INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 3, Pages 342–354

SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHODS FOR LINEAR HYPERBOLIC EQUATIONS WITH SINGULAR INITIAL DATA

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Abstract. In this paper, we consider the discontinuous Galerkin (DG) methods to solve linear hyperbolic equations with singular initial data. With the help of weight functions, the superconvergence properties outside the pollution region will be investigated. We show that, by using piecewise polynomials of degree k and suitable initial discretizations, the DG solution is $(2k+1)$-th order accurate at the downwind points and $(k+2)$-th order accurate at all the other downwind-biased Radau points. Moreover, the derivative of error between the DG and exact solutions converges at a rate of $k+1$ at all the interior upwind-biased Radau points. Besides the above, the DG solution is also $(k+2)$-th order accurate towards a particular projection of the exact solution and the numerical cell averages are $(2k+1)$-th order accurate. Numerical experiments are presented to confirm the theoretical results.

Key words. Discontinuous Galerkin (DG) method, singular initial data, linear hyperbolic equations, superconvergence, weight function, weighted norms.

1. Introduction

In this paper, we apply discontinuous Galerkin (DG) methods to solve linear hyperbolic equation with non-smooth solution in one space dimension

$$u_t + u_x = 0, \quad (x,t) \in [0,2\pi] \times (0,T], \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in [0,2\pi], \tag{2}$$

where the initial solution $u_0(x)$ has a discontinuity at $x = c$, but is otherwise smooth. We consider problem with suitable Dirichlet boundary condition

$$u(0,t) = g(t) \tag{3}$$

such that the exact solution is smooth except along the characteristic line $x = t + c$. It is well known that the numerical solution has spurious oscillations around the discontinuity line, which is regarded as "pollution region". The early works studying error estimates of DG methods for hyperbolic problems with discontinuities were given by Johnson et. al. [16, 17, 18]. They proved that the width of the pollution region is of the size at most $\mathcal{O}(h^{\frac{1}{2}} \log(1/h))$ with linear space-time elements. Later, similar results were also obtained by Cockburn and Guzmán [10] and Zhang and Shu [26] with the RKDG methods. The main idea is to introduce special weight functions which are very small near the singularity and are close to 1 outside the pollution region. More recently, Yang and Shu [25] applied the same idea and proved the $(2k+1)$-th superconvergence in negative-order norms outside the pollution region. To our best knowledge of the authors, this is the only superconvergence result for DG methods applied to hyperbolic equations with singular exact solutions.

Received by the editors September 21, 2016 and, in revised form, November 28, 2016.
2000 *Mathematics Subject Classification.* 65M15, 65M60, 65N15, 60N30.

The DG method was first introduced in 1973 by Reed and Hill [21], in the framework of neutron linear transport. Later, the method was applied by Johnson and Pitkäranta to a scalar linear hyperbolic equation and the L^p-norm error estimate was proved [17]. Subsequently, Cockburn et al. developed Runge-Kutta discontinuous Galerkin (RKDG) methods for hyperbolic conservation laws in a series of papers [13, 12, 11, 14]. Generally, we choose completely discontinuous piecewise polynomial space for DG methods. Hence, DG methods have several advantages such as high parallel efficiency, efficient h-p adaptivity, arbitrary order of accuracy and so on.

Superconvergence properties of DG methods for hyperbolic equations have been studied intensively, see [1, 2, 3, 4, 27, 20, 28, 8, 9, 24, 5, 6, 7] and the references therein. Many of the previous works are based on local error estimates or Fourier analysis, and the results only work for some special problems. In 2010, Cheng and Shu [9] applied energy analysis to obtain a $(k + 3/2)$-th superconvergence rate for the error between the DG solution and the particular projection of the exact solution. However, numerical experiments demonstrated the rate should be $k + 2$. Recently, Yang and Shu [24] extended the results in [9] to show that, with suitable initial discretization and upwind fluxes, the DG solution is $(k+2)$-th order superclose to the exact solution at the downwind-biased Radau points. The same convergence rate also works for the numerical cell averages. Subsequently, Cao et al. [5] proved a $(2k + 1)$-th order convergence rate of the error at the downwind point by constructing a special interpolation function. After that, in [7] and [6], the idea was applied to problems in two space dimensions and those in one space dimension with upwind-biased fluxes.

One of the most significant applications of the superconvergence is the construction of adaptive methods. The key point is to use the superconvergence properties to introduce a new numerical approximation which is superclose to the exact solution. Then the error between the two numerical approximations can be used as an error indicator to detect the regions with poor resolutions or singularities [19]. In this paper, we would like to analyze the error of the DG method for linear hyperbolic conservation law (1) outside the pollution region. The basic idea is to construct a suitable interpolation function u_I such that the DG solution u_h is $(2k + 1)$-th order accurate towards u_I under some weighted norms. By using the special properties of the weight functions we can prove several superconvergence results between the DG solution and the exact solution outside the pollution region. We will show that, under suitable initial discretizations, the DG solution is $(2k+1)$-th order accurate at the downwind points and $(k+2)$-th order accurate at all the other downwind-biased Radau points. Moreover, the derivative converges at a rate of $k + 1$ at all the interior upwind-biased Radau points. Besides the above, the DG solution is $(k + 2)$-th order accurate towards a particular projection of the exact solution and the numerical cell averages are $(2k + 1)$-th order accurate.

The organization of this paper is as follows. In Section 2, we will present preliminaries, including an introduction of DG scheme, some special projections, several elementary lemmas as well as the weight functions. In Section 3, we prove the main superconvergence results. Numerical experiments will be given in Section 4 to validate our theoretical results. Finally, we will end in Section 5 with concluding remarks and remarks on future works.

2. Preliminaries

In this section, we present some preliminaries that will be used throughout the paper. For simplicity, we use C to represent a generic positive constant that does not depend on the mesh size h, but may take different values at difference occurrences.

2.1. The DG scheme. In this subsection, we demonstrate the formulation of DG methods for (1) with Dirichlet boundary condition (3) on the computational domain $\Omega = [0, 2\pi]$.

First, we divide Ω into N cells

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi,$$

and denote

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$$

as the cells. Let $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ denote the length of cell I_j and $h = \max_j h_j$, $h_{min} = \min_j h_j$ be the length of the largest and smallest cells, respectively. Moreover, we define $\lambda = h/h_{min}$. In this paper, we consider regular meshes, i.e. there exists a constant C, such that $\lambda \le C$.

Next, we define

$$V_h = \{v : v|_{I_j} \in \mathcal{P}^k(I_j),\ j = 1, \cdots, N\}$$

as the finite element space, where $\mathcal{P}^k(I_j)$ denotes the space of polynomials of degree at most k in I_j. The DG scheme for (1) can be formulated as: Find $u_h \in V_h$, such that for any $v \in V_h$,

$$a_j(u_h, v) = ((u_h)_t, v)_j - (u_h, v_x)_j + u_h^- v^-|_{j+\frac{1}{2}} - u_h^- v^+|_{j-\frac{1}{2}} = 0, \tag{4}$$

where $(w, v)_j = \int_{I_j} wvdx$ and $v^-_{j+\frac{1}{2}} = v(x^-_{j+\frac{1}{2}})$ denotes the left limit of v at $x_{j+\frac{1}{2}}$. Likewise for v^+. Moreover, we define $[v]_{j+\frac{1}{2}} = v^+_{j+\frac{1}{2}} - v^-_{j+\frac{1}{2}}$ as the jump of v across $x_{j+\frac{1}{2}}$. For simplicity of presentation, we denote

$$\begin{aligned} \mathcal{H}_j(w, v) &= (w, v_x)_j - w^- v^-|_{j+\frac{1}{2}} + w^- v^+|_{j-\frac{1}{2}} \\ &= -(w_x, v)_j - [w]_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}, \end{aligned} \tag{5}$$

which further yields

$$(w_t, v)_j = a_j(w, v) + \mathcal{H}_j(w, v), \tag{6}$$

Finally, we introduce the bilinear form a and $\mathcal{H}$ on the whole computational domain as

$$a(w, v) = \sum_{j=1}^{N} a_j(w, v),$$

and

$$\mathcal{H}(w, v) = \sum_{j=1}^{N} \mathcal{H}_j(w, v).$$

2.2. Norms. In this subsection, we give the definition of weighted norms that we will use throughout the paper. For any function u and any positive function ψ, we define the weighted L^p-norm of u as

$$\|u\|_{0,p,\psi,D} = \begin{cases} \left(\int_D |u|^p \psi dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{x\in D} |u\psi|, & p = \infty. \end{cases}$$

Moreover, the weighted $W^{m,p}(D)$-norm of u is defined as

$$\|u\|_{m,p,\psi,D} = \begin{cases} \left(\sum_{j=0}^{m} \|D^j u\|_{0,p,\psi,D}^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{j\le m} \|D^j u\|_{0,\infty,\psi,D}, & p = \infty. \end{cases}$$

If $\psi = 1$, the weighted norms will degenerate to the standard Sobolev norms. For convenience, if $p = 2$, $\psi = 1$ and $D = \Omega$, then the corresponding subscripts will be omitted. For example, $\|u\|_0$ is the standard L^2-norm of u on Ω.

2.3. Special projections and properties of the DG discretization. In this subsection, we present special projections and demonstrate the properties of the DG discretization to be used in the proof of the main theorem.

First, we define $\mathbb{P}_\ell(w)$ as the ℓ-th order standard L^2 projection of function w into V_h, such that

$$(\mathbb{P}_\ell(w), v)_j = (w, v)_j, \quad \forall v \in \mathcal{P}^\ell(I_j). \tag{7}$$

In addition, we also construct two Gauss-Radau projections $\mathbb{P}_+$ and $\mathbb{P}_-$ by

$$(\mathbb{P}_+(w), v)_j = (w, v)_j, \quad \forall v \in \mathcal{P}^{k-1}(I_j) \quad \text{and} \quad \mathbb{P}_+(w)(x^+_{j-\frac{1}{2}}) = w(x^+_{j-\frac{1}{2}}), \tag{8}$$

$$(\mathbb{P}_-(w), v)_j = (w, v)_j, \quad \forall v \in \mathcal{P}^{k-1}(I_j) \quad \text{and} \quad \mathbb{P}_-(w)(x^-_{j+\frac{1}{2}}) = w(x^-_{j+\frac{1}{2}}). \tag{9}$$

Subsequently, for the above projection $\mathbb{P}_h$, which is either $\mathbb{P}_+$ or $\mathbb{P}_-$, we denote the error operator by $\mathbb{P}_h^\perp = \mathbb{I} - \mathbb{P}_h$, where $\mathbb{I}$ is the identity operator.

For the bilinear form $\mathcal{H}_j$, it is easy to check the following Lemma.

Lemma 1. *Suppose $q \in W^{1,2}(I_j)$ and $v \in V_h$, the two Gauss-Radau projections satisfy*

$$\mathcal{H}_j(\mathbb{P}_-^\perp q(x), v) = 0 \quad \textit{and} \quad \mathcal{H}_j(v, \mathbb{P}_+^\perp q(x)) = 0. \tag{10}$$

Next, we would like to roughly introduce a special interpolation function u_I which is $(2k+1)$-th order superclose to u_h. More details of the construction of u_I can be found in [5].

First, suppose $u(x,t)$ has the following Radau expansion in each cell I_j, $j = 1, 2, \cdots, N$:

$$u(x,t) = u(x^-_{j+\frac{1}{2}}, t) + \sum_{m=1}^{\infty} u_{j,m}(t)(L_{j,m} - L_{j,m-1})(x),$$

where $L_{j,m}$ is the Legendre polynomial of degree m in I_j, and $u_{j,m}$ is the coefficient.

Second, for all $v \in W^{1,2}(I_j)$, we define

$$D^{-1}v(x) = \frac{2}{h_j}\int_{x_{j-\frac{1}{2}}}^{x} v(\tau)d\tau, \quad x \in I_j,$$

and

$$F_1 = \mathbb{P}_- D^{-1} L_{j,k}, \quad F_i = -\mathbb{P}_- D^{-1} F_{i-1} = -(-\mathbb{P}_- D^{-1})^i L_{j,k}, \quad i \geq 2.$$

The special interpolation function is defined as

$$u_I^\ell = \mathbb{P}_- u - \sum_{i=1}^{\ell} \left(\frac{h_j}{2}\right)^i \partial_t^i u_{j,k+1}(t) F_i(x), \quad 1 \leq \ell \leq k. \tag{11}$$

For the bilinear form a_j, we have the following theorem (see [5] Theorem 3.2).

Theorem 2. *If $u \in W^{k+\ell+2,\infty}(\Omega)$ and $k \geq 1$, then for $1 \leq \ell \leq k$ and any $v \in V_h$, we have*

$$|a_j(u - u_I^\ell, v)| \leq C h^{k+\ell+1} \|u\|_{k+\ell+2,\infty,I_j} \|v\|_{0,1,I_j}. \tag{12}$$

2.4. The weight function. In this paper, we will consider two weight functions $\psi^1(x,t)$ and $\psi^{-1}(x,t)$ which will be used to determine the left and right boundaries of the pollution region, respectively. First, we define the cut-off exponential function $\phi(r) \in C^1 : \mathbb{R} \to \mathbb{R}$ as

$$\phi(r) = \begin{cases} 2 - e^r, & r < 0, \\ e^{-r}, & r \geq 0. \end{cases}$$

Then, $\psi^\beta(x,t)$ for $\beta = \pm 1$ are defined as the solutions of the linear hyperbolic problem

$$\psi_t^\beta + \psi_x^\beta = 0,$$
$$\psi^\beta(x,0) = \phi\left(\frac{\beta(x - x_c)}{\gamma h^\sigma}\right).$$

Following [25], we choose $\sigma = \frac{1}{2}$ and $x_c = 2\beta s \log(1/h) \gamma h^{1/2}$, with s and γ to be sufficiently large. It is very easy to check the following property.

Proposition 3. *For each of the weight function $\psi^\beta(x,t)$, the following properties hold*

$$1 \leq \psi^\beta(x,t) \leq 2, \quad \beta(x - x_c - t) \leq 0, \tag{13}$$

$$0 \leq \psi^\beta(x,t) < h^s, \quad \beta(x - x_c - t) > s \log(1/h) \gamma h^{1/2}, \tag{14}$$

$$\max_{d \leq \gamma h^{\frac{1}{2}}} \left| \frac{D_x^m \psi(x+d,t)}{D_x^m \psi(x,t)} \right| \leq e, \quad m = 0,1,2. \tag{15}$$

Besides that above, we will use several lemmas about the weight functions in [26]. For simplicity, we will drop the superscript and use ψ for both ψ^β.

Lemma 4. *For any $v \in V_h$, we have the following identity*

$$2\mathcal{H}(v, v\psi) = -\sum_j [v]_{j-\frac{1}{2}}^2 \psi_{j-\frac{1}{2}} - (v_{N+\frac{1}{2}}^-)^2 \psi_{N+\frac{1}{2}} + (v_{\frac{1}{2}}^-)^2 \psi_{\frac{1}{2}} + (v, v\psi_x). \tag{16}$$

Lemma 5. *Let $\mathbb{P}_h$ be a Gauss-Radau projection, either $\mathbb{P}_-$ or $\mathbb{P}_+$. There exists a positive constant C independent of h, such that for any $v \in V_h$, we have*

$$\|\mathbb{P}_h^\perp(\psi v)\|_{0,\psi^{-1},D} \leq C \gamma^{-1} h^{1/2} \|v\|_{0,\psi,D}, \tag{17}$$

$$\|\mathbb{P}_h(\psi v)\|_{0,\psi^{-1},D} \leq C \|v\|_{0,\psi,D}, \tag{18}$$

where D is either the single cell I_j or the whole computational domain Ω.

3. Superconvergence

In this section, we proceed to discuss the superconvergence properties of DG solution at some special points including downwind points and Radau points, and the superconvergence of the cell averages, etc. We begin with the following energy inequality.

Lemma 6. *For any $v \in V_h$ with $v^-_{\frac{1}{2}} = 0$, define $w = v_t - \mathbb{P}_{k-1} v_t \in V_h$, then we have*

$$\frac{d\|v\|^2_{0,\psi}}{dt} \le C\Big(\sum_{j=1}^{N} \Big| a_j(v,w)\Big(\frac{w}{\|w\|^2_{0,I_j}}, \mathbb{P}^{\perp}_{+}(v\psi)\Big)_j \Big| + \sum_{j=1}^{N} |a_j(v, \mathbb{P}_{+}(v\psi))| + \|v\|^2_{0,\psi}\Big). \tag{19}$$

Proof. For any $v \in V_h$ and weight functions ψ given in Subsection 2.4, we have

$$\begin{aligned} \frac{d\|v\|^2_{0,\psi}}{dt} &= 2(v_t, v\psi) + (v, v\psi_t) \\ &= 2a(v, v\psi) + 2\mathcal{H}(v, v\psi) + (v, v\psi_t) \\ &= 2a(v, v\psi) + (v, v\psi_t) + (v, v\psi_x) \\ &\quad - \sum_{j=1}^{N} [v]^2_{j-\frac{1}{2}} \psi_{j-\frac{1}{2}} + (v^-_{\frac{1}{2}})^2 \psi_{\frac{1}{2}} - (v^-_{N+\frac{1}{2}})^2 \psi_{N+\frac{1}{2}} \\ &= 2a(v, v\psi) - \sum_{j=1}^{N} [v]^2_{j-\frac{1}{2}} \psi_{j-\frac{1}{2}} + (v^-_{\frac{1}{2}})^2 \psi_{\frac{1}{2}} - (v^-_{N+\frac{1}{2}})^2 \psi_{N+\frac{1}{2}} \\ &\le \sum_{j=1}^{N} \Big(2a_j(v, v\psi) - [v]^2_{j-\frac{1}{2}} \psi_{j-\frac{1}{2}} \Big), \end{aligned} \tag{20}$$

where the second step follows from (6), the third step requires Lemma 4, the fourth step holds because $\psi_t + \psi_x = 0$. Next by using Lemma 1, we obtain

$$\begin{aligned} a_j(v, v\psi) &= (v_t, \mathbb{P}^{\perp}_{+}(v\psi))_j + a_j(v_t, \mathbb{P}_{+}(v\psi)) - \mathcal{H}_j(v, \mathbb{P}^{\perp}_{+}(v\psi)) \\ &= (v_t, \mathbb{P}^{\perp}_{+}(v\psi))_j + a_j(v, \mathbb{P}_{+}(v\psi)). \end{aligned} \tag{21}$$

To estimate the first term on the right-hand side, we define $w = v_t - \mathbb{P}_{k-1}v_t$, then

$$
\begin{aligned}
(22)\qquad (v_t, \mathbb{P}_+^{\perp}(v\psi))_j &= \left(\frac{(v_t,w)_j}{(w,w)_j} w, \mathbb{P}_+^{\perp}(v\psi)\right)_j \\
&= (a_j(v,w) + \mathcal{H}_j(v,w)) \left(\frac{w}{\|w\|_{0,I_j}^2}, \mathbb{P}_+^{\perp}(v\psi)\right)_j \\
&\le \left| a_j(v,w)\left(\frac{w}{\|w\|_{0,I_j}^2}, \mathbb{P}_+^{\perp}(v\psi)\right)_j \right| + \left| [v]_{j-\frac12} w_{j-\frac12}^+ \left(\frac{w}{\|w\|_{0,I_j}^2}, \mathbb{P}_+^{\perp}(v\psi)\right)_j \right| \\
&\le \left| a_j(v,w)\left(\frac{w}{\|w\|_{0,I_j}^2}, \mathbb{P}_+^{\perp}(v\psi)\right)_j \right| \\
&+ |[v]_{j-\frac12}| \sqrt{\psi_{j-\frac12}}\, h^{-\frac12} \|\mathbb{P}_+^{\perp}(v\psi)\|_{0,I_j} (\psi_{j-\frac12})^{-\frac12}, \\
&\le \left| a_j(v,w)\left(\frac{w}{\|w\|_{0,I_j}^2}, \mathbb{P}_+^{\perp}(v\psi)\right)_j \right| + \frac{C}{\gamma}([v]_{j-\frac12}^2 \psi_{j-\frac12} + \|v\|_{0,\psi,I_j}^2),
\end{aligned}
$$

where the first equality holds because of the property of projection $\mathbb{P}_{k-1}$, the last second inequality holds with trace inequality as well as the last inequality arises from (17) and (15). Plugging (22) into (21), we obtain

$$
\begin{aligned}
a_j(v, v\psi) \le & \left| a_j(v,w)\left(\frac{w}{\|w\|_{0,I_j}^2}, \mathbb{P}_+^{\perp}(v\psi)\right)_j \right| + \frac{C}{\gamma}([v]_{j-\frac12}^2 \psi_{j-\frac12} + \|v\|_{0,\psi,I_j}^2) \\
& + a_j(v, \mathbb{P}_+(v\psi)).
\end{aligned}
$$

Summing up the above equation in j and then plugging into (20), we obtain

$$
\begin{aligned}
(23)\quad \frac{d\|v\|_{0,\psi}^2}{dt} \le & \, 2\sum_{j=1}^N \left| a_j(v,w)\left(\frac{w}{\|w\|_{0,I_j}^2}, \mathbb{P}_+^{\perp}(v\psi)\right)_j \right| \\
& + \frac{C}{\gamma}(\sum_{j=1}^N [v]_{j-\frac12}^2 \psi_{j-\frac12} + \sum_{j=1}^N \|v\|_{0,\psi,I_j}^2) + 2\sum_{j=1}^N a_j(v, \mathbb{P}_+(v\psi)) - \sum_{j=1}^N [v]_{j-\frac12}^2 \psi_{j-\frac12} \\
\le & C\left(\sum_{j=1}^N \left| a_j(v,w)\left(\frac{w}{\|w\|_{0,I_j}^2}, \mathbb{P}_+^{\perp}(v\psi)\right)_j \right| + \sum_{j=1}^N |a_j(v, \mathbb{P}_+(v\psi))| + \|v\|_{0,\psi}^2 \right),
\end{aligned}
$$

where the last step requires γ to be sufficiently large. Now, we complete our proof. □

Next, we would like to modify the exact solution u of (1) into $\tilde{u} \in W^{k+\ell+2,\infty}(\Omega)$ without changing the DG solution. More details of the construction can be found in [25, 26]. This modification is only used for the theoretical analysis, since we need the high-order derivatives of the exact solution. For simplicity, we also use u instead of $\tilde{u}$ in the rest of the paper. Now, we proceed to estimate the error between u_I^ℓ and u_h.

Theorem 7. *For the above u and u_h and the special interpolation $u_I^\ell \in V_h$, $1 \le \ell \le k$, we have*

$$\|u_h - u_I^\ell\|_{0,\psi}(t) \le C(\|u_h - u_I^\ell\|_{0,\psi}(0) + th^{k+\ell+1}\|u\|_{k+\ell+2,\infty,\psi}). \tag{24}$$

Proof. Since $(u_h - u_I^\ell)^-_{\frac{1}{2}} = 0$, let $v = u_h - u_I^\ell$ in Lemma 6, we have

$$\begin{aligned}
&\frac{d\|u_h - u_I^\ell\|^2_{0,\psi}}{dt}\\
\le& C\Big(\sum_{j=1}^N \Big|a_j(u_h - u_I^\ell, w)\Big(\frac{w}{\|w\|^2_{0,I_j}}, \mathbb{P}_+^\perp(v\psi)\Big)_j\Big| + \sum_{j=1}^N |a_j(u_h - u_I^\ell, \mathbb{P}_+(v\psi))| + \|v\|^2_{0,\psi}\Big)\\
=& C\Big(\sum_{j=1}^N \Big|a_j(u - u_I^\ell, w)\Big(\frac{w}{\|w\|^2_{0,I_j}}, \mathbb{P}_+^\perp(v\psi)\Big)_j\Big| + \sum_{j=1}^N |a_j(u - u_I^\ell, \mathbb{P}_+(v\psi))| + \|v\|^2_{0,\psi}\Big)\\
\le& C\Big(\sum_{j=1}^N h_j^{k+\ell+1}\|u\|_{0,k+\ell+2,\infty,I_j}\Big(\|w\|_{0,1,I_j}\Big|\Big(\frac{w}{\|w\|^2_{0,I_j}}, \mathbb{P}_+^\perp(v\psi)\Big)_j\Big| + \|\mathbb{P}_+(v\psi)\|_{0,1,I_j}\Big)\\
&+ \|v\|^2_{0,\psi}\Big)\\
\le& C\Big(\sum_{j=1}^N h_j^{k+\ell+1}\|u\|_{k+\ell+2,\infty,\psi,I_j}\|v\|_{0,\psi,I_j}h^{\frac{1}{2}} + \|v\|^2_{0,\psi}\Big)\\
\le& C(h^{k+\ell+1}\|u\|_{k+\ell+2,\infty,\psi}\|u_h - u_I^\ell\|_{0,\psi} + \|u_h - u_I^\ell\|^2_{0,\psi}),
\end{aligned} \tag{25}$$

where the second step is due to the Galerkin orthogonality, step three follows from Theorem 2, step four requires Lemma 5 and step five holds based on the Cauchy-Schwarz inequality. From (25) it is easy to obtain

$$\frac{d\|u_h - u_I^\ell\|_{0,\psi}}{dt} \le C(h^{k+\ell+1}\|u\|_{k+\ell+2,\infty,\psi} + \|u_h - u_I^\ell\|_{0,\psi}), \tag{26}$$

which further implies

$$\|u_h - u_I^\ell\|_{0,\psi}(t) \le C(\|u_h - u_I^\ell\|_{0,\psi}(0) + th^{k+\ell+1}\|u\|_{k+\ell+2,\infty,\psi}), \tag{27}$$

by Gronwall's inequality. □

Remark 8. *From Theorem 7, a natural choice of the initial discretization is*

$$u_h(x,0) = u_I^k(x,0), \quad \forall x \in \Omega, \tag{28}$$

then we have

$$\|u_h - u_I^k\|_{0,\psi}(t) \le Cth^{2k+1}\|u\|_{2k+2,\infty,\psi}.$$

We will demonstrate the right-hand side of the above estimate is indeed $(2k+1)$-th order accurate, i.e. $\|u\|_{2k+2,\infty,\psi}$ is bounded. Following [25, 26], we define $x_L(t) = t - |x_c| = t - 2s\log(1/h)\gamma h^{1/2}$ as the left boundary of the pollution region and consider the weight function $\psi = \psi^1$. Moreover, we define

$$w(t) = \max\Big\{x_{j+\frac{1}{2}} : x_{j-\frac{1}{2}} < t - \frac{1}{2}|x_c|\Big\}$$

and $\mathcal{R}_1(t) = (0, w(t))$, $\mathcal{R}_2 = [0, 2\pi] \setminus \mathcal{R}_1(t)$. It is easy to check that $\mathcal{R}_1$ stays away from the bad interval $[t-h, t+h]$ which contains the discontinuity of the exact solution and on $\mathcal{R}_2$ we have $\psi^1 \leq h^s$. Therefore, we can take s to be sufficiently large, such that $\|u\|_{2k+2,\infty,\psi^1} \leq C$, which further yields

$$\|u_h - u_I^k\|_{0,\psi^1,\mathcal{R}_L(t)} \leq \|u_h - u_I^k\|_{0,\psi^1} \leq Ch^{2k+1},$$

where $\mathcal{R}_L(t) = [0, x_L(t)]$. Since $1 \leq \psi^1 \leq 2$ on $\mathcal{R}_L(t)$, we have

$$\|u_h - u_I^k\|_{0,\mathcal{R}_L(t)} \leq C\|u_h - u_I^k\|_{0,\psi^1,\mathcal{R}_L(t)} \leq Ch^{2k+1}.$$

Similarly, we also define $t + |x_c| = t + 2s\log(1/h)\gamma h^{1/2}$ to be the right boundary of the pollution region and $\mathcal{R}_R(t) = [t + |x_c|, 2\pi]$. Following the same analysis above and replace ψ^1 by ψ^{-1}, we have

$$\|u_h - u_I^k\|_{0,\mathcal{R}_R(t)} \leq Ch^{2k+1}.$$

Combine the above two inequalities together to obtain

$$\|u_h - u_I^k\|_{0,\mathcal{R}(t)} \leq Ch^{2k+1}, \tag{29}$$

where $\mathcal{R}(t) = \mathcal{R}_L(t) \cup \mathcal{R}_R(t)$. Then, we can present the superconvergence of the DG solution on $\mathcal{R}$. The results are given in the following theorem. For simplicity of presentation, we define

$$S = \{j : I_j \in \mathcal{R}(t)\} \quad \text{and} \quad N_s = |S|.$$

Now we are ready to demonstrate the main result of this paper, i.e. superconvergence properties at some special cases outside the pollution region in the following theorem. The proof of the theorem depends on the superconvergence of $u - u_I$ which were given in [5], so we omit the proof.

Theorem 9. *With the initial discretization* (28)*, we can claim the following superconvergence properties at several special points (including downwind point, downwind-biased Radau points and upwind-biased Radau points), for special projection $\mathbb{P}_-$ and cell averages.*

(I) The superconvergence at the downwind point:

$$|(u - u_h)(x_{j+\frac{1}{2}}^-, t)| \leq Ch^{2k+\frac{1}{2}}, \quad \forall\, j \in S,$$

$$\left(\frac{1}{N_s}\sum_{j\in S}(u - u_h)^2(x_{j+\frac{1}{2}}^-, t)\right)^{\frac{1}{2}} \leq Ch^{2k+1}.$$

(II) The superconvergence at downwind-biased Radau points:

$$|(u - u_h)(R_{j,\ell}^r, t)| \leq Ch^{k+2},$$

where $R_{j,\ell}^r$, with $\ell = 0, 1, \cdots, k$, are the $k+1$ zeros of the downwind-biased Radau polynomial $L_{j,k+1} - L_{j,k}$ in I_j, $\forall\, j \in S$, except the point $R_{j,0}^r = x_{j+\frac{1}{2}}$.

(III) The superconvergence of the derivative at upwind-biased Radau points:

$$|(u - u_h)_x(R_{j,\ell}^l, t)| \leq Ch^{k+1},$$

where $R_{j,\ell}^l$, with $\ell = 0, \cdots, k$, are the k interior zeros of the upwind-biased Radau polynomial $L_{j,k} + L_{j,k+1}$ in I_j, $\forall\, j \in S$, except the point $R_{j,0}^l = x_{j-\frac{1}{2}}$.

(IV) The DG solutions are superclose to $\mathbb{P}_-$***:***

$$\|u_h - \mathbb{P}_- u\|_{0,\mathcal{R}(t)} \le Ch^{k+2}.$$

(V) The superconvergence for cell averages:

$$\left(\frac{1}{N_s} \sum_{j\in S} \left(\frac{1}{h_j} \int_{I_j} (u - u_h) dx \right)^2 \right)^{\frac{1}{2}} \le Ch^{2k+1}.$$

Remark 10. *To obtain the above theorem, we need to start from* (29)*. For example, it is easy to obtain*

$$\left(\frac{1}{N_s} \sum_{j\in S} (u_I^k - u_h)^2 (x^-_{j+\frac{1}{2}}, t) \right)^{\frac{1}{2}} \le Ch^{2k+1}.$$

Moreover, for any $j \in S$, the error between u and u_I^k at the downwind point is given as [5]

$$|(u_I^k - u)(x^-_{j+\frac{1}{2}}, t)| \le Ch^{2k+1}.$$

The above two inequalities further yield the second result in part I of Theorem 9. We can basically follow the same line to obtain other estimates in Theorem 9.

4. Numerical experiments

Before we proceed to the numerical experiments, we introduce the high order time discrezation. There exists kinds of time discretiztions (see e.g. [23, 22, 15]) to solve the ODE system $\mathbf{u}_t = \mathbf{L}(\mathbf{u})$. In this paper, we would like to apply third-order Runge-Kutta method [23]

$$\begin{aligned} \mathbf{u}^{(1)} &= \mathbf{u}^n + \Delta t \mathbf{L}(\mathbf{u}^n), \\ \mathbf{u}^{(2)} &= \frac{3}{4}\mathbf{u}^n + \frac{1}{4}\left(\mathbf{u}^{(1)} + \Delta t \mathbf{L}(\mathbf{u}^{(1)})\right), \\ \mathbf{u}^{n+1} &= \frac{1}{3}\mathbf{u}^n + \frac{2}{3}\left(\mathbf{u}^{(2)} + \Delta t \mathbf{L}(\mathbf{u}^{(2)})\right), \end{aligned} \tag{30}$$

where $\mathbf{u}^n$ is the coefficient vector of polynomials P^k at the time $t = n\Delta t$. Here, we choose the time step $\Delta t = 0.1h^{(2k+1)/3}$ to reduce the time error. Now, we obtain the fully discrete scheme.

In this section, we use numerical experiments to verify Theorem 9. We use P^3 polynomials to solve (1) with $u_0 = \sin(x) + \delta(x - 1.2)$, and $g(t) = \sin(-t)$ and compute up to $T = 0.1$. It is easy to see that, at $t = 0.1$, the δ-singularity is located at $x = 1.3$. In [26], the authors have numerically verified the size of the pollution region, and the main goal of this paper is to demonstrate the superconvergence rates outside. Therefore, we compute the following errors over a fixed interval $I = [0, 0.6] \cup [2, 2\pi]$.

$$e_f = \max_{j\in S} |(u - u_h)(x^-_{j-\frac{1}{2}}, T)|, \quad e_c = \left(\tfrac{1}{N_s} \sum_{j\in S} \left(\tfrac{1}{h_j} \int_{I_j} (u - u_h) dx \right)^2 \right)^{\frac{1}{2}},$$
$$e_r = \max_{j\in S} |(u - u_h)(R^r_{j,\ell}, T)|, \quad e_d = \max_{j\in S} |(u - u_h)_x(R^l_{j,\ell}, T)|,$$

where $S = \{j : I_j \in I\}$ and $N_s = |S|$.

The results are given in Table 1. The table demonstrates superconvergence rates

TABLE 1. Various errors with Radau interpolation of the Dirichlet boundary condition for $k = 3$.

	e_c		e_f		e_r		e_d	
n	error	order	error	order	error	order	error	order
200	4.51e-16	-	8.67e-16	-	1.47e-12	-	4.49e-10	-
400	1.69e-19	11.4	3.81e-19	11.2	4.60e-14	5.00	2.80e-11	4.00
800	1.32e-21	7.01	2.67e-21	7.16	1.44e-15	5.00	1.75e-12	4.00
1600	1.04e-23	6.98	2.11e-23	6.98	4.59e-17	5.00	1.10e-13	4.00

$2k+1$ (the polynomial degree is $k = 3$) for the numerical cell average and numerical approximation at downwind point (e_c and e_f), $k + 2$ for the numerical solution at right Radau points (e_r), and $k + 1$ for the derivative of the approximation at the interior left Radau points (e_d) between the DG approximation and the analytic solution, which confirm our theoretical results in Theorems 9.

5. Conclusion

In this paper, we use the DG method to solve hyperbolic conservation laws with singular initial data. We investigate the superconvergence properties outside the pollution region. We have shown the $(2k + 1)$-th order superconvergence rate for the DG solutions at downwind points and for the cell averages. Moreover, the DG solutions are also $(k + 2)$-th order accurate at the downwind-biased Radau points and the derivatives of the error are $(k+1)$-th order superconvergent at the upwind-biased Radau points. Numerical experiments were given to verify the theoretical results of DG method. In the future, we will consider the superconvergence of nonlinear conservation laws.

Acknowledgments

This research was supported by the National Natural Science Foundation of China under the grant 11571367 and 11601536, and Michigan Technological University Research Excellence Fund Scholarship and Creativity Grant 1605052.

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