20}\|\nabla\cdot\partial_t^2\tilde{\boldsymbol{\sigma}}^j\|\right)\right]E_2(t^{**}),
\end{aligned}$$

and

$$|I_4| \leq (t-t_{n-1})\left[\frac{k_n^3}{3}\|\nabla\cdot\partial_t\tilde{\boldsymbol{\sigma}}^n\|+\frac{k_n^4}{20}\|\nabla\cdot\partial_t^2\tilde{\boldsymbol{\sigma}}^n\|\right]E_2(t^{**}).$$

Now collect terms, replace $\nabla\cdot\tilde{\boldsymbol{\sigma}}^j$ by $-\mathbf{r}_2^j+\nabla\cdot\boldsymbol{\Sigma}^j$ using (46), and set $\mathcal{E}_{2,3} := M_1+M_2$, where

$$\begin{aligned}
M_1 = &\sum_{j=1}^{n-1}\left[k_j\left(\sum_{l=1}^{j-1}\frac{k_l^2}{2}\|\partial_t(\mathbf{r}_2^l-\nabla\cdot\boldsymbol{\Sigma}^l)\|\right)+\left((t-t_{n-1})\frac{k_j^2}{2}+\frac{k_j^3}{3}\right)\|\partial_t(\mathbf{r}_2^j-\nabla\cdot\boldsymbol{\Sigma}^j)\|\right] \\
&+(t-t_{n-1})\frac{k_n^3}{3}\|\partial_t(\mathbf{r}_2^n-\nabla\cdot\boldsymbol{\Sigma}^n)\|,
\end{aligned}$$

and

$$\begin{aligned}
M_2 = &\sum_{j=1}^{n-1}\left[k_j\left(\sum_{l=1}^{j-1}\frac{k_l^3}{12}\|\partial_t^2(\mathbf{r}_2^l-\nabla\cdot\boldsymbol{\Sigma}^l)\|\right)+\left((t-t_{n-1})\frac{k_j^3}{12}+\frac{k_j^4}{20}\right)\|\partial_t^2(\mathbf{r}_2^j-\nabla\cdot\boldsymbol{\Sigma}^j)\|\right] \\
&+\frac{k_n^4}{20}(t-t_{n-1})\|\partial_t^2(\mathbf{r}_2^n-\nabla\cdot\boldsymbol{\Sigma}^n)\|
\end{aligned}$$

so that

$$|K_3(t)| \leq \mathcal{E}_{2,3}(t)E_2(t^{**}).$$

For the last term $K_4(t)$, one can repeat previous arguments to arrive at

$$\begin{aligned}
|K_4(t)| &\leq \left|\sum_{j=1}^{n-1}k_j\int_{t_{j-1}}^{t_j}\|\bar{f}^j-f(\tau)\|\,d\tau+(t-t_{n-1})\int_{t_{n-1}}^{t}\|\bar{f}^n-f(\tau)\|\,d\tau\right|E_2(t^{**}) \\
&=: \mathcal{E}_{2,4}(t)E_2(t^{**}).
\end{aligned}$$

On substituting in (50), it follows that

$$\|\xi_u(t)\| \leq \|\xi_u(0)\|+2\sum_{j=1}^{4}\mathcal{E}_{2,j}(t)+2\int_0^t\|\eta_{u,t}(s)\|\,ds,$$

which completes the rest of the proof. □

In order to present the final theorem in this paper, we introduce some notations: For $D := \Omega$ or K, let

$$
\begin{aligned}
\mathcal{E}_1^0(D) &= \|h_0(\alpha\mathbf{\Sigma}^0 + \nabla_h U^0)\|_{L^2(D)},\\
\mathcal{E}_2^n(D) &= \Big(\|h_n^{\ell+1}\mathbf{r}_2^n\|_{L^2(D)} + \|h_n(\alpha\mathbf{\Sigma}^n + \nabla_h U^n)\|_{L^2(D)}\Big),\\
\mathcal{E}_3^n(D) &= \Big(\|h_n^{\ell+1}\partial_t\mathbf{r}_2^n\|_{L^2(D)} + \|h_n\partial_t(\alpha\mathbf{\Sigma}^n + \nabla_h U^n)\|_{L^2(D)}\Big),\\
\mathcal{E}_4^0(D) &= \|h_0(\alpha\partial_t\mathbf{\Sigma}^0 + \nabla_h\partial_t U^0)\|_{L^2(D)},\\
\mathcal{E}_5^0(D) &= \Big(\|h_0^{1/2}J(\alpha\mathbf{\Sigma}^0\cdot\mathbf{t})\|_{0,\Gamma_h,D} + \|h_0 \operatorname{curl}_h(\alpha\mathbf{\Sigma}^0)\|_{L^2(D)}\Big),\\
\mathcal{E}_6^n(D) &= \Big(\|h_n\mathbf{r}_2^n\|_{L^2(D)} + \|h_n^{1/2}J(\alpha\mathbf{\Sigma}^n\cdot\mathbf{t})\|_{0,\Gamma_h,D} + \|h_n \operatorname{curl}_h(\alpha\mathbf{\Sigma}^n)\|_{L^2(D)}\Big),\\
\mathcal{E}_7^n(D) &= \Big(\|h_n^{\ell+1}\partial_t\mathbf{r}_2^n\|_{L^2(D)} + \|h_n\partial_t(\alpha\mathbf{\Sigma}^n + \nabla_h U^n)\|_{L^2(D)}\Big),\\
\mathcal{E}_8^n(D) &= \Big(\|h_n^{\ell+1}\partial_t^2\mathbf{r}_2^n\|_{L^2(D)} + \|h_n\partial_t^2(\alpha\mathbf{\Sigma}^n + \nabla_h U^n)\|_{L^2(D)}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_{1,1}^n &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} ||(1+\mu^j)(I-P_h^j)\partial_t^2 U^j||\,ds,\\
\mathcal{E}_{1,2}^n &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} ||\mu^j\partial_t^2 U^j||\,ds,\\
\mathcal{E}_{1,3}^n &= \sum_{j=1}^n \left(\frac{k_j^2}{2}||\partial_t(\mathbf{r}_2^j - \nabla\cdot\mathbf{\Sigma}^j)|| + \frac{k_j^3}{12}||\partial_t^2(\mathbf{r}_2^j - \nabla\cdot\mathbf{\Sigma}^j)||\right),\\
\mathcal{E}_{1,4}^n &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} ||\bar{f}^j - f(s)||\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{E}_{2,1}^n &= \sum_{j=1}^n \left(k_j\sum_{l=0}^{j-1}\|(P_h^{l+1}-P_h^l)\partial_t U^l\| + \int_{t_{j-1}}^{t_j}\|(I-P_h^j)U_t(s)\|\,ds\right)\\
&\quad + k_n\|(P_h^1 - P_h^0)\partial_t U^0\| + t^n\|(I-P_h^0)\partial_t U^0\|,\\
\mathcal{E}_{2,2}^n &= \sum_{j=1}^n k_j^2||\partial_t^2 U^j||,\\
\mathcal{E}_{2,3}^n &= \sum_{j=1}^n k_j\sum_{l=1}^{j-1}\left(\frac{k_l^2}{2}\|\partial_t(\mathbf{r}_2^l - \nabla\cdot\mathbf{\Sigma}^l) + \frac{k_l^3}{12}\|\partial_t^2(\mathbf{r}_2^l - \nabla\cdot\mathbf{\Sigma}^l)\|\right),\\
\mathcal{E}_{2,4}^n &= \sum_{j=1}^n k_j\int_{t_{j-1}}^{t_j}\|\bar{f}^j - f(s)\|\,ds.
\end{aligned}
$$

Using estimates of η_u and $\eta_{\boldsymbol{\sigma}}$ in Theorems 4.1 and 4.2, the final theorem of this section can be written as

Theorem 4.3. *Let* $(u,\boldsymbol{\sigma})$ *be the solution of* (6)-(7) *and* $(U,\mathbf{\Sigma})$ *be the solution of* (26)-(27). *Then for* $m \in [1;N]$, *the following estimates hold for the RT index* $\ell = 0,1$*:*

$$\|U^m - u(t_m)\| \le \|e_u(0)\| + C_1\mathcal{E}_1^0(\Omega) + C_2\mathcal{E}_2^m(\Omega) + C_3\sum_{n=1}^m k_n\mathcal{E}_3^n + \sum_{i=1}^4 c_i\mathcal{E}_{2,i}^n(\Omega), \tag{51}$$

and

$$\begin{aligned}\|\mathbf{\Sigma}^m - \boldsymbol{\sigma}(t_m))\|_{A^{-1}} \quad \leq \quad & \|e_{u,t}(0)\| + \|e_{\boldsymbol{\sigma}}(0)\|_{A^{-1}} + C_4\mathcal{E}_4^0(\Omega) + C_5\mathcal{E}_5^0(\Omega) \\ & + C_6\mathcal{E}_6^m(\Omega) + C_7\mathcal{E}_7^m(\Omega) + C_8\sum_{n=1}^{m} k_n\mathcal{E}_8^n(\Omega) + \sum_{i=1}^{4} c_i\mathcal{E}_{1,i}^n(\Omega), \end{aligned} \tag{52}$$

where C_i's and c_i's are constants which depend only on the coefficient matrix A, the domain Ω, the shape regularity of the elements, polynomial degree ℓ and interpolation constants.

Remark 4.2. *The last term in* (52)*, that is, $\mathcal{E}_{1,4}^n(\Omega)$ (also in* (51)*, that is, $\mathcal{E}_{2,4}^n(\Omega)$) measures the effect of approximating the forcing function f at discrete points in time. This bound is of similar form as the bound in* [26]-[29]*, where the same discretization has been used in the context of parabolic problems. However, a modification of $\bar{f}^n$ as*

$$\bar{f}^n = \frac{1}{k_n}\int_{k_{n-1}}^{t_n} f(s)\,ds$$

will specially improve the estimate $\mathcal{E}_{1,4}^n(\Omega)$ in (52)*.*

Remark 4.3. *The numerical implementation of the proposed a posteriori estimators in the adaptive algorithm deserves special attention and will be considered elsewhere.*

5. Conclusion

The current work presents a first step towards true *a posteriori* estimate in the $L^\infty(L^2)$-norm for mixed finite element approximations of second order wave equations. While Baker's technique is usually used to derive $L^\infty(L^2)$ estimates for the displacement u, in this paper, we resort to an application of integration for deriving these estimates. For adaptive algorithm, we need efficiency bounds, which would be an interesting direction for further research. Moreover, the numerical implementation of the adaptive algorithm based on the proposed estimators will be a part of our future work.

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References

[1] S. Adjerid, A posteriori finite element error estimation for second-order hyperbolic problems. Comput. Methods Appl. Mech. Engrg. 191 (2002), pp. 4699-4719.

[2] S. Adjerid, A posteriori error estimation for the method of lumped masses applied to second-order hyperbolic problems. Comput. Methods Appl. Mech. Engrg. 195 (2006), pp. 4203-4219.

[3] M. Ainsworth, J. T. Oden, A Posteriori Error Estimation in Finite Element Analysis. Pure and Applied Mathematics. New York: Wiley-Interscience, 2000.

[4] G. Akrivis, C. Makridakis, R. H. Nochetto, A posteriori error estimates for the Crank-Nicolson method for parabolic equations. Math. Comp. 75 (2006), pp. 511-531.

[5] I. Babuška, S. Ohnimus, A posteriori error estimation for the semi discrete finite element method of parabolic partial differential equations. Comput. Meth. Appl. Mech. Engrg. 190 (2001), pp. 4691–4712.

[6] G. A. Baker, Error estimates for finite element methods for second order hyperbolic equations. SIAM J. Numer. Anal. 13 (1976), pp.564-576.

[7] C. Bernardi, E. Süli, Time and space adaptivity for the second-order wave equation. Math. Models Methods Appl. Sci. 15 (2005), pp.199-225.

[8] M. Bieterman, I. Babuška, The finite element method for parabolic equations I: A posteriori error estimation. Numer. Math. 40 (1982), pp. 339–372.

[9] M. Bieterman, I. Babuška, The finite element method for parabolic equations II: A posteriori error estimation and adaptive approach. Numer. Math. 40 (1982), pp. 373–406.

[10] D. Braess, Finite elements. Theory, Fast Solvers, and Applications in Elasticity Theory. Third edition. Cambridge University Press, Cambridge, 2007.

[11] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Second edition, Springer-Verlag, New York, 1991.

[12] W. Bangerth, M. Geiger, and R. Rannacher, Adaptive Galerkin finite element methods for the wave equation. Comput. Meth. Appl. Math. 10 (2010), pp. 3-48.

[13] W. Bangerth, R. Rannacher, Finite element approximation of the acoustic wave equation: Error control and mesh adaptivity. East-West J. Numer. Math. 7 (1999), pp. 263-282.

[14] W. Bangerth, R. Rannacher, Adaptive finite element techniques for the acoustic wave equation. J. Comput. Acoust. 9 (2001), pp. 575-591.

[15] C. Carstensen, A posteriori error estimate for the mixed finite element method. Math. Comp. 66 (1997), pp. 465–476.

[16] J. M. Cascon, L. Ferragut, M. I. Asensio, Space-time adaptive algorithm for a mixed parabolic problem. Numer. Math. 103 (2006), pp. 367–392.

[17] A. Demlow, O. Lakkis, C. Makridakis, A posteriori error estimates in the maximum norm for parabolic problems. SIAM J. Numer. Anal. 47 (2009), pp. 2157–2176.

[18] K. Eriksson, C. Johnson, Adaptive finite element methods for parabolic problems I : A linear model problem. SIAM J. Numer. Anal. 28 (1991), pp. 43–77.

[19] L.C. Evans, Partial Differential Equations, Second Indian Edition, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence,Rhode Island, USA, 2014.

[20] E. H. Georgoulis, O. Lakkis, C. Makridakis, A posteriori $L^\infty(L^2)$-error bounds for finite element approximations to the wave equation. IMA J. Numer. Anal. 33 (2013), pp. 1245-1264.

[21] E. H. Georgoulis, O. Lakkis, J.M. Virtanen, A posteriori error control for discontinuous Galerkin methods for parabolic problems. SIAM J. Numer. Anal. 49 (2011), pp. 427-458.

[22] E. H. Georgoulis, O. Lakkis, C. Makridakis, J. M. Virtanen, A posteriori error estimates for leap-frog and cosine methods for second order evolution problems. SIAM J. Numer. Anal. 54 (2016), pp. 120-136.

[23] J. G. Huang, J. Lai, T. Tang, An adaptive time stepping method with efficient error control for second-order evolution problems. Sci. China Math. 56 (2013), pp. 2753-2771.

[24] C. Johnson, Discontinuous Galerkin finite element methods for second order hyperbolic problems. Comput. Meth. Appl. Mech. Engrg. 107 (1993), pp. 117-129.

[25] C. Johnson, Y. Y. Nie, V. Thomée, An a posteriori error estimate and adaptive time step control for a backward Euler discretization of a parabolic problem. SIAM J. Numer. Anal. 27 (1990), pp. 277–291.

[26] O. Lakkis, C. Makridakis, Elliptic reconstruction and a posteriori error estimates for fully discrete linear parabolic problems. Math. Comp. 75 (2006), pp. 1627–1658.

[27] M. G. Larson and A. Malqvist, A posteriori error estimates for mixed finite element approximations of parabolic problems. Numer. Math. 118 (2011), pp. 33-48.

[28] C. Makridakis, R. H. Nochetto, Elliptic reconstruction and a posteriori error estimates for parabolic problems. SIAM J. Numer. Anal. 41 (2003), pp. 1585–1594.

[29] S. Memon, Neela Nataraj, Amiya K. Pani, An a posteriori error analysis of mixed finite element Galerkin approximations to second order linear parabolic problems. SIAM J. Numer. Anal. 50 (2012), pp. 1367-1393.

[30] R. H. Nochetto, G. Savaré and C. Verdi, A posteriori error estimates for variable time step discretizations of nonlinear evolution equations. Comm. Pure Appl. Math. 53 (2000), pp. 525–589.

[31] M. Picasso, Numerical study of an anisotropic error estimator in the $L^2(H^1)$ norm for the finite element discretization of the wave equation. SIAM J. Sci. Comput. 32 (2010), pp. 2213-2234.

[32] R. Verfürth, A Posteriori Error Estimation Techniques for Finite Element Methods. Oxford Science Publications, Oxford University Press, Oxford, 2013.

[33] R. Verfürth, A posteriori error estimates for nonlinear problems: $L^r(0,T;L^\rho(\Omega))$- error estimates for finite element discretization of parabolic equations. Math. Comp. 67 (1998), pp. 1335–1360.

[34] M. F. Wheeler, A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations. SIAM J. Numer. Anal. 10 (1973), pp. 723–759.

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