INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 4-5, Pages 550–570

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Computing and Information

OVERLAPPING DOMAIN DECOMPOSITION PRECONDITIONERS FOR UNCONSTRAINED ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we propose several overlapping domain decomposition preconditioners for solving the unconstrained elliptic optimal control problem, based on the two level additive Schwarz algorithm. We consider the cases with controls on the whole domain and controls from a local subset. The latter case can be viewed as the subproblems when we solve the control-constrained control problem by using semi-smooth Newton method. When the controls act on the whole domain, we construct a symmetric and positive definite preconditioner which is proved to be robust combined with preconditioned MINRES method, and a symmetric and indefinite preconditioner which can be used in the preconditioned GMRES method and shows better numerical performance than the positive definite one. When the controls act on a local subset, we also construct a similar symmetric and indefinite preconditioner, the numerical experiments show its efficiency when combined with preconditioned GMRES method.

Key words. Overlapping domain decomposition method, elliptic optimal control problem, preconditioned MINRES method, preconditioned GMRES method.

1. Introduction

In this paper, we consider the following unconstrained elliptic optimal control problem with distributed control:

$$\min_{u\in L^2(\Omega_0)} J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega_0)} \tag{1}$$

subject to

$$\begin{cases} -\Delta y = f + B_0 u & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$, B_0 is the extension by zero operator from $L^2(\Omega_0)$ to $L^2(\Omega)$ with $\Omega_0 \subseteq \Omega$ the control domain and $\text{meas}(\Omega_0) > 0$, $f \in L^2(\Omega)$ is a given function, $y_d \in L^2(\Omega)$ is the desired state or observation, $\alpha > 0$ is the regularization parameter.

It can be proved by standard arguments (see e.g., [14]) that the optimal control problem (1)-(2) admits a unique solution $u \in L^2(\Omega_0)$, which can be characterized by its first order necessary (also sufficient) optimality system

$$\begin{cases} -\Delta y = f + B_0 u & \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \\ -\Delta p = y - y_d & \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega, \\ \alpha u + B_0' p = 0 & \text{in } \Omega_0, \end{cases} \tag{3}$$

where p is the adjoint state and B_0' is the adjoint operator of B_0 associated with L^2 inner product. The three equations in (3) serve as the state equation, the

Received by the editors January 1, 2017 and, in revised form, March 20, 2017.
2000 *Mathematics Subject Classification.* 49K20, 65F10.

adjoint equation and the first order optimality condition. By eliminating the control variable u through the third equation in (3) we arrive at the equivalent form

$$(4) \qquad \begin{cases} -\Delta y = f - \dfrac{1}{\alpha} B_0 B_0' p \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega; \\ -\Delta p = y - y_d \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We note that (4) formulates a saddle point problem involving the state and the adjoint state variables. With appropriate finite dimensional discretization, e.g., finite element method, we are led to a symmetric and strongly indefinite linear system whose efficient solving is challenging in considering the dependence of the linear system on the mesh size h and the regularization parameter α, which will be the focus of current paper.

Optimal control problems governed by partial differential equations play an important role in science and engineering. For the theoretical results we refer to [14]. Recently, the increasing requirement of efficient simulations for such kind of problems stimulates the research of both discretization schemes and efficient solvers for optimization problems with PDEs constraints. A lot of achievements have been made on this subject in the decades. For recent developments on optimization algorithms and convergence analysis of numerical schemes we refer to [12] for more details.

In addition to the convergence analysis of discretization schemes and the design of optimization algorithms, the fast and robust solving of the resulting algebraic system is also very important for efficient simulations of PDE-constrained optimal control problems. There are a lot of attempts to study the efficient solution of optimal control problems which differ from the approaches utilized. For example, in [27, 20] the authors used the preconditioned Krylov subspace method to solve the first order optimality system by constructing some block preconditioners. In [3, 22, 19] the authors used the mutigrid method to design fast solvers. Another strategy is to use the domain decomposition methods to deal with the optimal control problem, see e.g., [1, 2, 10, 11, 15]. We also mention [16, 17, 26] for the parallel implementations of domain decomposition type algorithms.

Domain decomposition methods (DDM for short) have been successfully used to construct fast solvers for the self-adjoint and positive definite partial differential equations, the essential parallel ability makes them attractive in applications. For more details on the design and convergence of DDM we refer to the monograph [23], the review papers [24, 25] and the references cited therein. As to the design of DDM for nonselfadjoint or indefinite problems, we refer to [6] and the references therein.

There are also some contributions of DDM to solving PDE-constrained optimal control problems. The applications of DDM for optimal control problems can roughly be divided into two categories, depending on how the domain decomposition strategy is integrated with the optimization. One category is that the domain decomposition strategy is used only at the PDE level. That is to use the domain decomposition methods to solve the state equation and the adjoint equation, respectively. We refer to [8] for such kind of approach where a projected gradient method serves as the outer optimization algorithm. An obvious weakness of this approach is that the robustness of the algorithm with respect to the regularization parameter α can not be verified. Another type of DDM is that the domain decomposition methods are used to solve the optimal control problem directly, either reformulating the original problem or decomposing the original problem to some local optimal control problems. In this case, both overlapping and nonoverlapping DDM can be used. We refer to [2] for a Robin type DDM and [11, 15] for a Neumann-Neumann type method. We point out that again the robustness of the algorithm with respect to the mesh size h and the regularization parameter α has not been generally proved.

In current paper we regard the problem (4) as a symmetric and indefinite linear system and use the domain decomposition strategy to design some preconditioners. We focus on the construction of preconditioners based on the two level additive Schwarz algorithm. One can define different kind of preconditioners based on different local solvers in the additive Schwarz algorithm. To be more specific, we can solve this problem by preconditioned MINRES method with a symmetric and positive definite preconditioner or preconditioned GMRES method with a general preconditioner. In the case $\Omega_0 = \Omega$, we can choose the symmetric and positive definite inexact local solvers which induce a symmetric and positive definite preconditioner, or the symmetric and indefinite exact local solvers which induce a symmetric and indefinite preconditioner. In the case $\Omega_0 \subset \Omega$, we can define a symmetric and indefinite preconditioner with the exact local solvers as well. For the symmetric and positive definite preconditioner, we prove that the spectral condition number of the preconditioned system can be bounded by a constant which is independent of the parameter α and the mesh size h. This shows the robustness of our algorithm over [2, 8, 11], at least theoretically. The numerical experiments illustrate the robustness of this preconditioner. The symmetric and indefinite preconditioner we proposed is associated with some local control problems, so it can preserve the structure of the problem under consideration, at least from the algebraic point of view. The numerical experiments also show that the symmetric and indefinite preconditioner is robust, although the theoretical analysis is still missing at this moment. We remark that a similar symmetric and indefinite preconditioner can be found in [15] when the control acts on the whole domain. Here we consider a particular case of local control and analyze the above two algorithms within a unified framework. Moreover, the local control problems ($\Omega_0 \subset \Omega$) can be viewed as the subproblems when we solve the control problem with pointwise control constraints by using semi-smooth Newton method ([13]). Thus, the combination of our proposed DDM with semi-smooth Newton method can be used to efficiently solve control problems with pointwise control constraints. We also expect that our method can be used to solve nonlinear control problems when combined with SQP method. For the comparison of the two DDM algorithms we proposed, we remark that the iteration number of the preconditioned GMRES method with the symmetric and indefinite preconditioner is notably less than that of the preconditioned MINRES method with the symmetric and definite preconditioner.

The remaining part of this paper is organized as follows. In section 2 we give the finite element discretization of the control problem and the space decomposition which will be used to design the preconditioners. In section 3 we present the uniform framework of constructing our preconditioners in the operator form. We prove in section 4 that the symmetric and positive definite preconditioner is robust with respect to the parameter α and the mesh size h. In section 5 we discuss some issues related to the symmetric and indefinite preconditioner. The paper is ended with some numerical experiments to illustrate the efficiency of our proposed preconditioners.

Denote $tL^2(\Omega) \cap wH_0^1(\Omega)$ the Hilbert space $H_0^1(\Omega)$ with inner product $t^2(\cdot,\cdot) + w^2(\nabla\cdot,\nabla\cdot)$ for some positive constants t and w and $tL^2(\Omega)$ the Hilbert space $L^2(\Omega)$ with inner product $t^2(\cdot,\cdot)$ for some positive constant t. Define $U \times V$ as the standard product space of Hilbert spaces U and V with inner product $(\cdot,\cdot)_U + (\cdot,\cdot)_V$. B' is the adjoint operator of B associated with L^2 inner product and $\mathcal{Q}^T$ is the transpose of matrix $\mathcal{Q}$. Throughout this paper, c and C, with or without subscripts, denote generic and strictly positive constants. They are assumed to be independent of the mesh parameters h, H and the regularization parameter α.

2. Finite element discretization and the space decomposition

We study the first order optimality system in the compact form (4). The weak form can be stated as

$$\begin{cases} (y,\phi) - (\nabla\phi,\nabla p) = (y_d,\phi), \ \forall \phi \in H_0^1(\Omega), \\ -(\nabla y,\nabla\psi) - \dfrac{1}{\alpha}(Dp,\psi) = -(f,\psi), \ \forall \psi \in H_0^1(\Omega), \end{cases} \tag{5}$$

where $D = B_0 B_0'$.

Let $Z = H_0^1(\Omega) \times H_0^1(\Omega)$. Define the bilinear form $k(\cdot,\cdot)$ and the linear functional g on Z as follows

$$k((\theta,\omega),(\phi,\psi)) = (\theta,\phi) - a(\phi,\omega) - \frac{1}{\alpha}(D\omega,\psi) - a(\theta,\psi), \tag{6}$$

$$g((\phi,\psi)) = (y_d,\phi) - (f,\psi), \tag{7}$$

where $a(\cdot,\cdot)$ is defined by

$$a(\phi,\psi) = \int_\Omega \nabla\phi\nabla\psi dx, \quad \forall \phi,\psi \in H_0^1(\Omega).$$

The problem of finding solutions of (5) is equivalent to find $(y,p) \in Z$ such that

$$k((y,p),(\phi,\psi)) = g((\phi,\psi)), \ \forall (\phi,\psi) \in Z. \tag{8}$$

Now we consider the finite element approximations to problems (5) and (8). To begin with, we firstly assume a shape-regular and quasi-uniform triangulation $\mathcal{T}_H$ of Ω with grid parameter H. Then after regular refinement we obtain a triangulation $\mathcal{T}_h$ with grid parameter h. Associated with $\mathcal{T}_H$ we define the piecewise linear and continuous finite element space V_H such that $V_H \subset H_0^1(\Omega)$. Let $V_h \subset H_0^1(\Omega)$ be the piecewise linear and conforming finite element space over $\mathcal{T}_h$.

The finite element approximations of (5) and (8) can be defined as follows: Find $(y_h,p_h) \in V_h \times V_h$ such that

$$\begin{cases} (y_h,\phi_h) - a(\phi_h,p_h) = (y_d,\phi_h), \ \forall \phi_h \in V_h, \\ -a(y_h,\psi_h) - \dfrac{1}{\alpha}(Dp_h,\psi_h) = -(f,\psi_h), \ \forall \psi_h \in V_h \end{cases} \tag{9}$$

and

$$k((y_h,p_h),(\phi_h,\psi_h)) = g((\phi_h,\psi_h), \ \forall (\phi_h,\psi_h) \in Z_h := V_h \times V_h. \tag{10}$$

We denote the elements of $\mathcal{T}_H$ by Ω_i $(i = 1, \cdots, N)$ with diameter H_i. We extend each Ω_i to a larger region $\hat{\Omega}_i$, which can be done by repeatedly adding a layer of the fine elements with a generous overlap, such that

$$\text{dist } (\partial\hat{\Omega}_i \cap \Omega, \partial\Omega_i \cap \Omega) \geq \delta_i \geq \beta H_i, \quad \forall\, i = 1, \cdots, N$$

for some $\beta > 0$.

For each $i = 1, \cdots, N$, we make the following assumptions

(1) $\hat{\Omega}_i \subset \Omega$;
(2) diam $(\hat{\Omega}_i) \leq C_\beta H$, where $H = \max\limits_{i=1,\cdots,N} H_i$ and C_β is some constant independent of α and the mesh size;
(3) $\partial\hat{\Omega}_i$ does not cut through any fine elements of $\mathcal{T}_h$.

For the partition $\{\hat{\Omega}_i\}$ we give the following assumption (see [23, P. 57]).

Assumption 2.1. *The partition $\{\hat{\Omega}_i\}$ can be colored using at most N^C colors with N^C the color number of this partition, in such a way that subregions with the same color are disjoint.*

The finite element space $V_H \subset H_0^1(\Omega)$ associated with triangulation $\mathcal{T}_H$ will act as the coarse space in the two level additive Schwarz algorithm and we denote it by $V_h^{(0)}$. For each $i = 1, \cdots, N$, define the finite element space $V_h^{(i)} = \tilde{V}_h(\hat{\Omega}_i) \cap H_0^1(\hat{\Omega}_i)$, where $\tilde{V}_h(\hat{\Omega}_i)$ is the restriction of V_h over $\hat{\Omega}_i$. Let $R'_{i,V}$ be the interpolation operators from $V_h^{(i)}$ to V_h. Then we can define the space decomposition of V_h as

$$V_h = \sum_{i=0}^{N} R'_{i,V} V_h^{(i)}. \tag{11}$$

In our case $R'_{i,V}(i = 1, \cdots, N)$ can be defined by

$$R'_{i,V} w_i = \begin{cases} w_i, \text{ in } \hat{\Omega}_i, \\ 0, \text{ otherwise} \end{cases} \tag{12}$$

for any $w_i \in V_h^{(i)}$. $R'_{0,V}$ is the interpolation operator from $V_h^{(0)} = V_H$ to V_h satisfying $R'_{0,V}\phi = \phi$ for any $\phi \in V_h^{(0)}$.

Denote $Z_h^{(i)} = V_h^{(i)} \times V_h^{(i)}$ and let R'_i be the interpolation operators from $Z_h^{(i)}$ to Z_h. Then the space decomposition of Z_h is given by

$$Z_h = \sum_{i=0}^{N} R'_i Z_h^{(i)}. \tag{13}$$

For each $i = 0, \cdots, N$, if we assume that $w_i = (\phi_i, \psi_i) \in Z_h^{(i)}$, then $R'_i w_i = (R'_{i,V}\phi_i, R'_{i,V}\psi_i) \in Z_h$.

3. Definition of the preconditioners

Firstly, we give in this part the procedure of constructing two level overlapping additive Schwarz preconditioner based on the space decomposition (13), and propose several specific local solvers. Secondly, we define the inexact and exact local solvers which will be used to define proper DDM based preconditioners, for more details we refer to [23].

We introduce the local bilinear form on the subspaces $Z_h^{(i)}(i=0,1,\cdots,N)$:

$$k_i(\cdot,\cdot) : Z_h^{(i)} \times Z_h^{(i)} \to \mathbb{R}, \quad i=0,\cdots,N. \tag{14}$$

One can define different local solvers associated with the specific definition of the bilinear form $k_i(\cdot,\cdot)(i=0,\cdots,N)$. In this paper, our definition of $k_i(\cdot,\cdot)$ is based on some bilinear forms

$$b(\cdot,\cdot) : Z_h \times Z_h \to \mathbb{R}.$$

Once $b(\cdot,\cdot)$ is defined, we can define $k_i(\cdot,\cdot)(i=0,\cdots,N)$ by

$$k_i(w_i,v_i) = b(R_i'w_i, R_i'v_i) \ \forall w_i, v_i \in Z_h^{(i)}, \ i=0,\cdots,N. \tag{15}$$

Based on this definition of local solvers, we can define the projection-like operators

$$T_i = R_i'\tilde{T}_i : Z_h \to R_i'Z_h^{(i)} \subset Z_h, \ i=0,\cdots,N,$$

where $\tilde{T}_i : Z_h \to Z_h^{(i)}(i=0,\cdots,N)$ satisfies

$$k_i(\tilde{T}_iw, v_i) = k(w, R_i'v_i), \ \forall v_i \in Z_h^{(i)}, \forall w \in Z_h.$$

Then the additive operator is given by

$$T = \sum_{i=0}^{N} T_i. \tag{16}$$

Define the operators K, B, $K_i(i=0,\cdots,N)$ as follows

$$K : Z_h \to Z_h, (Kw,v) = k(w,v), \ \forall w,v \in Z_h,$$

$$B : Z_h \to Z_h, (Bw,v) = b(w,v), \ \forall w,v \in Z_h,$$

$$K_i : Z_h^{(i)} \to Z_h^{(i)}, (K_iw_i,v_i) = k_i(w_i,v_i), \ \forall w_i,v_i \in Z_h^{(i)}, i=0,\cdots,N.$$

Then it follows from (15) that

$$K_i = R_iBR_i', \ i=0,\cdots,N, \tag{17}$$

$$T_i = R_i'K_i^{-1}R_iK, \ i=0,\cdots,N, \tag{18}$$

$$T = \sum_{i=0}^{N} R_i'K_i^{-1}R_iK. \tag{19}$$

We will show the invertibility of the additive operator T and the operators $K_i(i=0,\cdots,N)$ latter, which is associated with the solvability of the local solvers in our case.

If we denote

$$P = \sum_{i=0}^{N} R_i'K_i^{-1}R_i, \tag{20}$$

then $T = PK$. Hence, we can regard P as a preconditioner of the operator K and T is the preconditioned operator. That is to say we can construct our preconditioner P which has the form (20) by the procedure presented above.

3.1. The inexact local solvers. The inexact local solvers defined here are only for the case $\Omega_0 = \Omega$. We choose the bilinear form $b(\cdot,\cdot)$ as

$$b((w,q),(\phi,\psi)) = \alpha^{1/2}a(w,\phi) + (w,\phi) + \alpha^{-1/2}a(q,\psi) + \alpha^{-1}(q,\psi) \tag{21}$$

for any $(w,q),(\phi,\psi) \in Z_h = V_h \times V_h$.

We note that b is a symmetric and positive definite bilinear form and the local solvers $k_i(\cdot,\cdot)(i = 0,\cdots,N)$ are the exact local solvers of b. According to the domain decomposition theory of the selfadjoint and positive definite problem, we know that the additive operator T and the operators $K_i(i = 0,\cdots,N)$ are invertible and the preconditioner is symmetric and positive definite as well. We denote the preconditioner (20) as P_{SPD}. Hence, we can use preconditioned MINRES method to solve the problem (10) with preconditioner P_{SPD}. We will prove the robustness of this preconditioner combined with MINRES method in next section.

3.2. The exact local solvers. For the general case $\Omega_0 \subset \Omega$, we choose

$$b(w,v) = k(w,v),\ \forall w,v \in Z_h.$$

The precontioner generated in this case is symmetric and indefinite, we will prove in section 5 the solvability of the local problems and the invertibility of the operator T. We denote this preconditioner as P_{SI}. We can choose preconditioned GMRES method with this preconditioner to solve the problem (10).

4. The robustness of the preconditioner P_{SPD}

Recall the convergence theory of preconditioned MINRES with a symmetric and positive definite preconditioner, it suffices to prove that the spectral condition number of the operator $T = P_{\mathrm{SPD}}K$ can be bounded by some constant C independent of the parameter α and the mesh size h and H, in order to show the robustness of the preconditioner. Before doing that, we collect some useful observations.

Consider the variational problem: Find $w \in V$ such that

$$s(w,v) = (f,v),\ \forall v \in V,$$

where $f \in V'$ and V is a Hilbert space with V' its dual space, $s(\cdot,\cdot)$ is a symmetric and continuous bilinear form on V.

Firstly, we assume that $s(\cdot,\cdot)$ is positive definite. The convergence analysis of additive Schwarz type domain decomposition method for this kind of problems can be divided into the following several steps.

(1) Find a proper symmetric and positive definite bilinear form $b(\cdot,\cdot)$ in V, which induces an inner product $(\cdot,\cdot)_b$ such that

$$\inf_{0\neq w\in V}\frac{s(w,w)}{b(w,w)} = \inf_{0\neq w\in V}\sup_{0\neq v\in V}\frac{s(w,v)}{b(w,w)^{1/2}b(v,v)^{1/2}} \geq c > 0,$$

$$\sup_{0\neq w\in V}\frac{s(w,w)}{b(w,w)} = \sup_{0\neq w\in V}\sup_{0\neq v\in V}\frac{s(w,v)}{b(w,w)^{1/2}b(v,v)^{1/2}} \leq C,$$

where c and C are some constants which are independent of the parameter of the problem and the mesh size.

(2) Define an equivalent bilinear form $\tilde{b}(\cdot,\cdot)$ of $b(\cdot,\cdot)$ associated with the given domain decomposition, i.e.

$$c_1 b(w,w) \leq \tilde{b}(w,w) \leq C_1 b(w,w).$$

(3) Define the preconditioner $\tilde{B}$ such that

$$\tilde{b}(w,v) = (\tilde{B}w, v), \ \forall w, v \in V.$$

Then the preconditioner $\tilde{B}$ associated with the problem

$$(Sw, v) = s(w, v) = (f, v) \ \ \forall v \in V$$

satisfies

$$\inf_{0\neq w\in V} \sup_{0\neq v\in V} \frac{(Sw,v)}{\|w\|_{\tilde{B}}\|v\|_{\tilde{B}}} \geq \frac{c}{C_1},$$

$$\sup_{0\neq w\in V} \sup_{0\neq v\in V} \frac{(Sw,v)}{\|w\|_{\tilde{B}}\|v\|_{\tilde{B}}} \leq \frac{C}{c_1}.$$

Hence, $\kappa(\tilde{B}^{-1}S) \leq \frac{CC_1}{cc_1}$. By taking $b(\cdot,\cdot) = s(\cdot,\cdot)$, it follows that $c = 1$ and $C = 1$. Then we have $\kappa(\tilde{B}^{-1}S) \leq \frac{C_1}{c_1}$.

Now we consider the case that $s(\cdot,\cdot)$ is symmetric but indefinite. The procedure mentioned above can also be used to design and analyze the preconditioner associated with the domain decomposition method in this case, by using the preconditioned MINRES method.

By using these observations, we firstly prove that the bilinear form (21) induces a proper inner product for the problem (10).

Since $\Omega_0 = \Omega$, B_0 is the identity operator, i.e., $B_0 = I$. The weak form of the optimality conditions (3) is

$$\begin{cases} \alpha(u,v) + (p,v) = 0, & \forall v \in L^2(\Omega) \\ (y,\phi) - (\nabla\phi, \nabla p) = (y_d, \phi), & \forall \phi \in H_0^1(\Omega), \\ (u,\psi) - (\nabla y, \nabla\psi) = -(f,\psi), & \forall \psi \in H_0^1(\Omega). \end{cases} \tag{22}$$

Consider a general problem: Find $(u, y, p) \in L^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{cases} \alpha(u,v) + (p,v) = (g,v), & \forall v \in L^2(\Omega) \\ (y,\phi) - (\nabla\phi, \nabla p) = (h, \phi), & \forall \phi \in H_0^1(\Omega), \\ (u,\psi) - (\nabla y, \nabla\psi) = (l,\psi), & \forall \psi \in H_0^1(\Omega), \end{cases} \tag{23}$$

where g, h, l are given right hand sides.

Lemma 4.1. *The problem (23) induces an isomorphism between*

$$U \times V = \left(\alpha^{1/2}L^2(\Omega) \times (L^2(\Omega) \cap \alpha^{1/4}H_0^1(\Omega))\right) \times \left(\alpha^{-1/2}L^2(\Omega) \cap \alpha^{-1/4}H_0^1(\Omega)\right)$$

and its dual. Namely, $\|(u,y,p)\|_{U\times V} \cong \|(g,h,l)\|_{(U\times V)'}$.

Proof. Denote

$$\mathcal{A}((u,y),(v,\phi)) := \alpha(u,v) + (y,\phi) \tag{24}$$

for any $(u,y),(v,\phi) \in U$ and

$$\mathcal{B}((u,y),\psi) := (u,\psi) - (\nabla y, \nabla\psi) \tag{25}$$

for any $(u,y) \in U, \psi \in V$. Then we can rewrite the problem (23) as: Find $(u,y,p) \in U \times V$ such that

$$\begin{cases} \mathcal{A}((u,y),(v,\phi)) + \mathcal{B}((v,\phi),p) & = (g,v) + (h,\phi) & \forall\ (v,\phi) \in U, \\ \mathcal{B}((u,y),\psi) & = (l,\psi) & \forall\ \psi \in V. \end{cases} \tag{26}$$

This is a standard saddle point problem. We follow the standard approach of [4] to prove the existence of a unique solution.

Obviously, we have

$$\begin{aligned}\mathcal{A}((u,y),(v,\phi)) &= \alpha(u,v)+(y,\phi)\\ &\le \alpha\|u\|_0\|v\|_0+\|y\|_0\|\phi\|_0\\ &\le (\alpha\|u\|_0^2+\|y\|_0^2)^{1/2}(\alpha\|v\|_0^2+\|\phi\|_0^2)^{1/2}\\ &\le \|(u,y)\|_U\|(v,\phi)\|_U\end{aligned} \tag{27}$$

and

$$\begin{aligned}\mathcal{B}((u,y),\psi) &= (u,\psi)-(\nabla y,\nabla\psi)\\ &\le \|u\|_0\|\psi\|_0+\|\nabla y\|_0\|\nabla\psi\|_0\\ &\le (\alpha\|u\|_0^2+\alpha^{1/2}\|\nabla y\|_0^2)^{1/2}(\alpha^{-1}\|\psi\|_0^2+\alpha^{-1/2}\|\nabla\psi\|_0^2)^{1/2}\\ &\le \|(u,y)\|_U\|\psi\|_V,\end{aligned}$$

this verifies the continuity of the bilinear forms $\mathcal{A}$ and $\mathcal{B}$.

Now we only need to verify the Babuška-Brezzi conditions.

(1) Let $(u,y)\in\ker(\mathcal{B}):=\{(v,\phi)\in U:\mathcal{B}((v,\phi),\psi)=0,\ \forall\ \psi\in V\}$. Then $\alpha^{1/2}(\nabla y,\nabla y)=\alpha^{1/2}(y,u)$. Therefore, we have

$$\alpha\|u\|_0^2+\alpha^{1/2}\|\nabla y\|_0^2+\|y\|_0^2\le\frac{3}{2}(\alpha\|u\|_0^2+\|y\|_0^2), \tag{28}$$

i.e.,

$$\mathcal{A}((u,y),(u,y))\ge\frac{2}{3}\|(u,y)\|_U^2,\ \forall(u,y)\in\ker(\mathcal{B}). \tag{29}$$

Namely, the coercivity of $\mathcal{A}$ on $\ker(\mathcal{B})$ is verified.

(2) We then establish the inf-sup condition. That is, for each $\psi\in V$ there holds

$$\sup_{(u,y)\in U}\frac{\mathcal{B}((u,y),\psi)}{\|(u,y)\|_U}\ge\gamma\|\psi\|_V,$$

i.e.,

$$\sup_{(u,y)\in U}\frac{(\psi,u)-(\nabla y,\nabla\psi)}{(\alpha\|u\|_0^2+\alpha^{1/2}\|\nabla y\|_0^2+\|y\|_0^2)^{1/2}}\ge\gamma(\alpha^{-1/2}\|\nabla\psi\|_0^2+\alpha^{-1}\|\psi\|_0^2)^{1/2}$$

for some constant $\gamma>0$.

Indeed, for each $\psi\ne0$, let $u=\alpha^{-1}\psi$, $y=-\alpha^{-1/2}\psi$, we have

$$(\psi,u)-(\nabla y,\nabla\psi)=\alpha^{-1}\|\psi\|_0^2+\alpha^{-1/2}\|\nabla\psi\|_0^2=\|\psi\|_V^2$$

and

$$\begin{aligned}\alpha\|u\|_0^2+\alpha^{1/2}\|\nabla y\|_0^2+\|y\|_0^2&=\alpha^{-1}\|\psi\|_0^2+\alpha^{-1/2}\|\nabla\psi\|_0^2+\alpha^{-1}\|\psi\|_0^2\\ &\le 2\|\psi\|_V^2.\end{aligned}$$

Therefore,

$$\sup_{(u,y)\in U}\frac{\mathcal{B}((u,y),\psi)}{\|(u,y)\|_U}\ge\frac{\sqrt{2}}{2}\|\psi\|_V.$$

This finishes the proof. □

For the reduced problem (5) involving only the state y and adjoint state p we have the similar result.

Corollary 4.2. *The problem (5) induces an isomorphism between*

$$U_r = L^2(\Omega) \cap \alpha^{1/4} H_0^1(\Omega) \times \left(\alpha^{-1/2} L^2(\Omega) \cap \alpha^{-1/4} H_0^1(\Omega)\right)$$

and its dual.

It follows from Corollary 4.2 and the definitions of bilinear forms b and k that

$$\inf_{0 \neq w \in Z} \sup_{0 \neq v \in Z} \frac{k(w,v)}{b(w,w)^{1/2} b(v,v)^{1/2}} \geq c > 0,$$

$$\sup_{0 \neq w \in Z} \sup_{0 \neq v \in Z} \frac{k(w,v)}{b(w,w)^{1/2} b(v,v)^{1/2}} \leq C,$$

where $Z = H_0^1(\Omega) \times H_0^1(\Omega)$, c and C are some constants independent of the parameter of the problem and the mesh size.

Since $V_h \subset H_0^1(\Omega)$, we have $Z_h = V_h \times V_h \subset Z$ and thus

$$\inf_{0 \neq w_h \in Z_h} \sup_{0 \neq v_h \in Z_h} \frac{k(w_h,v_h)}{b(w_h,w_h)^{1/2} b(v_h,v_h)^{1/2}} \geq c > 0,$$

$$\sup_{0 \neq w_h \in Z_h} \sup_{0 \neq v_h \in Z_h} \frac{k(w_h,v_h)}{b(w_h,w_h)^{1/2} b(v_h,v_h)^{1/2}} \leq C.$$

It is clear that the inner product induced by $b(\cdot,\cdot)$ on Z_h is exactly the one we are looking for. Next, we prove that the inner product induced by $\tilde{b}(\cdot,\cdot)$, associated with P_{PSD}, is equivalent to that of $b(\cdot,\cdot)$.

To begin with, we introduce the elliptic partial differential equations with Dirichlet boundary condition associated with $b(\cdot,\cdot)$: Find $z = (y,p) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$b((y,p),(\phi,\psi)) = (\bar{f},\phi) + (\bar{g},\psi), \ \forall (\phi,\psi) \in H_0^1(\Omega) \times H_0^1(\Omega) \tag{30}$$

for given right hand sides $\bar{f}$ and $\bar{g}$. For each $i = 0, \cdots, N$, let $a_i(\cdot,\cdot)$ be the local approximations of $a(\cdot,\cdot)$:

$$a_i(\theta_i,\phi_i) = a(R'_{i,V}\theta_i, R'_{i,V}\phi_i), \ \forall \theta_i.\phi_i \in V_h^{(i)}.$$

The condition number of the preconditioned system $P_{\mathrm{SPD}}B$ associated with (30) mainly depends on the constants arising from the stable decomposition and the strengthened Cauchy-Schwarz inequalities or alternatively the finite covering assumption. The color number N^C, which gives an upper bound for the spectral radius of the matrix in the strengthened Cauchy-Schwarz inequalities (see [23, Lemmas 2.10 and 3.11]), is associated with the partition $\{\hat{\Omega}_i\}$ in the finite covering assumption and can be bounded by a positive constant easily. So we only focus on the stable decomposition assumption in this paper.

Lemma 4.3. [23, Lemma 3.9] *Let ϕ_h be a continuous and piecewise quadratic function defined on $\mathcal{T}_h$ and $I^h\phi_h \in V^h$ be its piecewise linear interpolation on the same mesh. Then there exists a constant C, independent of h, such that*

$$|I^h\phi_h|_{H^1(K)} \leq C|\phi_h|_{H^1(K)}, \ K \in \mathcal{T}_h.$$

Lemma 4.4. *There exists a constant $C > 0$, such that each ξ in V_h admits a decomposition*

$$\xi = \sum_{i=0}^{N} R'_{i,V}\xi_i \quad \{\xi_i \in V_h^{(i)} : i = 0, \cdots, N\},$$

and satisfies

$$\sum_{i=0}^{N}[\alpha^{1/2}a_i(\xi_i,\xi_i)+(\xi_i,\xi_i)]\le C[\alpha^{1/2}a(\xi,\xi)+(\xi,\xi)].$$

Proof. The proof follows the standard approach. Let $\tilde{I}^H : H_0^1(\Omega)\to V_H = V_h^{(0)}$ be the quasi-interpolation operator defined in [21, 23], $I^h : C(\bar{\Omega})\to V_h$ be the Lagrangian interpolation operator ([7]). The standard stability results and error estimates give (see e.g., [23, Lemma 3.6])

$$\begin{aligned}
\|\tilde{I}^H\phi\|_0 &\le C\|\phi\|_0, \quad \forall\phi\in H_0^1(\Omega),\\
\|\phi-\tilde{I}^H\phi\|_0 &\le C\|\phi\|_0, \quad \forall\phi\in H_0^1(\Omega),\\
|\tilde{I}^H\phi|_1 &\le C|\phi|_1, \quad \forall\phi\in H_0^1(\Omega),\\
\|\phi_i-I^h\phi_i\|_{0,\hat{\Omega}_i} &\le Ch|\phi_i|_{1,\hat{\Omega}_i}, \quad \forall\phi_i\in H_0^1(\hat{\Omega}_i)\cap C(\overline{\hat{\Omega}_i}).
\end{aligned}$$

For each $\xi\in V_h$, we firstly define

$$\xi_0=\tilde{I}^H\xi\in V_h^{(0)} \tag{31}$$

and denote the remainder by

$$w=\xi-R'_{0,V}\xi_0=\xi-I^h\xi_0.$$

We next define the local components by

$$\xi_i=R_{i,V}(I^h(\theta_i w))\in V_h^{(i)}, \quad i=1,\cdots,N. \tag{32}$$

Here $R_{i,V}$ is the adjoint operator of $R'_{i,V}$ associated with L^2 inner product and the operator $R'_{i,V}$ is defined in (12), $\{\theta_i\}$ is the piecewise linear partition of unity associated with the overlapping partition (see e.g. [23]). Then we have

$$\begin{aligned}
\xi &= I^h\xi_0+I^h(\xi-I^h\xi_0)=I^h\xi_0+I^h(\sum_{i=1}^{N}\theta_i w)\\
&= I^h\xi_0+\sum_{i=1}^{N}R'_{i,V}R_{i,V}(I^h(\theta_i w))=R'_{0,V}\xi_0+\sum_{i=1}^{N}R'_{i,V}\xi_i.
\end{aligned} \tag{33}$$

For the coarse component ξ_0 we have

$$\begin{aligned}
\alpha^{1/2}a_0(\xi_0,\xi_0)+(\xi_0,\xi_0) &= \alpha^{1/2}|\tilde{I}^H\xi|_1^2+\|\tilde{I}^H\xi\|_0^2\\
&\le C(\alpha^{1/2}|\xi|_1^2+\|\xi\|_0^2)\\
&= C[\alpha^{1/2}a(\xi,\xi)+(\xi,\xi)].
\end{aligned} \tag{34}$$

For the local components ξ_i $(i=1,\cdots,N)$, since $\theta_i w$ is continuous and piecewise quadratic, it follows from Lemma 4.3 that

$$a_i(\xi_i,\xi_i)=|I^h(\theta_i w)|_{1,\hat{\Omega}_i}^2\le C|\theta_i w|_{1,\hat{\Omega}_i}^2$$

and

$$\begin{aligned}(\xi_i, \xi_i) &= \|I^h(\theta_i w)\|^2_{0,\hat{\Omega}_i} = \|I^h(\theta_i w) - \theta_i w + \theta_i w\|^2_{0,\hat{\Omega}_i} \\ &\le C(\|I^h(\theta_i w) - \theta_i w\|^2_{0,\hat{\Omega}_i} + \|\theta_i w\|^2_{0,\hat{\Omega}_i}) \\ &\le C(\|\theta_i w\|^2_{0,\hat{\Omega}_i} + h^2|\theta_i w|^2_{1,\hat{\Omega}_i}) \\ &\le C\|\theta_i w\|^2_{0,\hat{\Omega}_i} \le C\|w\|^2_{0,\hat{\Omega}_i}.\end{aligned}$$

Hence,

$$\alpha^{1/2}a_i(\xi_i, \xi_i) + (\xi_i, \xi_i) \le C(\alpha^{1/2}|\theta_i w|^2_{1,\hat{\Omega}_i} + \|w\|^2_{0,\hat{\Omega}_i}).$$

Therefore, we are led to

$$\sum_{i=1}^{N}\big[\alpha^{1/2}a_i(\xi_i, \xi_i) + (\xi_i, \xi_i)\big] \le C\sum_{i=1}^{N}(\alpha^{1/2}|\theta_i w|^2_{1,\hat{\Omega}_i} + \|w\|^2_{0,\hat{\Omega}_i}).$$

Recall that ([23, P. 69])

$$\sum_{i=1}^{N}|\theta_i w|^2_{1,\hat{\Omega}_i} \le C(1 + \frac{H}{\delta})a(\xi, \xi)$$

and

$$\sum_{i=1}^{N}\|w\|^2_{0,\hat{\Omega}_i} \le C\|w\|^2_{0,\Omega} \le C\ (\xi, \xi).$$

Here in our case, $\delta \ge \beta H$, i.e., $\frac{H}{\delta} \le \frac{1}{\beta}$ (see e.g., [23]). We obtain

$$\sum_{i=1}^{N}\big[\alpha^{1/2}a_i(\xi_i, \xi_i) + (\xi_i, \xi_i)\big] \le C(1 + \frac{1}{\beta})(\alpha^{1/2}a(\xi, \xi) + (\xi, \xi)). \tag{35}$$

Combining (34) and (35), we arrive at

$$\sum_{i=0}^{N}\big[\alpha^{1/2}a_i(\xi_i, \xi_i) + (\xi_i, \xi_i)\big] \le C(1 + \frac{1}{\beta})\big[\alpha^{1/2}a(\xi, \xi) + (\xi, \xi)\big].$$

This completes the proof of the lemma. □

Lemma 4.5. *There exists a constant $C > 0$, such that each (ξ, ζ) in Z_h admits a decomposition*

$$(\xi, \zeta) = \sum_{i=0}^{N} R'_i(\xi_i, \zeta_i) \quad \{(\xi_i, \zeta_i) \in Z_h^{(i)} : i = 0, \cdots, N\},$$

and satisfies

$$\begin{aligned}&\sum_{i=0}^{N}\big[\alpha^{1/2}a_i(\xi_i, \xi_i) + (\xi_i, \xi_i) + \alpha^{-1/2}a_i(\zeta_i, \zeta_i) + \alpha^{-1}(\zeta_i, \zeta_i)\big] \\ \le\ & C\big[\alpha^{1/2}a(\xi, \xi) + (\xi, \xi) + \alpha^{-1/2}a(\zeta, \zeta) + \alpha^{-1}(\zeta, \zeta)\big].\end{aligned}$$

Proof. Notice that

$$\alpha^{-1/2}a(\zeta, \zeta) + \alpha^{-1}(\zeta, \zeta) = \alpha^{-1}(\alpha^{1/2}a(\zeta, \zeta) + (\zeta, \zeta)).$$

Since ξ and ζ are independent variables, the desired result follows from Lemma 4.4 immediately. □

Using the standard procedure of Schwarz framework in domain decomposition theory, we can obtain the following spectral estimation (see, e.g. [23])

$$\lambda_{\min}(P_{\mathrm{SPD}}B) \geq c_0^{-1}, \quad \lambda_{\max}(P_{\mathrm{SPD}}B) \leq C,$$

where $c_0 = C(1+\beta^{-1})$ and P_{SPD} is the the symmetric and positive definite preconditioner defined in (20).

We define

$$\tilde{b}(w,v) = (P_{\mathrm{SPD}}^{-1}w, v) \ \ \forall w, v \in Z_h. \tag{36}$$

Then we have

$$C^{-1}b(w,w) \leq \tilde{b}(w,w) \leq c_0 b(w,w) \ \ \forall w \in Z_h.$$

Hence, according to the above analysis we have

Theorem 4.6. *Let $\tilde{b}(\cdot,\cdot)$ be defined in (36). Then we have*

$$\inf_{0\neq w\in Z_h} \sup_{0\neq v\in Z_h} \frac{k(w,v)}{\tilde{b}(w,w)^{1/2}\tilde{b}(v,v)^{1/2}} \geq c,$$

$$\sup_{0\neq w\in Z_h} \sup_{0\neq v\in Z_h} \frac{k(w,v)}{\tilde{b}(w,w)^{1/2}\tilde{b}(v,v)^{1/2}} \leq C.$$

Furthermore, if we take P_{SPD} as a preconditioner for the problem (10) with MINRES method, the spectral condition number satisfies

$$\kappa(P_{\mathrm{SPD}}K) \leq C.$$

5. Some issues related to the preconditioner P_{SI}

In this section, we prove the solvability of the exact local solvers and the nonsingularity of the additive Schwarz operator $T = P_{\mathrm{SI}}K$.

By the definition of the exact local solvers and (15), we have

$$k_i(w_i, v_i) = k(R_i'w_i, R_i'v_i), \ \forall w, v \in Z_h^{(i)}, \ i = 0, \cdots, N.$$

For each $i = 0, 1, \cdots, N$, let $D_i = R_{i,V}DR_{i,V}'$ with $D = B_0B_0'$. Then we have

$$(D_i\phi_i, \psi_i) = (DR_{i,V}'\phi_i, R_{i,V}'\psi_i), \ \forall \phi_i, \psi_i \in V_h^{(i)}$$

and

$$k_i((y_i,p_i),(\phi,\psi)) = (y_i,\phi) - a_i(\phi,p_i) - \frac{1}{\alpha}(D_ip_i,\psi) - a_i(y_i,\psi), \ \forall (y_i,p_i),(\phi,\psi) \in Z_h^{(i)}.$$

Lemma 5.1. *For each $(y,p) \in Z_h$, there exists a unique $(y_i,p_i) \in Z_h^{(i)}$ ($i = 0, 1, \cdots, N$) such that*

$$k_i((y_i,p_i),(\phi,\psi)) = k((y,p), R_i'(\phi,\psi)), \ \forall (\phi,\psi) \in Z_h^{(i)}.$$

Proof. For $i = 0$, the result holds obviously. We only focus on the case $i = 1, \cdots, N$ in the proof below.

For a fixed $(y,p) \in Z_h$, according to the definition of $k(\cdot,\cdot)$ we have

$$\begin{aligned} k((y,p), R_i'(\phi,\psi)) &= (y, R_{i,V}'\phi) - a(R_{i,V}'\phi, p) - \frac{1}{\alpha}(Dp, R_{i,V}'\psi) - a(y, R_{i,V}'\psi) \\ &= (R_{i,V}y, \phi) - a(\phi, R_{i,V}p) - \frac{1}{\alpha}(R_{i,V}Dp, \psi) - a(R_{i,V}y, \psi), \end{aligned}$$

which defines a linear functional on $Z_h^{(i)}$. By using the Riesz representation theorem, there exists a unique $(f_i, y_d^{(i)}) \in V_h^{(i)} \times V_h^{(i)}$ such that

$$k((y,p), R_i'(\phi,\psi)) = (y_d^{(i)}, \phi) - (f_i, \psi),\ \forall(\phi,\psi) \in Z_h^{(i)}.$$

(i) If $\hat{\Omega}_i \cap \Omega_0 = \emptyset$, then $R_{i,V} D = 0$ and $D_i = 0$. The equation we need to solve

$$k_i((y_i,p_i),(\phi,\psi)) = k((y,p), R_i'(\phi,\psi)) = (y_d^{(i)}, \phi) - (f_i, \psi),\ \forall(\phi,\psi) \in Z_h^{(i)}$$

is equivalent to the equation

$$\begin{cases} a_i(\phi, p_i) = (y_i - y_d^{(i)}, \phi),\ \forall \phi \in V_h^{(i)}, \\ a_i(y_i, \psi) = (f_i, \psi),\ \forall \phi \in V_h^{(i)}. \end{cases}$$

Since $a_i(\cdot,\cdot)$ is a continuous, symmetric and coercive bilinear form on $V_h^{(i)}$, by using Lax-Milgram theorem we get the existence of a unique $(y_i, p_i) \in Z_h^{(i)}$.

(ii) If $\hat{\Omega}_i \cap \Omega_0 \neq \emptyset$, then $D_i = B_i B_i'$, where $B_i = R_{i,V} B_0$ is the extension by zero operator from $\hat{\Omega}_i \cap \Omega_0$ to $\hat{\Omega}_i$. The equation we need to solve

$$k_i((y_i,p_i),(\phi,\psi)) = k((y,p), R_i'(\phi,\psi)) = (y_d^{(i)}, \phi) - (f_i, \psi),\ \forall(\phi,\psi) \in Z_h^{(i)} \tag{37}$$

is equivalent to the following equation

$$\begin{cases} a_i(\phi, p_i) = (y_i - y_d^{(i)}, \phi),\ \forall \phi \in V_h^{(i)}, \\ a_i(y_i, \psi) = (f_i - \dfrac{1}{\alpha} D_i p_i, \psi),\ \forall \psi \in V_h^{(i)}, \end{cases}$$

which is exactly the weak form of the optimality conditions of the following control problem:

$$\min_{u_i \in L^2(\hat{\Omega}_i \cap \Omega_0)} J(y_i, u_i) = \tfrac{1}{2}\|y_i - y_d^{(i)}\|^2_{L^2(\hat{\Omega}_i)} + \tfrac{\alpha}{2}\|u_i\|^2_{L^2(\hat{\Omega}_i \cap \Omega_0)} \tag{38}$$

subject to

$$a_i(y_i, \phi) = (f_i + B_i u_i, \phi)_{L^2(\hat{\Omega}_i)},\ \forall \phi \in V_h^{(i)}.$$

The existence of a unique solution to the equation (37) is a direct consequence of the existence of a unique solution to optimal control problem (38). This completes the proof. □

According to this lemma, we see that for each $i = 0, 1, \cdots, N$, the definition of the projection-like operator $T_i (i = 0, \cdots, N)$ is proper. Next, we show that the operator T is non-singular.

Lemma 5.2. *The additive Schwarz operator T given by (19) is nonsingular.*

Proof. Since T is a linear operator on the finite dimensional space Z_h, in order to prove T is nonsingular, it suffices to prove that $\ker(T) = \{0\}$. Let $z = (y,p) \in Z_h$ and suppose that $Tz = 0$, we intend to prove that $z = 0$.

Let $T_i z = R_i' \tilde{T}_i z = (R_{i,V}' y_i, R_{i,V}' p_i)$ and $(y_i, p_i) \in Z_h^{(i)}$ for $i = 0, \cdots, N$, then $Tz = \sum_{i=0}^N T_i z = (\sum_{i=0}^N R_{i,V}' y_i, \sum_{i=0}^N R_{i,V}' p_i)$. We note that $Tz = 0$ is equivalent to $\sum_{i=0}^N R_{i,V}' y_i = 0$ and $\sum_{i=0}^N R_{i,V}' p_i = 0$.

The definition of the operator T_i $(i = 0, \cdots, N)$ implies that

$$(39)\quad \begin{cases} (y_i, \phi) - a_i(\phi, p_i) = (y, R'_{i,V}\phi) - a(R'_{i,V}\phi, p),\ \forall \phi \in V_h^{(i)}, \\ -a_i(y_i, \psi) - \dfrac{1}{\alpha}(D_i p_i, \psi) = -a(y, R'_{i,V}\psi) - \dfrac{1}{\alpha}(Dp, R'_{i,V}\psi),\ \forall \phi \in V_h^{(i)}. \end{cases}$$

For each $i = 0, \cdots, N$, taking $\phi = y_i$, $\psi = p_i$ in above two equations and subtracting the second equality from the first one, summing over $i = 0, \cdots, N$ results in

$$\begin{aligned} \sum_{i=0}^{N}(\frac{1}{\alpha}(D_i p_i, p_i) + (y_i, y_i)) &= (y, \sum_{i=0}^{N} R'_{i,V} y_i) + \frac{1}{\alpha}(Dp, \sum_{i=0}^{N} R'_{i,V} p_i) \\ &\quad + a(y, \sum_{i=0}^{N} R'_{i,V} p_i) - a(\sum_{i=0}^{N} R'_{i,V} y_i, p) \\ &= 0. \end{aligned} \tag{40}$$

Noticing that $(D_i p_i, p_i) \geq 0,\ (y_i, y_i) \geq 0$ for each $i = 0, \cdots, N$, we have $(y_i, y_i) = 0$, i.e., $y_i = 0$ for $i = 0, \cdots, N$.

Furthermore, from the first equation in (39) we can derive

$$\sum_{i=0}^{N}((y_i, p_i) - a_i(p_i, p_i)) = (y, \sum_{i=0}^{N} R'_{i,V} p_i) - a(\sum_{i=0}^{N} R'_{i,V} p_i, p) = 0.$$

Since $y_i = 0$ for $i = 0, \cdots, N$, we arrive at

$$\sum_{i=0}^{N} a_i(p_i, p_i) = 0.$$

Then $p_i = 0$ follows immediately from the coercivity of $a_i(\cdot, \cdot)$ on $V_h^{(i)}$, $i = 0, \cdots, N$. This shows $T_i z = 0$ $(i = 0, \cdots, N)$.

According to the space decomposition (11), we have for $\forall \phi, \psi \in V_h$, there exist $\phi_i, \psi_i \in V_h^{(i)}$ $(i = 0, \cdots, N)$ such that $\phi = \sum_{i=0}^{N} R'_{i,V}\phi_i$ and $\psi = \sum_{i=0}^{N} R'_{i,V}\psi_i$. Then, we have for $\forall \phi, \psi \in V_h$ that

$$\begin{aligned} (y, \phi) - a(\phi, p) &= (y, \sum_{i=0}^{N} R'_{i,V}\phi_i) - a(\sum_{i=0}^{N} R'_{i,V}\phi_i, p) \\ &= \sum_{i=0}^{N}((y, R'_{i,V}\phi_i) - a(R'_{i,V}\phi_i, p)) \\ &= \sum_{i=0}^{N}((y_i, \phi_i) - a_i(\phi_i, p_i)) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} a(y,\psi)+\frac{1}{\alpha}(Dp,\psi) &= a(y,\sum_{i=0}^{N}R'_{i,V}\psi_i)+\frac{1}{\alpha}(Dp,\sum_{i=0}^{N}R'_{i,V}\psi_i) \\ &= \sum_{i=0}^{N}(a(y,R'_{i,V}\psi_i)+\frac{1}{\alpha}(Dp,R'_{i,V}\psi_i)) \\ &= \sum_{i=0}^{N}(a_i(y_i,\psi_i)+\frac{1}{\alpha}(D_ip_i,\psi_i)) \\ &= 0. \end{aligned}$$

This implies that $z=(y,p)=(0,0)$. We thus complete the proof. □

6. Numerical experiments

In this section, we present some numerical results to illustrate the efficiency of the preconditioners we proposed. At first, we give the matrix form of the linear system to be solved and the matrix form of our preconditioners. Then we give some numerical results.

Suppose n is the number of degrees of freedom of V_h and $\{\tilde{\phi}_i : i=1,\cdots,n\}$ are the node basis functions of V_h. Then $V_h=\text{span}\{\tilde{\phi}_i|\ i=1,\cdots,n\}$. For the linear system (9) we can obtain the following equivalent matrix form

$$\begin{pmatrix} M & -A \\ -A & -\alpha^{-1}M_0 \end{pmatrix}\begin{pmatrix} Y \\ P \end{pmatrix}=\begin{pmatrix} Y_d \\ -F \end{pmatrix},$$

where $A=(a(\tilde{\phi}_i,\tilde{\phi}_j))_{n\times n}$ is the stiffness matrix, $M=((\tilde{\phi}_i,\tilde{\phi}_j))_{n\times n}$ is the mass matrix, $M_0=((D\tilde{\phi}_i,\tilde{\phi}_j))_{n\times n}$, $Y=(y_i)_{n\times 1}$, $P=(p_i)_{n\times 1}$, $y_h=\sum_{i=1}^{n}y_i\tilde{\phi}_i$, $p_h=\sum_{i=1}^{n}p_i\tilde{\phi}_i$, $F=((f,\tilde{\phi}_i))_{n\times 1}$, $Y_d=((y_d,\tilde{\phi}_i))_{n\times 1}$. We remark that $M_0=M$ in the case $\Omega_0=\Omega$.

Let

$$\mathcal{K}=\begin{pmatrix} M & -A \\ -A & -\alpha^{-1}M_0 \end{pmatrix}$$

and $\mathcal{Z}=(Y,P)^T$, $\mathcal{G}=(Y_d,-F)^T$. Then the above linear system can be rewritten in the compact form

$$\mathcal{K}\mathcal{Z}=\mathcal{G}.$$

In our numerical implementation, we obtain the partition $\{\hat{\Omega}_i\}$ by repeatedly adding a layer of fine elements. In this setting, for each $i=0,\cdots,N$ we have $V_h^{(i)}=\text{span}\{\hat{\phi}_k : k=1,\cdots,n_i\}$ where $n_i=\dim(V_h^{(i)})$ and $\hat{\phi}_k$ $(k=1,\cdots,n_i)$ are the node basis functions of $V_h^{(i)}$. Then the matrix representation of the operator $R'_{i,V}$ is $\mathcal{I}_{i,V}^T$ where $\mathcal{I}_{i,V}=(r_{k,j})_{n_i\times n}$ and $r_{k,j}$ satisfies $R'_{i,V}\hat{\phi}_k=\sum_{j=1}^{n}r_{k,j}\tilde{\phi}_j$. Then we denote the matrix representation of R'_i by $\mathcal{I}_i^T$ where

$$\mathcal{I}_i=\begin{pmatrix} \mathcal{I}_{i,V} & 0 \\ 0 & \mathcal{I}_{i,V} \end{pmatrix}.$$

TABLE 1. Iteration number versus α and δ with preconditioned MINRES (I).

α	H \ h	$\delta = h$						$\delta = 2h$					
		1/4	1/8	1/16	1/32	1/64	1/128	1/4	1/8	1/16	1/32	1/64	1/128
1.0	1/2	8	28	38	54	74	122	8	30	40	46	60	86
	1/4		32	72	96	130	220		26	62	78	104	148
	1/8			86	98	114	164			50	90	102	118
	1/16				102	104	114				78	104	106
	1/32					98	100					108	100
	1/64						96						118
10^{-4}	1/2	8	28	44	63	86	125	8	33	59	65	78	100
	1/4		32	67	93	129	207		28	71	83	107	145
	1/8			78	95	113	161			47	85	103	117
	1/16				99	101	111				75	103	105
	1/32					99	97					125	106
	1/64						97						149
10^{-8}	1/2	8	28	41	47	49	61	8	35	67	63	69	73
	1/4		32	69	63	60	71		29	89	94	93	93
	1/8			65	81	73	79			53	93	95	99
	1/16				75	77	85				53	89	95
	1/32					73	83					43	85
	1/64						79						43

TABLE 2. Iteration number versus α and δ with preconditioned MINRES (II).

α	H \ h	$\delta = 4h$						$\delta = H/2$					
		1/4	1/8	1/16	1/32	1/64	1/128	1/4	1/8	1/16	1/32	1/64	1/128
1.0	1/2	8	10	38	42	52	66	8	30	38	40	42	46
	1/4		24	36	66	80	108		32	62	66	68	72
	1/8			50	52	92	102			86	90	92	94
	1/16				62	78	106				102	104	106
	1/32					112	116					98	100
	1/64						200						96
10^{-4}	1/2	8	19	70	73	77	85	8	33	70	79	81	83
	1/4		28	49	81	89	113		32	71	81	83	87
	1/8			53	51	89	105			78	85	89	93
	1/16				61	79	107				99	103	107
	1/32					107	131					99	106
	1/64						205						97
10^{-8}	1/2	8	15	78	88	81	85	8	35	78	99	105	111
	1/4		29	55	104	107	111		32	89	104	109	117
	1/8			60	63	105	107			65	93	105	115
	1/16				63	57	99				75	89	99
	1/32					57	51					73	85
	1/64						53						79

The matrix representation of the operator relating to the bilinear form (21) is

$$\mathbf{B} = \begin{pmatrix} \alpha^{1/2}A + M & 0 \\ 0 & \alpha^{-1/2}A + \alpha^{-1}M \end{pmatrix}.$$

Hence, the matrix form of our preconditioners are

$$\mathcal{P}_{\mathrm{SPD}} = \sum_{i=0}^{N} \mathcal{I}_i^T (\mathcal{I}_i \mathbf{B} \mathcal{I}_i^T)^{-1} \mathcal{I}_i,$$

$$\mathcal{P}_{\mathrm{SI}} = \sum_{i=0}^{N} \mathcal{I}_i^T (\mathcal{I}_i \mathcal{K} \mathcal{I}_i^T)^{-1} \mathcal{I}_i.$$

We consider the optimal control problem

$$\min_{u \in L^2(\Omega_0)} \ J(y,u) \quad = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega_0)}$$

TABLE 3. Iteration number versus α and δ with preconditioned GMRES (I).

α	H \ h	$\delta = h$						$\delta = 2h$					
		1/4	1/8	1/16	1/32	1/64	1/128	1/4	1/8	1/16	1/32	1/64	1/128
1.0	1/2	8	13	15	18	21	25	2	12	14	16	19	22
	1/4		16	17	19	21	25		12	17	18	19	21
	1/8			17	17	17	19			14	17	17	17
	1/16				16	15	15				17	16	15
	1/32					14	13					16	14
	1/64						12						14
10^{-4}	1/2	8	12	15	18	22	27	2	12	14	16	18	22
	1/4		16	18	20	23	27		11	16	18	20	23
	1/8			19	18	17	21			13	18	18	19
	1/16				16	15	16				19	16	16
	1/32					14	13					19	14
	1/64						12						15
10^{-8}	1/2	6	11	12	12	13	16	2	10	11	11	12	14
	1/4		13	15	15	15	17		10	14	15	14	15
	1/8			14	15	15	17			11	14	14	15
	1/16				14	15	16				11	14	14
	1/32					13	14					10	14
	1/64						15						12

TABLE 4. Iteration number versus α and δ with preconditioned GMRES (II).

α	H \ h	$\delta = 4h$						$\delta = H/2$					
		1/4	1/8	1/16	1/32	1/64	1/128	1/4	1/8	1/16	1/32	1/64	1/128
1.0	1/2	2	2	12	14	16	19	8	12	12	13	13	13
	1/4		9	11	16	18	19		16	17	16	16	16
	1/8			13	14	17	17			17	17	17	17
	1/16				17	17	16				16	16	16
	1/32					22	17					14	14
	1/64						24						12
10^{-4}	1/2	2	2	12	14	16	18	8	12	12	11	11	11
	1/4		9	11	16	18	20		16	16	16	16	16
	1/8			12	13	18	18			19	18	18	18
	1/16				15	19	16				16	16	16
	1/32					23	19					14	14
	1/64						30						12
10^{-8}	1/2	2	2	9	11	11	12	6	10	9	9	9	10
	1/4		9	11	13	15	14		13	14	13	13	13
	1/8			12	11	13	14			14	14	13	13
	1/16				12	11	14				14	14	14
	1/32					11	10					13	14
	1/64						11						15

subject to

$$\begin{cases} -\Delta y = f + B_0 u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (0,1) \times (0,1)$, $f = 2\pi^2 \sin(\pi x)\sin(\pi y)$ and $y_d = \sin(\pi x)\sin(\pi y)$. In the case that the control acts on a local set, we take $\Omega_0 = (\frac{1}{4}, \frac{3}{4}) \times (\frac{1}{4}, \frac{3}{4})$, $f = 2\pi^2 \sin(\pi x)\sin(\pi y)$ and $y_d = \sin(\pi x) + \sin(\pi y)$. We set the tolerance for the residual of MINRES or GMRES algorithm as $1.0e-8$. We test the numerical performance of our proposed preconditioners with respect to the regularization parameter α, the mesh size H relating to the number of subdomains, the mesh size h and the overlaps. More precisely, we have done the following tests and list the results below.

(1) We test the numerical performance of the preconditioners with different mesh sizes H, h for $\alpha = 1.0$, 10^{-4}, 10^{-8} and the overlaps $\delta = h$, $2h$, $4h$.
(2) We test the numerical performance of the preconditioners with different mesh sizes H, h and fixed $\delta = H/2$ for $\alpha = 1.0,\ 10^{-4},\ 10^{-8}$.

Table 5. Iteration number versus α and δ with preconditioned GMRES for local controls (I).

		$\delta = h$						$\delta = 2h$					
	H \ h	1/4	1/8	1/16	1/32	1/64	1/128	1/4	1/8	1/16	1/32	1/64	1/128
1.0	1/2	8	13	15	18	21	25	2	12	14	16	19	22
	1/4		16	18	19	20	25		12	17	18	19	21
	1/8			17	17	17	20			14	17	17	17
	1/16				16	15	15				17	16	15
	1/32					14	13					18	14
	1/64						12						18
10^{-4}	1/2	8	13	15	17	21	27	2	12	14	16	18	22
	1/4		17	19	20	24	29		11	17	19	21	24
	1/8			19	19	19	23			15	19	19	19
	1/16				17	17	17				21	18	17
	1/32					15	15					22	15
	1/64						13						20
10^{-8}	1/2	6	11	14	16	19	26	2	10	12	14	16	20
	1/4		15	20	22	29	46		10	15	19	22	29
	1/8			20	23	27	36			14	19	23	26
	1/16				22	24	28				20	21	24
	1/32					22	24					25	21
	1/64						21						31

Table 6. Iteration number versus α and δ with preconditioned GMRES for local controls (II).

α		$\delta = 4h$						$\delta = H/2$					
	H \ h	1/4	1/8	1/16	1/32	1/64	1/128	1/4	1/8	1/16	1/32	1/64	1/128
1.0	1/2	2	2	12	14	16	19	8	12	12	13	13	13
	1/4		9	12	17	18	19		16	17	17	16	16
	1/8			13	14	17	17			17	17	17	17
	1/16				17	17	16				16	16	16
	1/32					23	18					14	14
	1/64						26						12
10^{-4}	1/2	2	2	12	13	16	18	8	12	12	12	12	12
	1/4		9	12	17	18	21		17	17	17	17	17
	1/8			13	15	19	19			19	19	19	19
	1/16				17	21	17				17	18	17
	1/32					28	23					15	15
	1/64						35						13
10^{-8}	1/2	2	2	11	12	14	16	6	10	11	11	10	10
	1/4		9	12	16	18	22		15	15	16	16	16
	1/8			13	14	19	23			20	19	19	19
	1/16				15	20	20				22	21	20
	1/32					24	25					22	21
	1/64						36						21

In the following tables we show the results for fixed number of subdomains with different scale of the linear system and the results for the fixed scale of the linear system with different numbers of subdomains. Firstly, we illustrate the performance of our proposed DDM preconditioners for controls from the whole domain Ω. In Table 1 and 2 we list the results by using the preconditioned MINRES algorithm with preconditioner $P_{\rm SPD}$. In Table 3 and 4 we list the results by using preconditioned GMRES algorithm with preconditioner $P_{\rm SI}$. Secondly, in Table 5 and 6 we give the results by using preconditioned GMRES algorithm with preconditioner $P_{\rm SI}$ for the case that controls act on a subset $\Omega_0 \subset \Omega$.

The above results show that the iteration numbers are relatively stable with respect to the regularization parameter α, the scale of the linear system measure by h, the number of the subdomains measured by H. Moreover, we can observe that the iteration numbers decrease as the overlap parameter β increase ($\delta = h, 2h, 4h$ and $H/2$ in the tables). In the case that the control acts on the whole domain, the numerical results for the preconditioner $P_{\rm SPD}$ are consistent with our theoretical prediction, while the performance of the preconditioner $P_{\rm SI}$ is very similar to that when domain decomposition method is used to solve the self-adjoint and positive definite problem. We note that this is consistent with our expectations and $P_{\rm SI}$ shows relatively better performance than that of $P_{\rm SPD}$ for solving optimal control problems. In the case that the control acts on a subset $\Omega_0 \subset \Omega$, the performance of the preconditioner $P_{\rm SI}$ is similar to the previous case that control acts on the whole domain.

Acknowledgements

The first author was supported by the National Natural Science Foundation of China under grant 11671391. The second author was supported by the National Natural Science Foundation of China under grants 91530204, 11201464 and 11671391. The third author was supported by the National Natural Science Foundation of China under grants 91530204 and 11571356.

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