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NUMERICAL ANALYSIS OF A THERMOELASTIC DIFFUSION PROBLEM WITH VOIDS

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Abstract. In this paper we consider, from the numerical point of view, a thermoelastic diffusion porous problem. This is written as a coupled system of two hyperbolic equations, for the displacement and porosity fields, and two parabolic equations, for the temperature and chemical potential fields. Its variational formulation leads to a coupled system of four parabolic variational equations in terms of the velocity, porosity speed, temperature and chemical potential. The existence and uniqueness of weak solutions, as well as an energy decay property, are recalled. Then, the numerical approximation is introduced by using the finite element method for the spatial approximation and the implicit Euler scheme to discretize the time derivatives. A stability property is proved and some a priori error estimates are obtained, from which the convergence of the algorithm is derived and, under suitable additional regularity conditions, its linear convergence is deduced. Finally, some numerical approximations are presented to demonstrate the accuracy of the algorithm and to show the behaviour of the solution.

Key words. Thermoelasticity, porosity, diffusion, finite element approximations, error estimates and numerical simulations.

1. Introduction

The field of diffusion in solids has been a topic with an increasing interest, after the second world war, in the development of high technologies. In this way, the problems dealing with the diffusion of matter in thermoelastic bodies, and the interaction of mechano-diffusion processes, have become the subject of research of many authors since the first thermoelastic diffusion theory introduced by Nowacki ([19]). Nowadays, thermodiffusion problems are very interesting to oil companies for more efficient extraction of oil from oil deposits. In such problems, diffusion can be defined as the random walk, of an ensemble of particles, from regions of high concentrations to regions of lower concentrations ([1]).

There is a number of theories about mechanical properties of granular materials. Here, we follow the theory developed by Goodman and Cowin (see [14]), who established that the mass material admits a decomposition into the density of the matrix material and the volume fraction field (the porosity), introducing a new kinematic variable. Then, few years later Nunziato and Cowin proposed in [20] a new theory to describe the properties of homogeneous elastic materials with voids free of fluid. This has been shown to be useful for the study of rocks, soils and manufactured porous materials as ceramics and pressed powders. Moreover, Ieşan developed a linear theory of thermoelastic materials with voids generalizing some ideas of [13] (see, e.g., [15] or the recent monograph [16]).

In this paper, we consider, from the numerical point of view, a thermoelastic diffusion problem with voids presented in [1, 4]. The linear thermoelasticity theory with voids is extended to include diffusion effects, leading to a new and interesting problem. In this case, thermodiffusion is due to the coupling of the fields of temperature, porosity, mass diffusion and that of strain. Related problems have been considered by other authors in, for instance, [9, 10, 17, 21] (without the diffusion effect) or [2, 3, 5, 6]. Here, we concentrate on

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the numerical approximation of this problem, introducing fully discrete approximations by using the finite element method and the implicit Euler scheme, proving a stability property and some a priori error estimates, and providing some numerical simulations.

The outline of this paper is as follows. In Section 2, the mathematical model and its variational formulation are presented. An existence and uniqueness result and an energy decay property, proved in [4], are recalled. Then, fully discrete approximations are introduced in Section 3 by using the finite element method for the spatial approximation and the backward Euler scheme for the discretization of the time derivatives. A priori error estimates are obtained, from which the convergence of the algorithm is derived and its linear convergence is deduced under suitable additional regularity conditions. Finally, in Section 4 some numerical examples are shown to demonstrate the accuracy of the algorithm and the behaviour of the solution.

2. The model and its variational formulation

Let us denote by $[0,\ell]$, $\ell>0$, and $[0,T]$, $T>0$, the one-dimensional rod of length ℓ and the time interval of interest, respectively. Moreover, let $x\in[0,\ell]$ and $t\in[0,T]$ be the spatial and time variables. In order to simplify the writing, we do not indicate the dependence of the functions on x and t, and the subscript under a variable represents its derivative with respect to the prescribed variable.

According to [4], a porous themoelastic rod with diffusion is considered assuming that the matrix material is elastic and that interstices are void of material, constituting a generalization of the classical theory of elasticity (see, for instance, [12, 13]). Therefore, let u, ϕ, θ, $P\in\mathbb{R}$ be the displacement field of the solid elastic material, the porosity (or volume fraction), the difference of the temperature between the actual state and a reference temperature, and the chemical potential, respectively.

Therefore, following [1, 4] the mechanical problem of a one-dimensional porous thermoelastic rod with diffusion is written as follows.

Problem P. *Find the displacement field $u:[0,\ell]\times[0,T]\to\mathbb{R}$, the porosity field $\phi:[0,\ell]\times[0,T]\to\mathbb{R}$, the temperature field $\theta:[0,\ell]\times[0,T]\to\mathbb{R}$ and the chemical potential field $P:[0,\ell]\times[0,T]\to\mathbb{R}$ such that*

$$\rho u_{tt}=\alpha u_{xx}+b\phi_x-\gamma_1\theta_x-\gamma_2 P_x \quad in \quad (0,\ell)\times(0,T), \tag{1}$$

$$J\phi_{tt}=\eta\phi_{xx}-bu_x-\delta\phi+m_1\theta+m_2P \quad in \quad (0,\ell)\times(0,T), \tag{2}$$

$$c\theta_t=k^*\theta_{xx}-\gamma_1 u_{tx}-m_1\phi_t-\kappa P_t \quad in \quad (0,\ell)\times(0,T), \tag{3}$$

$$\nu P_t=h^*P_{xx}-\gamma_2 u_{tx}-m_2\phi_t-\kappa\theta_t \quad in \quad (0,\ell)\times(0,T), \tag{4}$$

$$\begin{aligned}&u(0,t)=u(\ell,t)=0,\quad \phi(0,t)=\phi(\ell,t)=0,\quad \theta(0,t)=\theta(\ell,t)=0,\\ &\qquad\qquad P(0,t)=P(\ell,t)=0\quad for\ a.e.\quad t\in(0,T),\end{aligned} \tag{5}$$

$$u(x,0)=u_0(x),\quad \phi(x,0)=\phi_0(x),\quad \theta(x,0)=\theta_0(x)\quad for\ a.e.\ x\in(0,\ell), \tag{6}$$

$$u_t(x,0)=v_0(x),\quad \phi_t(x,0)=e_0(x),\quad P(x,0)=P_0(x)\quad for\ a.e.\ x\in(0,\ell). \tag{7}$$

In this system of equations, ρ is the mass density, α is the elastic coefficient, γ_1, γ_2 represent thermal expansion coefficients, b is a porosity coefficient, $J=\rho\, k$, where k is the equilibrated inertia, m_1, m_2 are thermal expansion coefficients, η, δ represent porosity diffusion coefficients, κ is a thermal expansion coefficient, c is the heat capacity, ν denotes the diffusion relaxation time, k^* represents a thermal diffusion coefficient, h^* is a diffusion coefficient, and u_0, v_0, ϕ_0, e_0, θ_0 and P_0 are given initial conditions. Moreover, homogeneous Dirichlet boundary conditions have been assumed for the sake of simplicity for all the variables, but we point out that other boundary conditions could be used with some minor modifications in the numerical analysis presented in the next section.

Remark 2.1. *We note that this analysis could be extended to a multidimensional setting but the modifications are not straightforward. Analogously, we could also consider the analysis of a related contact problem (using, for instance, the normal compliance contact condition).*

In order to obtain the variational formulation of Problem P, let $Y = L^2(0,\ell)$ and $E = H^1(0,\ell)$ and denote by $(\cdot,\cdot)$ the scalar product in the space Y, with corresponding norm $\|\cdot\|$. Moreover, let us define the variational space V as follows,

$$V = \{v \in H^1(0,\ell)\,;\, v(0) = 0 \quad \text{and} \quad v(\ell) = 0\},$$

with scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$.

By using the integration by parts and the Dirichlet boundary conditions at $x = 0,\, \ell$, we write the variational formulation of Problem P in terms of the velocity $v = u_t$, the porosity speed $e = \phi_t$, the temperature θ and the chemical potential P.

Problem VP. *Find the velocity field $v : [0,T] \to V$, the porosity speed field $e : [0,T] \to V$, the temperature field $\theta : [0,T] \to V$ and the chemical potential field $P : [0,T] \to V$ such that $v(0) = v_0$, $e(0) = e_0$, $\theta(0) = \theta_0$, $P(0) = P_0$ and, for a.e. $t \in (0,T)$ and for all $z,\, r,\, l,\, m \in V$,*

$$\rho(v_t(t),z) + \alpha(u_x(t),z_x) = b(\phi_x(t),z) - \gamma_1(\theta_x(t),z) - \gamma_2(P_x(t),z), \tag{8}$$

$$\begin{aligned} &J(e_t(t),r) + \eta(\phi_x(t),r_x) + \delta(\phi(t),r) = -b(u_x(t),r)\\ &\qquad +m_1(\theta(t),r) + m_2(P(t),r), \end{aligned} \tag{9}$$

$$c(\theta_t(t),l) + k^*(\theta_x(t),l_x) = -\gamma_1(v_x(t),l) - m_1(e(t),l) - \kappa(P_t(t),l), \tag{10}$$

$$\nu(P_t(t),m) + h^*(P_x(t),m_x) = -\gamma_2(v_x(t),m) - m_2(e(t),m) - \kappa(\theta_t(t),m), \tag{11}$$

where the displacement and the porosity fields are then recovered from the relations

$$u(t) = \int_0^t v(s)\,ds + u_0, \quad \phi(t) = \int_0^t e(s)\,ds + \phi_0. \tag{12}$$

The existence and uniqueness of weak solutions to Problem VP were considered in [4]. They are stated in the following.

Theorem 2.2. *Assume that the coefficients satisfy the properties*

$$\rho > 0, \quad \alpha > 0, \quad J > 0, \quad \eta > 0, \quad \delta > 0, \quad c > 0, \tag{13}$$

$$k^* > 0, \quad \nu > 0, \quad h^* > 0, \tag{14}$$

$$c\nu > \kappa^2, \tag{15}$$

and that the initial conditions have the regularity

$$u_0 \in V, \quad v_0 \in Y, \quad \phi_0 \in V, \quad e_0 \in Y, \quad \theta_0,\, P_0 \in Y.$$

Therefore, Problem VP has a unique solution with the regularity,

$$u, \phi \in C([0,T];V) \cap C^1([0,T];Y), \quad \theta,\, P \in C([0,T];Y) \cap L^2([0,T];V).$$

Moreover, if the initial conditions have the additional regularity

$$u_0 \in H^2(0,\ell), \quad v_0 \in E, \quad \phi_0 \in H^2(0,\ell), \quad e_0 \in E, \quad \theta_0,\, P_0 \in V, \tag{16}$$

then Problem VP has a unique solution with the regularity,

$$\begin{aligned} &u,\, \phi \in C([0,T]; V \cap H^2(0,\ell)) \cap C^1([0,T];V) \cap C^2([0,T];Y),\\ &\theta,\, P \in C([0,T];V) \cap C^1([0,T];Y). \end{aligned}$$

The following energy decay property was also proved in [4].

Theorem 2.3. *Let the assumptions of Theorem 2.2 hold and define the energy of the system by*

$$
\begin{aligned}
E(t) = \frac{1}{2}\big[&\rho\|v(t)\|^2 + J\|e(t)\|^2 + \alpha\|u_x(t)\|^2 + \eta\|\phi_x(t)\|^2 + \delta\|\phi(t)\|^2 \\
&+c\|\theta(t)\|^2 + \|P(t)\|^2 + 2\kappa(\theta(t), P(t)) + 2b(\phi(t), u_x(t))\big].
\end{aligned}
\tag{17}
$$

Under the additional regularity on the initial conditions (16), there exists a positive constant $C > 0$ such that, for all $t > 0$,

$$E(t) \leq \frac{C}{t}.$$

3. Fully discrete approximations: a priori error estimates

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the interval $[0, \ell]$ is divided into M subintervals $a_0 = 0 < a_1 < \ldots < a_M = \ell$ of length $h = a_{i+1} - a_i = \ell/M$ and so, we construct the finite dimensional space $V^h \subset V$, approximating the variational space V, given by

$$V^h = \{v^h \in C([0, \ell]); v^h_{|[a_i,a_{i+1}]} \in P_1([a_i, a_{i+1}])\ i = 0, \ldots, M-1,\ v^h(0) = v^h(\ell) = 0\}, \tag{18}$$

where $P_1([a_i, a_{i+1}])$ represents the space of polynomials of degree less or equal to one in the subinterval $[a_i, a_{i+1}]$; i.e. the finite element space V^h is composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by u^h_0, v^h_0, ϕ^h_0, e^h_0, θ^h_0 and P^h_0, are given by

$$u^h_0 = \mathcal{P}^h u_0, \quad v^h_0 = \mathcal{P}^h v_0, \quad \phi^h_0 = \mathcal{P}^h \phi_0, \quad e^h_0 = \mathcal{P}^h e_0, \quad \theta^h_0 = \mathcal{P}^h \theta_0, \quad P^h_0 = \mathcal{P}^h P_0, \tag{19}$$

where $\mathcal{P}^h$ is the $L^2(0, \ell)$-projection operator over V^h.

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $\Delta t = T/N$ and nodes $t_n = n\,\Delta t$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/\Delta t$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP$^{h,\Delta t}$. *Find the discrete velocity field $v^{h,\Delta t} = \{v^{h,\Delta t}_n\}_{n=0}^N \subset V^h$, the discrete porosity speed field $e^{h,\Delta t} = \{e^{h,\Delta t}_n\}_{n=0}^N \subset V^h$, the discrete temperature field $\theta^{h,\Delta t} = \{\theta^{h,\Delta t}_n\}_{n=0}^N \subset V^h$ and the discrete chemical potential field $P^{h,\Delta t} = \{P^{h,\Delta t}_n\}_{n=0}^N \subset V^h$ such that $v^{h,\Delta t}_0 = v^h_0$, $e^{h,\Delta t}_0 = e^h_0$, $\theta^{h,\Delta t}_0 = \theta^h_0$, $P^{h,\Delta t}_0 = P^h_0$ and, for $n = 1, \ldots, N$,*

$$
\begin{aligned}
\rho(\delta v^{h,\Delta t}_n, z^h) + \alpha((u^{h,\Delta t}_n)_x, z^h_x) &= b((\phi^{h,\Delta t}_n)_x, z^h) - \gamma_1((\theta^{h,\Delta t}_n)_x, z^h) \\
&\quad -\gamma_2((P^{h,\Delta t}_n)_x, z^h) \quad \forall z^h \in V^h,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
J(\delta e^{h,\Delta t}_n, r^h) + \eta((\phi^{h,\Delta t}_n)_x, r^h_x) + \delta(\phi^{h,\Delta t}_n, r^h) &= -b((u^{h,\Delta t}_n)_x, r^h) \\
&\quad +m_1(\theta^{h,\Delta t}_n, r^h) + m_2(P^{h,\Delta t}_n, r^h) \quad \forall r^h \in V^h,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
c(\delta\theta^{h,\Delta t}_n, l^h) + k^*((\theta^{h,\Delta t}_n)_x, l^h_x) &= -\gamma_1((v^{h,\Delta t}_n)_x, l^h) - m_1(e^{h,\Delta t}_n, l^h) \\
&\quad -\kappa(\delta P^{h,\Delta t}_n, l^h) \quad \forall l^h \in V^h,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\nu(\delta P^{h,\Delta t}_n, m^h) + h^*((P^{h,\Delta t}_n)_x, m^h_x) &= -\gamma_2((v^{h,\Delta t}_n)_x, m^h) - m_2(e^{h,\Delta t}_n, m^h) \\
&\quad -\kappa(\delta\theta^{h,\Delta t}_n, m^h) \quad \forall m^h \in V^h,
\end{aligned}
\tag{23}
$$

where the discrete displacement and porosity fields $u_n^{h,\Delta t}$ and $\phi_n^{h,\Delta t}$ are now recovered from the relations

$$u_n^{h,\Delta t} = \Delta t \sum_{j=1}^{n} v_j^{h,\Delta t} + u_0^h, \quad \phi_n^{h,\Delta t} = \Delta t \sum_{j=1}^{n} e_j^{h,\Delta t} + \phi_0^h. \tag{24}$$

Under the assumption $c\nu > \kappa^2$ (see (15)), using the well-known Lax-Milgram lemma, we can prove that there exists a unique discrete solution to Problem $VP^{h,\Delta t}$.

The following stability result is proved using some algebraic manipulations and a discrete version of Gronwall's inequality.

Theorem 3.1. *Under the assumptions of Theorem 2.2, it follows that the sequences* $\{u^{h,\Delta t}, v^{h,\Delta t}, \phi^{h,\Delta t}, e^{h,\Delta t}, \theta^{h,\Delta t}, P^{h,\Delta t}\}$ *generated by Problem* $VP^{h,\Delta t}$ *satisfy the stability estimate:*

$$\|v_n^{h,\Delta t}\|^2 + \|(u_n^{h,\Delta t})_x\|^2 + \|e_n^{h,\Delta t}\|^2 + \|(\phi_n^{h,\Delta t})_x\|^2 + \|\phi_n^{h,\Delta t}\|^2 + \|\theta_n^{h,\Delta t}\|^2 + \|P_n^{h,\Delta t}\|^2 \le C,$$

where C *is a positive constant assumed to be independent of the discretization parameters* h *and* Δt.

Proof. In order to simplify the writing of this proof, we remove the superscripts h and Δt in all the variables.

Taking $z^h = v_n$ as a test function in discrete variational equation (20) it follows that

$$\rho(\delta v_n, v_n) + \alpha((u_n)_x, (v_n)_x) = b((\phi_n)_x, v_n) - \gamma_1((\theta_n)_x, v_n) - \gamma_2((P_n)_x, v_n),$$

and keeping in mind that

$$(\delta v_n, v_n) \ge \frac{1}{2\Delta t}\left\{\|v_n\|^2 - \|v_{n-1}\|^2\right\},$$
$$((u_n)_x, (v_n)_x) \ge \frac{1}{2\Delta t}\left\{\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2\right\},$$

we have

$$\begin{aligned}\frac{\rho}{2\Delta t}\left\{\|v_n\|^2 - \|v_{n-1}\|^2\right\} &+ \frac{\alpha}{2\Delta t}\left\{\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2\right\}\\ &\le C(\|(\phi_n)_x\|^2 + \|v_n\|^2) - \gamma_1((\theta_n)_x, v_n) - \gamma_2((P_n)_x, v_n).\end{aligned} \tag{25}$$

Taking $r^h = e_n$ as a test function in discrete variational equation (21) we obtain

$$J(\delta e_n, e_n) + \eta((\phi_n)_x, (e_n)_x) + \delta(\phi_n, e_n) = -b((u_n)_x, e_n) + m_1(\theta_n, e_n) + m_2(P_n, e_n).$$

Thus, keeping in mind that $e_n = (\phi_n - \phi_{n-1})/\Delta t$ and using the estimates

$$(\delta e_n, e_n) \ge \frac{1}{2\Delta t}\left\{\|e_n\|^2 - \|e_{n-1}\|^2\right\},$$
$$((\phi_n)_x, (e_n)_x) \ge \frac{1}{2\Delta t}\left\{\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2\right\},$$
$$(\phi_n, e_n) \ge \frac{1}{2\Delta t}\left\{\|\phi_n\|^2 - \|\phi_{n-1}\|^2\right\},$$

it follows that

$$\begin{aligned}&\frac{J}{2\Delta t}\left\{\|e_n\|^2 - \|e_{n-1}\|^2\right\} + \frac{\eta}{2\Delta t}\left\{\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2\right\}\\ &\quad + \frac{\delta}{2\Delta t}\left\{\|\phi_n\|^2 - \|\phi_{n-1}\|^2\right\} \le C(\|(u_n)_x\|^2 + \|\theta_n\|^2 + \|e_n\|^2 + \|P_n\|^2).\end{aligned} \tag{26}$$

Next, taking $l^h = \theta_n$ as a test function in discrete variational equation (22) we have

$$c(\delta\theta_n, \theta_n) + k^*((\theta_n)_x, (\theta_n)_x) = -\gamma_1((v_n)_x, \theta_n) - m_1(e_n, \theta_n) - \kappa(\delta P_n, \theta_n),$$

and, taking into account that

$$(\delta\theta_n, \theta_n) = \frac{1}{2\Delta t}\left\{\|\theta_n\|^2 - \|\theta_{n-1}\|^2 + \|\theta_n - \theta_{n-1}\|^2\right\},$$

we find that

$$\begin{aligned}&\frac{c}{2\Delta t}\left\{\|\theta_n\|^2 - \|\theta_{n-1}\|^2 + \|\theta_n - \theta_{n-1}\|^2\right\} + \kappa(\delta P_n, \theta_n)\\ &\quad\leq C(\|\theta_n\|^2 + \|e_n\|^2) - \gamma_1((v_n)_x, \theta_n).\end{aligned} \tag{27}$$

Finally, taking $m^h = P_n$ as a test function in discrete variational equation (23) we obtain

$$\nu(\delta P_n, P_n) + h^*((P_n)_x, (P_n)_x) = -\gamma_2((v_n)_x, P_n) - m_2(e_n, P_n) - \kappa(\delta\theta_n, P_n).$$

Therefore, keeping in mind that

$$(\delta P_n, P_n) = \frac{1}{2\Delta t}\left\{\|P_n\|^2 - \|P_{n-1}\|^2 + \|P_n - P_{n-1}\|^2\right\},$$

it follows that

$$\begin{aligned}&\frac{\nu}{2\Delta t}\left\{\|P_n\|^2 - \|P_{n-1}\|^2 + \|P_n - P_{n-1}\|^2\right\} + \kappa(\delta\theta_n, P_n)\\ &\quad\leq C(\|P_n\|^2 + \|e_n\|^2) - \gamma_2((v_n)_x, P_n).\end{aligned} \tag{28}$$

Observing that

$$\kappa(\delta P_n, \theta_n) + \kappa(\delta\theta_n, P_n) = \frac{\kappa}{\Delta t}\left[(P_n, \theta_n) - (P_{n-1}, \theta_{n-1}) + (P_n - P_{n-1}, \theta_n - \theta_{n-1})\right],$$

and that, using the assumed condition $c\nu > \kappa^2$ (see (15)),

$$\frac{c}{2\Delta t}\|\theta_n - \theta_{n-1}\|^2 + \frac{\nu}{2\Delta t}\|P_n - P_{n-1}\|^2 + \frac{\kappa}{\Delta t}(P_n - P_{n-1}, \theta_n - \theta_{n-1}) \geq 0,$$

combining estimates (25), (26), (27) and (28), and keeping in mind that

$$\begin{aligned}-((v_n)_x, \theta_n) &= ((\theta_n)_x, v_n),\\ -((v_n)_x, P_n) &= ((P_n)_x, v_n),\end{aligned}$$

we find that

$$\begin{aligned}&\frac{\rho}{2\Delta t}\left\{\|v_n\|^2 - \|v_{n-1}\|^2\right\} + \frac{\alpha}{2\Delta t}\left\{\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2\right\}\\ &\quad + \frac{J}{2\Delta t}\left\{\|e_n\|^2 - \|e_{n-1}\|^2\right\} + \frac{\eta}{2\Delta t}\left\{\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2\right\}\\ &\quad + \frac{\delta}{2\Delta t}\left\{\|\phi_n\|^2 - \|\phi_{n-1}\|^2\right\} + \frac{\kappa}{\Delta t}\left\{(P_n, \theta_n) - (P_{n-1}, \theta_{n-1})\right\}\\ &\quad + \frac{c}{2\Delta t}\left\{\|\theta_n\|^2 - \|\theta_{n-1}\|^2 + \|\theta_n - \theta_{n-1}\|^2\right\}\\ &\quad + \frac{\nu}{2\Delta t}\left\{\|P_n\|^2 - \|P_{n-1}\|^2 + \|P_n - P_{n-1}\|^2\right\}\\ &\leq C\big(\|(\phi_n)_x\|^2 + \|v_n\|^2 + \|(u_n)_x\|^2 + \|\theta_n\|^2 + \|e_n\|^2 + \|P_n\|^2\big).\end{aligned}$$

Summing up to n the previous estimates, it follows that

$$\begin{aligned}\rho\|v_n\|^2 &+ \alpha\|(u_n)_x\|^2 + J\|e_n\|^2 + \eta\|(\phi_n)_x\|^2 \\ &+ \delta\|\phi_n\|^2 + 2\kappa(P_n,\theta_n) + c\|\theta_n\|^2 + \nu\|P_n\|^2 \\ \le& C\Delta t\sum_{j=1}^{n}\big(\|(\phi_j)_x\|^2 + \|v_j\|^2 + \|(u_j)_x\|^2 + \|\theta_j\|^2 + \|e_j\|^2 + \|P_j\|^2\big) \\ &+ C\big(\|v_0\|^2 + \|(u_0)_x\|^2 + \|e_0\|^2 + \|(\phi_0)_x\|^2 + \|\phi_0\|^2 + \|\theta_0\|^2 + \|P_0\|^2\big).\end{aligned} \tag{29}$$

Now, using again condition (15), we can choose $\zeta > 0$ such that $\kappa/c < \zeta < \nu/\kappa$, which implies that

$$c\|\theta_n\|^2 + \nu\|P_n\|^2 + 2\kappa(P_n,\theta_n) \ge \left(c - \frac{\kappa}{\zeta}\right)\|\theta_n\|^2 + (\nu - \kappa\zeta)\|P_n\|^2.$$

Finally, using a discrete version of Gronwall's inequality (see, for instance, [8]), we obtain the desired stability property. □

From Theorem 3.1, we obtain the following discrete version of the energy decay property.

Corollary 3.2. *If we define the discrete energy at time $t = t_n$, $E_n^{h,\Delta t}$, as follows*

$$\begin{aligned}E_n^{h,\Delta t} =& \rho\|v_n^{h,\Delta t}\|^2 + J\|e_n^{h,\Delta t}\|^2 + c\|\theta_n^{h,\Delta t}\|^2 + \alpha\|(u_n^{h,\Delta t})_x\|^2 + \delta\|\phi_n^{h,\Delta t}\|^2 \\ &+\eta\|(\phi_n^{h,\Delta t})_x\|^2 + \nu\|P_n^{h,\Delta t}\|^2 + 2\kappa(P_n^{h,\Delta t},\theta_n^{h,\Delta t}) + 2b(\phi_n^{h,\Delta t},(u_n^{h,\Delta t})_x),\end{aligned} \tag{30}$$

then we have

$$\frac{E_n^{h,\Delta t} - E_{n-1}^{h,\Delta t}}{\Delta t} \le 0.$$

Remark 3.3. *We note that this result only implies that the energy decreases along with the time. Even if we could try to investigate its decay rate, this is beyond the scope of this paper. For instance, in [18, 22] the authors discuss long time decay for different numerical approaches of locally damped wave equations.*

Now, we will obtain some a priori error estimates for the numerical errors $u_n - u_n^{h,\Delta t}$, $v_n - v_n^{h,\Delta t}$, $\phi_n - \phi_n^{h,\Delta t}$, $e_n - e_n^{h,\Delta t}$, $\theta_n - \theta_n^{h,\Delta t}$ and $P_n - P_n^{h,\Delta t}$.

We have the following theorem which gives some a priori error estimates.

Theorem 3.4. *Under the assumptions of Theorem 2.2 with the additional regularity (16) on the initial conditions, if we denote by (v,e,θ,P) the solution to problem VP and by $(v^{h,\Delta t}, e^{h,\Delta t}, \theta^{h,\Delta t}, P^{h,\Delta t})$ the solution to problem $VP^{h,\Delta t}$, then we have the following a priori error estimates, for all $z^h = \{z_j^h\}_{j=0}^N \subset V^h$, $r^h = \{r_j^h\}_{j=0}^N \subset V^h$, $l^h = \{l_j^h\}_{j=0}^N \subset V^h$*

and $m^h = \{m_j^h\}_{j=0}^N \subset V^h$,

$$
\begin{aligned}
&\max_{0\le n\le N} \Big\{\|\theta_n-\theta_n^{h,\Delta t}\|^2+\|P_n-P_n^{h,\Delta t}\|^2+\|v_n-v_n^{h,\Delta t}\|^2+\|(u_n-u_n^{h,\Delta t})_x\|^2\\
&\qquad\qquad +\|e_n-e_n^{h,\Delta t}\|^2+\|(\phi_n-\phi_n^{h,\Delta t})_x\|^2+\|\phi_n-\phi_n^{h,\Delta t}\|^2\Big\}\\
&\le C\Delta t\sum_{j=1}^N \Big(\|(\theta_t)_j-\delta\theta_j\|^2+\|\theta_j-l_j^h\|^2+\|(\theta_j-l_j^h)_x\|^2+\|(v_t)_j-\delta v_j\|^2\\
&\quad+\|(P_t)_j-\delta P_j\|^2+\|P_j-m_j^h\|^2+\|(P_j-m_j^h)_x\|^2\\
&\quad+\|v_j-z_j^h\|^2+\|(v_j-z_j^h)_x\|^2+\|((u_t)_j-\delta u_j)_x\|^2+\|e_j-r_j^h\|^2\\
&\quad+\|(e_t)_j-\delta e_j\|^2+\|((\phi_t)_j-\delta\phi_j)_x\|^2+\|(e_j-r_j^h)_x\|^2\Big)\\
&\quad+C\max_{0\le n\le N}\Big\{\|v_n-z_n^h\|^2+\|e_n-r_n^h\|^2+\|\theta_n-l_n^h\|^2+\|P_n-m_n^h\|^2\Big\}\\
&\quad+\frac{C}{\Delta t}\sum_{j=1}^{N-1}\|v_j-z_j^h-(v_{j+1}-z_{j+1}^h)\|^2+\frac{C}{\Delta t}\sum_{j=1}^{N-1}\|e_j-r_j^h-(e_{j+1}-r_{j+1}^h)\|^2\\
&\quad+\frac{C}{\Delta t}\sum_{j=1}^{N-1}\|\theta_j-l_j^h-(\theta_{j+1}-l_{j+1}^h)\|^2+\frac{C}{\Delta t}\sum_{j=1}^{N-1}\|P_j-m_j^h-(P_{j+1}-m_{j+1}^h)\|^2\\
&\quad+C\Big(\|\theta_0-\theta_0^h\|^2+\|v_0-v_0^h\|^2+\|(u_0-u_0^h)_x\|^2+\|e_0-e_0^h\|^2\\
&\quad+\|P_0-P_0^h\|^2+\|(\phi_0-\phi_0^h)_x\|^2+\|\phi_0-\phi_0^h\|^2\Big),
\end{aligned}
\tag{31}
$$

where $C>0$ *is a positive constant assumed to be independent of the discretization parameters* h *and* Δt*, but depending on the continuous solution, and* $\delta\theta_j=(\theta_j-\theta_{j-1})/\Delta t$*,* $\delta P_j=(P_j-P_{j-1})/\Delta t$*,* $\delta v_j=(v_j-v_{j-1})/\Delta t$*,* $\delta u_j=(u_j-u_{j-1})/\Delta t$*,* $\delta\phi_j=(\phi_j-\phi_{j-1})/\Delta t$ *and* $\delta e_j=(e_j-e_{j-1})/\Delta t$*.*

Proof. First, we obtain some estimates for the temperature field. Then, we subtract variational equation (10) at time $t=t_n$ for a test function $l=l^h\in V^h\subset V$ and discrete variational equation (22) to obtain, for all $l^h\in V^h$,

$$
\begin{aligned}
&c((\theta_t)_n-\delta\theta_n^{h,\Delta t},l^h)+k^*((\theta_n-\theta_n^{h,\Delta t})_x,l_x^h)\\
&\quad=-\gamma_1((v_n-v_n^{h,\Delta t})_x,l^h)-m_1(e_n-e_n^{h,\Delta t},l^h)-\kappa((P_t)_n-\delta P_n^{h,\Delta t},l^h),
\end{aligned}
$$

and so, we have, for all $l^h\in V^h$,

$$
\begin{aligned}
&c((\theta_t)_n-\delta\theta_n^{h,\Delta t},\theta_n-\theta_n^{h,\Delta t})+k^*((\theta_n-\theta_n^{h,\Delta t})_x,(\theta_n-\theta_n^{h,\Delta t})_x)\\
&\qquad+\gamma_1((v_n-v_n^{h,\Delta t})_x,\theta_n-\theta_n^{h,\Delta t})+m_1(e_n-e_n^{h,\Delta t},\theta_n-\theta_n^{h,\Delta t})\\
&\qquad+\kappa((P_t)_n-\delta P_n^{h,\Delta t},\theta_n-\theta_n^{h,\Delta t})\\
&\quad=c((\theta_t)_n-\delta\theta_n^{h,\Delta t},\theta_n-l^h)+k^*((\theta_n-\theta_n^{h,\Delta t})_x,(\theta_n-l^h)_x)\\
&\qquad+\gamma_1((v_n-v_n^{h,\Delta t})_x,\theta_n-l^h)+m_1(e_n-e_n^{h,\Delta t},\theta_n-l^h)\\
&\qquad+\kappa((P_t)_n-\delta P_n^{h,\Delta t},\theta_n-l^h).
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
&((\theta_t)_n-\delta\theta_n^{h,\Delta t},\theta_n-\theta_n^{h,\Delta t})=((\theta_t)_n-\delta\theta_n,\theta_n-\theta_n^{h,\Delta t})\\
&\quad+\frac{1}{2\Delta t}\Big\{\|\theta_n-\theta_n^{h,\Delta t}\|^2-\|\theta_{n-1}-\theta_{n-1}^{h,\Delta t}\|^2+\|\theta_n-\theta_n^{h,\Delta t}-(\theta_{n-1}-\theta_{n-1}^{h,\Delta t})\|^2\Big\},\\
&((v_n-v_n^{h,\Delta t})_x,\theta_n-l^h)=-(v_n-v_n^{h,\Delta t},(\theta_n-l^h)_x),
\end{aligned}
$$

where $\delta\theta_n = (\theta_n - \theta_{n-1})/\Delta t$, using Cauchy-Schwarz inequality and the inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$, $a, b, \epsilon \in \mathbb{R}$, $\epsilon > 0$, it follows that

$$\begin{aligned}
&\frac{c}{2\Delta t}\Big\{\|\theta_n - \theta_n^{h,\Delta t}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{h,\Delta t}\|^2 + \|\theta_n - \theta_n^{h,\Delta t} - (\theta_{n-1} - \theta_{n-1}^{h,\Delta t})\|^2\Big\}\\
&\qquad +\kappa(\delta P_n - \delta P_n^{h,\Delta t}, \theta_n - \theta_n^{h,\Delta t})\\
&\quad \leq C\Big(\|(\theta_t)_n - \delta\theta_n\|^2 + \|\theta_n - l^h\|^2 + \|(\theta_n - l^h)_x\|^2 + \|e_n - e_n^{h,\Delta t}\|^2\\
&\quad +\|(P_t)_n - \delta P_n\|^2 + \|\theta_n - \theta_n^{h,\Delta t}\|^2 + (\delta\theta_n - \delta\theta_n^{h,\Delta t}, \theta_n - l^h) + \|v_n - v_n^{h,\Delta t}\|^2\\
&\quad +(\delta P_n - \delta P_n^{h,\Delta t}, \theta_n - l^h)\Big) - \gamma_1((v_n - v_n^{h,\Delta t})_x, \theta_n - \theta_n^{h,\Delta t}) \quad \forall l^h \in V^h,
\end{aligned} \tag{32}$$

where $\delta P_n = (P_n - P_{n-1})/\Delta t$.

Secondly, we get some estimates for the chemical potential field. Then, we subtract variational equation (11) at time $t = t_n$ for a test function $m = m^h \in V^h \subset V$ and discrete variational equation (23) to obtain, for all $m^h \in V^h$,

$$\begin{aligned}
&\nu((P_t)_n - \delta P_n^{h,\Delta t}, m^h) + h^*((P_n - P_n^{h,\Delta t})_x, m_x^h)\\
&\quad = -\gamma_2((v_n - v_n^{h,\Delta t})_x, m^h) - m_2(e_n - e_n^{h,\Delta t}, m^h) - \kappa((\theta_t)_n - \delta\theta_n^{h,\Delta t}, m^h),
\end{aligned}$$

and so, we have, for all $m^h \in V^h$,

$$\begin{aligned}
&\nu((P_t)_n - \delta P_n^{h,\Delta t}, P_n - P_n^{h,\Delta t}) + h^*((P_n - P_n^{h,\Delta t})_x, (P_n - P_n^{h,\Delta t})_x)\\
&\quad +\gamma_2((v_n - v_n^{h,\Delta t})_x, P_n - P_n^{h,\Delta t}) + m_2(e_n - e_n^{h,\Delta t}, P_n - P_n^{h,\Delta t})\\
&\quad +\kappa((\theta_t)_n - \delta\theta_n^{h,\Delta t}, P_n - P_n^{h,\Delta t})\\
&\quad = \nu((P_t)_n - \delta P_n^{h,\Delta t}, P_n - m^h) + h^*((P_n - P_n^{h,\Delta t})_x, (P_n - m^h)_x)\\
&\quad +\gamma_2((v_n - v_n^{h,\Delta t})_x, P_n - m^h) + m_2(e_n - e_n^{h,\Delta t}, P_n - m^h)\\
&\quad +\kappa((\theta_t)_n - \delta\theta_n^{h,\Delta t}, P_n - m^h).
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&((P_t)_n - \delta P_n^{h,\Delta t}, P_n - P_n^{h,\Delta t}) = ((P_t)_n - \delta P_n, P_n - P_n^{h,\Delta t}) + \frac{1}{2\Delta t}\\
&\quad \times\Big\{\|P_n - P_n^{h,\Delta t}\|^2 - \|P_{n-1} - P_{n-1}^{h,\Delta t}\|^2 + \|P_n - P_n^{h,\Delta t} - (P_{n-1} - P_{n-1}^{h,\Delta t})\|^2\Big\},\\
&((v_n - v_n^{h,\Delta t})_x, P_n - m^h) = -(v_n - v_n^{h,\Delta t}, (P_n - m^h)_x),
\end{aligned}$$

using Cauchy-Schwarz inequality and the inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$, $a, b, \epsilon \in \mathbb{R}$, $\epsilon > 0$, it follows that

$$\begin{aligned}
&\frac{\nu}{2\Delta t}\Big\{\|P_n - P_n^{h,\Delta t}\|^2 - \|P_{n-1} - P_{n-1}^{h,\Delta t}\|^2 + \|P_n - P_n^{h,\Delta t} - (P_{n-1} - P_{n-1}^{h,\Delta t})\|^2\Big\}\\
&+\kappa(\delta\theta_n - \delta\theta_n^{h,\Delta t}, P_n - P_n^{h,\Delta t})\\
&\leq C\Big(\|(P_t)_n - \delta P_n\|^2 + \|P_n - m^h\|^2 + \|(P_n - m^h)_x\|^2 + \|e_n - e_n^{h,\Delta t}\|^2\\
&\quad +\|(\theta_t)_n - \delta\theta_n\|^2 + \|P_n - P_n^{h,\Delta t}\|^2 + (\delta P_n - \delta P_n^{h,\Delta t}, P_n - m^h)\\
&\quad +\|v_n - v_n^{h,\Delta t}\|^2 + (\delta\theta_n - \delta\theta_n^{h,\Delta t}, P_n - m^h)\Big)\\
&\quad -\gamma_2((v_n - v_n^{h,\Delta t})_x, P_n - P_n^{h,\Delta t}) \quad \forall m^h \in V^h.
\end{aligned} \tag{33}$$

Now, using again assumption (15) we easily find that

$$\begin{aligned}
&\kappa(\delta P_n - \delta P_n^{h,\Delta t}, \theta_n - \theta_n^{h,\Delta t}) + \kappa(\delta\theta_n - \delta\theta_n^{h,\Delta t}, P_n - P_n^{h,\Delta t})\\
&\quad = \frac{\kappa}{\Delta t}\big[(P_n - P_n^{h,\Delta t}, \theta_n - \theta_n^{h,\Delta t}) - (P_{n-1} - P_{n-1}^{h,\Delta t}, \theta_{n-1} - \theta_{n-1}^{h,\Delta t})\\
&\qquad +(P_n - P_n^{h,\Delta t} - (P_{n-1} - P_{n-1}^{h,\Delta t}), \theta_n - \theta_n^{h,\Delta t} - (\theta_{n-1} - \theta_{n-1}^{h,\Delta t}))\big],
\end{aligned}$$

and also,

$$\frac{c}{2\Delta t}\|\theta_n-\theta_n^{h,\Delta t}-(\theta_{n-1}-\theta_{n-1}^{h,\Delta t})\|^2+\frac{\nu}{2\Delta t}\|P_n-P_n^{h,\Delta t}-(P_{n-1}-P_{n-1}^{h,\Delta t})\|^2$$
$$+\frac{\kappa}{\Delta t}(P_n-P_n^{h,\Delta t}-(P_{n-1}-P_{n-1}^{h,\Delta t}),\theta_n-\theta_n^{h,\Delta t}-(\theta_{n-1}-\theta_{n-1}^{h,\Delta t}))\geq 0.$$

Combining (32) and (33) and taking into account the previous two estimates we get, for all l^h, $m^h\in V^h$,

$$\begin{aligned}
&\frac{c}{2\Delta t}\left\{\|\theta_n-\theta_n^{h,\Delta t}\|^2-\|\theta_{n-1}-\theta_{n-1}^{h,\Delta t}\|^2\right\}+\\
&\quad\frac{\nu}{2\Delta t}\left\{\|P_n-P_n^{h,\Delta t}\|^2-\|P_{n-1}-P_{n-1}^{h,\Delta t}\|^2\right\}\\
&\qquad+\frac{\kappa}{\Delta t}\left\{(P_n-P_n^{h,\Delta t},\theta_n-\theta_n^{h,\Delta t})-(P_{n-1}-P_{n-1}^{h,\Delta t},\theta_{n-1}-\theta_{n-1}^{h,\Delta t})\right\}\\
&\quad\leq C\Big(\|(\theta_t)_n-\delta\theta_n\|^2+\|\theta_n-l^h\|^2+\|(\theta_n-l^h)_x\|^2+\|e_n-e_n^{h,\Delta t}\|^2\\
&\qquad+\|(P_t)_n-\delta P_n\|^2+\|\theta_n-\theta_n^{h,\Delta t}\|^2+(\delta\theta_n-\delta\theta_n^{h,\Delta t},\theta_n-l^h)+\|v_n-v_n^{h,\Delta t}\|^2\\
&\qquad+(\delta P_n-\delta P_n^{h,\Delta t},\theta_n-l^h)+\|(P_t)_n-\delta P_n\|^2+\|P_n-m^h\|^2+\|(P_n-m^h)_x\|^2\\
&\qquad+\|P_n-P_n^{h,\Delta t}\|^2+(\delta\theta_n-\delta\theta_n^{h,\Delta t},P_n-m^h)+(\delta P_n-\delta P_n^{h,\Delta t},P_n-m^h)\Big)\\
&\qquad-\gamma_1((v_n-v_n^{h,\Delta t})_x,\theta_n-\theta_n^{h,\Delta t})-\gamma_2((v_n-v_n^{h,\Delta t})_x,P_n-P_n^{h,\Delta t}).
\end{aligned}\tag{34}$$

Next, we get the estimates for the velocity field. Thus, subtracting variational equation (8) at time $t=t_n$ for a test function $z=z^h\in V^h\subset V$ and discrete variational equation (20) we obtain, for all $z^h\in V^h$,

$$\rho((v_t)_n-\delta v_n^{h,\Delta t},z^h)+\alpha((u_n-u_n^{h,\Delta t})_x,z_x^h)=b((\phi_n-\phi_n^{h,\Delta t})_x,z^h)$$
$$-\gamma_1((\theta_n-\theta_n^{h,\Delta t})_x,z^h)-\gamma_2((P_n-P_n^{h,\Delta t})_x,z^h),$$

and so we find that, for all $z^h\in V^h$,

$$\begin{aligned}
&\rho((v_t)_n-\delta v_n^{h,\Delta t},v_n-v_n^{h,\Delta t})\\
&\quad+\alpha((u_n-u_n^{h,\Delta t})_x,(v_n-v_n^{h,\Delta t})_x)-b((\phi_n-\phi_n^{h,\Delta t})_x,v_n-v_n^{h,\Delta t})\\
&\quad+\gamma_1((\theta_n-\theta_n^{h,\Delta t})_x,v_n-v_n^{h,\Delta t})+\gamma_2((P_n-P_n^{h,\Delta t})_x,v_n-v_n^{h,\Delta t})\\
&\;=\rho((v_t)_n-\delta v_n^{h,\Delta t},v_n-z^h)\\
&\quad+\alpha((u_n-u_n^{h,\Delta t})_x,(v_n-z^h)_x)-b((\phi_n-\phi_n^{h,\Delta t})_x,v_n-z^h)\\
&\quad-\gamma_1((\theta_n-\theta_n^{h,\Delta t})_x,v_n-z^h)-\gamma_2((P_n-P_n^{h,\Delta t})_x,v_n-z^h).
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&((v_t)_n-\delta v_n^{h,\Delta t},v_n-v_n^{h,\Delta t})\geq((v_t)_n-\delta v_n,v_n-v_n^{h,\Delta t})\\
&\qquad+\frac{1}{2\Delta t}\left[\|v_n-v_n^{h,\Delta t}\|^2-\|v_{n-1}-v_{n-1}^{h,\Delta t}\|^2\right],\\
&((u_n-u_n^{h,\Delta t})_x,(v_n-v_n^{h,\Delta t})_x)\geq((u_n-u_n^{h,\Delta t})_x,((u_t)_n-\delta u_n)_x)\\
&\qquad+\frac{1}{2\Delta t}\left[\|(u_n-u_n^{h,\Delta t})_x\|^2-\|(u_{n-1}-u_{n-1}^{h,\Delta t})_x\|^2\right],\\
&((\theta_n-\theta_n^{h,\Delta t})_x,v_n-z^h)=-(\theta_n-\theta_n^{h,\Delta t},(v_n-z^h)_x),\\
&((P_n-P_n^{h,\Delta t})_x,v_n-z^h)=-(P_n-P_n^{h,\Delta t},(v_n-z^h)_x)
\end{aligned}$$

where $\delta u_n = (u_n - u_{n-1})/\Delta t$, $\delta v_n = (v_n - v_{n-1})/\Delta t$ and we recall that $v_n^{h,\Delta t} = \delta u_n^{h,\Delta t} = (u_n^{h,\Delta t} - u_{n-1}^{h,\Delta t})/\Delta t$, we have, for all $z^h \in V^h$,

$$\begin{aligned}
&\frac{1}{2\Delta t}\left[\|v_n - v_n^{h,\Delta t}\|^2 - \|v_{n-1} - v_{n-1}^{h,\Delta t}\|^2\right] \\
&\quad + \frac{1}{2\Delta t}\left[\|(u_n - u_n^{h,\Delta t})_x\|^2 - \|(u_{n-1} - u_{n-1}^{h,\Delta t})_x\|^2\right] \\
&\quad \le C\Big(\|v_n - v_n^{h,\Delta t}\|^2 + \|(v_t)_n - \delta v_n\|^2 + \|(u_n - u_n^{h,\Delta t})_x\|^2 \\
&\qquad + \|(\phi_n - \phi_n^{h,\Delta t})_x\|^2 + \|v_n - z^h\|^2 + \|(v_n - z^h)_x\|^2 + \|P_n - P_n^{h,\Delta t}\|^2 \\
&\qquad + \|((u_t)_n - \delta u_n)_x\|^2 + (\delta v_n - \delta v_n^{h,\Delta t}, v_n - z^h) + \|\theta_n - \theta_n^{h,\Delta t}\|^2\Big) \\
&\qquad - \gamma_1((\theta_n - \theta_n^{h,\Delta t})_x, v_n - v_n^{h,\Delta t}) - \gamma_2((P_n - P_n^{h,\Delta t})_x, v_n - v_n^{h,\Delta t}).
\end{aligned} \tag{35}$$

Finally, we obtain the estimates for the porosity speed. Then, we subtract variational equation (9) at time $t = t_n$ for a test function $r = r^h \in V^h \subset V$ and discrete variational equation (21) to obtain

$$\begin{aligned}
&J((e_t)_n - \delta e_n^{h,\Delta t}, r^h) + \eta((\phi_n - \phi_n^{h,\Delta t})_x, r_x^h) + \delta(\phi_n - \phi_n^{h,\Delta t}, r^h) \\
&= -b((u_n - u_n^{h,\Delta t})_x, r^h) + m_1(\theta_n - \theta_n^{h,\Delta t}, r^h) + m_2(P_n - P_n^{h,\Delta t}, r^h) \quad \forall r^h \in V^h,
\end{aligned}$$

and so we find that, for all $r^h \in V^h$,

$$\begin{aligned}
&J((e_t)_n - \delta e_n^{h,\Delta t}, e_n - e_n^{h,\Delta t}) + \eta((\phi_n - \phi_n^{h,\Delta t})_x, (e_n - e_n^{h,\Delta t})_x) \\
&\quad + \delta(\phi_n - \phi_n^{h,\Delta t}, e_n - e_n^{h,\Delta t}) + b((u_n - u_n^{h,\Delta t})_x, e_n - e_n^{h,\Delta t}) \\
&\quad + m_1(\theta_n - \theta_n^{h,\Delta t}, e_n - e_n^{h,\Delta t}) + m_2(P_n - P_n^{h,\Delta t}, e_n - e_n^{h,\Delta t}) \\
&= J((e_t)_n - \delta e_n^{h,\Delta t}, e_n - r^h) + \eta((\phi_n - \phi_n^{h,\Delta t})_x, (e_n - r^h)_x) \\
&\quad + \delta(\phi_n - \phi_n^{h,\Delta t}, e_n - r^h) + b((u_n - u_n^{h,\Delta t})_x, e_n - r^h) \\
&\quad + m_1(\theta_n - \theta_n^{h,\Delta t}, e_n - r^h) + m_2(P_n - P_n^{h,\Delta t}, e_n - r^h).
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&((e_t)_n - \delta e_n^{h,\Delta t}, e_n - e_n^{h,\Delta t}) \ge ((e_t)_n - \delta e_n, e_n - e_n^{h,\Delta t}) \\
&\quad + \frac{1}{2\Delta t}\left[\|e_n - e_n^{h,\Delta t}\|^2 - \|e_{n-1} - e_{n-1}^{h,\Delta t}\|^2\right], \\
&((\phi_n - \phi_n^{h,\Delta t})_x, (e_n - e_n^{h,\Delta t})_x) \ge ((\phi_n - \phi_n^{h,\Delta t})_x, ((\phi_t)_n - \delta\phi_n)_x) \\
&\quad + \frac{1}{2\Delta t}\left[\|(\phi_n - \phi_n^{h,\Delta t})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{h,\Delta t})_x\|^2\right], \\
&(\phi_n - \phi_n^{h,\Delta t}, e_n - e_n^{h,\Delta t}) \ge (\phi_n - \phi_n^{h,\Delta t}, (\phi_t)_n - \delta\phi_n) \\
&\quad + \frac{1}{2\Delta t}\left[\|\phi_n - \phi_n^{h,\Delta t}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{h,\Delta t}\|^2\right],
\end{aligned}$$

where $\delta\phi_n = (\phi_n - \phi_{n-1})/\Delta t$, $\delta e_n = (e_n - e_{n-1})/\Delta t$ and we recall that $e_n^{h,\Delta t} = \delta\phi_n^{h,\Delta t} = (\phi_n^{h,\Delta t} - \phi_{n-1}^{h,\Delta t})/\Delta t$, we have, for all $r^h \in V^h$,

$$
\begin{aligned}
&\frac{1}{2\Delta t}\Big[\|e_n - e_n^{h,\Delta t}\|^2 - \|e_{n-1} - e_{n-1}^{h,\Delta t}\|^2\Big]\\
&\quad+\frac{1}{2\Delta t}\Big[\|(\phi_n - \phi_n^{h,\Delta t})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{h,\Delta t})_x\|^2\Big]\\
&\quad+\frac{1}{2\Delta t}\Big[\|\phi_n - \phi_n^{h,\Delta t}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{h,\Delta t}\|^2\Big]\\
&\le C\Big(\|e_n - e_n^{h,\Delta t}\|^2 + \|(e_t)_n - \delta e_n\|^2 + \|\phi_n - \phi_n^{h,\Delta t}\|^2 + \|(\phi_n - \phi_n^{h,\Delta t})_x\|^2\\
&\quad+\|((\phi_t)_n - \delta\phi_n)_x\|^2 + \|\theta_n - \theta_n^{h,\Delta t}\|^2 + \|e_n - r^h\|^2 + \|(e_n - r^h)_x\|^2\\
&\quad+\|P_n - P_n^{h,\Delta t}\|^2 + \|(u_n - u_n^{h,\Delta t})_x\|^2 + (\delta e_n - \delta e_n^{h,\Delta t}, e_n - r^h)\Big).
\end{aligned}
\tag{36}
$$

Keeping in mind that

$$
\begin{aligned}
((\theta_n - \theta_n^{h,\Delta t})_x, v_n - v_n^{h,\Delta t}) &= -(\theta_n - \theta_n^{h,\Delta t}, (v_n - v_n^{h,\Delta t})_x),\\
((P_n - P_n^{h,\Delta t})_x, v_n - v_n^{h,\Delta t}) &= -(P_n - P_n^{h,\Delta t}, (v_n - v_n^{h,\Delta t})_x),
\end{aligned}
$$

combining estimates (34), (35) and (36) we find that

$$
\begin{aligned}
&\frac{c}{2\Delta t}\Big[\|\theta_n - \theta_n^{h,\Delta t}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{h,\Delta t}\|^2\Big]\\
&\quad+\frac{\nu}{2\Delta t}\Big[\|P_n - P_n^{h,\Delta t}\|^2 - \|P_{n-1} - P_{n-1}^{h,\Delta t}\|^2\Big]\\
&\quad+\frac{\kappa}{\Delta t}\Big[(P_n - P_n^{h,\Delta t}, \theta_n - \theta_n^{h,\Delta t}) - (P_{n-1} - P_{n-1}^{h,\Delta t}, \theta_{n-1} - \theta_{n-1}^{h,\Delta t})\Big]\\
&\quad+\frac{1}{2\Delta t}\Big[\|v_n - v_n^{h,\Delta t}\|^2 - \|v_{n-1} - v_{n-1}^{h,\Delta t}\|^2\Big]\\
&\quad+\frac{1}{2\Delta t}\Big[\|(u_n - u_n^{h,\Delta t})_x\|^2 - \|(u_{n-1} - u_{n-1}^{h,\Delta t})_x\|^2\Big]\\
&\quad+\frac{1}{2\Delta t}\Big[\|e_n - e_n^{h,\Delta t}\|^2 - \|e_{n-1} - e_{n-1}^{h,\Delta t}\|^2\Big]\\
&\quad+\frac{1}{2\Delta t}\Big[\|(\phi_n - \phi_n^{h,\Delta t})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{h,\Delta t})_x\|^2\Big]\\
&\quad+\frac{1}{2\Delta t}\Big[\|\phi_n - \phi_n^{h,\Delta t}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{h,\Delta t}\|^2\Big]\\
&\le C\Big(\|(\theta_t)_n - \delta\theta_n\|^2 + \|\theta_n - l^h\|^2 + \|(\theta_n - l^h)_x\|^2 + \|e_n - e_n^{h,\Delta t}\|^2\\
&\quad+\|(P_t)_n - \delta P_n\|^2 + \|P_n - m^h\|^2 + \|(P_n - m^h)_x\|^2 + \|P_n - P_n^{h,\Delta t}\|^2\\
&\quad+\|\theta_n - \theta_n^{h,\Delta t}\|^2 + (\delta\theta_n - \delta\theta_n^{h,\Delta t}, \theta_n - l^h) + \|v_n - v_n^{h,\Delta t}\|^2 + \|(v_t)_n - \delta v_n\|^2\\
&\quad+\|(u_n - u_n^{h,\Delta t})_x\|^2 + \|(\phi_n - \phi_n^{h,\Delta t})_x\|^2 + \|v_n - z^h\|^2 + \|(v_n - z^h)_x\|^2\\
&\quad+\|((u_t)_n - \delta u_n)_x\|^2 + (\delta v_n - \delta v_n^{h,\Delta t}, v_n - z^h) + \|\phi_n - \phi_n^{h,\Delta t}\|^2 + \|e_n - r^h\|^2\\
&\quad+\|(e_t)_n - \delta e_n\|^2 + \|((\phi_t)_n - \delta\phi_n)_x\|^2 + \|(e_n - r^h)_x\|^2\\
&\quad+(\delta P_n - \delta P_n^{h,\Delta t}, P_n - m^h) + (\delta P_n - \delta P_n^{h,\Delta t}, \theta_n - l^h)\\
&\quad+(\delta\theta_n - \delta\theta_n^{h,\Delta t}, P_n - m^h)\\
&\quad+(\delta e_n - \delta e_n^{h,\Delta t}, e_n - r^h)\Big) \quad \forall z^h,\ r^h,\ l^h,\ m^h \in V^h.
\end{aligned}
$$

Therefore, multiplying the previous estimates by Δt and summing up to n, we obtain, for all $z^h=\{z_j^h\}_{j=0}^n\subset V^h$, $r^h=\{r_j^h\}_{j=0}^n\subset V^h$, $l^h=\{l_j^h\}_{j=0}^n\subset V^h$ and $m^h=\{m_j^h\}_{j=0}^n\subset V^h$,

$$\begin{aligned}
&c\|\theta_n-\theta_n^{h,\Delta t}\|^2+2\kappa(P_n-P_n^{h,\Delta t},\theta_n-\theta_n^{h,\Delta t})+\nu\|P_n-P_n^{h,\Delta t}\|^2+\|v_n-v_n^{h,\Delta t}\|^2\\
&\quad+\|(u_n-u_n^{h,\Delta t})_x\|^2+\|e_n-e_n^{h,\Delta t}\|^2+\|(\phi_n-\phi_n^{h,\Delta t})_x\|^2+\|\phi_n-\phi_n^{h,\Delta t}\|^2\\
&\quad\le C\Delta t\sum_{j=1}^n\Big(\|(\theta_t)_j-\delta\theta_j\|^2+\|\theta_j-l_j^h\|^2+\|(\theta_j-l_j^h)_x\|^2+\|e_j-e_j^{h,\Delta t}\|^2\\
&\quad+\|(P_t)_j-\delta P_j\|^2+\|P_j-m_j^h\|^2+\|(P_j-m_j^h)_x\|^2+\|P_j-P_j^{h,\Delta t}\|^2\\
&\quad+\|\theta_j-\theta_j^{h,\Delta t}\|^2+(\delta\theta_j-\delta\theta_j^{h,\Delta t},\theta_j-l_j^h)+\|v_j-v_j^{h,\Delta t}\|^2+\|(v_t)_j-\delta v_j\|^2\\
&\quad+\|(u_j-u_j^{h,\Delta t})_x\|^2+\|(\phi_j-\phi_j^{h,\Delta t})_x\|^2+\|v_j-z_j^h\|^2+\|(v_j-z_j^h)_x\|^2\\
&\quad+\|((u_t)_j-\delta u_j)_x\|^2+(\delta v_j-\delta v_j^{h,\Delta t},v_j-z_j^h)+\|\phi_j-\phi_j^{h,\Delta t}\|^2+\|e_j-r_j^h\|^2\\
&\quad+\|(e_t)_j-\delta e_j\|^2+\|((\phi_t)_j-\delta\phi_j)_x\|^2+\|(e_j-r_j^h)_x\|^2\\
&\quad+(\delta P_j-\delta P_j^{h,\Delta t},P_j-m_j^h)+(\delta P_j-\delta P_j^{h,\Delta t},\theta_j-l_j^h)+(\delta\theta_j-\delta\theta_j^{h,\Delta t},P_j-m_j^h)\\
&\quad+(\delta e_j-\delta e_j^{h,\Delta t},e_j-r_j^h)\Big)+C\Big(\|\theta_0-\theta_0^h\|^2+\|v_0-v_0^h\|^2+\|P_0-P_0^h\|^2\\
&\quad+\|(u_0-u_0^h)_x\|^2+\|e_0-e_0^h\|^2+\|(\phi_0-\phi_0^h)_x\|^2+\|\phi_0-\phi_0^h\|^2\Big).
\end{aligned}$$

Using again assumption (15), as in the proof of Theorem 3.1, we can choose $\zeta>0$ such that $\kappa/c<\zeta<\nu/\kappa$, and therefore,

$$\begin{aligned}
&c\|\theta_n-\theta_n^{h,\Delta t}\|^2+\nu\|P_n-P_n^{h,\Delta t}\|^2+2\kappa(P_n-P_n^{h,\Delta t},\theta_n-\theta_n^{h,\Delta t})\\
&\quad\ge\left(c-\frac{\kappa}{\zeta}\right)\|\theta_n-\theta_n^{h,\Delta t}\|^2+(\nu-\kappa\zeta)\,\|P_n-P_n^{h,\Delta t}\|^2.
\end{aligned}$$

Thus, we have, for all $z^h=\{z_j^h\}_{j=0}^n\subset V^h$, $r^h=\{r_j^h\}_{j=0}^n\subset V^h$, $m^h=\{m_j^h\}_{j=0}^n\subset V^h$ and $l^h=\{l_j^h\}_{j=0}^n\subset V^h$,

$$\begin{aligned}
&\|\theta_n-\theta_n^{h,\Delta t}\|^2+\|P_n-P_n^{h,\Delta t}\|^2+\|v_n-v_n^{h,\Delta t}\|^2\\
&\quad+\|(u_n-u_n^{h,\Delta t})_x\|^2+\|e_n-e_n^{h,\Delta t}\|^2+\|(\phi_n-\phi_n^{h,\Delta t})_x\|^2+\|\phi_n-\phi_n^{h,\Delta t}\|^2\\
&\quad\le C\Delta t\sum_{j=1}^n\Big(\|(\theta_t)_j-\delta\theta_j\|^2+\|\theta_j-l_j^h\|^2+\|(\theta_j-l_j^h)_x\|^2+\|e_j-e_j^{h,\Delta t}\|^2\\
&\quad+\|(P_t)_j-\delta P_j\|^2+\|P_j-m_j^h\|^2+\|(P_j-m_j^h)_x\|^2+\|P_j-P_j^{h,\Delta t}\|^2\\
&\quad+\|\theta_j-\theta_j^{h,\Delta t}\|^2+(\delta\theta_j-\delta\theta_j^{h,\Delta t},\theta_j-l_j^h)+\|v_j-v_j^{h,\Delta t}\|^2+\|(v_t)_j-\delta v_j\|^2\\
&\quad+\|(u_j-u_j^{h,\Delta t})_x\|^2+\|(\phi_j-\phi_j^{h,\Delta t})_x\|^2+\|v_j-z_j^h\|^2+\|(v_j-z_j^h)_x\|^2\\
&\quad+\|((u_t)_j-\delta u_j)_x\|^2+(\delta v_j-\delta v_j^{h,\Delta t},v_j-z_j^h)+\|\phi_j-\phi_j^{h,\Delta t}\|^2+\|e_j-r_j^h\|^2\\
&\quad+\|(e_t)_j-\delta e_j\|^2+\|((\phi_t)_j-\delta\phi_j)_x\|^2+\|(e_j-r_j^h)_x\|^2\\
&\quad+(\delta P_j-\delta P_j^{h,\Delta t},P_j-m_j^h)+(\delta P_j-\delta P_j^{h,\Delta t},\theta_j-l_j^h)\\
&\quad+(\delta\theta_j-\delta\theta_j^{h,\Delta t},P_j-m_j^h)\\
&\quad+(\delta e_j-\delta e_j^{h,\Delta t},e_j-r_j^h)\Big)+C\Big(\|\theta_0-\theta_0^h\|^2+\|v_0-v_0^h\|^2+\|P_0-P_0^h\|^2\\
&\quad+\|(u_0-u_0^h)_x\|^2+\|e_0-e_0^h\|^2+\|(\phi_0-\phi_0^h)_x\|^2+\|\phi_0-\phi_0^h\|^2\Big),
\end{aligned}$$

and, keeping in mind that

$$
\begin{aligned}
&\Delta t\sum_{j=1}^{n}(\delta v_j-\delta v_j^{h,\Delta t},v_j-z_j^h)\\
&=\sum_{j=1}^{n}(v_j-v_j^{h,\Delta t}-(v_{j-1}-v_{j-1}^{h,\Delta t}),v_j-z_j^h)\\
&=(v_n-v_n^{h,\Delta t},v_n-z_n^h)+(v_0^h-v_0,v_1-z_1^h)+\sum_{j=1}^{n-1}(v_j-v_j^{h,\Delta t},v_j-z_j^h-(v_{j+1}-z_{j+1}^h)),\\
&\Delta t\sum_{j=1}^{n}(\delta e_j-\delta e_j^{h,\Delta t},e_j-r_j^h)\\
&=\sum_{j=1}^{n}(e_j-e_j^{h,\Delta t}-(e_{j-1}-e_{j-1}^{h,\Delta t}),e_j-r_j^h)\\
&=(e_n-e_n^{h,\Delta t},e_n-r_n^h)+(e_0^h-e_0,e_1-r_1^h)+\sum_{j=1}^{n-1}(e_j-e_j^{h,\Delta t},e_j-r_j^h-(e_{j+1}-r_{j+1}^h)),\\
&\Delta t\sum_{j=1}^{n}(\delta\theta_j-\delta\theta_j^{h,\Delta t},\theta_j-l_j^h)=\sum_{j=1}^{n}(\theta_j-\theta_j^{h,\Delta t}-(\theta_{j-1}-\theta_{j-1}^{h,\Delta t}),\theta_j-l_j^h)\\
&=(\theta_n-\theta_n^{h,\Delta t},\theta_n-l_n^h)+(\theta_0^h-\theta_0,\theta_1-l_1^h)+\sum_{j=1}^{n-1}(\theta_j-\theta_j^{h,\Delta t},\theta_j-l_j^h-(\theta_{j+1}-l_{j+1}^h)),\\
&\Delta t\sum_{j=1}^{n}(\delta P_j-\delta P_j^{h,\Delta t},P_j-m_j^h)=\sum_{j=1}^{n}(P_j-P_j^{h,\Delta t}-(P_{j-1}-P_{j-1}^{h,\Delta t}),P_j-m_j^h)\\
&=(P_n-P_n^{h,\Delta t},P_n-m_n^h)+(P_0^h-P_0,P_1-m_1^h)\\
&\quad+\sum_{j=1}^{n-1}(P_j-P_j^{h,\Delta t},P_j-m_j^h-(P_{j+1}-m_{j+1}^h)),\\
&\Delta t\sum_{j=1}^{n}(\delta P_j-\delta P_j^{h,\Delta t},\theta_j-l_j^h)=\sum_{j=1}^{n}(P_j-P_j^{h,\Delta t}-(P_{j-1}-P_{j-1}^{h,\Delta t}),\theta_j-l_j^h)\\
&=(P_n-P_n^{h,\Delta t},\theta_n-l_n^h)+(P_0^h-P_0,\theta_1-l_1^h)+\sum_{j=1}^{n-1}(P_j-P_j^{h,\Delta t},\theta_j-l_j^h-(\theta_{j+1}-l_{j+1}^h)),\\
&\Delta t\sum_{j=1}^{n}(\delta\theta_j-\delta\theta_j^{h,\Delta t},P_j-m_j^h)=\sum_{j=1}^{n}(\theta_j-\theta_j^{h,\Delta t}-(\theta_{j-1}-\theta_{j-1}^{h,\Delta t}),P_j-m_j^h)\\
&=(\theta_n-\theta_n^{h,\Delta t},P_n-m_n^h)+(\theta_0^h-\theta_0,P_1-m_1^h)\\
&\quad+\sum_{j=1}^{n-1}(\theta_j-\theta_j^{h,\Delta t},P_j-m_j^h-(P_{j+1}-m_{j+1}^h)),
\end{aligned}
$$

using a discrete version of Gronwall's inequality (see, for instance, [8]), we have the a priori error estimates (31). □

From estimates (31), keeping in mind the density of $C^3([0,T];C^\infty(0,\ell))$ into $C^2([0,T];Y)$ and $C^1([0,T];C^\infty(0,\ell))$ into $C^1([0,T];V)$, we obtain the following corollary which states the convergence of the algorithm.

Corollary 3.5. *Let the assumptions of Theorem 3.4 still hold. Due to the regularity on the initial conditions (16) we have (see [11])*

$$\begin{aligned}&\|\theta_0-\theta_0^h\|+\|v_0-v_0^h\|+\|(u_0-u_0^h)_x\|+\|e_0-e_0^h\|\\&\quad+\|P_0-P_0^h\|+\|(\phi_0-\phi_0^h)_x\|+\|\phi_0-\phi_0^h\|\to 0 \quad as \quad h\to 0,\end{aligned}$$

and so the fully discrete solution to Problem $VP^{h,\Delta t}$ converges; i.e.

$$\begin{aligned}\max_{0\le n\le N}&\Big\{\|\theta_n-\theta_n^{h,\Delta t}\|+\|P_n-P_n^{h,\Delta t}\|+\|v_n-v_n^{h,\Delta t}\|+\|(u_n-u_n^{h,\Delta t})_x\|\\&+\|e_n-e_n^{h,\Delta t}\|+\|(\phi_n-\phi_n^{h,\Delta t})_x\|+\|\phi_n-\phi_n^{h,\Delta t}\|\Big\}\to 0 \quad as \quad h,\Delta t\to 0.\end{aligned}$$

The proof of the above result is done using classical results of approximation by finite elements, algebraic manipulations and taking into account that, for a function $w\in C^1([0,T];C^\infty(0,\ell))$, we find that (see [7,8]),

$$\frac{1}{\Delta t}\sum_{j=1}^{N-1}\|w_j-\mathcal{P}^hw_j-(w_{j+1}-\mathcal{P}^hw_{j+1})\|^2\le Ch\|w\|^2_{C^1([0,T];C^\infty(0,\ell))}.$$

Now, we point out that estimates (31) can be used to obtain the convergence order under some additional regularity conditions. Hence, we have the following result which states the linear convergence of the algorithm under suitable additional regularity.

Corollary 3.6. *Let the assumptions of Theorem 3.4 still hold. If we assume that the solution to Problem VP has the additional regularity,*

$$\begin{aligned}&u\in H^2(0,T;H^1(0,\ell))\cap H^3(0,T;Y)\cap C^1([0,T];H^2(0,\ell)),\\&\phi\in H^2(0,T;H^1(0,\ell))\cap H^3(0,T;Y)\cap C^1([0,T];H^2(0,\ell)),\\&\theta\in H^1(0,T;H^1(0,\ell))\cap H^2(0,T;Y)\cap C([0,T];H^2(0,\ell)),\\&P\in H^1(0,T;H^1(0,\ell))\cap H^2(0,T;Y)\cap C([0,T];H^2(0,\ell)),\end{aligned}\tag{37}$$

and we use the finite element space V^h defined in (18) and the discrete initial conditions u_0^h, v_0^h, ϕ_0^h, e_0^h, θ_0^h and P_0^h given in (19), the linear convergence of the algorithm is deduced; i.e. there exists a positive constant $C>0$, independent of the discretization parameters h and Δt, such that

$$\begin{aligned}\max_{0\le n\le N}&\Big\{\|\theta_n-\theta_n^{h,\Delta t}\|+\|P_n-P_n^{h,\Delta t}\|+\|v_n-v_n^{h,\Delta t}\|+\|(u_n-u_n^{h,\Delta t})_x\|+\|e_n-e_n^{h,\Delta t}\|\\&+\|(\phi_n-\phi_n^{h,\Delta t})_x\|+\|\phi_n-\phi_n^{h,\Delta t}\|\Big\}\le C(h+\Delta t).\end{aligned}$$

The proof of the linear convergence is done proceeding in a classical way. According to [11], it follows the well-known approximation property by finite elements:

$$\begin{aligned}\Delta t\sum_{j=1}^{N}&\Big(\inf_{z_j^h\in V^h}\|v_j-z_j^h\|^2+\inf_{z_j^h\in V^h}\|(v_j-z_j^h)_x\|^2+\inf_{r_j^h\in V^h}\|e_j-r_j^h\|^2\\&+\inf_{r_j^h\in V^h}\|(e_j-r_j^h)_x\|^2+\inf_{l_j^h\in V^h}\|\theta_j-l_j^h\|^2+\inf_{l_j^h\in V^h}\|(\theta_j-l_j^h)_x\|^2\\&+\inf_{m_j^h\in V^h}\|P_j-m_j^h\|^2+\inf_{m_j^h\in V^h}\|(P_j-m_j^h)_x\|^2\Big)+\max_{0\le n\le N}\inf_{z_n^h\in V^h}\|v_n-z_n^h\|^2\\&+\max_{0\le n\le N}\inf_{r_n^h\in V^h}\|e_n-r_n^h\|^2+\max_{0\le n\le N}\inf_{l_n^h\in V^h}\|\theta_n-l_n^h\|^2+\max_{0\le n\le N}\inf_{m_n^h\in V^h}\|P_n-m_n^h\|^2\\&\le Ch^2\big(\|u\|^2_{C^1([0,T];H^2(0,\ell))}+\|\phi\|^2_{C^1([0,T];H^2(0,\ell))}+\|\theta\|^2_{C([0,T];H^2(0,\ell))}+\|P\|^2_{C([0,T];H^2(0,\ell))}\big).\end{aligned}$$

From the additional regularity (37), we find that

$$
\begin{aligned}
&\|v_0 - v_0^h\|^2 + \|(u_0 - u_0^h)_x\|^2 + \|e_0 - e_0^h\|^2 + \|\phi_0 - \phi_0^h\|^2 + \|(\phi_0 - \phi_0^h)_x\|^2 \\
&\quad + \|\theta_0 - \theta_0^h\|^2 + \|P_0 - P_0^h\|^2 \\
&\le Ch^2\big(\|u_0\|^2_{H^2(0,\ell)} + \|v_0\|^2_{H^2(0,\ell)} + \|\phi_0\|^2_{H^2(0,\ell)} + \|e_0\|^2_{H^2(0,\ell)} + \|\theta_0\|^2_{H^2(0,\ell)} \\
&\quad + \|P_0\|^2_{H^2(0,\ell)}\big), \\
&\quad \Delta t \sum_{j=1}^{N} \Big[\|((u_t)_j - \delta u_j)_x\|^2 + \|(v_t)_j - \delta v_j\|^2 + \|((\phi_t)_j - \delta\phi_j)_x\|^2 + \|(e_t)_j - \delta e_j\|^2 \\
&\quad + \|(\theta_t)_j - \delta\theta_j\|^2 + \|(P_t)_j - \delta P_j\|^2\Big] \\
&\le C(\Delta t)^2\big(\|u\|^2_{H^2(0,T;H^1(0,\ell))} + \|u\|^2_{H^3(0,T;Y)} + \|\phi\|^2_{H^2(0,T;H^1(0,\ell))} + \|\phi\|^2_{H^3(0,T;Y)} \\
&\quad + \|\theta\|^2_{H^2(0,T;Y)} + \|P\|^2_{H^2(0,T;Y)}\big).
\end{aligned}
$$

Finally, the remaining terms in estimates (31) can be bounded as follows (see [7, 8] for details),

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{j=1}^{N-1} \|v_j - z_j^h - (v_{j+1} - z_{j+1}^h)\|^2 + \frac{1}{\Delta t}\sum_{j=1}^{N-1} \|e_j - r_j^h - (e_{j+1} - r_{j+1}^h)\|^2 \\
&\quad + \frac{1}{\Delta t}\sum_{j=1}^{N-1} \|\theta_j - l_j^h - (\theta_{j+1} - l_{j+1}^h)\|^2 + \frac{1}{\Delta t}\sum_{j=1}^{N-1} \|P_j - m_j^h - (P_{j+1} - m_{j+1}^h)\|^2 \\
&\le Ch^2\big(\|u\|^2_{H^2(0,T;H^1(0,\ell))} + \|\phi\|^2_{H^2(0,T;H^1(0,\ell))} + \|\theta\|^2_{H^1(0,T;H^1(0,\ell))} + \|P\|^2_{H^1(0,T;H^1(0,\ell))}\big).
\end{aligned}
$$

Thus, keeping in mind the previous estimates and using the a priori error estimates (31), we derive the linear convergence of the algorithm.

4. Numerical results

In this final section, we present the numerical scheme which we have implemented in MATLAB in order to obtain the solutions to Problem VP$^{h,\Delta t}$ and then, we show some numerical examples to demonstrate its accuracy and the behaviour of the solutions.

Considering the finite element space defined in (18), for $n = 1, 2, \dots, N$ and given $u_{n-1}^{h,\Delta t}$, $v_{n-1}^{h,\Delta t}$, $\phi_{n-1}^{h,\Delta t}$, $e_{n-1}^{h,\Delta t}$, $\theta_{n-1}^{h,\Delta t}$, $P_{n-1}^{h,\Delta t} \in V^h$, the discrete velocity field $v_n^{h,\Delta t}$, the discrete porosity speed $e_n^{h,\Delta t}$, the discrete temperature field $\theta_n^{h,\Delta t}$ and the discrete chemical potential field $P_n^{h,\Delta t}$, at time $t = t_n$, are then calculated solving the following problem, obtained from equations (20)-(23), for all z^h, r^h, l^h, $m^h \in V^h$,

$$\begin{aligned}&\rho(v_n^{h,\Delta t}, z^h) + \alpha\,(\Delta t)^2((v_n^{h,\Delta t})_x, z^h_x) = \rho(v_{n-1}^{h,\Delta t}, z^h) - \alpha\,\Delta t((u_{n-1}^{h,\Delta t})_x, z^h_x)\\ &\qquad +b\,\Delta t((\phi_n^{h,\Delta t})_x, z^h) - \gamma_1\,\Delta t((\theta_n^{h,\Delta t})_x, z^h) - \gamma_2\,\Delta t((P_n^{h,\Delta t})_x, z^h),\end{aligned}$$

$$\begin{aligned}&J(e_n^{h,\Delta t}, r^h) + \eta\,(\Delta t)^2((e_n^{h,\Delta t})_x, r^h_x) + \delta\,(\Delta t)^2(e_n^{h,\Delta t}, r^h)\\ &\qquad = J(e_{n-1}^{h,\Delta t}, r^h) - \eta\,\Delta t((\phi_{n-1}^{h,\Delta t})_x, r^h_x) - \delta\,\Delta t(\phi_{n-1}^{h,\Delta t}, r^h) - b\,\Delta t((u_n^{h,\Delta t})_x, r^h)\\ &\qquad +m_1\,\Delta t(\theta_n^{h,\Delta t}, r^h) + m_2\,\Delta t(P_n^{h,\Delta t}, r^h),\end{aligned}$$

$$\begin{aligned}&c(\theta_n^{h,\Delta t}, l^h) + k^*\,\Delta t((\theta_n^{h,\Delta t})_x, l^h_x) + \kappa(P_n^{h,\Delta t}, l^h) = c(\theta_{n-1}^{h,\Delta t}, l^h) + \kappa\,(P_{n-1}^{h,\Delta t}, l^h)\\ &\qquad -\gamma_1\,\Delta t((v_n^{h,\Delta t})_x, l^h) - m_1\,\Delta t(e_n^{h,\Delta t}, l^h),\end{aligned}$$

$$\begin{aligned}&\nu(P_n^{h,\Delta t}, m^h) + h^*\,\Delta t((P_n^{h,\Delta t})_x, m^h_x) + \kappa(\theta_n^{h,\Delta t}, m^h) = \nu(P_{n-1}^{h,\Delta t}, m^h) + \kappa(\theta_{n-1}^{h,\Delta t}, m^h)\\ &\qquad -\gamma_2\,\Delta t((v_n^{h,\Delta t})_x, m^h) - m_2\,\Delta t(e_n^{h,\Delta t}, m^h),\end{aligned}$$

where the discrete displacement and porosity fields $u_n^{h,\Delta t}$ and $\phi_n^{h,\Delta t}$ are now recovered from the relations

$$u_n^{h,\Delta t} = \Delta t\, v_n^{h,\Delta t} + u_{n-1}^{h,\Delta t}, \quad \phi_n^{h,\Delta t} = \Delta t\, e_n^{h,\Delta t} + \phi_{n-1}^{h,\Delta t}.$$

This problem leads to a linear system for a variable U in an adequate product space which is solved by using classical Cholesky's method. This numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical run ($h = \Delta t = 0.01$) took about 0.590 seconds of CPU time.

4.1. First example: numerical convergence. As an academical example, in order to show the accuracy of the approximations, we consider the following problem.

Problem $\mathbf{P^{ex}}$ *Find the displacement field $u : [0,1]\times[0,1] \to \mathbb{R}$, the porosity field $\phi : [0,1]\times[0,1] \to \mathbb{R}$, the temperature field $\theta : [0,1]\times[0,1] \to \mathbb{R}$ and the chemical potential field $P : [0,1]\times[0,1] \to \mathbb{R}$ such that*

$$\begin{aligned}&u_{tt} = 2u_{xx} + \phi_x - \theta_x - P_x + F_1 \quad \textit{in} \quad (0,1)\times(0,1),\\ &\phi_{tt} = \phi_{xx} - u_x - \phi + \theta + P + F_2 \quad \textit{in} \quad (0,1)\times(0,1),\\ &2\theta_t = \theta_{xx} - u_{tx} - \phi_t - P_t + F_3 \quad \textit{in} \quad (0,1)\times(0,1),\\ &2P_t = P_{xx} - u_{tx} - \phi_t - \theta_t + F_4 \quad \textit{in} \quad (0,1)\times(0,1),\\ &u(0,t) = u(1,t) = 0, \quad \phi(0,t) = \phi(1,t) = 0, \quad \theta(0,t) = \theta(1,t) = 0,\\ &\qquad\qquad P(0,t) = P(1,t) = 0 \quad \textit{for a.e.} \quad t \in (0,1),\\ &u(x,0) = 0, \quad \phi(x,0) = 0, \quad \theta(x,0) = 0 \quad \textit{for a.e. } x \in (0,1),\\ &u_t(x,0) = 0, \quad \phi_t(x,0) = 0, \quad P(x,0) = 0 \quad \textit{for a.e. } x \in (0,1),\end{aligned}$$

where the artificial volume forces F_1, F_2, F_3 and F_4 are given by

$$\begin{aligned}&F_1(x,t) = 6tx(x-1)^2 - 2t^3(6x-4) - t^3(3x^2-4x+1) + 2t(2x-1),\\ &F_2(x,t) = 6tx(x-1)^2 + t^3(3x^2-4x+1) - 2t^3(6x-4) + t^3x(x-1)^2 - 2tx(x-1),\\ &F_3(x,t) = 3t^2(3x^2-4x+1) - 2t + 3t^2x(x-1)^2 + 3x(x-1),\\ &F_4(x,t) = 3t^2(3x^2-4x+1) - 2t + 3t^2x(x-1)^2 + 3x(x-1).\end{aligned}$$

Moreover Problem P^{ex} corresponds to Problem P with the following data:

$$\begin{aligned}&\ell = 1, \quad T = 1, \quad \rho = 1, \quad \alpha = 2, \quad b = 1, \quad \gamma_1 = 1, \quad \gamma_2 = 1, \quad J = 1, \quad \eta = 2,\\ &\delta = 1, \quad m_1 = 1, \quad m_2 = 1, \quad k^* = 1, \quad c = 2, \quad \kappa = 1, \quad h^* = 1, \quad \nu = 2,\end{aligned}$$

and the initial conditions, for $x \in (0,1)$,

$$u(x,0) = v(x,0) = 0, \quad \phi(x,0) = e(x,0) = 0, \quad \theta(x,0) = 0, \quad P(x,0) = 0.$$

We note that the exact solution to Problem P^{ex} can be easily calculated and it has the form, for $(x,t) \in (0,1) \times (0,1)$,

$$u(x,t) = \phi(x,t) = t^3\, x(x-1)^2, \quad \theta(x,t) = P(x,t) = t\, x(x-1).$$

To show the numerical convergence and the asymptotic behaviour of the algorithm, the numerical errors given by

$$\max_{0\le n\le N} \Big\{ \|\theta_n - \theta_n^{h,\Delta t}\| + \|P_n - P_n^{h,\Delta t}\| + \|v_n - v_n^{h,\Delta t}\| + \|(u_n - u_n^{h,\Delta t})_x\| + \|e_n - e_n^{h,\Delta t}\| \\ + \|(\phi_n - \phi_n^{h,\Delta t})_x\| + \|\phi_n - \phi_n^{h,\Delta t}\| \Big\}$$

are calculated and presented (multiplied by 10^3) in Table 1 for several values of the discretization parameters h and Δt. Finally, the evolution of the error depending on the parameter $h + \Delta t$ is plotted in Figure 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 3.6, is achieved.

Table 1. Example 1: Numerical errors (x10^3) for some h and Δt.

$h \downarrow \Delta t \rightarrow$	0.01	0.005	0.001	0.0005	0.0002	0.0001	0.00005
0.01	1.104676	0.553370	0.116852	0.073133	0.056590	0.054401	0.054196
0.005	1.111073	0.558472	0.111200	0.055656	0.024493	0.016599	0.005693
0.001	1.113422	0.560713	0.112783	0.056391	0.022514	0.011218	0.005579
0.0005	1.113498	0.560787	0.112856	0.056463	0.022582	0.011281	0.005629
0.0002	1.113519	0.560808	0.112877	0.056484	0.022603	0.011301	0.005649
0.0001	1.113522	0.560811	0.112880	0.056487	0.022606	0.011304	0.005652
0.00005	1.113522	0.560812	0.112881	0.056488	0.022606	0.011305	0.005652

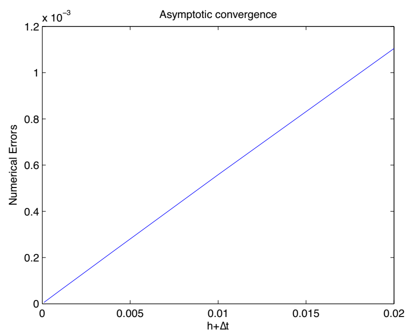

Figure 1. Example 1: Asymptotic constant error.

If we assume now that there are no volume forces, and we use the final time $T = 60$ and the following initial conditions:

$$u_0 = v_0 = \phi_0 = e_0 = \theta_0 = P_0 = x^2(x-1),$$

being the remaining data the same than in the previous simulation and taking the discretization parameters $h = 0.001$ and $\Delta t = 0.01$, the evolution in time of the discrete energy $E_n^{h,\Delta t}$, defined in (30), is plotted in Figure 2. As can be seen, it converges to zero and an exponential decay seems to be achieved, although theoretically there has been only possible to prove a polynomial decay of the energy functional.

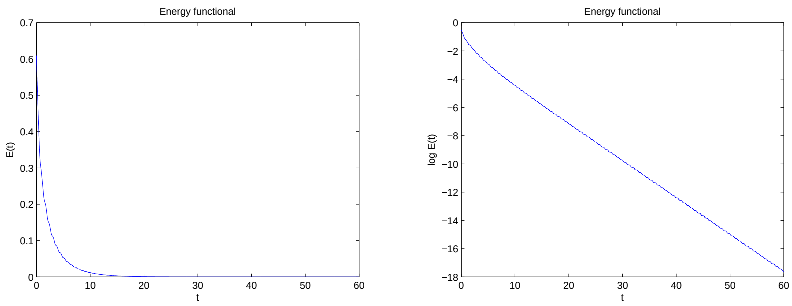

FIGURE 2. Example 1: Evolution of the discrete energy in the usual plot (left) and the semi-log plot (right).

4.2. Second example: application of a force. As a second example we employ the following data:

$$\begin{gathered}\ell = 1, \quad T = 1, \quad \rho = 1.5, \quad \alpha = 2.6, \quad b = 10, \quad \gamma_1 = 4.25, \quad \gamma_2 = 2.76,\\ J = 4.9, \quad \eta = 5.1, \quad \delta = 7.2, \quad m_1 = 1.3, \quad m_2 = 3.1, \quad k^* = 2, \quad c = 2.4,\\ \kappa = 0.7, \quad h^* = 1.75, \quad \nu = 3,\end{gathered}$$

and the initial conditions $u_0 = v_0 = \phi_0 = e_0 = \theta_0 = P_0 = 0$. Moreover, homogeneous Neumann conditions are used at the right end for all the variables, except for the displacement field, for which a compressive force $g(t) = -3t^2$ is applied, and homogeneous Dirichlet boundary conditions are imposed at the left end for all the variables.

Taking $h = \Delta t = 0.001$ as the discretization parameters, in Figure 3 the evolution in time of the displacements, the porosity, the temperature and the chemical potential are shown at the right point $x = 1$. As can be seen, the displacements are negative and this behaviour agrees with the fact that we have applied a compressive force. Meanwhile, the porosity, the temperature and the chemical potential, generated by the resulting deformation, increase with respect to the time.

4.3. Third example: influence of the temperature and the diffusion effect. As a third example we consider the following data:

$$\begin{gathered}\ell = 1, \quad T = 1, \quad \rho = 3, \quad \alpha = 5, \quad b = 1, \quad \gamma_1 = 2.3, \quad \gamma_2 = 0.2, \quad J = 1,\\ \eta = 4, \quad \delta = 7, \quad m_1 = 3, \quad m_2 = 2, \quad k^* = 4, \quad c = 2, \quad \kappa = 3 \quad h^* = 3, \quad \nu = 5,\end{gathered}$$

and the initial conditions $u_0 = v_0 = \phi_0 = e_0 = P_0 = 0$ and $\theta_0 = 5x^3(x-1)(4x^2-3)$. Moreover, homogeneous Dirichlet boundary conditions are used at both ends for all the variables.

Using the discretization parameters $h = \Delta t = 0.001$, in Figure 4 the displacements, porosity, temperature and chemical potential fields are plotted at final time. As expected, the initial temperature generates a deformation of the rod as well as porosity and chemical potential.

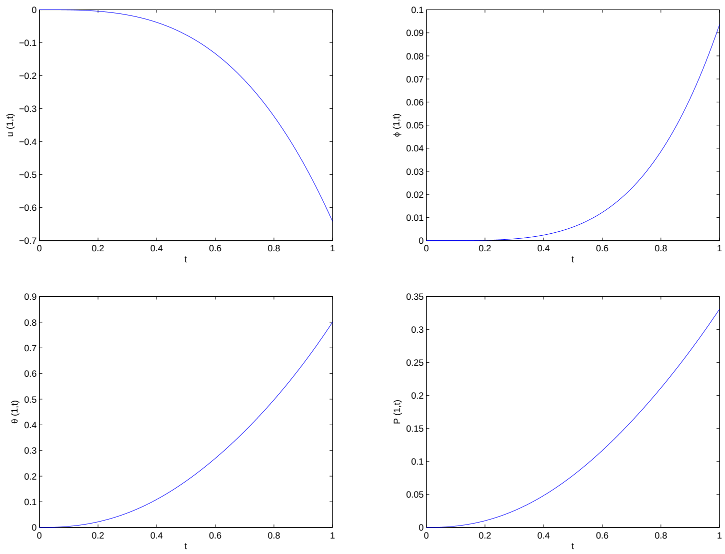

FIGURE 3. Example 2: Evolution in time of the displacement, porosity, temperature and chemical potential fields at the right point $x = 1$.

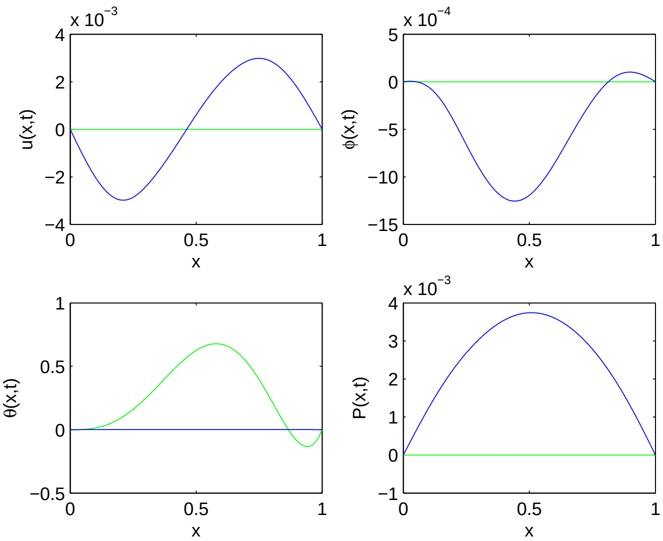

FIGURE 4. Example 3: Displacement, porosity, temperature and chemical potential fields at final time (green curves) and the respective initial conditions (blue curves).

Notice that the temperature field seems to decrease to zero. In Figure 5 the temperature field is shown at different times and it can be observed how it decreases due to the effect of the diffusion.

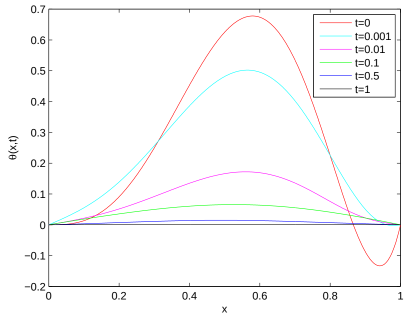

FIGURE 5. Example 3: Temperature field at different times.

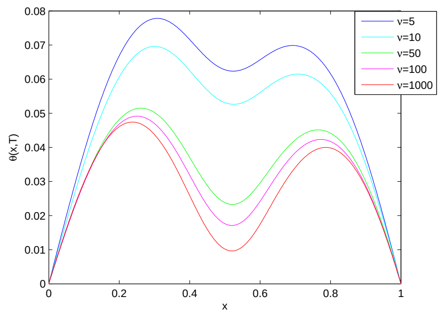

FIGURE 6. Example 4: Evolution of the temperature in time.

4.4. Fourth example: dependence on the diffusion coefficients. As a final example, the following data have been used:

$$\begin{aligned}&\ell = 1, \quad T = 1, \quad \rho = 1, \quad \alpha = 2, \quad b = 1, \quad \gamma_1 = 1, \quad \gamma_2 = 1, \quad J = 1,\\ &\eta = 2, \quad \delta = 1, \quad m_1 = 1, \quad m_2 = 1, \quad k^* = 1, \quad c = 3, \quad \kappa = 1, \quad h^* = 1,\end{aligned}$$

and the initial conditions $u_0 = v_0 = x^2(x-1)^3$, $\phi_0 = e_0 = x(x-1)$, $z_0 = x(x-1)(3x^2-7x-4)$ and $P_0 = 5x^3(x-1)(4x^2-3)$. Moreover, in this example we use again homogeneous Dirichlet boundary conditions and the diffusion coefficient ν varies between 5 and 1000 (values $\nu = 5, 10, 50, 100, 1000$ have been employed).

Taking $h = \Delta t = 0.001$ as the discretization parameters, in Figure 6 the temperature field is plotted for different values of the diffusion coefficient ν. As can be seen, the temperature decreases when the diffusion coefficient increases. We also have observed that an increase in the coefficient c results in a smoother solution, although we do not include the simulations for the sake of simplicity in the reading.

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