INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 1, Pages 63–75

DIFFERENTIAL QUADRATURE-BASED SIMULATION OF A CLASS OF FUZZY DAMPED FRACTIONAL DYNAMICAL SYSTEMS

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Abstract. In this paper, a numerical approach for the simulation of a dynamical model with damping defined by the Riemann–Liouville fractional derivative and with uncertainty, that is fuzziness, is discussed. The proposed method exploits differential quadrature rules and a Picard–like recursion. The convergence is formally discussed. Some example applications, in the linear and nonlinear regime, confirm the theoretical achievements.

Key words. Fractional derivatives, differential quadrature rules, Picard-like approach, fuzzy sets.

1. Introduction

Dynamical systems with uncertainty intended as fuzziness have been deeply investigated (e.g. see [14, 31, 6, 25]). Fuzziness is a way to take into account an uncertainty which cannot be identified as randomness [42].

In this paper, the following second order fuzzy initial value problem in presence of a Riemann-Liouville fractional derivative is considered

$$
\begin{aligned}
L_t^{(2)}\tilde{u}(t) + \delta \ {}^{RL}D^{\beta}(\tilde{u}(t)) &= f(\tilde{u}(t)) + \tilde{g}(t), \\
L_t^{(i)}\tilde{u}(0) &= \tilde{a}_i, \qquad i = 0, 1
\end{aligned} \tag{1}
$$

where $L_t^{(i)}$ is the ith-order derivative operator with respect to t, ${}^{RL}D^{\beta}$ is the Riemann–Liouville fractional derivative of order β, with $0 < \beta < 1$, $\delta \in \mathbb{R}$ a parameter. Besides, $f(\tilde{u}(t))$ is a functional form in $\tilde{u}$, $\tilde{g}(t)$ a given fuzzy-valued function and $\tilde{a}_i$ a fuzzy number, with $i = 0, 1$. Here, $\tilde{u}(t)$ represents the unknown fuzzy function for $t \in [0, T]$, with $T \in \mathbb{R}_+$. It should be pointed out that (1) may be seen as the fuzzification via the Zadeh's extension principle of the same problem but without fuzzy variables and parameters.

Broadly speaking, a fuzzy number can be seen as a more general representation of a real number, since it does not refer to a single value but to a set of possible values. From a computational perspective, a fuzzy quantity can be approximated by sets of closed intervals through the so–called α–cut approach. In a certain sense, this is congruent (even though different) with interval analysis (e.g. see [29]). Details about fuzzy numbers and all the relevant related issues will be provided in the next section. The problem corresponding to (1) in the non-fuzzy domain is relevant in structural dynamics, where fractional derivatives are mostly used for describing viscoelastic features of advanced materials [35]. Such kind of problems in the non-fuzzy domain has been mainly solved by means of perturbation techniques [35], which are known to involve demanding symbolic computations (e.g. [19]).

In this paper, fuzziness is considered in order to model an uncertain or imprecise system which is not a probabilistic dynamical system. Indeed, uncertainty in viscoelastic structures is a topic under discussion in the current literature [4].

Received by the editors on May 4, 2016, and accepted on July 15, 2016.

2000 *Mathematics Subject Classification.* 37M05, 65Q30, 65P99.

A numerical method is herein proposed to solve (1), by extending the approach presented in [37, 38] for solving linear integro–differential equations. More precisely, the proposed scheme foresees differential quadrature rules [9], which can be regarded as high-order finite-difference approximations [17, 28, 39, 40], and a Picard-like recursion into the fuzzy domain.

In the current literature, there are several examples of Picard-like approaches in the non-fuzzy domain (e.g. [15, 41, 34]), but not for problems like the one herein considered. Obviously, the method herein proposed reflects the transformation of the problem to be studied into a set of interval equations at α–levels. In particular, it follows some recent achievements in the field [7].

The approach herein proposed, in spite of its recursive nature, leads to a non–recursive approximate solution in the linear regime, by means of operational matrices and vectors of known quantities. The present scheme is different from the one proposed in [27], dealing with linear fuzzy partial differential equations. Convergence is herein formally discussed and some relevant example applications from the current literature, in the linear and nonlinear regime, are considered. Numerical results are in good agreement with the analytical solutions in the non-fuzzy domain, available in literature. The number of discrete points needed to get such solutions is small enough as well as the number of iterations for the nonlinear cases. Finally, it is worthwhile pointing out that the problem herein considered is different from the one in [13], where a linear system with a Caputo fractional derivative was numerically simulated by means of the homotopy perturbation method, without stating any property.

The paper is sectioned as follows. Section 2 introduces some basic notions. Section 3 presents the approach. In Section 4, some properties, and in particular convergence, are discussed. Section 5 is devoted to numerical experiments. The paper closes with some conluding remarks.

2. Preliminaries

In this section, some basic notions are provided (for more details, one can refer to classical textbooks, e.g. [30]). Throughout, $\mathbb{U}$ will denote a nonempty and closed set of $\mathbb{R}$.

Definition 2.1. *A* fuzzy number $\tilde{u}$ *is defined through a membership function* $\mu_u(x) : \mathbb{U} \to [0, 1]$ *and it satisfies* normality *and* convexity *on* $\mathbb{U}$*.*

The membership function maps each element from $\mathbb{U}$ to a membership value (or degree of membership), between a minimum, that is 0, and a maximum, that is 1, according to normality. Convexity requires that the membership function is piecewise continuous, but in compliance with normality. Arithmetic operations on fuzzy numbers can be approached either by means of the membership function, i.e. through the Zadeh extension principle, or by means of the α-cuts representation [21].

It should be pointed out that in a classical (or crisp) set, the membership function allows each element to have just 0 or 1 value, by meaning that a given element can belong or not to that set.

Definition 2.2. *An* α-cut *of the fuzzy number* $\tilde{u}$ *is the crisp set defined by*

$$[\tilde{u}]_\alpha = \{x \in \mathbb{U} : \mu_u(x) \geq \alpha\}, \qquad \alpha > 0. \tag{2}$$

Notice that for $\alpha = 0$, the α-cut of a fuzzy number $\tilde{u}$ is $[\tilde{u}]_0 = cl(\{x \in \mathbb{U} : \mu_u(x) > 0\})$, where cl denotes closure in the standard topology of $\mathbb{U}$.

The parametric representation of fuzzy numbers is intended as a generalization of the membership functions [24], allowing several shapes for the fuzzy numbers. This kind of representation is conceived to model the lower and upper extremal values of the α-cuts. Since an α-cut is a crisp interval, arithmetic operations can be performed by using the expressions of the lower and upper boundary of each α-cut.

Definition 2.3. *The* parametric form *of the fuzzy number $\tilde{u}$ is a pair $[\underline{u}(\alpha), \overline{u}(\alpha)]$ satisfying the following properties for each $\alpha \in [0,1]$:*

(1) $\underline{u}(\alpha)$ is a bounded, left-continuous, monotone increasing function over $[0,1]$,
(2) $\overline{u}(\alpha)$ is a bounded, left-continuous, monotone decreasing function over $[0,1]$,
(3) $\underline{u}(\alpha) \leq \overline{u}(\alpha)$.

The notation $[\tilde{u}]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$ is employed if such form exists.

So, a *triangular fuzzy number* (TFN), which is generally identified by an ordered triplet of numbers (u_C, d_L, d_R), can be represented through the closed interval

$$[\tilde{u}]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)] = [u_C + (\alpha - 1)d_L, u_C + (1-\alpha)d_R]\,, \tag{3}$$

for each $\alpha \in (0,1]$, being u_C the center, d_L and d_R the left and the right spreads.

The metric structure is given by the Hausdorff distance $D : \mathbb{U} \times \mathbb{U} \to \mathbb{R}_+ \bigcup \{0\}$,

$$D\left(\tilde{u}, \tilde{v}\right) = \sup_{\alpha \in [0,1]} \max\left\{\left|\underline{u}(\alpha) - \underline{v}(\alpha)\right|, \left|\overline{u}(\alpha) - \overline{v}(\alpha)\right|\right\}. \tag{4}$$

With regard to (1), the existence of sufficient derivatives of the solution is assumed, by referring to H-differentiability (in the sense of Hukuara) [33]. There are different notions of fuzzy derivatives [11], but H-differentiability has some advantages, such as providing fuzzy numbers as the solution of fuzzy differential equations (FDEs). An extensive work about the different ways for handling fuzzy derivatives can be found in [23].

In a recent paper, Bede [7] presented some characterization theorems in order to deal with the numerical solutions of FDEs in the setting of Hukuhara differentiability: under certain conditions it is possible to translate the original FDE equivalently into a system of ordinary differential equations. We will reformulate some theorems. To that end, let us rewrite the problem (1) as follows

$$\begin{array}{rcl} L_t^{(2)}\tilde{u}(t) & = & \phi(t, \tilde{u}(t), ^{RL}D^\beta(\tilde{u}(t))), \\ L_t^{(i)}\tilde{u}(0) & = & \tilde{a}_i, \qquad i = 0, 1. \end{array} \tag{5}$$

Theorem 2.4. [20] *Assume that $\phi : [0,T] \times \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ is continuous. A mapping $\tilde{u} : [0,T] \to \mathbb{U}$ is a solution to* (5) *if and only if $\tilde{u}(t)$ and $L_t^{(1)}\tilde{u}$ are continuous and satisfy the integral equation*

$$\tilde{u}(t) = \tilde{a}_0 + \tilde{a}_1 t + \int_0^t \int_0^s \phi(s, \tilde{u}(s), ^{RL}D^\beta(\tilde{u}(s)))dsds. \tag{6}$$

Theorem 2.4 yields the following theorem. For the sake of readibility, we let $\Phi(\tilde{u}(t)) = \phi(t, \tilde{u}(t), ^{RL}D^\beta(\tilde{u}(t)))$.

Theorem 2.5. [20] *Let $\phi : [0,T] \times \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ be a continuous map and assume that there exists $k > 0$ such that*

$$D(\Phi(\tilde{u}(t)), \Phi(\tilde{v}(t))) \leq kD(\tilde{u}(t), \tilde{v}(t)) \tag{7}$$

for all $t \in [0,T]$, $\tilde{u}, \tilde{v} : [0,T] \to \mathbb{U}$. Then the problem (5) *has a unique solution on $[0,T]$.*

Finally, by using Theorem 2.5, it is possible to prove the following result.

Theorem 2.6. [7] *Let $\phi : [0,T] \times \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ be such that for $\alpha \in [0,1]$*

$$
\begin{aligned}
&[\phi(t, \tilde{u}(t), {}^{RL}D^{\beta}(\tilde{u}(t)))]_{\alpha} \\
&= [\underline{\phi}(t, \underline{u}(t,\alpha), \overline{u}(t,\alpha), {}^{RL}D^{\beta}(\underline{u}(t,\alpha)), {}^{RL}D^{\beta}(\overline{u}(t,\alpha)), \alpha), \\
&\overline{\phi}(t, \underline{u}(t,\alpha), \overline{u}(t,\alpha), {}^{RL}D^{\beta}(\underline{u}(t,\alpha)), {}^{RL}D^{\beta}(\overline{u}(t,\alpha)), \alpha)].
\end{aligned} \tag{8}
$$

If the functions $\underline{\phi}(t, \ \underline{u}(t,\alpha), \ \overline{u}(t,\alpha), {}^{RL}D^{\beta}(\underline{u}(t,\alpha)), {}^{RL}D^{\beta}(\overline{u}(t,\alpha)), \alpha)$ and $\overline{\phi}(t, \underline{u}(t,\alpha), \overline{u}(t,\alpha), {}^{RL}D^{\beta}(\underline{u}(t,\alpha)), {}^{RL}D^{\beta}(\overline{u}(t,\alpha)), \alpha)$ are equicontinuous and uniformly Lipschitz in the second and third arguments, then the problem (5) and the following system of differential equations are equivalent, with a unique level-wise solution $[\tilde{u}(t)]_{\alpha} = [\underline{u}(t,\alpha), \overline{u}(t,\alpha)]$ in the interval $[0,T]$:

$$
\begin{cases}
L_t^{(2)}\underline{u}(t,\alpha) &= \underline{\phi}(t, \underline{u}(t,\alpha), \overline{u}(t,\alpha), {}^{RL}D^{\beta}(\underline{u}(t,\alpha)), {}^{RL}D^{\beta}(\overline{u}(t,\alpha)), \alpha), \\
L_t^{(2)}\overline{u}(t,\alpha) &= \overline{\phi}(t, \underline{u}(t,\alpha), \overline{u}(t,\alpha), {}^{RL}D^{\beta}(\underline{u}(t,\alpha)), {}^{RL}D^{\beta}(\overline{u}(t,\alpha)), \alpha), \\
L_t^{(i)}\underline{u}(0,\alpha) &= \underline{a}_i(\alpha), \quad i = 0,1, \\
L_t^{(i)}\overline{u}(0,\alpha) &= \overline{a}_i(\alpha), \quad i = 0,1.
\end{cases} \tag{9}
$$

The fuzzy Riemann-Liouville fractional derivative was first introduced in [1] and discussed in [5].

By recalling the definition of the Riemann-Liouville fractional derivative of order β [32] (provided it exists), one has

$$
\begin{aligned}
&[{}^{RL}D^{\beta}(\tilde{u}(t))]_{\alpha} = [{}^{RL}D^{\beta}(\underline{u}(t,\alpha)), {}^{RL}D^{\beta}(\overline{u}(t,\alpha)] \\
&= [\frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \underline{u}(s,\alpha)(t-s)^{-\beta} ds, \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \overline{u}(s,\alpha)(t-s)^{-\beta} ds],
\end{aligned} \tag{10}
$$

where $\Gamma(.)$ represents the Gamma function. One has to mention that there is a definition of fuzzy Riemann-Liouville fractional derivative in the settings of generalized differentiability [2], but in light of Theorem 2.6, herein we refer to H-differentiability.

Assuming that the derivatives exist, in light of Theorem 2.6 and [12, 33], one obtains

$$
\begin{cases}
L_t^{(2)}\mathbf{U}(t,\alpha) + \delta \ {}^{RL}D^{\beta}\mathbf{U}(t,\alpha) &= \mathbf{F}(\mathbf{U}(t,\alpha),\alpha) + \mathbf{G}(t,\alpha), \\
L_t^{(i)}\mathbf{U}(0,\alpha) &= \mathbf{a}_i(\alpha), \quad i = 0,1,
\end{cases} \tag{11}
$$

where

$$\mathbf{U}(t,\alpha)^T = (\underline{u}(t,\alpha), \ \overline{u}(t,\alpha)), \tag{12}$$

$$\mathbf{G}(t,\alpha)^T = \big(\underline{g}(t,\alpha), \ \overline{g}(t,\alpha)\big), \tag{13}$$

$$\mathbf{a}_i(\alpha)^T = (\underline{a}_i(\alpha), \ \overline{a}_i(\alpha)), \tag{14}$$

$$\mathbf{F}(\mathbf{U}(t,\alpha),\alpha)^T = \big(\underline{f}(\underline{u}(t,\alpha), \overline{u}(t,\alpha), \alpha), \ \overline{f}(\underline{u}(t,\alpha), \overline{u}(t,\alpha), \alpha)\big). \tag{15}$$

In the following definition, quadrature and differential quadrature, which allows to write integrals and derivatives as weighted sums, are referred to the fuzzy domain [27].

Definition 2.7. *Let $0 = t_1 < t_2 < \ldots < t_N = T$ be a fixed (though arbitrary) partition of the interval $[0,T]$. The parametric form of fuzzy quadrature and fuzzy* *differential quadrature rules is respectively:*

$$\left[\int_0^T \tilde{u}(t)dt\right]_\alpha = \left[\int_0^T \underline{u}(t)dt, \int_0^T \overline{u}(t)dt\right]_\alpha = \left[\sum_{i=1}^N C_i \underline{u}_i, \sum_{i=1}^N C_i \overline{u}_i\right], \tag{16}$$

$$\left[\frac{d^r \tilde{u}}{dt^r}\right]_\alpha = \left[\frac{d^r \underline{u}}{dt^r}, \frac{d^r \overline{u}}{dt^r}\right]_\alpha = \left[\sum_{i=1}^N A_i^{(r)}(t)\underline{u}_i(\alpha), \sum_{i=1}^N A_i^{(r)}(t)\overline{u}_i(\alpha)\right]. \tag{17}$$

where $\underline{u}_i(\alpha) = \underline{u}(t_i, \alpha)$ *and* $\overline{u}_i(\alpha) = \overline{u}(t_i, \alpha)$.

Finally, in order to support the discussion in Section 4, the following definition, taken from [3], is recalled.

Definition 2.8. *Let $f(x)$ be a bounded function $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ a closed interval, and h a positive real number. The* modulus *of smoothness of f, $\omega(f, .) : [0, \infty) \to [0, \infty)$, is defined by*

$$\omega(f, h) = \sup\{|f(x) - f(y)| : x, y \in I, |x - y| \leq h\}. \tag{18}$$

If f is continuous, then $\omega(f, h)$ is called the uniform modulus of continuity.

In order to support the discussion in Section 4, we recall the following theorem, giving the properties of $\omega(f, h)$.

Theorem 2.9. [3] *The following properties hold true*

1. *$|f(x) - f(y)| \leq \omega(f, d(x, y))$ for any $x, y \in I$;*
2. *$\omega(f, h)$ is nondecreasing in h;*
3. *$\omega(f, 0) = 0$;*
4. *$\omega(f, h_1 + h_2) \leq \omega(f, h_1) + \omega(f, h_2)$ for any $h_1, h_2 \in [0, \infty)$;*
5. *$\omega(f, rh) \leq r\omega(f, h)$ for any $h \in [0, \infty)$ and $r \in \mathbb{N}$;*
6. *$\omega(f, zh) \leq (z + 1)\omega(f, h)$ for any $z, h \in [0, \infty)$;*
7. *if f is continuous then $\lim_{h \to 0} \omega(f, h) = 0$.*

3. Methodology

In this section, the approach discussed in [37, 38] is extended in order to solve (11). For the remainder of this paper, I will represent the problem domain.

Let us consider the second order inverse operator

$$L_t^{-1}(\cdot) = \int_0^t \int_0^t (\cdot)dtdt. \tag{19}$$

In light of Theorems 5–8 in [22] , (19) may be applied on both sides of (11). So, one obtains

$$\mathbf{U}(t, \alpha) = \mathbf{A}(\alpha)\mathbf{w} + L_t^{-1}\big(\mathbf{F}(\mathbf{U}(t, \alpha), \alpha) + \mathbf{G}(t, \alpha) - \delta\ {}^{RL}D^\beta \mathbf{U}(t, \alpha)\big), \tag{20}$$

where

$$\mathbf{w}^T = (1, t), \qquad \mathbf{A}(\alpha) = \begin{pmatrix} \underline{a}_0(\alpha) & \underline{a}_1(\alpha), \\ \overline{a}_0(\alpha) & \overline{a}_1(\alpha) \end{pmatrix} \tag{21}$$

(see Eq. (12), Eq. (13), Eq. (15) for $\mathbf{U}$, $\mathbf{G}$, $\mathbf{F}$ respectively). Here, for $i = 0, 1$, $\underline{a}_i(\alpha)$ and $\overline{a}_i(\alpha)$ will represent the ith derivative with respect to t of $\underline{u}$ and $\overline{u}$ respectively, at the point $(0, \alpha)$.

Using successive approximations, the solution $\mathbf{U}(t, \alpha)$ can be expressed as

$$\mathbf{U}(t, \alpha) = \sum_{k=0}^\infty \mathbf{U}_k(t, \alpha), \tag{22}$$

where $\mathbf{U}_k^T(t,\alpha) = (\underline{u}_k(t,\alpha), \overline{u}_k(t,\alpha))$ has to be determined recursively using the formulas

$$\begin{aligned} \mathbf{U}_0(t,\alpha) &= \mathbf{A}(\alpha)\mathbf{w} + L_t^{-1}(\mathbf{G}(t,\alpha)), && (23)\\ \mathbf{U}_{k+1}(t,\alpha) &= L_t^{-1}\left(-\delta\ {}^{RL}D^{\beta}\mathbf{U}_k(t,\alpha) + \mathbf{F}(\mathbf{U}_k(t,\alpha),\alpha)\right). && (24) \end{aligned}$$

Consider a fixed (though arbitrary) partition $t_1 < t_2 < \ldots < t_{N-1} < t_N$ of the interval I. Let $h = max_i|t_{i+1} - t_i|$ be the norm of the partition.

Besides, if we let $\underline{u}_{k,j}(\alpha) = \underline{u}_k(t_j,\alpha)$ and $\overline{u}_{k,j}(\alpha) = \overline{u}_k(t_j,\alpha)$, then

$$\mathbf{U}_k^T(\alpha) = (\underline{\mathbf{u}}_k(\alpha), \overline{\mathbf{u}}_k(\alpha)) = \left(\underline{u}_{k,1}(\alpha), \ldots, \underline{u}_{k,N}(\alpha), \overline{u}_{k,1}(\alpha), \ldots, \overline{u}_{k,N}(\alpha)\right). \quad (25)$$

Using numerical integration and the differential quadrature (DQ) rules, one obtains that

$$\begin{aligned} \mathbf{U}_0(t,\alpha) &= \mathbf{A}(\alpha)\mathbf{w} + \mathbf{C}(t)\mathbf{Q}(\alpha), && (26)\\ \mathbf{U}_{k+1}(t,\alpha) &= \mathbf{C}(t)\left[-\gamma\mathbf{B}\mathbf{U}_k(\alpha) + \mathbf{F}(\mathbf{U}_k(\alpha),\alpha)\right], && (27) \end{aligned}$$

where $\gamma = \delta/\Gamma(1-\beta)$ and

$$\begin{aligned} \mathbf{Q}^T(\alpha) &= \left(\underline{\mathbf{Q}}_0^T(\alpha), \overline{\mathbf{Q}}_0^T(\alpha)\right) && (28)\\ &= \left(\underline{g}(t_1,\alpha), \ldots, \underline{g}(t_N,\alpha), \overline{g}(t_1,\alpha), \ldots, \overline{g}(t_N,\alpha)\right), \\ \mathbf{C}(t) &= \begin{pmatrix} \mathbf{C}_0(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_0(t) \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_0 \end{pmatrix}. && (29) \end{aligned}$$

with $\mathbf{C}_0(t) = (C_1(t), \ldots, C_N(t))$ and $\mathbf{B}_0 = \mathbf{A}^{(1)}\mathbf{C}^{(\beta)}$. Here, $\mathbf{A}^{(1)}$ denotes the $N \times N$ matrix of the DQ weighting coefficients and $\mathbf{C}^{(\beta)}$ the $N \times N$ matrix of which entries are $C_i^{\beta}(t) = \int_0^t l_i(s)(t-s)^{-\beta}ds$, evaluated at the grid points $t_1, \ldots, t_N$ for each $i = 1, \ldots, N$, with $l_i(s)$ being the ith Lagrange basis polynomial. The approximate solution after p terms in (22) is

$$\mathbf{U}^{[p]}(t,\alpha) = \mathbf{U}_0(t,\alpha) + \mathbf{C}(t)\sum_{k=0}^{p-1}\left[-\gamma\mathbf{B}\mathbf{U}_k(\alpha) + \mathbf{F}\left(\mathbf{U}_k(\alpha),\alpha\right)\right]. \quad (30)$$

For the linear case, i.e. $f(\tilde{u}(t)) = -\Omega_0^2\tilde{u}(t)$ (being Ω_0 the natural frequency), one gets

$$\mathbf{U}_{k+1}(t,\alpha) = \mathbf{H}(t)\mathbf{U}_k(\alpha), \quad (31)$$

where

$$\mathbf{H}(t) = \begin{pmatrix} \mathbf{H}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_0 \end{pmatrix}, \quad (32)$$

with the N–sized vector $\mathbf{H}_0(t) = -\mathbf{C}_0(t)\left(\gamma\mathbf{B}_0 + \Omega_0^2\mathbf{I}\right)$, $\mathbf{I}$ being the identity matrix of size N.

Let $\mathbf{D}$ be the matrix

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_0 \end{pmatrix}, \quad (33)$$

with the $N \times N$ sub–matrix $\mathbf{D}_0^T = \left[\mathbf{H}_0(t_1), \ldots, \mathbf{H}_0(t_N)\right]^T$. Since

$$\mathbf{U}_k(\alpha) = \mathbf{D}\mathbf{U}_{k-1}(\alpha) = \mathbf{D}^k\mathbf{U}_0(\alpha), \quad (34)$$

the truncation of (22) after p terms becomes in the linear case

$$\mathbf{U}^{[p]}(t,\alpha) = \mathbf{U}_0(t,\alpha) + \mathbf{H}(t) \sum_{k=0}^{p-1} \mathbf{D}^k \mathbf{U}_0(\alpha), \tag{35}$$

where $\mathbf{U}_0^T(\alpha) = (\underline{\mathbf{u}}_0(\alpha), \overline{\mathbf{u}}_0(\alpha))^T = \big(\underline{u}_{0,1}(\alpha), \ldots, \underline{u}_{0,N}(\alpha), \overline{u}_{0,1}(\alpha), \ldots, \overline{u}_{0,N}(\alpha)\big)$.

4. Properties

Let $\rho(\mathbf{D_0})$ be the spectral radius of the matrix $\mathbf{D_0}$. The following Lemma is stated.

Lemma 4.1. *Suppose that $\rho(\mathbf{D}_0) < 1$. Then the solution $\mathbf{U}(t,\alpha)$ in (22) is given by*

$$\mathbf{U}(t,\alpha) = \mathbf{U}_0(t,\alpha) + \mathbf{H}(t)(\mathbf{I} - \mathbf{D})^{-1} \mathbf{U}_0(\alpha). \tag{36}$$

Proof. The proof follows as that of Theorem 1 in [37], by considering that the spectral radius of $\mathbf{D}$ is equal to that of $\mathbf{D}_0$. More precisely, by observing that (35) can be written as

$$\mathbf{U}^{[p]}(t,\alpha) = \mathbf{U}_0(t,\alpha) + \mathbf{H}(t) \left[(\mathbf{I} - \mathbf{D})^{-1} (\mathbf{I} - \mathbf{D}^p)\right] \mathbf{U}_0(\alpha), \tag{37}$$

then (36) is readily derived when $\rho(\mathbf{D}_0) < 1$ and $p \to \infty$. $\square$

Let $\mathbf{V}(t,\alpha) = [\underline{v}(t,\alpha), \overline{v}(t,\alpha)]$ be the exact solution of (11). The following theorem is proved.

Theorem 4.2. *Let $t_1 < t_2 < \ldots < t_{N-1} < t_N$ be a partition of the interval I having the norm h . Suppose that $\underline{v}(t,\alpha)$ and $\overline{v}(t,\alpha)$ are bounded and continuous functions in I. Then the following error estimates hold true*

$$|\underline{v}(t,\alpha) - \underline{u}(t,\alpha)| \le \sigma(\underline{v}, h), \quad |\overline{v}(t,\alpha) - \overline{u}(t,\alpha)| \le \sigma(\overline{v}, h) \tag{38}$$

where

$$\sigma(\underline{v}, h) = r\omega(\underline{v}, h) + |\underline{a_1} rh + \mathbf{C}_0(rh)\underline{\mathbf{Q}}_0(\alpha) + \mathbf{H}_0(rh) \sum_{k=0}^{p-1} \mathbf{D}_0^k \underline{\mathbf{u}}_0(\alpha)|, \tag{39}$$

$$\sigma(\overline{v}, h) = r\omega(\overline{v}, h) + |\overline{a_1} rh + \mathbf{C}_0(rh)\overline{\mathbf{Q}}_0(\alpha) + \mathbf{H}_0(rh) \sum_{k=0}^{p-1} \mathbf{D}_0^k \overline{\mathbf{u}}_0(\alpha)|, \tag{40}$$

with the integer $1 \le r << N$.

Proof. Let $t \in I$ be arbitrarily fixed. Let us consider the error $|\underline{v}(t,\alpha) - \underline{u}(t,\alpha)|$.

If $t \in [t_i, t_{i+r}]$, with $1 \le r << N$ and $1 \le i \le N - r$, then $|t - t_i| \le rh$. Besides, in such arbitrarily small interval, $\underline{u}(t,\alpha) \approx \underline{u}_L(t,\alpha)$ is assumed, being $\underline{u}_L(t,\alpha)$ the linear part of $\underline{u}(t,\alpha)$. So one can refer to Eq. (35).

In particular, without any restriction, one may assume $t_i = 0$ and by recalling that $\underline{v}(0,\alpha) = \underline{a}_0(\alpha)$, one gets in reason of Theorem 2.9

$$|\underline{v}(t,\alpha) - \underline{v}(0,\alpha) - \underline{R}(t,\alpha)| \le r\omega(\underline{v}, h) + |\underline{R}(rh,\alpha)| \tag{41}$$

where $\underline{R}(t,\alpha) = \underline{a_1} t + \mathbf{C}_0(t)\underline{\mathbf{Q}}_0(\alpha) + \mathbf{H}_0(t) \sum_{k=0}^{p-1} \mathbf{D}_0^k \underline{\mathbf{u}}_0(\alpha)$. Since a similar result holds for $|\overline{v}(t,\alpha) - \overline{u}(t,\alpha)|$, this completes the proof. $\square$

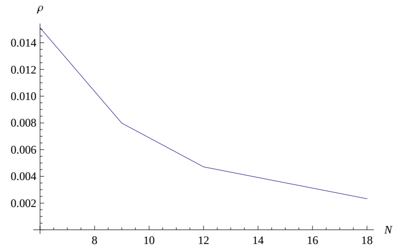


FIGURE 1. First example. Graph of $\rho(\mathbf{D}_0)$ versus N.

Theorem 4.2 makes evidence of consistency, since $\lim_{h\to 0} \sigma(\underline{v}, h) = \lim_{h\to 0} \sigma(\overline{v}, h) = 0$ (see Theorem 2.9). It is straightforward to observe that errors computed at successive points are bounded by the right–hand side of the inequality (38), by letting $r = 1$. On the other hand, in such Picard-like approaches the iterates are expected not to grow unlimitedly, as outlined in the following remark.

Remark 4.3. *Let* $\mathbb{M}$ *be the metric space given by the set of continuous real valued functions defined on the compact interval* I*. If* $\underline{f}$ *and* $\overline{f}$ *are Lipschitz continuous functions in* $\mathbb{M}$*, than the mapping derived from the integral equation* (20) *is a contraction on* $\mathbb{M}$ [10]*. As a consequence, by virtue of the fixed point theorem, the iterates* $\{\underline{u}_k\}$ *and* $\{\overline{u}_k\}$ *have the property that* $\lim_{k\to\infty} \underline{u}_k = \underline{u}^*$ *and* $\lim_{k\to\infty} \overline{u}_k = \overline{u}^*$*, with* $\underline{u}^*, \overline{u}^* \in \mathbb{M}$.

5. Numerical simulations

As a preamble to this section, one has to consider that, in view of the continuity of the functions involved, one can refer to $L_\infty = \|V(t,0) - U(t,0)\|_\infty$ as the maximum norm (see [16]), being $\mathbf{V}(t,0)$ and $\mathbf{U}(t,0)$ the exact and the approximate solution (see Eq. (22)) respectively for $\alpha = 0$. In the following examples, because of comparative purposes (i.e. by considering existing solutions in the non-fuzzy domain), $T = 1$ has been fixed, but this is not a restriction (e.g. see [38]). Grid coordinates are here obtained through the Gauss-Tchebychev-Lobatto (GCL) distribution [26]. For nonlinear cases, iterations stop when $|\mathbf{U}^{[p+1]} - \mathbf{U}^{[p]}| \le \epsilon$, with ϵ arbitrarily small. All the numerical computations were perfomed by means of MATHEMATICA 6.0.

5.1. Example 1: free damped vibrations of a linear oscillator in a medium with small viscosity.

As a first example application, a linear case is considered. The analytical solution of this problem in the non-fuzzy domain was obtained in [35] by means of the method of multiple time scales.

Null initial velocity and a triangular fuzzy number (see Eq. (3)) as initial displacement, that is $[\tilde{a}_0]_\alpha = [\alpha - 1, 1 - \alpha]$, were assumed. In absence of a forcing term it is $\tilde{g}(t) = 0$. Besides, $\delta = 10^{-4}$, $\Omega_0 = 1$ and $N = 9$ GCL points were adopted. In view that the condition $\rho(\mathbf{D}_0) < 1$ holds, Eq. (36) was used. The behaviour of $\rho(\mathbf{D}_0)$ in the range $6 \le N \le 18$ is shown in Fig. 1. It is clear that by increasing N, the value of $\rho(\mathbf{D}_0)$ approaches zero; values $N < 6$ have not been considered, because they are not relevant to the accuracy of the solution. Fig. 2a shows the approximate solution for $\beta = 0.65$.

With regard to the same value of β, the graphs of the approximate and analytical solutions obtained for some values of α are depicted in Fig. 2b and 2c. The graphs are referred to the functions $\underline{u}$ (below) and $\overline{u}$ (above). The L_∞ norm for this case is

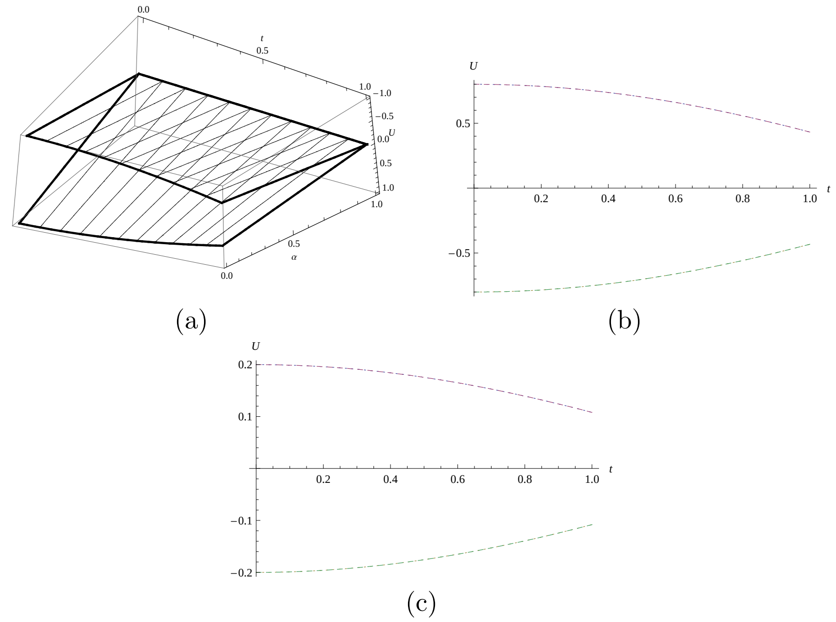


FIGURE 2. First example. Graphs for $\beta = 0.65$: (a) approximate solution; numerical (dotted line) and the analytical solutions (dashed line) for (b) $\alpha = 0.2$, (c) $\alpha = 0.8$. The graphs correspond to the functions $\underline{u}$ (below) and $\overline{u}$ (above).

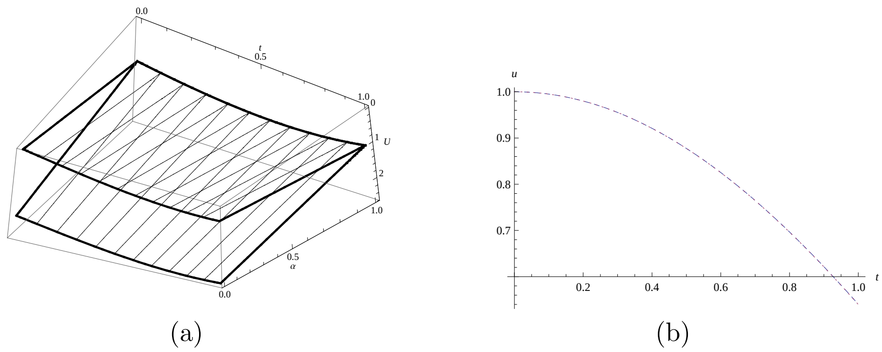
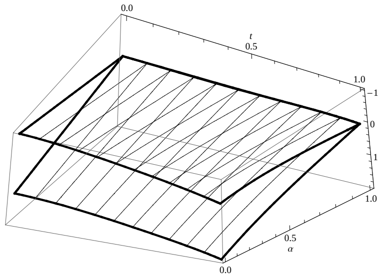


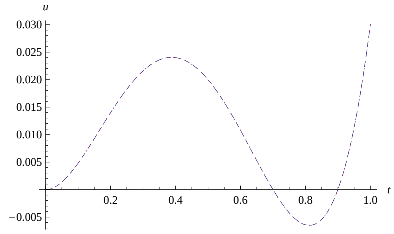
FIGURE 3. Second example. Graphs for $\beta = 0.75$: (a) approximate solution, (b) numerical (dotted line) and the analytical solutions (dashed line) for $\alpha = 1$.

$5.40E-05$. The same value about for $N = 6$ ($5.41E-05$) and $N = 18$ ($5.37E-05$) was found.

5.2. Example 2: free damped vibrations of a weakly nonlinear oscillator in a medium with small viscosity. As for the linear case, the analytical solution of this problem in the non-fuzzy domain was obtained by means of the method of multiple time scales. The nonlinear part is cubic, that is $f(\tilde{u}) = -k\tilde{u}$. As in the previous example, $\delta = 10^{-4}$, $\Omega_0 = 1$ were fixed and in addition $k = 10^{-3}$. Besides,



(a)



(b)

FIGURE 4. Third example. Graphs for $\beta = 0.5$: (a) approximate solution, (b) numerical (dotted line) and the analytical solutions (dashed line) for $\alpha = 1$.

$\beta = 0.75$. Null initial velocity and the following triangular fuzzy number as initial displacement $[\tilde{a}_0]_\alpha = [\alpha, 2 - \alpha]$ were assumed.

The approximate solution, obtained through $N = 9$ GCL points and $p = 6$ (that is five iterations), is depicted in Fig. 3a, showing how the initial TFN evolves.

Fig. 3b shows the analytical solution (in the non-fuzzy domain) and the approximate solution for $\alpha = 1$.

The maximum error between the two solutions is $5.209E-04$. By increasing the number of iterations, this result substantially does not improve. By using $N = 13$ GCL points, the maximum error between the two solutions decreases by $5.52E-08$.

5.3. Example 3: forced damped vibrations of a nonlinear oscillator. The problem herein considered was investigated (in the non-fuzzy domain) in [43, 36] and mentioned in [35]. More precisely, in [43] the analytical solution is provided, whereas in [36] the Adomian decomposition method (ADM) is used. The nonlinear part is again cubic. As for the case discussed in [36], $\beta = 0.5$, $\delta = 0.8$, $k = 1$, $\Omega_0 = 0$ were fixed, in addition to null initial velocity. Besides, the following triangular fuzzy number as initial displacement $[a_0]_\alpha = [\alpha - 1, 1 - \alpha]$ and the following fuzzy function as forcing term $[\tilde{g}(t)]_\alpha = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha] g_0(t)$ were assumed, being

$$\begin{aligned} g_0(t) = &2\left(t - \frac{9}{10}\right)\left(t - \frac{7}{10}\right) + 4t\left(t - \frac{7}{10}\right) + 4t\left(t - \frac{9}{10}\right) + 2t^2 + \\ &+ \frac{8}{10\Gamma(1/2)}\left(\frac{128}{35}\sqrt{t^7} - \frac{128}{25}\sqrt{t^5} + \frac{42}{25}\sqrt{t^3}\right) + \left[t^2(t - \frac{9}{10})(t - \frac{7}{10})\right]^3, \end{aligned}$$

that is the same function adopted in [36].

By means of these assumptions, the case studied in [43, 36] can be reproduced, by considering $\alpha = 1$. The approximate solution is depicted in Fig. 4a. Fig. 4b shows the numerical and the analytical solution in the non-fuzzy domain for $\alpha = 1$.

In [36] the four–terms approximate solution exhibits a maximum error in the domain $[0, 1]$ equal to $9.95E - 05$. By reproducing the same case in the non-fuzzy domain by the proposed method, one finds, with $N = 17$ GCL points and $p = 4$ (i.e. after three iterations), a maximum error equal to $5.18E - 06$ and after two more iterations ($p = 6$) $5.41E - 07$. The latter can be assumed as the best result, because for $p = 9$ one gets a similar one, that is $5.27E - 07$.

Information about running time are not available in [36], but some comments about this (in the non-fuzzy domain) are possible. It is well–known that the p–term approximate solution in the ADM requires the computation of $p-1$ Adomian polynomials in the nonlinear case. Different algorithms for the Adomian polynomials have been proposed in order to improve computational efficiency and a comparative study was proposed in [18]. The running times for generating the first p one–variable Adomian polynomials, with $p = 21$, can vary between 0.11 s and 0.41 s, depending on the adopted algorithm and by using MATHEMATICA 7.0 and a CPU clocking in at 2.53 GHz [18].

By means of a CPU with similar performances (2.4 GHz), we found that the generation of $p = 21$ terms for our approximate solution, in this example with $N = 17$, takes 0.016 s.

6. Conclusions

In this note, a Picard-like numerical scheme, formerly employed to solve linear integro–differential equations, has been extended to handle fuzzy initial value problems involving Riemann–Liouville fuzzy fractional derivatives. In the linear regime, the technique proposed is a non-recursive scheme. The convergence of the method has been formally discussed. Some example applications show the effectiveness of the proposed approach. Future research activities will focus on the application of the method to long–term simulation of multi-degree-of-freedom fractional systems, moving towards piecewise solutions as in [27].

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