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AN EFFICIENT APPROXIMATION FOR OPTIMAL DAMPING IN MECHANICAL SYSTEMS

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Abstract. This paper is concerned with an efficient algorithm for damping optimization in mechanical systems with a prescribed structure. Our approach is based on the minimization of the total energy of the system which is equivalent to the minimization of the trace of the corresponding Lyapunov equation. Thus, the prescribed structure in our case means that a mechanical system is close to a modally damped system. Although our approach is very efficient (as expected) for mechanical systems close to modally damped system, our experiments show that for some cases when systems are not modally damped, the proposed approach provides efficient approximation of optimal damping.

Key words. Linear vibration system, damped vibration, multi-variable optimization of dampers' viscosities, passive damping, Lyapunov equation, relative residual bound, linear residual bounds, quadratic residual bounds.

1. Nomenclature

We will use the following notation. Matrices written in the simple Roman fonts, M, D or K, for example, will have $\mathcal{O}(n^2)$ entries. Matrices written in mathematical bold fonts, $\mathbf{A}$, $\mathbf{B}$ will have $\mathcal{O}(m^2)$ entries, where $m = 2n$. The symbol $\|\cdot\|$ stands for the standard 2-norm.

2. Introduction

We consider a damped linear vibration system described by the differential equation

$$M\ddot{x} + D\dot{x} + Kx = 0,$$
$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0,$$

where M, D, K (called mass, damping and stiffness matrix, respectively) are real, symmetric matrices of order n with M, K positive definite and $D = C_u + C$, where C_u is positive definite and presents internal damping, while C represents external damping and it is positive semidefinite. The matrix C_u is usually taken as a small multiple of critical damping or proportional damping. In this paper, we assume that internal damping is a small multiple of mass matrix, that is, $C_u = \alpha M$.

The problem of deriving *optimal damping* in some sense is an old and widely investigated problem which has been considered by many authors.

For example, in [1] the question of placement of damping elements was investigated, while in [2] the problem of periodic optimal control, which maximizes energy dissipation, was considered.

On the other hand, the optimization problem, which considers only viscosity optimization, was considered in the following papers [3], [4], [5], [6], [7] and [8].

In papers [9], [10] and [11], the authors have recemtly considered approximations based on modal eigenvectors which provide an efficient calculation of objective functions. The case of mechanical systems with a given force was considered in papers [12] and [13], where the authors derived explicit formulas for objective functions for particular types of mechanical systems, while in [14] it was shown how to compute eigenfrequencies of structures composed of a series of inclined cables.

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The purpose of this paper is to present new results of approximation algorithms for deriving optimal damping. As we will show, in some cases determination of optimal damping can be given by an explicit formula, while in some other cases we present a numerical approach to determination of optimal damping which can be efficiently implemented.

We are going to use optimization criterion considered in many papers, like [15],[3], [6], [5], [7]. This optimization criterion is given by the minimization requirement of the total energy of the system, that is,

$$\int_0^\infty E(t)\,dt \to \min\,. \tag{1}$$

Since criterion (1) depends on the initial condition, the simplest way to correct this is to take the average of (1) over all initial states of the unit total energy and a given frequency range. It can be shown that this average corresponds to the trace of the solution of the corresponding Lyapunov equation.

Since up to date an efficient general algorithm for the optimization of damping does not exist, that is, available algorithms optimize only viscosities of dampers, not their positions, we propose a simple and efficient approach to the overall damping optimization. With this new approach, one can find optimal positions and corresponding damper viscosities efficiently with satisfactory accuracy.

Our approach is based on the fact that for a modally damped mechanical system all three matrices M, D and K can be simultaneously diagonalized. Thus, the main assumption here will be that we have the case where M, D and K are simultaneously diagonalizable or that they are close to the case when all three matrices can be simultaneously diagonalized. Although this approach has been widely used by different scientific communities, especially in engineering, in this paper we propose a slightly different perspective, which will allow us to determine optimal damping very efficiently for a certain structure of mechanical systems, as will be demonstrated later.

Moreover, since only the damping matrix $D(v)$ depends on parameters, usual approaches to viscosity optimization (v) assume preprocessing based on diagonalization of the mass and stiffness matrices, M and K. On the other hand, in this paper we propose a new approach, which is based on diagonalization of the damping matrix $D(v)$, and then calculation of optimal viscosities. As we will show in this paper, this approach can be very efficient for structured systems which allow us to determine optimal viscosities, explicitly or numerically considerably faster.

For estimation of optimal viscosity for given damper positions we propose a new algorithm based on the simple "reduction" (truncation) of the corresponding Lyapunov equation, which usually speeds up the procedure by at least 40 times, while for the optimization of damper positions we propose a new heuristic. Both algorithms are based on a certain heuristic and unfortunately we do not have bounds for their accuracy, but as exemplified in the last section by the Lyapunov equation of modest dimensions ($n \leq 100$), they perform very well, thus we assume that the obtained results will be even better for bigger dimensions.

Currently, two types of algorithms are in use for the estimation of optimal viscosity (for given damper positions). The first type are the Newton-type algorithms for one-dimensional problems, which use some Lyapunov solvers, and the second type are the algorithms which explicitly calculate the trace of the solution of the corresponding Lyapunov equation.

Algorithms of the second type are presented in [15], [6] or [7], and they consider the case with one or more dampers with the same viscosity.

On the other hand, the Newton-type algorithm for the case with $r \geq 1$ different dampers was proposed in [16]. As shown in [7], the algorithm proposed in [16] can produce a poor result due to the problems with determination of the starting point.

The paper is organized as follows. In Section 2, we precisely define problem setting, while in Section 3, we present an approximation for our objective function. The problem of

damping optimization with particular emphasis the structured case was studied in Section 4. The efficiency and performance of the proposed approach are illustrated in Section 5.

3. Problem definition

As mentioned in the introduction, the minimization of the total energy (1) is equivalent to the minimization of the trace of the solution of the corresponding Lyapunov equation (more details can be found in [17], [6], [4], [3], [5]).

Thus, let

$$M\ddot{x} + D\dot{x} + Kx = 0 \tag{2}$$

be the differential equation describing a damped linear vibration system, where M, D, K are the mass, damping and stiffness matrices, respectively.

The eigenvalue problem

$$(\lambda^2 M + \lambda D + K)x = 0 \tag{3}$$

corresponds to (2).

Just for the purpose of recapitulation of some basic properties of the eigenvalue problem (3), we will use the eigenvalue decomposition

$$\Phi^T K \Phi = \Omega^2, \quad \Phi^T M \Phi = I, \tag{4}$$

where

$$\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_n), \qquad \omega_1 \leq \ldots \leq \omega_n. \tag{5}$$

By setting

$$y_1 = \Omega\, \Phi^T x, \quad y_2 = \Phi^T \dot{x}, \tag{6}$$

(2) can be written as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}, \tag{7}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix}, \tag{8}$$

(we are now in the $2n$-dimensional phase space), with the solution

$$\mathbf{y} = e^{\mathbf{A}t}\mathbf{y}_0, \quad \text{where } \mathbf{y}_0 \text{ is the initial data.} \tag{9}$$

It can be shown that the criterion of the minimization of the total energy (1) is equivalent to

$$\operatorname{tr}(\mathbf{Z}\mathbf{X}_{\mathbf{\Phi}}) \to \min, \tag{10}$$

where $\mathbf{X}_{\mathbf{\Phi}}$ is the solution of the following Lyapunov equation

$$\mathbf{A}^T \mathbf{X}_{\mathbf{\Phi}} + \mathbf{X}_{\mathbf{\Phi}} \mathbf{A} = -\mathbf{I},$$

and $\mathbf{Z}$ is a symmetric positive semidefinite matrix which may be normalized to have a unit trace. If for the measure σ we take the measure generated by the Lebesgue measure on $\mathbb{R}^{2n}$, we obtain $\mathbf{Z} = \frac{1}{2n}\mathbf{I}$. Without loss of generality, hereinafter we omit the factor $\frac{1}{2n}$ from the definition of the matrix $\mathbf{Z}$.

While we have internal damping which is not trivial, it can be shown that all eigenvalues of (3) lie in the left complex plane. This means that the matrix A from (8) is asymptotically stable.

Further, it is easy to show that

$$\operatorname{tr}(\mathbf{Z}\mathbf{X}_{\mathbf{\Phi}}) = \operatorname{tr}(\mathbf{Y}),$$

where $\mathbf{Y}$ is a solution of the so-called "dual Lyapunov equation"

$$\mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T = -\mathbf{Z}. \tag{11}$$

The structure of the matrix $\mathbf{Z}$ has been studied in detail in [17] and some of these results are presented in [6].

Throughout this paper we will assume that the matrix $\mathbf{Z}$ has the following form

$$\mathbf{Z} = \begin{bmatrix} 0_{t_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{t_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{t_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{t_2} \end{bmatrix}, \tag{12}$$

where I_s is the s-dimensional identity matrix, and 0_{t_i} is the t_i-dimensional ($i = 1, 2$) zero matrix, with t_1 and s defined such that the eigenfrequencies from (5) smaller than ω_{t_1} and greater than ω_{t_1+s} are not dangerous (observe that $t_2 = n - t_1 - s$).

Note that the solution of Lyapunov equation (11) is a function of several variables, damper positions and corresponding viscosities. Thus, simultaneous optimization of damper positions and viscosity can be computationally very demanding . In what follows, we will propose a new approach to dampers optimization.

First, in the next section we will present a new algorithm which approximates the solution (as well as its trace) of the corresponding Lyapunov equation, and after that in Section 5 we propose a new algorithm for finding optimal damper positions.

4. Approximation of the solution of the Lyapunov equation

It is well known that linearization from (8) is not unique. Thus for our purpose we will rewrite (2) using the following linearization:

$$\dot{\mathbf{y}} = \mathbf{A}_* \mathbf{y}, \tag{13}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{A}_* = \begin{bmatrix} 0 & K^{\frac{1}{2}} M^{-\frac{1}{2}} \\ -M^{-\frac{1}{2}} K^{\frac{1}{2}} & -M^{-\frac{1}{2}} D M^{-\frac{1}{2}} \end{bmatrix}. \tag{14}$$

Following the exposure from the previous section it follows that we are interested in minimizing the trace

$$\operatorname{tr}(\mathbf{ZY}),$$

where $\mathbf{Y}$ is a solution of the Lyapunov equation

$$\mathbf{A}_*{}^T \mathbf{Y} + \mathbf{Y} \mathbf{A}_* = -\mathbf{I}, \tag{15}$$

and $\mathbf{Z}$ is defined in (12).

Up to this point we have not introduced any new approaches or new ideas described in the introduction. Thus, next we proceed with the presentation of the new approach to approximation of the solution of Lyapunov equation (15), which is different from the standard ones (mostly used by engineers), that are based on modal approximation of mechanical systems. Our approach will combine two aspects, one is a modal approximation approach and the other is an approach based on the improved error estimates, see e.g. [9], [10] and [11].

For that purpose let

$$M^{-\frac{1}{2}} D M^{-\frac{1}{2}} = U_0 \Delta U_0^T, \qquad \Delta = \operatorname{diag}(\delta_1, \ldots, \delta_n), \tag{16}$$

be the eigenvalue decomposition of the "damping matrix" $M^{-\frac{1}{2}} D M^{-\frac{1}{2}}$.

Let

$$\mathbf{T} = \begin{bmatrix} I & 0 \\ 0 & U_0 \end{bmatrix} \tag{17}$$

be the orthogonal matrix, where U_0 is defined in (16). If one multiplies Lyapunov equation (15) from the left and from right by $\mathbf{T}^T$ and $\mathbf{T}$, respectively, then one gets

$$\mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} = -\mathbf{I}, \tag{18}$$

where

$$\mathbf{A} = \mathbf{T}^T \mathbf{A}_* \mathbf{T} = \begin{bmatrix} 0 & B^T \\ -B & -\Delta \end{bmatrix}, \tag{19}$$

where $B = U_0^T M^{-\frac{1}{2}} K^{\frac{1}{2}}$ and

$$\mathbf{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}. \tag{20}$$

Now equation (18) can be written as

$$\begin{bmatrix} 0 & -B^T \\ B & -\Delta \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} 0 & B^T \\ -B & -\Delta \end{bmatrix} = - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{21}$$

where

$$\mathbf{Y} = \mathbf{T} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \mathbf{T}^T. \tag{22}$$

It obviously holds $\mathrm{tr}(\mathbf{ZY}) = \mathrm{tr}(\mathbf{Z_U X})$, where

$$\mathbf{Z_U} = \mathbf{T}^T \mathbf{Z} \mathbf{T}. \tag{23}$$

Now, from (21) one gets

$$B^T X_{12}^T + X_{12} B = I, \tag{24}$$

$$-B^T X_{22} + X_{11} B^T - X_{12} \Delta = 0, \tag{25}$$

$$B X_{11} - X_{22} B - \Delta X_{12}^T = 0, \tag{26}$$

$$B X_{12} + X_{12}^T B^T - \Delta X_{22} - X_{22} \Delta = -I. \tag{27}$$

From (24) it follows

$$X_{12} = \frac{1}{2} B^{-1} + S B^{-1}, \quad \text{where} \quad S = -S^T. \tag{28}$$

Thus, if one knows the skew-symmetric matrix S from (28), then the solution $\mathbf{X}$ is known.

A new approach: As described in the introduction, our approach is based on some interesting properties of a modally damped system. As is well known (see e.g. [18, Theorem 2.3]), the modally damped system satisfies the so-called commuting condition

$$DK^{-1}M = MK^{-1}D.$$

It can also be shown, provided that inverses exist, that the above equality is equivalent to

$$KD^{-1}M = MD^{-1}K \text{ and also to } DM^{-1}K = KM^{-1}D. \tag{29}$$

All this means that the mechanical system can be modally damped, that is, that all three matrices M, D and K can be simultaneously diagonalized, even if some of them are singular.

Next we will show that this assumption is equivalent to the assumption on commuting X_{12} and B, that is, we will show that if

$$X_{12} B = B X_{12}, \tag{30}$$

then (29) holds, and the mechanical system is modally damped.

If X_{12} and B commute, that is, if (30) holds, then (27) and (26) imply that

$$\Delta X_{22} + X_{22} \Delta = 2I, \tag{31}$$

$$X_{11} = B^{-1} X_{22} B + B^{-1} \Delta X_{12}^T. \tag{32}$$

Here we have used the fact that if (30) holds, then $B X_{12} + X_{12}^T B^T = I$.

On the other hand, the assumption that (30) holds implies that S from (28) is a zero matrix.

The main idea: Now we do not assume that (30) holds, that is, our mechanical system is no longer modally damped.

But if it is still "good in some sense", or "close" to a modally damped system, we can use the above conclusions to approximate solution $\mathbf{X}$ of Lyapunov equation (18).

For that purpose, first we will approximate X_{12} from (28) with

$$\widetilde{X}_{12} = \frac{1}{2}B^{-1}. \tag{33}$$

Further, from (31) follows that

$$\widetilde{X}_{22} = \Delta^{-1}. \tag{34}$$

Once we have derived $\widetilde{X}_{22}$, it is easy to derive the last unknown approximation $\widetilde{X}_{11}$. Indeed, from (32) it follows

$$\widetilde{X}_{11} = B^{-1}\widetilde{X}_{22}B + \frac{1}{2}B^{-1}\Delta B^{-T}. \tag{35}$$

In the next theorem, we will present the residual error

$$R_{er} = \|\mathbf{A}^T\widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}\mathbf{A} + \mathbf{I}\| \tag{36}$$

made by the approximation

$$\widetilde{\mathbf{X}} = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix}, \tag{37}$$

which is equivalent as if we inserted approximations $\widetilde{X}_{11}$, $\widetilde{X}_{22}$ and $\widetilde{X}_{12}$ into (25).

The following theoretical results will be used for error estimation made by the above approximation of the solution of the Lyapunov equation. It will also be used for the a priori estimation, whether the considered mechanical system is approximated with a modally damped one.

Theorem 4.1. *Let $\widetilde{\mathbf{X}}$ be the approximation of solution (20) of Lyapunov equation (21) obtained by (33), (34) and (35). Then the residual error R_{er} is given by*

$$R_{er} = \|B^T\Delta^{-1} - B^{-1}\Delta^{-1}BB^T\|. \tag{38}$$

Proof. The proof simply follows by inserting $\widetilde{\mathbf{X}}$ into (21). Indeed, from (37) and (21) one gets

$$\begin{bmatrix} 0 & -B^T \\ B & -\Delta \end{bmatrix}\begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix} + \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix}\begin{bmatrix} 0 & B^T \\ -B & -\Delta \end{bmatrix} = -\begin{bmatrix} I & Err \\ 0 & I \end{bmatrix}, \tag{39}$$

where

$$-Err = -B^T\widetilde{X}_{22} + B^{-1}\widetilde{X}_{22}BB^T + \frac{1}{2}\left(-B^{-1}\Delta + B^{-1}\Delta\right).$$

Now, since

$$R_{er} = \|Err\|,$$

(38) holds, which completes the proof. □

Lemma 4.1. *Let $B = U_0^T M^{-\frac{1}{2}}K^{\frac{1}{2}}$ and $\Delta = U_0^T M^{-\frac{1}{2}}DM^{-\frac{1}{2}}U_0$. Then $BB^T\Delta^{-1} = \Delta^{-1}BB^T$ if and only if the mechanical system from (2) is modally damped, that is, equality (29) holds.*

Proof. Since $B = U_0^T M^{-\frac{1}{2}}K^{\frac{1}{2}}$ and $\Delta = U_0^T M^{-\frac{1}{2}}DM^{-\frac{1}{2}}U_0$, it follows that

$$BB^T = U_0^T M^{-\frac{1}{2}}KM^{-\frac{1}{2}}U_0 \quad \text{and} \quad \Delta^{-1} = U_0^T M^{\frac{1}{2}}D^{-1}M^{\frac{1}{2}}U_0.$$

Now, simple multiplication gives

$$BB^T\Delta^{-1} = U_0^T M^{-\frac{1}{2}}KD^{-1}M^{\frac{1}{2}}U_0,$$
$$\Delta^{-1}BB^T = U_0^T M^{\frac{1}{2}}D^{-1}KM^{-\frac{1}{2}}U_0,$$

which together implies that if $BB^T\Delta^{-1} = \Delta^{-1}BB^T$, then

$$KD^{-1}M = MD^{-1}K.$$

□

As a consequence of the above theorem, we have the following corollary.

Corollary 4.1. *Let the assumptions of Theorem 4.1 hold, that is,*

$$BB^T\Delta^{-1} = \Delta^{-1}BB^T,$$

holds if and only (29) holds. Then $\widetilde{\mathbf{X}}$ *is a solution of Lyapunov equation (21).*

Proof. If $BB^T\Delta^{-1} = \Delta^{-1}BB^T$, then from (38) it follows that $R_{er} = 0$, which implies that $\widetilde{\mathbf{X}} = \mathbf{X}$. □

5. Damping optimization

Using the approximation from the previous section, here we will present a new approach to damping optimization. Thus, we assume that the considered mechanical system is close to the perturbed modally damped system, that is, further on we assume that the residual error R_{er} from (38) is small enough, which means that $KD^{-1}M \approx MD^{-1}K$ in some sense.

For that purpose, let

$$M = U_M\Lambda_M U_M^T, \quad U_M = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}, \quad \Lambda_M = \begin{bmatrix} \mu_1 & \dots & \mu_n \end{bmatrix},$$

be the eigenvalue decomposition of the mass matrix M.

We will distinguish two different cases. In the first case, we assume that the damping matrix D has the same eigenvector structure as the mass matrix M, that is, we will assume that

$$D_I = \nu_1 u_1 u_1^T + \nu_2 u_2 u_2^T + \ldots + \nu_n u_n u_n^T, \tag{40}$$

where $\nu_i = v_i + \alpha$, $i = 1, \ldots, n$.

For the damping matrix close to D_I, and in the case when the number of dampers is equal to the dimension, that is, when $r = n$, we will be able to derive the explicit formula for optimal damping viscosities v_i, $i = 1, \ldots, n$. On the other hand, for the case when the number of dampers is less than the dimension or some viscosities are the same, we will present a formula that covers these cases in a more general setting.

Thus, back to the first case, we will assume that the damping matrix D is close to D_I from (40), that is,

$$D \approx (v_1 + \alpha)u_1 u_1^T + (v_2 + \alpha)u_2 u_2^T + \ldots + (v_n + \alpha)u_n u_n^T. \tag{41}$$

Below we will derive a simple formula for calculation of optimal viscosities $v_1, \ldots, v_n$, for which the trace of the approximation $\widetilde{\mathbf{X}}$ from (37) is minimal.

If we are interested in damping the s undamped frequencies, then using the matrix Z from (12) we obtain that for the matrix Z_U from (23) it can be written as

$$\mathbf{Z_U} \doteq \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix},$$

where

$$Z_1 = \operatorname{diag}(0_{t_1}, I_s, 0_{t_2}), \tag{42}$$

$$Z_2 = U_0^T \operatorname{diag}(0_{t_1}, I_s, 0_{t_2}) U_0. \tag{43}$$

Since our penalty function is a trace of the solution of the corresponding Lyapunov equation, note that for the approximation of the trace holds

$$\begin{aligned} \operatorname{tr}(\mathbf{Z_U X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) &= \operatorname{tr}(Z_1\widetilde{X}_{11}) + \operatorname{tr}(Z_2\widetilde{X}_{22}) \\ &= \operatorname{tr}(Z_1\widetilde{X}_{22}) + \operatorname{tr}(Z_2\widetilde{X}_{22}) + \frac{1}{2}\operatorname{tr}(B^{-1}Z_1\Delta B^{-T}), \end{aligned}$$

that is,

$$\operatorname{tr}(\mathbf{Z_U X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \operatorname{tr}(Z_1\Delta^{-1}) + \operatorname{tr}(Z_2\Delta^{-1}) + \frac{1}{2}\operatorname{tr}(Z_1\Delta B^{-T}B^{-1}). \tag{44}$$

Approximation (44) will be our starting point, which will allow us to derive an approximation for optimal $v_1^*, \dots, v_n^*$.

Note that from (41) and (16) it follows that

$$M^{-\frac{1}{2}} D M^{-\frac{1}{2}} = U_0 \Delta U_0^T, \quad \text{where } \Delta = \operatorname{diag}(v_1+\alpha, v_2+\alpha, \dots, v_n+\alpha). \tag{45}$$

Now from (44) and (45) one gets

$$\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\dots,v_n)) = \sum_{i=1}^{n} \frac{(Z_1)_{ii}+(Z_2)_{ii}}{v_i+\alpha} + \frac{1}{2}\sum_{i=1}^{n}(v_i+\alpha)(Z_1)_{ii}b_i, \tag{46}$$

where $b_i = \|T(:,i)\|^2$, $\quad T = B^{-1}$ for $i = 1, \dots, n$.

Using the fact that all quantities in (46) are nonnegative, simply by using partial derivatives

$$\frac{\delta}{\delta v_i}\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\dots,v_n)) = -\frac{(Z_1)_{ii}+(Z_2)_{ii}}{(v_i+\alpha)^2} + \frac{1}{2}(Z_1)_{ii}b_i, \quad i = 1,\dots,n, \tag{47}$$

and equalizing to zeros, one gets that

$$v_i^* = \sqrt{\frac{2(Z_1)_{ii}+2(Z_2)_{ii}}{(Z_1)_{ii}b_i}} - \alpha, \qquad i = 1,\dots,n.$$

are minimum for the trace, that is

$$(v_1^*,\dots,v_n^*) = \arg\min \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\dots,v_n)).$$

The setting given by (40) was just a motivation for a more general case that will be considered in the next section.

5.1. Damping optimization for the structured case. Troughout this section we assume that the eigenvalue decomposition of the matrix $M^{-\frac{1}{2}} D M^{-\frac{1}{2}}$ is given by

$$M^{-\frac{1}{2}} D M^{-\frac{1}{2}} = U_0 \Delta U_0^T, \quad \Delta = v_1 D_1 \oplus v_2 D_2 \oplus \cdots \oplus v_d D_d, \tag{48}$$

where each matrix D_i, $i = 1, \dots, d$ is a diagonal matrix and it has a dimension d_i, $i = 1, \dots, d$, respectively, with $\sum_{i=1}^{d} d_i = n$.

The above assumption means that the matrix Δ is a direct sum of smaller matrices that correspond to the same viscosities and it arises from the fact that very often the damping matrix D can have blocks of dampers which have the same viscosities. Moreover, in the assumed setting, damping blocks with different viscosities do not interlace with each other.

Note that the setting included in (40) is also covered by (48) since we can also use this approach in the case when $d_i = 1$, $\forall i = 1, \dots, n$ considering that all viscosities are different. On the other hand, we would like to emphasize that damping of the form (48) generalizes dampings studied in the previous section. Moreover, it also includes more general cases in which the damping matrix D is permutation similar to the block diagonal matrix where each block corresponds to damping parts with its own viscosity parameter.

Similarly to the above, if we are interested in damping the first s most important eigenfrequencies, then the matrix $\mathbf{Z_U}$ from (23) can be written as

$$\mathbf{Z_U} = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix},$$

where Z_1 and Z_2 are given by (42-43).

Also, for the approximation of the trace it holds

$$\operatorname{tr}(\mathbf{Z_U}\mathbf{X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \operatorname{tr}(Z_1\widetilde{X}_{22}) + \operatorname{tr}(Z_2\widetilde{X}_{22}) + \frac{1}{2}\operatorname{tr}(B^{-1}Z_1\Delta B^{-T}),$$

that is,

$$\operatorname{tr}(\mathbf{Z_U}\mathbf{X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \operatorname{tr}(Z_1\Delta^{-1}) + \operatorname{tr}(Z_2\Delta^{-1}) + \frac{1}{2}\operatorname{tr}(Z_1\Delta B^{-T}B^{-1}). \tag{49}$$

Now using approximation (49) together with (48) we can derive approximate optimal parameters $v_1^*, \ldots, v_d^*$.

In particular, from (48) and (49) one gets

$$\begin{aligned}\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_d)) &= \sum_{i=1}^{d}\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{v_i(D_i)_{k_jk_j}+\alpha}\\ &\quad+\frac{1}{2}\sum_{i=1}^{d}\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}(Z_1)_{jj}b_j(v_i(D_i)_{k_jk_j}+\alpha),\end{aligned}\tag{50}$$

where k_j determines the index that depends on j and it holds that $k_j = j-(d_1+\cdots+d_{j-1})$. Moreover, b_i is the 2-norm of the column of the matrix $T = B^{-1}$, that is, $b_i = \|T(:,i)\|^2$.

In general, for this function we are not able to derive an explicit formula for optimal viscosities. But since in this case, where the matrix which diagonalizes the matrix $M^{-\frac{1}{2}}DM^{-\frac{1}{2}}$ is the same for all viscosities, we can determine optimal viscosities efficiently by a numerical optimization procedure which will be described in the next section.

Additionally, we are also able to derive an explicit formula for a global minimum if $\sum_{i=1}^{d}\operatorname{rank}(D_i) = n$ and $\alpha = 0$. In that case, our objective function has the following form:

$$\begin{aligned}\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_d)) &= \sum_{i=1}^{d}\frac{1}{v_i}\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{(D_i)_{k_jk_j}}\\ &\quad+\frac{1}{2}\sum_{i=1}^{d}v_i\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}(Z_1)_{jj}b_j(D_i)_{k_jk_j}.\end{aligned}\tag{51}$$

Using the fact that all quantities in (50) are nonnegative, one easily obtains partial derivatives

$$\begin{aligned}\frac{\delta}{\delta v_i}\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_d)) &= -\frac{\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{(D_i)_{k_jk_j}}}{v_i^2}\\ &\quad+\frac{1}{2}\sum_{j=d_1+\cdots+d_{i-1}}^{d_1+\cdots+d_i}(Z_1)_{jj}b_j(D_i)_{k_jk_j},\end{aligned}\tag{52}$$

for $i = 1,\ldots,d$. Now, by equalizing the above derivations with zero, one gets that

$$v_i^* = \sqrt{\frac{2\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{(D_i)_{k_jk_j}}}{\sum_{j=d_1+\cdots+d_{i-1}}^{d_1+\cdots+d_i}(Z_1)_{jj}b_j(D_i)_{k_jk_j}}},\qquad i=1,\ldots,d,\tag{53}$$

are optimal viscosities, that is,

$$(v_1^*,\ldots,v_d^*) = \arg\min\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_d)).$$

Remark 1 The objective function given by (50) for the parameter $\alpha \neq 0$ can be efficiently optimized using a numerical optimization procedure. In particular, in this case we deal with the minimization of d functions where the ith function f_i is given by

$$\begin{aligned}f_i(v_i) &= \sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{v_i(D_i)_{k_jk_j}+\alpha}\\ &\quad+1/2\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}(Z_1)_{jj}b_j(v_i(D_i)_{k_jk_j}+\alpha),\end{aligned}\tag{54}$$

for $i = 1,\ldots,d$. Here the function f_i is a strictly convex function with global minima v_i^*, for $i = 1,\ldots,d$ respectively, where minima v_i^*, for $i = 1,\ldots,d$ can be efficiently determined

using iterative solvers. By this approach we are able to determine optimal parameters v_i^*, for $i = 1, \dots, d$ that minimize $\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1, \dots, v_d))$.

5.2. Damping optimization for the general case. In this section, we consider a more general case, than the two cases from previous sections, but still we assume that our system corresponds to the configuration where (38) is small enough, or that approximation $KD^{-1}M \approx MD^{-1}K$ holds in a certain sense.

Since M is a positive definite and D is a positive semidefinite matrix, there exists an orthogonal matrix U such that

$$M^{-\frac{1}{2}} D M^{-\frac{1}{2}} = U \Delta U^T, \quad \Delta = \mathrm{diag}(\delta_1, \dots, \delta_n). \tag{55}$$

Apart from previous cases where we are able to derive an explicit formula for global minima, in this section we will present a numerical approach to calculation of an approximation of optimal viscosities. The main problem within this *general case* is that the matrix U, which diagonalizes the matrix $M^{-\frac{1}{2}} D M^{-\frac{1}{2}}$, depends on viscosities that determine the damping matrix D, contrary to the cases from previous sections.

Thus, let us assume that d dampers with corresponding viscosities v_i, $i = 1, \dots, d$ are given, which determine our external damping matrix $C(v_1, \dots, v_d)$, that is, the damping matrix is given by $D(v_1, \dots, v_d) = C_u + C(v_1, \dots, v_d)$. Since in general the matrix U from (55) depends on viscosities, let U_0 be a unitary matrix which diagonalizes $D(v_1^0, \dots, v_d^0)$ for initial viscosities $(v_1^0, \dots, v_d^0)$.

Now, similarly to the beginning of this section, we can calculate approximation of the trace of the solution of the corresponding Lyapunov equation for the given viscosities $(v_1^0, \dots, v_d^0)$. That is, similarly to the above, we can show that

$$\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1^0, \dots, v_d^0)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii} + (Z_1)_{ii}}{\delta_i} + \frac{1}{2} \sum_{i=1}^{n} \delta_i (Z_1)_{ii} b_i, \tag{56}$$

where Z_1 and Z_2 are given by (42-43) and

$$M^{-\frac{1}{2}} (D(v_1^0, \dots, v_d^0)) M^{-\frac{1}{2}} = U_0 \Delta U_0^T, \quad \Delta = \mathrm{diag}(\delta_1, \dots, \delta_n)\,, \tag{57}$$

$$b_i = \|T_0(:, i)\|^2, \quad T_0 = B^{-1}, \quad i = 1, \dots, n. \tag{58}$$

Now, we do not have an explicit formula for optimal viscosities, thus we propose the following numerical approach to viscosity optimization.

During the optimization process, the next iteration (for viscosities) $(v_1^1, \dots, v_d^1)$ can be calculated using corresponding matrix U_1, given as

$$M^{-\frac{1}{2}} (D(v_1^1, \dots, v_d^1)) M^{-\frac{1}{2}} = U_1 \Delta U_1^T, \quad \Delta = \mathrm{diag}(\delta_1^1, \dots, \delta_n^1),$$

which insures the corresponding trace approximation. It is important to notice here that very often, during the optimization process, the same subspace U is also good for several iteration steps (that is, for several viscosity updates). Thus, during the optimization process, we first check if the same subspace is good enough, meaning that the residual error

$$er_U = \|M^U - \mathrm{diag}(M_{11}^U, M_{22}^U, \dots, M_{nn}^U)\| < tol_U, \tag{59}$$

where $M^U = U_0^T M^{-\frac{1}{2}} (C_u + C(v_1^1, \dots, v_d^1)) M^{-\frac{1}{2}} U_0$ and tol_U is some given tolerance.

This means that in the optimization process, if er_U defined in (59) for viscosities $(v_1^1, \dots, v_d^1)$ is smaller than the tolerance tol_U, we will use the unitary matrix U_0 instead of U_1 for approximation of the trace $\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1^1, \dots, v_d^1))$.

The algorithm for the optimization of viscosities is summarized in Algorithm 1.

We would like to emphasize that the main cost in trace approximation (56) belongs to the calculation of the matrix U; thus in Algorithm 1, by using the residual tolerance tol_U we can avoid calculation of the matrix U for some viscosities $(v_1^i, \dots, v_d^i)$ (for some i's), which significantly accelerates the optimization process. Moreover, as one could expect, the matrix U does not need to be calculated (up to tolerance tol_U) in each step of iterations

Algorithm 1 Computation of optimal viscosities

Require: System matrices; tolerance tol_U for updating eigensubspace U; starting viscosities $(v_1^0, \ldots, v_d^0)$.

Ensure: Approximation of optimal viscosities.

1: Calculate approximation of the trace given in (56) and U_0 given in (57). Set $U = U_0$.

2: Find optimal viscosities by using an appropriate optimization procedure (e.g. the Nelder-Mead algorithm). Evaluate the function value using trace approximation at the given viscosities $(v_1^i, \ldots, v_d^i)$ as in Steps 3 to 8:

3: Calculate the error for the subspace U from

$$er_U = \|M^U - \mathrm{diag}(M_{11}^U, M_{22}^U, \ldots, M_{nn}^U)\|,$$

where $M^U = U^T M^{-\frac{1}{2}}(C_u + C(v_1^i, \ldots, v_d^i))M^{-\frac{1}{2}}U$.

4: **if** $err_U < tol_U$, **then**

5: Compute the function value at viscosities $(v_1^i, \ldots, v_d^i)$ using

$$\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1^i, \ldots, v_d^i)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii} + (Z_1)_{ii}}{\delta_i} + \frac{1}{2}\sum_{i=1}^{n} \delta_i (Z_1)_{ii} b_i,$$

where Z_1 and Z_2 are given by (42-43) and

$$\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n), \quad \delta_i = (U^T M^{-\frac{1}{2}}(C_u + C(v_1^i, \ldots, v_d^i))M^{-\frac{1}{2}}U)_{ii}\,,\ i = 1, \ldots, n.$$

6: **else**

7: Compute new U and Δ, such that

$$M^{-\frac{1}{2}} D M^{\frac{1}{2}} = U \Delta U^T, \quad \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n).$$

Compute the function value at viscosities $(v_1^i, \ldots, v_d^i)$ using formula

$$\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1^i, \ldots, v_d^i)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii} + (Z_1)_{ii}}{\delta_i} + \frac{1}{2}\sum_{i=1}^{n} \delta_i (Z_1)_{ii} b_i,$$

where $b_i = \|T_0(:, i)\|^2$, $T_0 = B^{-1}$, $i = 1, \ldots, n$ and Z_1, Z_2 are given by (42-43).

8: **end if**

(for various viscosities $(v_1^i, \ldots, v_d^i)$) if a mechanical system has some special structure or if the changes in viscosities are small (which often appears during the optimization process).

Remark 2 Note that by using Algorithm 1, we can also minimize objective functions given by (50). In that case, the err_U will be zero (up to machine tolerance) and we will be able to calculate approximation of the objective function without calculating the matrix U in each step. On the other hand, since in this case the objective function consists of d independent functions, it is even more efficient to use an approach described in Remark 1.

6. Numerical experiments

In this section, we will illustrate the performance of the new approach on two examples which consider the mechanical system, the so-called the *n-mass oscillator*. In all examples, we will take $\mathbf{Z_U} = \mathbf{I}$.

Example 1 In this example, we consider the system from (2) with dimension $n = 20$, where the mass and stiffness matrices are defined as:

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n),$$
$$m_i = \begin{cases} 200 - 20(i-1) & , i = 1, \ldots, 10 \\ 201 + 20(i-11) & , i = 11, \ldots, 20 \end{cases}$$

$$K = \begin{bmatrix} 4 & -1 & -1 & & & \\ -1 & 4 & -1 & -1 & & \\ -1 & -1 & 4 & -1 & -1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & -1 & -1 & 4 & -1 & -1 \\ & & -1 & -1 & 4 & -1 \\ & & & -1 & -1 & 4 \end{bmatrix}.$$

The structure of masses and stiffness is shown in Figure 1. Further, the damping matrix

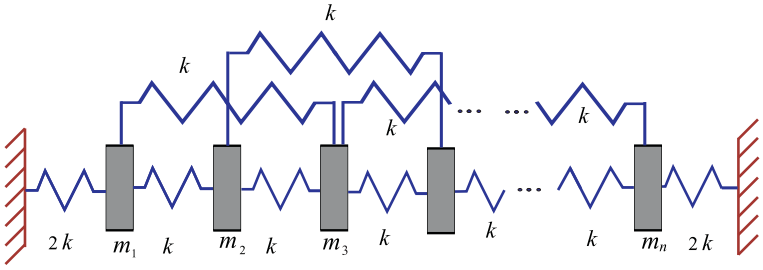

FIGURE 1. n-mass oscillator.

has a block diagonal structure $D = diag(D_1, D_2, \ldots, D_{10})$, where each block has its own viscosity v_j for $j = 1, \ldots, 10$. The block diagonal structure of the matrix D is shown in Figure 2.

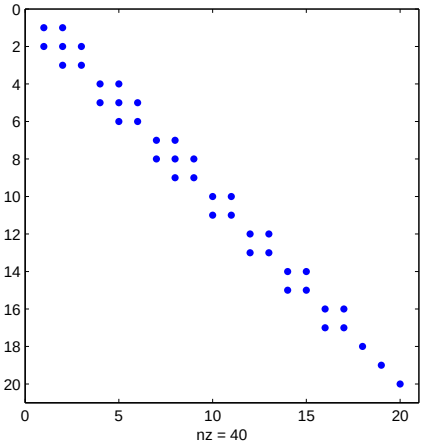

FIGURE 2. Block diagonal structure of matrix D.

The blocks are defined as:

$$D_i = \begin{bmatrix} v_i + v_i p & -v_i p & 0 \\ -v_i p & v_i + 2v_i p & -v_i p \\ 0 & -v_i p & v_i + v_i p \end{bmatrix}, \; i = 1, 2, 3,$$

$$D_i = \begin{bmatrix} v_i + v_i p & -v_i p \\ -v_i p & v_i + v_i p \end{bmatrix}, \; i = 4, 5, 6, 7,$$

$$D_i = \begin{bmatrix} v_i + v_i p \end{bmatrix}, \; i = 8, 9, 10,$$

where $p = 0.001$.

We will calculate optimal viscosities for two different cases:

Case 1. In the first case, we assume that there is no internal damping, that is, $\alpha = 0$.

For the purpose of comparison, we will present optimal viscosity and the corresponding minimal trace, denoted by $(v^*, \operatorname{tr}(\widetilde{\mathbf{X}}(v^*)))$, obtained by direct calculations using formula (53) and optimal viscosity and the corresponding minimal trace denoted by $(v, \operatorname{tr}(\mathbf{X}(v)))$, obtained by the minimization of the trace of the "dual Lyapunov equation" of equation (18) directly with MATLAB's function `fminsearch`, where we have used MATLAB's function `lyap` for solving Lyapunov equations.

For $(v^*, \operatorname{tr}(\widetilde{\mathbf{X}}(v^*)))$ and $(v, \operatorname{tr}(\mathbf{X}(v)))$, we have obtained the following:

$$v^* = \begin{bmatrix} 37.9626 \\ 23.3395 \\ 14.7396 \\ 19.4686 \\ 28.6084 \\ 32.6407 \\ 38.6879 \\ 45.7553 \\ 54.7100 \\ 64.6193 \end{bmatrix}, \quad \operatorname{tr}(\widetilde{\mathbf{X}}(v^*)) = 487.4226, \quad v = \begin{bmatrix} 38.1249 \\ 23.1773 \\ 14.5789 \\ 17.4601 \\ 28.4168 \\ 32.4962 \\ 38.5573 \\ 45.6625 \\ 55.0314 \\ 65.0329 \end{bmatrix}, \quad \operatorname{tr}(\mathbf{X}(v)) = 484.8125.$$

Thus, relative errors for the obtained approximations are:

$$err_v = \frac{||v - v^*||}{||v||} = 0.0171, \tag{1}$$

$$err_{tr} = \frac{||\operatorname{tr}(\widetilde{\mathbf{X}}(v^*)) - \operatorname{tr}(\mathbf{X}(v))||}{||\operatorname{tr}(\mathbf{X}(v))||} = 0.0054. \tag{2}$$

Here the residual error from (38) is $R_{er} = 0.3534$. This shows that even if the considered mechanical system is not very close to the modally damped one (R_{er} is not significantly smaller than 1), formula (53) still insures the satisfying result.

Case 2. Within the second case, we will assume the existence of internal damping; thus let $\alpha = 0.01$ be the coefficient of internal damping.

As emphasized in Remark 1 for the case $\alpha \neq 0$, one can not use formula (53) directly. Thus we will use Newton's method for optimization of the trace approximation given by formula (54). Again by $(v^*, \operatorname{tr}(\widetilde{\mathbf{X}}(v^*)))$ we denote the obtained approximation for optimal viscosity and the corresponding minimal trace. Similarly, by $(v, \operatorname{tr}(\mathbf{X}(v)))$ we denote optimal viscosity and the corresponding function value obtained by the minimization of the trace of the "dual Lyapunov equation" of equation (18) directly with MATLAB's function `fminsearch`, where the Lyapunov equation was solved by MATLAB's function `lyap`.

Here are the obtained quantities:

$$v^* = \begin{bmatrix} 36.3126 \\ 21.9638 \\ 13.9714 \\ 15.8175 \\ 26.1052 \\ 29.7869 \\ 35.4482 \\ 42.2551 \\ 51.4233 \\ 61.2265 \end{bmatrix}, \operatorname{tr}(\widetilde{\mathbf{X}}(v^*)) = 486.3990, \qquad v = \begin{bmatrix} 36.1512 \\ 22.1206 \\ 14.0986 \\ 17.6473 \\ 26.2969 \\ 29.9301 \\ 35.5781 \\ 42.3487 \\ 51.1036 \\ 60.8131 \end{bmatrix}, \operatorname{tr}(\mathbf{X}(v)) = 483.9260.$$

For relative errors defined in (1) and (2), here we have:

$$err_v = 0.0169, \quad err_{tr} = 0.0051.$$

In this example, the residual error from (38) has the similar magnitude, that is, $R_{er} = 0.3049$.

In the second example, we will consider a more general structure.

Example 2 In this example, we will consider the system from (2) with a dimension $n = 500$ and the matrices M and K defined as:

$$M = 10^3 \mathrm{diag}(m_1, m_2, \ldots, m_n),$$

$$m_i = \begin{cases} 200 - 20(i-1) & , i = 1, \ldots, 250 \\ 201 + 20(i-11) & , i = 251, \ldots, 500 \end{cases}$$

$$K = \begin{bmatrix} 10 & -1 & & & & \\ -1 & 10 & -1 & & & \\ & -1 & 10 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & -1 & 10 & -1 & \\ & & & -1 & 10 & \end{bmatrix}.$$

The damping matrix D has the block diagonal structure as follows

$$D = \begin{bmatrix} 0 & & & & \\ & D_1 & & & \\ & & 0 & & \\ & & & D_2 & \\ & & & & 0 \end{bmatrix} + C_u, \tag{3}$$

where $\alpha = 0.01$ and 0 represents a zero matrix of the corresponding dimension. The matrix D_i is defined as:

$$D_i = \begin{bmatrix} v_{i_1} + v_{i_1}p & -v_{i_1}p & & & \\ -v_{i_1}p & v_{i_1} + 2v_{i_1}p & -v_{i_1}p & & \\ & -v_{i_1}p & v_{i_1} + (v_{i_1} + v_{i_2})p + v_{i_2} & -v_{i_2}p & \\ & & -v_{i_2}p & v_{i_2} + v_{i_2}p & \end{bmatrix},$$

where $p = 0.01$. Thus, each block has 2 different viscosities, which means that we have 4 different viscosity parameters to optimize.

Again, we will compare the approximation of optimal viscosities obtained by our new approach proposed in Section 5.2 with optimal viscosity obtained by the minimization of the trace of the "dual Lyapunov equation" of equation (18) directly with MATLAB's function `fminsearch`, based on the MATLAB's function `lyap` for solving Lyapunov equations.

This comparison has been performed for different positions of matrices D_1 and D_2, that is, for in each new configuration we will change the position of matrices D_1 and D_2. The following configurations are taken into consideration:

$$\begin{aligned} (i,j) \quad \in \quad & \{(2,17),(2,67),(2,117),(2,267),(2,317),(52,67),(52,117),(52,167), \\ & (52,267),(52,317),(52,367),(52,417),(102,117),(102,217),(102,367), \\ & (152,167),(152,267),(152,317),(202,417),(252,267),(252,367),(252,417), \\ & (252,467),(302,367),(302,417),(352,417),(352,467)\}, \end{aligned}$$

where i represents the position of the matrix D_1 and j represents the position of the matrix D_2. Figure 3 shows the relative error

$$err_{tr} = \frac{||\operatorname{tr}(\mathbf{X}(v)) - \operatorname{tr}(\widetilde{\mathbf{X}}(v^*))||}{||\operatorname{tr}(\mathbf{X}(v))||},$$

for each configuration.

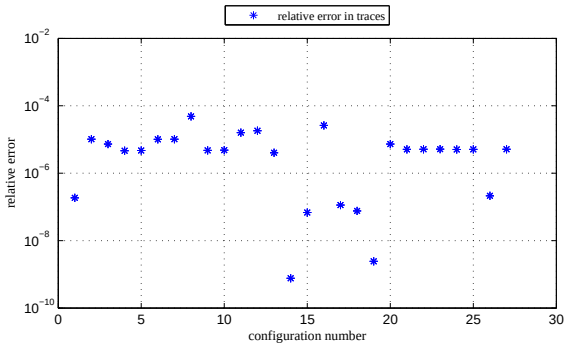

FIGURE 3. Relative errors err_{tr} for different positions of matrices D_1 and D_2.

During the optimization process, using Algorithm 1, we have calculated the percentage of updates of the matrix U with the tolerance $tol_U = 10^{-5}$. The number of updates for each configuration is shown in Figure 4.

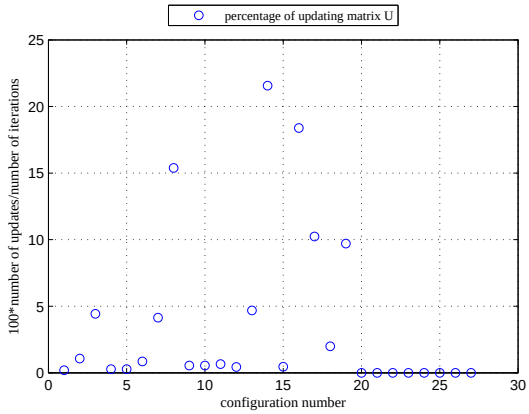

FIGURE 4. Percentage of updating matrix U for different positions of matrices D_1 and D_2.

Moreover, in order to illustrate the quality of the new approach using the surface plot in prescribed viscosities, we will set $v_1 = v_3$, $v_2 = v_4$, while v_1 and v_2 vary in segments $[40, 200]$, $[200, 340]$, respectively. The block with viscosities v_1 and v_2 starts at position 242 and the block with v_3 and v_4 starts at 470.

For the first step in iterations we have used the matrix U defined by optimal viscosities $v_1 = 101.4445$ and $v_2 = 268.3622$, while during the iteration process the matrix U has been updated with the tolerance $tol_U = 10^{-5}$.

In Figure 5, we can see the relative error

$$err_{rel} = \frac{||f_{lyap} - f_{aprox}||}{||f_{lyap}||},$$

in which f_{lyap} represents the trace of the solution of the Lyapunov equation for certain viscosities v_1 and v_2 and f_{aprox} represents the trace calculated by our algorithm. The relative error is less than 10^{-7}.

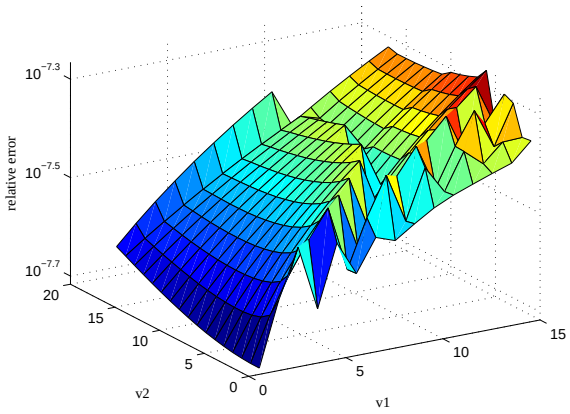

FIGURE 5. Relative error of the function value.

7. Conclusions

Troughout this paper we have considered damping optimization for the mechanical system $M\ddot{x} + D(v)\dot{x} + Kx = 0$. Since only the damping matrix $D(v)$ depends on parameters, the typical (or often used, standard) approach for viscosity optimization (v) assumes preprocessing based on the diagonalization of the mass and stiffness matrices, M and K.

Contrary to this approach, we propose the new approach, which is based on the diagonalization of the damping matrix $D(v)$, and then calculation of optimal viscosities. This is the main contribution of this paper, that is, we have shown that a slight change in the paradigm of damping optimization, for a certain structure, can significantly improve the performance of optimization methods.

Although, in general, the new approach can not be more efficient than the standard one, we have shown that in the case when M, D and K are closed to the case when all three can be simultaneously diagonalized (or when the mechanical system is closed to modally damped one) we can derive optimal viscosities, explicitly or numerically, very efficiently.

We have also provided the bounds which can be easily used to determine whether the considered mechanical system is suitable for applying the new approach, i.e., if the mechanical system under consideration is close to a modally damped one or not.

Our numerical examples show that with the proposed approach we can obtain satisfactory approximation for optimal parameters. Moreover, we illustrate that with our approach we can significantly accelerate the optimization process for the structured systems.

Acknowledgments

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