INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 3, Pages 419–436

A SYSTEMATIC METHOD TO CONSTRUCT MIMETIC FINITE-DIFFERENCE SCHEMES FOR INCOMPRESSIBLE FLOWS

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Abstract. We present a general procedure to construct a non-linear mimetic finite-difference operator. The method is very simple and general: it can be applied for any order scheme, for any number of grid points and for any operator constraints.

In order to validate the procedure, we apply it to a specific example, the Jacobian operator for the vorticity equation. In particular we consider a finite difference approximation of a second order Jacobian which uses a 9x9 uniform stencil, verifies the skew-symmetric property and satisfies physical constraints such as conservation of energy and enstrophy. This particular choice has been made in order to compare the present scheme with Arakawa's renowned Jacobian, which turns out to be a specific case of the general solution. Other possible generalizations of Arakawa's Jacobian are available in literature but only the present approach ensures that the class of solutions found is the widest possible. A simplified analysis of the general scheme is proposed in terms of truncation error and study of the linearised operator together with some numerical experiments. We also propose a class of analytical solutions for the vorticity equation to compare an exact solution with our numerical results.

Key words. Mimetic schemes, Arakawa's Jacobian, finite-difference, non-linear instability.

Introduction

We consider the vorticity equation for two-dimensional incompressible inviscid flow on a biperiodic domain D in the variables x and y,

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (\mathbf{v}\zeta) = 0 \tag{1a}$$

where

$$\nabla \cdot \mathbf{v} = 0 \tag{1b}$$

$$\mathbf{v} = \mathbf{k} \times \nabla \psi \tag{1c}$$

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v} = \nabla^2 \psi \tag{1d}$$

$\zeta = \zeta(x, y)$ is the vorticity, $\mathbf{v} = (u(x, y); v(x, y); 0)$ is the velocity field, $\psi = \psi(x, y)$ is the stream function and $\mathbf{k}$ is the unit vector normal to the plane of motion; $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ and ∇ denote respectively the standard three-dimensional dot and cross product of two vectors $\mathbf{A} = (A_1; A_2; A_3)$ and $\mathbf{B} = (B_1; B_2; B_3)$ and the gradient operator i.e. $\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{3} A_i B_i$, $\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2; A_3B_1 - A_1B_3; A_1B_2 - A_2B_1)$ and $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. Using eqs. (1b)-(1c) and recalling that we deal with a two-dimensional flow, eqs.(1a),(1d) simplifies to:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x} = 0; \quad \zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}.$$

Received by the editors November 16, 2015 and, in revised form, April 6, 2017.
2000 *Mathematics Subject Classification.* 35L65, 76M20.

We introduce the Jacobian operator

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} \tag{2}$$

with the following properties:

- Skew-symmetry:

$$J(a,b) = -J(b,a) \tag{3}$$

- Integral property:

$$\overline{aJ(b,c)} = \overline{cJ(a,b)} \tag{4}$$

where $\overline{f} = \int_D f dxdy$.

We can rewrite equation (1a) as:

$$\frac{\partial \zeta}{\partial t} = J(\zeta, \psi). \tag{5}$$

Conserved quantities

We start with two definitions.

Definition 1. *The mean kinetic energy for the equation (5) is defined as:*

$$K = \frac{1}{2}\overline{(\nabla\psi)^2}. \tag{6}$$

Definition 2. *The enstrophy (mean square vorticity) for the equation (5) is defined as:*

$$G = \frac{1}{2}\overline{\zeta^2}. \tag{7}$$

For any motion governed by equation (1) we have physical constraints such as conservation of energy,

$$\begin{aligned} \frac{\partial K}{\partial t} &\overset{(8a)}{=} \overline{\frac{1}{2}\frac{\partial(\nabla\psi)^2}{\partial t}} \overset{((8b)}{=} \overline{(\nabla\psi)\cdot\frac{\partial(\nabla\psi)}{\partial t}} \overset{((8c)}{=} \overline{-\psi\frac{\partial(\Delta\psi)}{\partial t}} \\ &\overset{((8d)}{=} \overline{-\psi\frac{\partial\zeta}{\partial t}} \overset{((8e)}{=} \overline{-\psi J(\zeta,\psi)} \overset{((8f)}{=} 0 \end{aligned} \tag{8}$$

and conservation of enstrophy,

$$\frac{\partial G}{\partial t} \overset{((9a)}{=} \overline{\frac{1}{2}\frac{\partial(\zeta^2)}{\partial t}} \overset{(9b)}{=} \overline{\zeta\frac{\partial\zeta}{\partial t}} \overset{(9c)}{=} \overline{\zeta J(\zeta,\psi)} \overset{(9d)}{=} 0 \tag{9}$$

where the RHS of both equations is zero thanks to the skew-symmetric (3) and the integral (4) properties with, respectively, $a = c = \psi$ and $a = b = \zeta$.

It is well known that non-linear problems as system (1) require the correct modeling of sub-grid terms (see, for example, J. Smagorinsky 1963 [17], J. W. Deardorff 1970 [2]); in this context special attention has been given when considering large-eddy simulations (LES) to the interaction between truncation error of the underlying discretization and the sub-grid scale modeling ([26], [27], [28]). The main issue is that a false transfer of energy between different scales can occur depending on different forms of truncation error, corresponding to different forms of discretization. In 1959 Phillips [14], treating non-linear numerical instability, proposed to add a smoothing term to equation (1a), but his solution resulted to be physically incorrect and to compromise the simulation. To overcome this problem, Arakawa [1]

introduced the use of a mimetic scheme and, alternatively, non-mimetic but higher order schemes have been developed ([10], [20], [19]). The latter choice is not necessarily preferable to a mimetic solution as pointed out in [16]: in this paper higher order schemes are compared with Arakawa's Jacobian which results to be the better candidate for under-resolved simulations. Moreover, in [8], Arakawa's scheme is compared with the fourth-order essentially non-oscillatory scheme (ENO-4) of Osher and Shu [13] for the equatorial barotropic equations. The authors showed how Arakawa's method is simpler to code and faster at run-time.
Arakawa's solution is a mimetic scheme able to conserve integral quantities and then satisfying other important constraints on the spectral distribution of the energy. His scheme has been widely used (see, for example [16],[9]) and studied: Dubinkina and Frank [3] examined the statistical properties of Arakawa's discretization, while Lilly [11] proposed a detailed paper based on a spectral analysis.
In our paper, a systematic method to construct mimetic finite difference schemes is presented. The method is explicitly applied to construct a mimetic Jacobian differential operator obtaining the widest generalization of Arakawa's solution. Other Arakawa's generalizations have been presented in literature: in 1974-1975 Jespersen [7] and Fix [5] studied different classes of conservative finite-element Jacobians; in 1989, Salmon and Talley [15] proposed a generalization of Arakawa's Jacobian in terms of independence of type of discretization (finite-differences, finite elements, spectral modes, or any mixture of the three); in 1998, McLachlan [12] using symmetry groups and skew-symmetric finite difference tensors, presented a systematic method for discretizing PDEs with a known list of integrals and, more recently, an extension of Arakawa's Jacobian in terms of SBP operators of arbitrary order has been proposed in [22] by Sorgentone, La Cognata and Nordström. However none of them is a general procedure to produce a complete set of solutions respect to a finite number of constraints.
The paper is organized as follows: in Section 1 the method to construct a general operator that satisfies special properties is presented, in Section 2 we solve the system generated by the general method applied to a specific class of problems and we provide with a compact form for the discrete operator, together with an expression for its truncation error. The modified wave number is shown for the linearised operator with a consequent stability analysis for time discretization. We also show some examples of different sets of solutions. In Section 3 two numerical experiments are proposed and in the Conclusions we summarize the work and show further possible developments.

1. The method

We start by considering a general finite differences discretization of a general non-linear operator L in N variables:

$$L_{\mathbf{k}}(\phi^1, ..., \phi^N) = \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \dots \sum_{\mathbf{k}_N} \widetilde{\phi}_{\mathbf{k}_1,\mathbf{k}_2,\dots,\mathbf{k}_N} \prod_{l=1}^{N} \phi^l_{\mathbf{k}+\mathbf{k}_l} \tag{10}$$

where $\mathbf{k}$ is a generic index and $\mathbf{k}_j$ is defined depending on the dimension of the domain and on the stencil we are using. This is a special form in writing a non-linear operator: the non-linearity is hidden in the products of sequences which contains all variables, while the coefficients are out of the product but depend on each variable. In such a way it's natural to construct a linear system for the coefficients $\widetilde{\phi}_{\mathbf{k}_1,\ldots,\mathbf{k}_N}$. As a matter of fact, once we specify domain and stencil, we will translate numerical and physical requests on the general operator in terms of linear equations for coefficients $\widetilde{\phi}_{\mathbf{k}_1,\ldots,\mathbf{k}_N}$.

Let us consider a specific case: we fix the set of solutions as the set of all non-linear discrete operators with a compact stencil, constant spatial step size h, with the following properties:

a) skew symmetry (eq. 3);
b) enstrophy conserving (eq. 9);
c) energy conserving (eq. 8);
d) second order consistency with the analytical Jacobian (eq. 2).

In order to impose such constraints, we started from Arakawa's work [1] to generalize his idea. We re-write equation (10) as

$$J_{\mathbf{k}}(\zeta,\psi)=\sum_{\mathbf{k'}}\sum_{\mathbf{k''}} c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}\zeta_{\mathbf{k}+\mathbf{k'}}\psi_{\mathbf{k}+\mathbf{k''}} \tag{11}$$

where now it is clear that the generic $\mathbf{k}$ index is defined as $\mathbf{k}=(i,j)$, $\mathbf{k'},\mathbf{k''}$ can assume values in the set $A\times A$ with $A=\{0,1,-1\}$, and of course $J_{\mathbf{k}}(\zeta,\psi)$ is an approximation of the exact Jacobian evaluated at grid point (i,j). In this explicit example it is easy to see the trick in writing linear equations: $c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}$ are coefficients of the non-linear variable $\zeta\psi$ to be determined (instead of having two different sets of coefficients, one for ζ and one for ψ), for this reason we don't have 9+9 non-linear unknowns but a linear system in the 81 unknowns $c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}$ (which correspond to the general coefficients $\widetilde{\phi}_{\mathbf{k}_1,\ldots,\mathbf{k}_N}$ of the original operator (10)).

It is useful to read again the Jacobian as Arakawa did:

$$J_{\mathbf{k}}(\zeta,\psi)=\sum_{\mathbf{k'}} a_{\mathbf{k},\mathbf{k}+\mathbf{k'}}\zeta_{\mathbf{k}+\mathbf{k'}},\quad \textit{where}\quad a_{\mathbf{k},\mathbf{k}+\mathbf{k'}}=\sum_{k''} c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}\psi_{\mathbf{k}+\mathbf{k''}} \tag{12}$$

or

$$J_{\mathbf{k}}(\zeta,\psi)=\sum_{\mathbf{k''}} b_{\mathbf{k},\mathbf{k}+\mathbf{k''}}\psi_{\mathbf{k}+\mathbf{k''}}\quad \textit{where}\quad b_{\mathbf{k},\mathbf{k}+\mathbf{k''}}=\sum_{\mathbf{k'}} c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}\zeta_{\mathbf{k}+\mathbf{k'}} \tag{13}$$

in order to translate properties $a)-b)-c)-d)$ in terms of coefficients $a_{\mathbf{k'},\mathbf{k''}}$, $b_{\mathbf{k'},\mathbf{k''}}$, $c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}$. Indeed, discrete analogues of requirements $a)-b)-c)-d)$ are:

a) $J_{\mathbf{k}}(\zeta,\psi)=-J_{\mathbf{k}}(\psi,\zeta),\ \forall\mathbf{k}$:

$$\begin{aligned}\sum_{\mathbf{k'}}\sum_{\mathbf{k''}} c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}\zeta_{\mathbf{k}+\mathbf{k'}}\psi_{\mathbf{k}+\mathbf{k''}} &= -\sum_{\mathbf{k'}}\sum_{\mathbf{k''}} c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}\zeta_{\mathbf{k}+\mathbf{k''}}\psi_{\mathbf{k}+\mathbf{k'}}\\ &\Rightarrow c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}=-c_{\mathbf{k},\mathbf{k''},\mathbf{k'}}\end{aligned} \tag{14}$$

b) $\sum_{\mathbf{k}}\zeta_{\mathbf{k}}J_{\mathbf{k}}(\zeta,\psi)=0$:

$$\begin{aligned}\sum_{\mathbf{k}}\zeta_{\mathbf{k}}J_{\mathbf{k}}(\zeta,\psi) &= \sum_{\mathbf{k}}\sum_{\mathbf{k'}} a_{\mathbf{k},\mathbf{k}+\mathbf{k'}}\zeta_{\mathbf{k}}\zeta_{\mathbf{k}+\mathbf{k'}}=0\\ &\Rightarrow a_{\mathbf{k}+\mathbf{k'},\mathbf{k}}=-a_{\mathbf{k},\mathbf{k}+\mathbf{k'}}\end{aligned} \tag{15}$$

c) $\sum_{\mathbf{k}}\psi_{\mathbf{k}}J_{\mathbf{k}}(\zeta,\psi)=0$:

$$\begin{aligned}\sum_{\mathbf{k}}\psi_{\mathbf{k}}J_{\mathbf{k}}(\zeta,\psi) &= \sum_{\mathbf{k}}\sum_{\mathbf{k''}} b_{\mathbf{k},\mathbf{k}+\mathbf{k''}}\psi_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{k''}}=0\\ &\Rightarrow b_{\mathbf{k}+\mathbf{k'},\mathbf{k}}=-b_{\mathbf{k},\mathbf{k}+\mathbf{k'}}\end{aligned} \tag{16}$$

d) to obtain consistency we use Taylor expansion (T_E) of equation (11) in each grid point (x_i, y_j):

$$T_E(J(\zeta,\psi)) = \sum_{i',j'}\sum_{i'',j''} c_{i,j,i',j',i'',j''}\Big[\Big(\sum_{l=0}^{n-1}\sum_{m=0}^{n-1}\frac{(i'h)^l(j'h)^m}{l!m!}\frac{\partial^{l+m}\zeta(x_i,y_j)}{\partial x^l\partial y^m}\Big) \Big(\sum_{l=0}^{n-1}\sum_{m=0}^{n-1}\frac{(i''h)^l(j''h)^m}{l!m!}\frac{\partial^{l+m}\psi(x_i,y_j)}{\partial x^l\partial y^m}\Big)\Big] \tag{17}$$

With the help of a symbolic manipulator, we obtain linear equations for coefficients $c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}$ nullifying any equation proportional to h^0, h^1, h^3, and we save only the contribution proportional to h^2 of the Jacobian, meaning $(\frac{\partial\zeta}{\partial x}\frac{\partial\psi}{\partial y})$ and $(\frac{\partial\zeta}{\partial y}\frac{\partial\psi}{\partial x})$.

Equations (14)-(16) are obtained using the same procedure proposed by Arakawa in [1], meaning that the quantity $a_{\mathbf{k},\mathbf{k+k'}}\zeta_{\mathbf{k}}\zeta_{\mathbf{k+k'}}$ can be interpreted as the square vorticity gain at the grid point $\mathbf{k}=(i,j)$ due to the interaction with the grid point $(\mathbf{k}+\mathbf{k}')=(i+i',j+j')$. It must have the same magnitude and opposite sign of the square vorticity gain at the grid point $(\mathbf{k}+\mathbf{k}')=(i+i',j+j')$ due to the interaction with the grid point $\mathbf{k}=(i,j)$ in order to avoid false production of enstrophy. Similar arguments hold for $b_{\mathbf{k},\mathbf{k+k'}}\psi_{\mathbf{k}}\psi_{\mathbf{k+k'}}$ in terms of conservation of the mean kinetic energy and for $c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}\zeta_{\mathbf{k+k''}}\psi_{\mathbf{k+k'}}$ in terms of skew-symmetry. Equations (14) and (17) are clearly written in terms of coefficients $c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}$ while equations (15) and (16) are, respectively, in terms of $a_{\mathbf{k},\mathbf{k+k'}}$ and $b_{\mathbf{k},\mathbf{k+k''}}$. By definition (12)-(13) coefficients $a_{\mathbf{k},\mathbf{k+k'}}$ and $b_{\mathbf{k},\mathbf{k+k''}}$ are linear combinations of ψ and ζ, so that we can read equations (15) and (16) as system of equations in the unknowns $c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}$.
We want to stress that what we are actually imposing are the integral constraints

$$\overline{\psi J(\zeta,\psi)} = 0 \tag{18}$$

and

$$\overline{\zeta J(\zeta,\psi)} = 0. \tag{19}$$

Analitically, they clearly coincide with the conservation of energy and enstrophy (eq. 8-9), numerically we have to take care of every single step.

- In (8a), (9a) we are just applying the definitions of energy and enstrophy
- (8b) and (9b) introduce an $O(\Delta t^2)$ error due to the time integration; indeed, we are using a Leap-Frog scheme (but similar arguments hold for general central schemes), then:

$$\begin{aligned}\frac{1}{2}\frac{\partial f^2}{\partial t} &\approx \frac{1}{2}\Big(\frac{f_{n+1}^2-f_{n-1}^2}{2\Delta t}\Big) = \frac{1}{2}\Big(\frac{(f_{n+1}-f_{n-1})(f_{n+1}+f_{n-1})}{2\Delta t}\\ &\overset{(exp\ f)}{=} \frac{1}{2}\Big(\frac{(f_{n+1}-f_{n-1})}{2\Delta t}(f_n+\Delta t f' + f_n - \Delta t f' + O(\Delta t^2))\\ &\approx \frac{1}{2}\frac{\partial f}{\partial t}2f + O(\Delta t^2) = \frac{\partial f}{\partial t}f + O(\Delta t^2).\end{aligned}$$

- (8c) is a bit more tricky because we go through integration by parts. Anyway it has be proven that the discrete property of integration by parts holds if we use SBP schemes (see [22],[23],[24]). We are using central finite difference for discretizing the Laplacian; it turns out that this scheme is an SBP scheme order 2 [22], then also this property holds

- (8d) follows from eq. (1d) introducing an error that depends on the system solver that we are using (in this work we used a Gauss-Seidel algorithm)
- (8e) and (9c) follow from eq. (5)
- (8f) and (9d) are the purpose of the paper and it is what we are actually imposing.

In the end, eqs. (18) and (19) are the discrete version of physical constraints of energy and enstrophy conservation.

2. Results

By imposing conditions $a) - b) - c) - d)$ we obtain a huge number of equations, but, using a symbolic manipulator, we can see that only 80 of them are linearly independent, meaning that the system exhibits ∞^1 solutions corresponding to have one free parameter. We proved the following theorem:

Theorem 2.1. ***Non-uniqueness of Arakawa's Jacobian***
There exists a set of solutions for the discrete 2^{nd} order Jacobian which satisfies conservation of energy and enstrophy and skew-symmetric property. This set is complete[1] *respect to the constraints (14)-(17) and it depends on one parameter; when the parameter is zero, we recover Arakawa's solution.*

This discrete set of solutions can be written in each grid point $\mathbf{k} = (i, j)$ as:

$$J^s_{\mathbf{k}}(\zeta, \psi) = \vec{\psi_{\mathbf{k}}}^T S \vec{\zeta_{\mathbf{k}}} \tag{20}$$

where $\vec{\psi_{\mathbf{k}}}$ and $\vec{\zeta_{\mathbf{k}}}$ are the column vectors containing the values of ψ and ζ respectively in the neighbourhood of the node $\mathbf{k}$. Explicitly, if we denote by a^T the transpose of the vector a:

$$\vec{\psi_{\mathbf{k}}} = (\psi_{i+1,j+1}, \psi_{i+1,j}, \psi_{i+1,j-1}, \psi_{i,j+1}, \psi_{i,j}, \psi_{i,j-1}, \psi_{i-1,j+1}, \psi_{i-1,j}, \psi_{i-1,j-1})^T;$$

$$\vec{\zeta_{\mathbf{k}}} = (\zeta_{i+1,j+1}, \zeta_{i+1,j}, \zeta_{i+1,j-1}, \zeta_{i,j+1}, \zeta_{i,j}, \zeta_{i,j-1}, \zeta_{i-1,j+1}, \zeta_{i-1,j}, \zeta_{i-1,j-1})^T;$$

and

$$S = \frac{1}{12h^2}\begin{pmatrix} 0 & -s^- & 0 & s^- & 0 & 0 & 0 & 0 & 0 \\ s^- & 0 & s^+ & -s^+ & 0 & -s^- & 0 & 0 & 0 \\ 0 & -s^+ & 0 & 0 & 0 & s^+ & 0 & 0 & 0 \\ -s^- & s^+ & 0 & 0 & 0 & 0 & -s^+ & s^- & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s^- & -s^+ & 0 & 0 & 0 & 0 & s^+ & -s^- \\ 0 & 0 & 0 & s^+ & 0 & 0 & 0 & -s^+ & 0 \\ 0 & 0 & 0 & -s^- & 0 & -s^+ & s^+ & 0 & s^- \\ 0 & 0 & 0 & 0 & 0 & s^- & 0 & -s^- & 0 \end{pmatrix}$$

with $s^- = s - 1$ and $s^+ = s + 1$.

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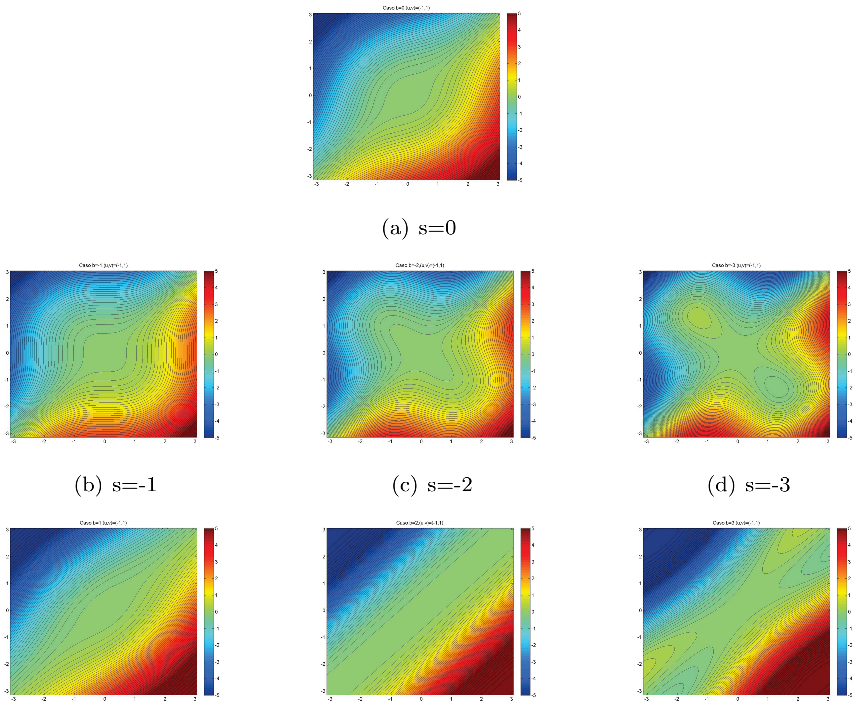
[1] By *complete*, we mean that solution (20) includes every possible discrete Jacobian satisfying the specified conditions.

2.1. Truncation error and Modified Wave number. Now we have the solution of the system a)-b)-c)-d), meaning that we can substitute the resulting values of $c_{\mathbf{k},\mathbf{k'},\mathbf{k''}}$ in eq. (17). In this way we can see how the parameter s acts on the truncation error:

$$\begin{aligned} &T_E(J_s(\zeta,\psi)) = \psi_y\zeta_x - \psi_x\zeta_y \\ &+ \frac{h^2}{6}[\psi_y\zeta_{xxx} - \psi_x\zeta_{xxy} - \psi_{xx}\zeta_{xy} + \psi_{xy}\zeta_{xx} - \psi_{xxx}\zeta_y + \psi_{xxy}\zeta_x \\ &- \psi_x\zeta_{yyy} + \psi_y\zeta_{yyx} - \psi_{xy}\zeta_{yy} + \psi_{yy}\zeta_{xy} - \psi_{yyx}\zeta_y + \psi_{yyy}\zeta_x \\ &+ \frac{s}{2}(2\psi_x\zeta_{yyx} - 2\psi_y\zeta_{xxy} + \psi_{xx}\zeta_{yy} - \psi_{yy}\zeta_{xx} + 2\psi_{xxy}\zeta_y - 2\psi_{yyx}\zeta_x)] + O(h^4). \end{aligned} \tag{21}$$

Remark 1. *Regarding the truncation error, eq. (21), we think of the parameter s as a constant not affecting the accuracy of the scheme. But this might not be the case, for example if we pick $s = \frac{1}{h^2}$. In this case we would lose consistency. Then the choice of s would, in general, depend on the resolution; for avoiding any problems, we restrict the choice of the parameter to the set $s = O(1)$.*



(a) s=0

(b) s=-1

(c) s=-2

(d) s=-3

(e) s=1

(f) s=2

(g) s=3

FIGURE 1. $\tilde{k} - k^*$ for $(u, v) = (-1, 1)$.

A possible way of choosing the best parameter may be to minimize the truncation error

$$R(J_s) = |J - T_E(J_s)|,$$

where J is defined in eq. (2) and $T_E(J_s)$ in eq. (21), but this is a difficult optimization problem and it is not the aim of this paper. For this reason, we will perform an analysis of the linearised operator to have a better understanding of the qualitative behaviour. In particular we will consider the discrete Jacobian itself and not as part of the vorticity equation, i.e. we will treat the non-physical case where the velocity field is constant but the stream function and the vorticity function are not correlated and they can be expressed as:

$$\psi(x,y) = -uy + vx$$
$$\zeta(x,y) = \overline{\zeta}e^{1i(mx+ny)}.$$

For this particular choice, the analytical Jacobian results to be:

$$J^{MWN}(\zeta,\psi) = 1i(-mu - nv)\zeta(x,y)$$

while the discrete Jacobian, J_s (with $mx \to mjh = j\alpha$, $ny \to nlh = l\beta$):

$$\begin{aligned} J_s^{MWN}(\zeta,\psi) =& 1ik^*\zeta(x,y) \\ =& 1i\frac{\zeta(x,y)}{3h}[-v(2sin(\beta) + sin(\beta)cos(\alpha)) \\ & - u(2sin(\alpha) + sin(\alpha)cos(\beta)) + s(v(-sin(\alpha) + sin(\alpha)cos(\beta)) \\ & - u(sin(\beta) - sin(\beta)cos(\alpha)))], \end{aligned} \tag{22}$$

where k^* is defined as the modified wave number. If we look at the modified wave number we observe that there are special choices where some terms nullify and the modified wave number k^* is close to the analytical one $\tilde{k} = (-mu - nv)$. For example, if we consider the case where u and v have the same constant value $\tilde{k}$, we can see that many terms in the MWN equation (22) nullify with $s = 1$, indeed the resulting modified wave number is:

$$J_1^{MWN}(\zeta,\psi) = \frac{-1i\tilde{k}\zeta(x,y)}{h}[sin(\beta) + sin(\alpha)];$$

or, in the case they have same absolute value but different sign, a similar result holds for $s = -1$, where the resulting MWN is:

$$J_{-1}^{MWN}(\zeta,\psi) = \frac{1i\tilde{k}\zeta(x,y)}{h}[sin(\beta) - sin(\alpha)].$$

Clearly, these choices are not meant to be the best, they are only qualitative directions made on the linearised operator, but they can give us an idea on how the parameter can be set in different ways for different problems, as we will see in the numerical experiments. To better understand this issue, we show the behaviour of the MWN in Fig. 1, where the projection of the difference between the exact and the discrete MWN ($\tilde{k} - k^*$) is plotted for a specific case, namely $(u, v) = (-1, 1)$. We can see that the choice $s = -2$ gives the better result, meaning a larger area where the difference between $\tilde{k}$ and k^* is close to zero.

2.2. Stability Analysis for the linearised case. All the previous analysis has been done for the semi-discretization in space; in this section we will consider the leap-frog scheme for the time discretization (following Arakawa, [25]) and perform the relative stability analysis. Particular attention must be paid to the choice of time-step in the case where the parameter is not zero. Indeed: let $Z = \frac{u}{v}$, in order to get stability, an analysis on the modified wave number gives us the following estimate:

$$\begin{aligned}\Omega &= \tfrac{1}{3h}[-v(2sin(\beta)+sin(\beta)cos(\alpha))-u(2sin(\alpha)+sin(\alpha)cos(\beta))\\ &\quad +s(v(-sin(\alpha)+sin(\alpha)cos(\beta))-u(sin(\beta)-sin(\beta)cos(\alpha)))]\\ &\le \tfrac{|v|}{3h}[2+1+|Z|(2+1)+|s|(1+1+|Z|(1+1))]\\ &= \tfrac{|v|}{3h}[3+3|Z|+|s|(2+2|Z|)] = \tfrac{|v|}{3h}[(1+|Z|)(3+2|s|)].\end{aligned}$$

Considering the stability region for the Leap-Frog scheme [21], we get:

$$\Delta t \le \frac{3h}{|v|}[(3+2|s|)(1+|Z|)]^{-1}$$

in the case s=0 (Arakawa) we get the classical condition:

$$\Delta t \le \frac{h}{|v|+|u|}.$$

2.3. Different sets of solutions. As outlined before, many classes of possible solutions arise: one possible choice is the family of skew-symmetric 2nd-order Jacobians. To obtain this special set of schemes we should impose only conditions a) and d), obtaining a wider family of solutions depending on many parameters. Here we show just two possible choices of this subset:

$$\begin{aligned}J^{s+}(\zeta,\psi) =&\frac{1}{4h^2}\{(\zeta_{i+1,j}-\zeta_{i-1,j})(\psi_{i,j+1}-\psi_{i,j-1})\\ &+s(\zeta_{i+1,j}+\zeta_{i-1,j})(\psi_{i,j+1}+\psi_{i,j-1})-(\psi_{i+1,j}-\psi_{i-1,j})(\zeta_{i,j+1}\\ &-\zeta_{i,j-1})-s(\psi_{i+1,j}+\psi_{i-1,j})(\zeta_{i,j+1}+\zeta_{i,j-1})\},\\ J^{s*}(\zeta,\psi) =&\frac{1}{8h^2}\{-(\psi_{i+1,j}[(\zeta_{i+1,j+1}-\zeta_{i+1,j-1})-s(\zeta_{i+1,j+1}+\zeta_{i+1,j-1})]\\ &-\psi_{i-1,j}[(\zeta_{i-1,j+1}-\zeta_{i-1,j-1})+s(\zeta_{i-1,j+1}+\zeta_{i-1,j-1})])\\ &+\psi_{i,j+1}[(\zeta_{i+1,j+1}-\zeta_{i-1,j+1})-s(\zeta_{i+1,j+1}+\zeta_{i-1,j+1})]\\ &-\psi_{i,j-1}[(\zeta_{i+1,j-1}-\zeta_{i-1,j-1})+s(\zeta_{i+1,j-1}+\zeta_{i-1,j-1})]\\ &+\zeta_{i+1,j}[(\psi_{i+1,j+1}-\psi_{i+1,j-1})-s(\psi_{i+1,j+1}+\psi_{i+1,j-1})]\\ &-\zeta_{i-1,j}[(\psi_{i-1,j+1}-\psi_{i-1,j-1})+s(\psi_{i-1,j+1}+\psi_{i-1,j-1})]\\ &-(\zeta_{i,j+1}[(\psi_{i+1,j+1}-\psi_{i-1,j+1})-s(\psi_{i+1,j+1}+\psi_{i-1,j+1})]\\ &-\zeta_{i,j-1}[(\psi_{i+1,j-1}-\psi_{i-1,j-1})+s(\psi_{i+1,j-1}+\psi_{i-1,j-1})])\}\end{aligned}$$

with Taylor expansion given by:

$$\begin{aligned}T_E(J^{s+}) =&(\psi_y\zeta_x-\psi_x\zeta_y)+h^2[\frac{1}{6}(\psi_y\zeta_{xxx}-\psi_{xxx}\zeta_y-\psi_x\zeta_{yyy}+\psi_{yyy}\zeta_x)\\ &-\frac{s}{4}(\psi_{xx}\zeta_{yy}+\psi_{yy}\zeta_{xx})]+O(h^4)\end{aligned}$$

$$\begin{aligned}T_E(J^{s*}) =&(\psi_y\zeta_x-\psi_x\zeta_y)+h^2[\frac{1}{6}(2\psi_y\zeta_{xxx}-3\psi_x\zeta_{xxy}-3\psi_{xx}\zeta_{xy}-2\psi_{xxx}\zeta_y\\ &-2\psi_x\zeta_{yyy}+3\psi_y\zeta_{yyx}+3\psi_{yy}\zeta_{xy}+2\psi_{yyy}\zeta_x)\\ &+\frac{s}{2}(\psi_{xx}\zeta_{yy}-\psi_{yy}\zeta_{xx}+\psi_x\zeta_{yyx}-\psi_y\zeta_{xxy}-\psi_{yyx}\zeta_x+\psi_{xxy}\zeta_y)]+O(h^4)\end{aligned}$$

We recognize that the general scheme (20) can be split into these two skew-symmetric schemes

$$J^s = \frac{J^{s+} + 2J^{s*}}{3}.$$

For the special case $s = 0$, which corresponds to the Arakawa's solution, we adopt his notation (following [1]) and we identify $J^{0+} = J^{++}$, $J^{0*} = \frac{J^{+\times}+J^{\times+}}{2}$:

$$\begin{aligned} J^{(\times+)}(\zeta,\psi) = \frac{1}{4h^2}\{&\psi_{i-1,j}(\zeta_{i-1,j+1} - \zeta_{i-1,j-1}) - \psi_{i+1,j}(\zeta_{i+1,j+1} - \zeta_{i+1,j-1}) \\ &+\psi_{i,j+1}(\zeta_{i+1,j+1} - \zeta_{i-1,j+1}) - \psi_{i,j-1}(\zeta_{i+1,j-1} - \zeta_{i-1,j-1})\} \\ J^{(+\times)}(\zeta,\psi) = \frac{1}{4h^2}\{&\zeta_{i+1,j}(\psi_{i+1,j+1} - \psi_{i+1,j-1}) - \zeta_{i-1,j}(\psi_{i-1,j+1} - \psi_{i-1,j-1}) \\ &-[\zeta_{i,j+1}(\psi_{i+1,j+1} - \psi_{i-1,j+1}) - \zeta_{i,j-1}(\psi_{i+1,j-1} - \psi_{i-1,j-1})]\} \end{aligned}$$

In this way we are back to Arakawa's main construction

$$J^0 = \frac{J^{++} + J^{+\times} + J^{\times+}}{3}$$

where each Jacobian has a specific property: respectively skew-symmetry, enstrophy and energy preserving. It's worth noting that we obtained this solution as a special case of our general scheme and not by a direct and specific combination of these three schemes as Arakawa did.

Formally we can split the general scheme into the analogous three Jacobians where each operator does have special properties as in Arakawa's case but this would be meaningless, because we lose separated consistency for the decomposition of J^{s*}. We summarize some examples of different subsets of solutions in table (1) and, in the next section, we will show their different behaviors.

TABLE 1. Examples of 2nd order Jacobians with different properties.

Skew-symmetric	Enstrophy Conserving	Energy Conserving
J^s	J^s	J^s
J^{s*}	$(J^{+\times} + J^{++})/2$	$(J^{\times+} + J^{++})/2$
J^{s+}	$J^{\times+}$	$J^{+\times}$

3. Numerical Experiments

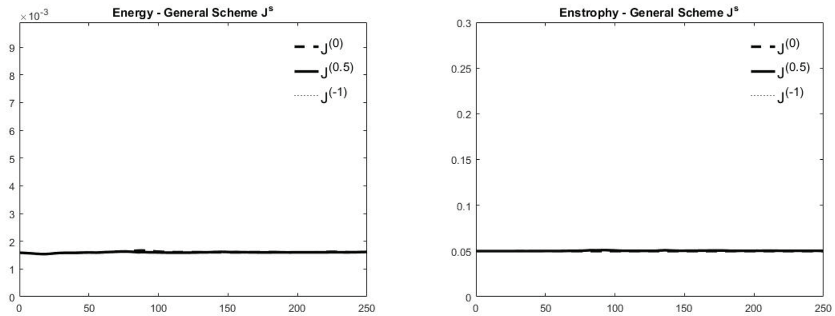
In the following numerical experiments, we will use the following norms:

- $\|f\|_1 = \sum_{i=1}^{n} |f_i|$
- $\|f\|_2 = (\sum_{i=1}^{n} |f_i|^2)^{\frac{1}{2}}$
- $\|f\|_\infty = max_i|f_i|$

a) *Comparing conservative and non-conservative schemes:*
In this section we consider a numerical experiment for the vorticity equation in the biperiodic domain $D = [a_x, b_x] \times [a_y, b_y]$ with initial condition:

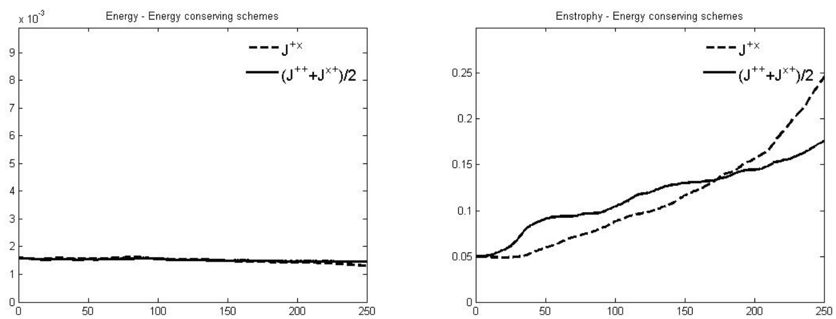
$$\zeta(x,y) = \sum_{k=4}^{12} \tilde{A} sin(\frac{2\pi kx}{b_x}) sin(\frac{2\pi ky}{b_y})$$

where we fixed $\tilde{A} = 0.15$, $a_x = a_y = 0$, $b_x = b_y = 16$ and N=129 (number of grid points per direction).



(a) Mean kinetic energy (b) Mean square vorticity

FIGURE 2. Energy and enstrophy for the general scheme J^s using different values of the parameter.



(a) Mean kinetic energy (b) Mean square vorticity

FIGURE 3. Energy and enstrophy for the energy-conserving schemes $J^{+\times}$ and $(J^{++} + J^{\times +})/2$.

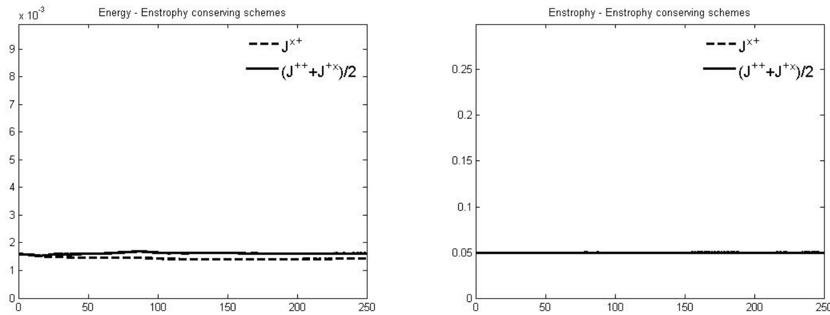
In the following experiment we want to show the different behaviours of Jacobians presented in Table 1. Since some of these, as expected, will not conserve energy and/or enstrophy and the solution will then explode, we decide to fix time-step as independent from the velocity field:

$$\Delta t = \frac{3Ch^2}{3 + 2|s|}$$

and we fix $C = 0.7$. In order to integrate system (1), at each time-step we need to solve also the equation for the stream function:

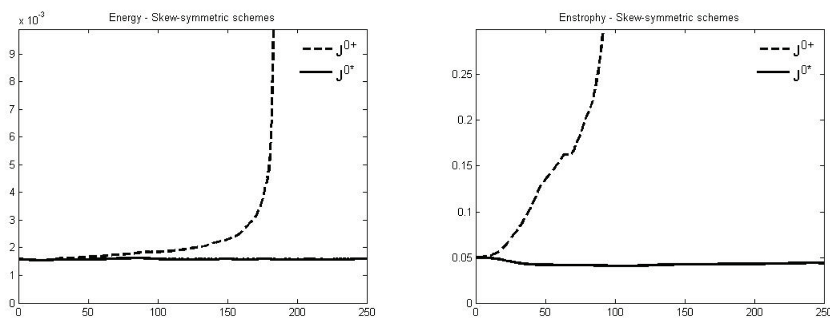
$$\Delta\psi = \zeta \tag{23}$$

Gauss-Seidel algorithm is used for equation (23) in combination with a Multi-Grid method in order to accelerate the convergence, as presented



(a) Mean kinetic energy (b) Mean square vorticity

FIGURE 4. Energy and enstrophy for the enstrophy-conserving schemes $J^{\times+}$ and $(J^{++} + J^{+\times})/2$.



(a) Mean kinetic energy (b) Mean square vorticity

FIGURE 5. Energy and enstrophy for the skew-symmetric schemes J^{0+} and J^{0*}.

in [18]. The behaviour of the energy and enstrophy (per area-element) over time is shown in figures (2)-(3)-(4)-(5). Figure 2 shows the parameter schemes' family (20) for s=0, s=0.5, s=-1 and, as expected, both integral constraints are satisfied. $J^{+\times}$ and $(J^{++} + J^{\times+})/2$ conserve only energy as we can see in Figure 3 where the enstrophy is going to diverge. The enstrophy-conserving property is a strong requirement as shown in Figures 4, where $J^{\times+}$ and $(J^{++} + J^{+\times})/2$ conserve the enstrophy while the energy, even if it's not diverging, shows some variation. In particular we analyzed the relative error for the energy and we found that, at the final time, the energy increased of about 4% for $(J^{++} + J^{+\times})/2$, and it shows a negative variation of about 10% for $J^{\times+}$. Finally some skew-symmetric cases are shown in Figures 5: J^{0+} does not conserve either energy or enstrophy and it quickly blows up. Conversely, the behaviour of J^{0*} is particular: it verifies only the skew-symmetric property as well as J^{0+}, but it gives rise to a stable simulation. J^{0*} is the combination of $J^{\times+}$ and $J^{+\times}$ which are, respectively, the discretization of the analytic forms $J(\zeta,\psi) = \nabla \cdot (\psi \nabla^{\perp} \zeta)$ and $J(\zeta,\psi) = -\nabla \cdot (\zeta \nabla^{\perp} \psi)$, with $\nabla^{\perp} = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$. It's possible to switch from one form to the other, thanks to the fact that the divergence of the curl is always zero, in our case divergence of velocity and of palinstrophy. This correspond to have $\psi_{xy} = \psi_{yx}$ and $\zeta_{xy} = \zeta_{yx}$. By simple inspection of numerical forms $J^{\times +}$ and $J^{+\times}$, it is possible to verify that we go through the same steps in the discrete case while, in general, this is not possible.

b) *A smoothed Rankine-vortex with advection:*
It is natural to define the classic Rankine Vortex in a cylindrical coordinate system (r,θ,z), where the coordinates represent respectively the radial distance (r), the azimuth (θ) and the height z. The unit normal vectors of the system are denoted by $(\hat{i},\hat{j},\hat{k})$. This vortex has a velocity field normal both to the z symmetry axes and to the radial vector r, meaning that the velocity field is parallel to the versor j. The length of the velocity depends only by the radius, in particular if we denote by M the characteristic distance, the inner part $(r < M)$ is proportional to r, while the outer $(r \geq M)$ is proportional to the inverse of the distance from the centre (1/r); the maximum value that the velocity can reach is at the characteristic distance M, where there is the shift, from the linear to the hyperbolic behaviour. We can analytically define the Rankine Vortex as:

$$\vec{v} = v_r \hat{i} + v_\theta \hat{j} + v_z \hat{k}$$

with

$$\begin{cases} v_r = 0 \\ v_\theta = \begin{cases} V_M \frac{r}{M} & if\ 0 \leq r < M \\ V_M \frac{M}{r} & if\ M \leq r \end{cases} \\ v_z = 0 \end{cases}$$

V_M is the maximum value of the velocity. It's easy to obtain the vorticity equation for this special field:

$$\begin{aligned} \vec{\zeta} = \nabla \times \vec{v} = \hat{k}\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r} &= \hat{k}(\frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}) \\ &= \hat{k}\begin{cases} 2\frac{V_M}{M} & if\ 0 \leq r < M \\ 0 & if\ M \leq r \end{cases} \end{aligned}$$

The Rankine Vortex is then characterized by a continuous velocity field but a discontinuous vorticity which could cause problems in the numerical simulations. For this reason we will construct a smoothed version of the Rankine vortex as follows:

$$\begin{cases} v_\theta(r) = r \quad if\ r < 1 \\ v_\theta(r) = r^{\alpha} e^{(1-r)\beta} \quad otherwise \end{cases} \tag{24}$$

so that

$$lim_{r\to\infty} v_\theta(r) = 0$$

and

$$v_\theta(1) = 1.$$

The vorticity is then:

$$\vec{\zeta} = \nabla \times \vec{v} = \hat{k}\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r} = \hat{k}(\frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}) \\ = \hat{k}\begin{cases} 2 \ \ if\ r < 1 \\ ((\alpha+1)r^{\alpha-1} - \beta r^{\alpha})e^{(1-r)\beta} \ \ otherwise \end{cases} \tag{25}$$

Now we fix the constants α and β such that

$$lim_{r\to\infty}\zeta(r) = 0$$

and

$$\zeta(1) = 2$$

obtaining

$$\alpha + 1 - \beta = 2$$

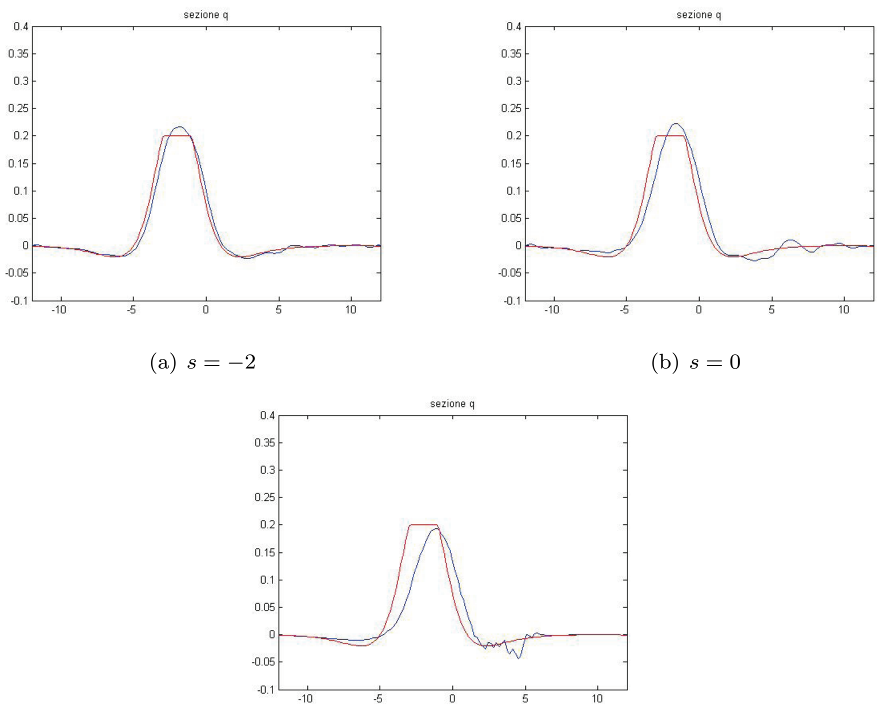
meaning the class of solution $\alpha = \beta + 1$.
We have a smooth class of analytical solution for the vorticity equation, we will add a constant advection so that the analysis that we have done for the linearised case is closer to the problem. As we can see from eqs. (24)-(25), the velocity and the vorticity for the Rankine Vortex depend only on $r = \sqrt{x^2 + y^2}$; for this reason we can solve the problem in our standard setting using 2D carthesian coordinates x and y. In the following simulations we consider the case $\alpha = 2$, $\beta = 1$ and we fix also the following data:

- $CFL = 0.7$
- Stream function tolerance $\epsilon = 10^{-11}$ or $k < 100$
- Domain $[-12, 12] \times [-12, 12]$
- Final time T=170
- Nodes NxN=129x129
- $(u, v) = (-1, 1)$ constant advection velocity

where $CFL = \frac{u_x \Delta t}{\Delta x} + \frac{u_y \Delta t}{\Delta x} = \frac{(u_x+u_y)\Delta t}{\Delta x}$ since $\Delta x = \Delta y$. In these simulations, the boundary conditions are fixed to be periodic in order to see the vortex turning around in our domain. In particular, at the final time the vortex has run for the whole domain passing through the diagonal ten times. In Fig. 6-7 we can see respectively sections of the vorticity and velocity at final time computed with our discrete Jacobian for different values of the parameter (in blue) compared with the analytical solution (in red), while in Table 2 we can appreciate the error for the vorticity with different norms, where we define e_ζ as the difference between the exact solution and the discrete one obtained by the numerical simulation. In this numerical example we can recover what we found in the qualitative analysis of Sec. 2.1, in particular in the discussion of the modified wave number with constant advection field of equal absolute value but opposite sign. Both from Table 2 and from Fig. 6-7 it is clear that the best choice of parameter is $s = -2$ (even considering a wider set of choices for the parameter), in agreement with the previous analysis on the MWN, Section 2.1 and Fig. 1. Energy and enstrophy are obviously conserved as already proved.

TABLE 2. Errors in the vorticity for different choice of the parameter.

Scheme parameter s	$\|e_\zeta\|_\infty$	$\|e_\zeta\|_1$	$\|e_\zeta\|_2$
0	7.5735e-002	2.8436e-001	1.6698e-001
-1	5.7278e-002	2.3686e-001	1.3008e-001
-2	4.0272e-002	1.5199e-001	8.8872e-002
1	1.0773e-001	3.5846e-001	2.3248e-001
2	1.2494e-001	4.4528e-001	2.8537e-001



(a) $s = -2$

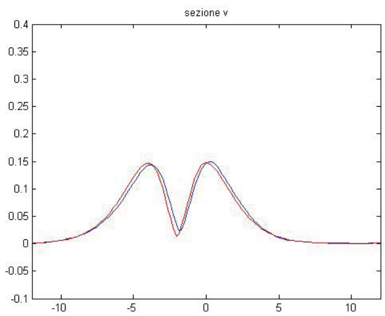
(b) $s = 0$

(c) $s = 2$

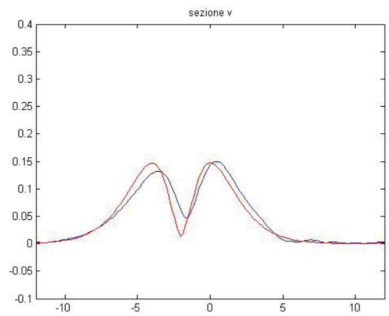
FIGURE 6. Comparison of the vorticity obtained with the discrete Jacobian (blue) with the analytical vorticity (red) given in (25) for the smoothed Rankine with advection.

4. Summary and Conclusions

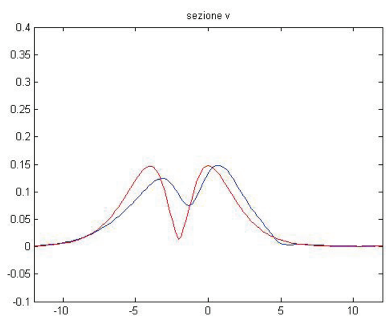
A systematic method to construct mimetic finite-difference schemes for two-dimensional incompressible flow is presented; with this method it is possible to produce any order non-linear operator on arbitrary stencils and with arbitrary properties. The discrete scheme's coefficients are identified by solving a linear system where the equations are the specific properties required; in this way we ensure selecting the whole generic family of operators we are looking for, no other operator with such properties can be left out. In this paper we applied the method to select



(a) $s = -2$



(b) $s = 0$



(c) $s = 2$

FIGURE 7. Comparison of the velocity obtained with the discrete Jacobian (blue) with the analytical velocity (red) given in (24) for the smoothed Rankine with advection.

the class of order two discrete Jacobians based on a 9x9 uniform stencil and with skew-symmetric, enstrophy and energy preserving properties and then compared this general solution with the one obtained by Akio Arakawa [1]. We showed there exists a whole set of solutions which satisfies all properties mentioned before and this set depends on one parameter, when the parameter is zero, Arakawa's scheme is recovered.

We performed a partial analysis of the scheme both in the physical and in the Fourier space to show how the parameter can affect the accuracy. To validate such analysis, we constructed an analytical solution smoothing the Rankine Vortex and showing how the optimal choice of the parameter can reduce the error.

We also proposed some examples of particular solutions: only skew-symmetric, only energy conserving and only enstrophy conserving schemes in order to compare them with the generic discrete Jacobian found by imposing all these conditions. The related numerical example underlines the difference between these operators: the general scheme is able to preserve numerical stability as well as the only enstrophy-conserving schemes and a particular skew-symmetric scheme, J^{0*}. In the near future we are going to apply this procedure to obtain higher order schemes and different classes of solutions.

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