INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 6, Pages 926–950

CONVERGENCE OF DISCONTINUOUS FINITE VOLUME DISCRETIZATIONS FOR A SEMILINEAR HYPERBOLIC OPTIMAL CONTROL PROBLEM

RUCHI SANDILYA AND SARVESH KUMAR

Abstract. In this paper, we discuss discontinuous finite volume approximations of the distributed optimal control problems governed by a class of semilinear hyperbolic partial differential equations with control constraints. The spatial discretization of the state and costate variables follows discontinuous finite volume schemes with piecewise linear elements, whereas three different strategies are used for the control approximation: variational discretization, piecewise constant and piecewise linear discretization. As the resulting semi-discrete optimal system is non-symmetric, we have employed *optimize then discretize* approach to approximate the control problem. *A priori* error estimates for control, state and costate variables are derived in suitable natural norms. The present analysis is an extension of the analysis given in Kumar and Sandilya [Int. J. Numer. Anal. Model. (2016), 13: 545-568]. Numerical experiments are presented to illustrate the performance of the proposed scheme and to confirm the predicted accuracy of the theoretical convergence rates.

Key words. Semilinear hyperbolic optimal control problems, variational discretization, piecewise constant and piecewise linear discretization, discontinuous finite volume methods, *a priori* error estimates, numerical experiments.

1. Introduction

It is well known that optimization problems governed by partial differential equations introduced in [25] have many applications in the field of science and technology. In particular, the hyperbolic optimal control problems arise in medical applications, acoustic problems as noise suppression and for optimal control in linear elasticity (cf. [1, 7, 33]). Although abundant literature is available on finite element analysis for elliptic and parabolic optimal control problems (see, e.g., [8, 10, 29, 30, 32]), there is relatively less work on hyperbolic optimal control problems (see, e.g., [15, 16, 31]). Most of these articles deal with conforming piecewise linear finite element discretizations for state and costate variables and control is discretized using piecewise constant or linear polynomials, and the rate of convergence for control is of $\mathcal{O}(h)$ and $\mathcal{O}(h^{3/2})$ when piecewise constant and linear polynomials are used, respectively. In order to improve the order of convergence, Hinze proposed a variational discretization approach for optimal control problems with control constraints in which control set is not discretized explicitly but discretized by a projection of the discrete costate variables, for details see [14]. For this new scheme, it has been shown that the rate of convergence for the control is of $\mathcal{O}(h^2)$.

Due to local conservation properties and other attractive features, finite volume methods have been extensively used for the approximation of partial differential equations obeying some conservation laws. They can also be considered as Petrov-Galerkin methods in which the finite dimensional trial and test spaces consist piecewise linear polynomials and piecewise constant functions, respectively. For more details on finite volume methods, kindly see [6, 12, 13, 24] and references

Received by the editors August 23, 2015 and, in revised form, July 27, 2016.
2000 *Mathematics Subject Classification.* 49M29, 49M05, 65M06, 65M08.

therein. Since the test space associated with the dual grid is piecewise constant, finite volume methods have some computational advantages over continuous finite element methods. Due to computational efficiency and simplicity, these methods are widely used for the approximation of linear elliptic, parabolic and hyperbolic optimal control problems (see e.g. [26, 27, 28]) and *a priori* error estimates have also been established. In these articles, variational discretization approach is used to approximate control variable and optimal order of convergence is obtained.

On the other hand, discontinuous Galerkin methods are very appealing to the scientific community because of their desirable properties like: mesh adaptivity, locally conservative, suitability for parallel computing, use of high order polynomials and no inter element continuity requirement (generally imposed on continuous and non-conforming finite element spaces) etc. A few contributions are available (cf. [11, 29, 30, 32]) which deal with discontinuous Galerkin methods for linear and semilinear parabolic optimal control problems. In order to utilize the desirable properties of both finite volume and discontinuous Galerkin methods, Ye in [35] proposed a hybrid scheme called discontinuous finite volume (DFV) methods to approximate linear elliptic problems. In DFV scheme, discontinuous piecewise linear functions are used in trial space whereas test space consists of piecewise constant functions. Later, with the appropriate modifications these methods have been applied to elliptic, parabolic and certain fluid flow problems (for details, see [3, 5, 18, 19, 20]). Recently, DFV methods have been applied to solve optimal control problems governed by elliptic [21], semilinear parabolic [22] and Brinkman [23] equations and here we extend these ideas to the case of semilinear hyperbolic optimal control problem.

For the numerical solution of optimal control problems, there are two different strategies- *optimize-then-discretize* and *discretize-then-optimize*. In *optimize-then-discretize* approach, the optimality conditions at the continuous level are formulated first and then one proceeds to the discretization step; whereas in *discretize-then-optimize* approach one first discretizes the continuous problem and then derives the optimality conditions accordingly. For non-symmetric discrete formulations, these two approaches need not coincide as they may lead to different discrete adjoint equations (see [2]). In general, finite volume element formulation is non-symmetric and the authors in [26, 27, 28] have employed *optimize-then-discretize* technique to discretize the optimal control problems. In the light of these articles and applicability of *optimize-then-discretize* approach, in this article, we will also undertake the same strategy (*optimize-then-discretize*) for the approximation of the concerned control problem.

The rest of this article is organized in the following manner. The remaining part of this section deals with some standard notations, statement of the governing problem and the corresponding optimality conditions. In Section 2, we apply DFV scheme to the considered optimal control problem and obtain its discrete formulation. Section 3 deals with *a priori* error estimates for different types of control discretization. In Section 4, we present numerical experiments to illustrate the theoretical results and performance of the method. Finally, based on theoretical and computational observations, some conclusions are drawn in Section 5.

Notations. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$ and T be a positive time that defines the time interval $I := (0, T]$. The standard notations are used for the Lebesgue spaces $L^p(\Omega)$ and the Sobolev Spaces $H^s(\Omega)$ and their associated norms $\|\cdot\|_{s,\Omega}$ and seminorms $|\cdot|_{s,\Omega}$. Also, we write $H^0(\Omega) := L^2(\Omega)$

and for simplicity we drop Ω whenever its possible. In addition, we use the notation $L^p(H^s)$, $1 \leq p \leq \infty$, $s \geq 0$, for the Banach space of all L^p integrable functions $\psi(t) : [0,T] \longrightarrow H^s(\Omega)$ with norm $\|\psi\|_{L^p(H^s)} := \left(\int_0^T \|\psi\|_{s,\Omega}^p\right)^{1/p}$ for $p \in [1,\infty)$ and standard modification at $p=\infty$. Throughout this article, C denotes a generic positive constant independent of the mesh size h (to be defined in the next section) but may depend on the size of Ω and can take different values at different places.

The optimal control problem. We consider the following distributed optimal control problem governed by a semilinear wave equation with control u and state y.

$$\min_{u \in U_{ad}} J(y,u) := \frac{1}{2}\int_0^T \left(\|y(t,x)-y_d(t,x)\|_{0,\Omega}^2 + \alpha\,\|u(t,x)\|_{0,\Omega}^2\right)\mathrm{dt}, \tag{1}$$

subject to

(2)
$$\begin{cases} \partial_{tt}y(t,x) - \nabla\cdot(\mathcal{A}\nabla y(t,x)) + \phi(y(t,x)) = Bu(t,x) + f(t,x), \quad \text{in} \quad (0,T)\times\Omega, \\ y(t,x) = 0, \quad \text{on} \quad (0,T)\times\partial\Omega, \\ y(0,x) = g(x),\ \partial_t y(0,x) = w(x), \quad x\in\Omega, \end{cases}$$

where,

$$U_{ad} := \{u(t,x) \in U = L^\infty(L^\infty) :\ a \leq u(t,x) \leq b,\ \ a.e.\,(t,x)\in(0,T)\times\Omega;\ a,b\in\mathbb{R}\}.$$

Here, $\alpha > 0$ is a given regularization parameter, B is a bounded linear operator and $\mathcal{A} = (a_{ij}(x))_{2\times 2}$ denotes a real valued, symmetric and uniformly positive definite matrix in Ω, i.e., there exists a positive constant α_0 such that

$$\xi^T\mathcal{A}\xi \geq \alpha_0\xi^T\xi, \quad \forall\xi\in\mathbb{R}^2.$$

The proposed model problem describes the optimal vibrations in a bridge with Ω as the domain of the bridge. The overall idea is to identify an additional force u acting in vertical direction giving rise to a transversal displacement y which best approximates the desired evolution y_d of transversal vibrations. In addition, for our further analysis we lay out the following assumptions on the given data. Let the desired state $y_d(t,\cdot)$ and the source term $f(t,\cdot) \in L^\infty(\Omega)$ or $H^1(\Omega)\cap L^\infty(\Omega)$ for $0 < t \leq T$. Also for nonlinear term ϕ, we impose the following assumptions given in [8, 32]. The function $\phi(t,x,y) : I\times\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is of class C^2 and its first derivative ϕ' is nonnegative. For $y=0$, ϕ and its derivatives upto second order are bounded by a positive constant. Moreover, on bounded sets, they are uniformly Lipschitz continuous with respect to y.

With the introduction of control-to-state mapping G with $G(u) = y$, the problem (1)-(2) reduces to:

$$\min_{u\in U_{ad}} j(u) := \min_{u\in U_{ad}} J(G(u),u). \tag{3}$$

Under some extra assumptions, the problem (3) exhibits at least one optimal control with associated state $y = G(u)$ (for details, see [25, 32, 34]). Due to nonlinearity of control-to-state operator the reduced objective functional need not be convex and hence the solutions may not be unique. Therefore, we will use the notion of local solution (defined in [8, 32]) in the sense of $L^2(L^2)$. A control $u \in U_{ad}$ is said to be the local solution of (3) in the sense of $L^2(L^2)$, if there exists a constant $\varepsilon > 0$ such that

$$j(u) \leq j(\tilde{u}), \quad \forall \tilde{u} \in U_{ad} \text{ with } \|\tilde{u} - u\|_{L^2(L^2)} \leq \varepsilon. \tag{4}$$

The local solution u of (3) in the sense of definition (4) satisfies the standard first order necessary optimality condition which can be formulated with the help of the following variational inequality.

$$j'(u)(\tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{ad},$$

and can be further rewritten in the form

$$(\alpha u + B^* p, \tilde{u} - u)_{L^2(L^2)} \geq 0, \quad \forall \tilde{u} \in U_{ad}. \tag{5}$$

Here, p is called the *adjoint state* (or *costate*) associated to local control u and solves the *adjoint state equation* which is given by

$$\begin{cases} \partial_{tt} p - \nabla \cdot (\mathcal{A} \nabla p) + \phi'(y) p = y - y_d, & \text{in} \quad I \times \Omega, \\ p = 0, \quad \text{in} \quad I \times \partial\Omega, \\ p(T, x) = 0, \ \partial_t p(T, x) = 0, \quad x \in \Omega. \end{cases}$$

If we use the following pointwise projection on the admissible set U_{ad},

$$P_{[a,b]} : L^2(L^2) \longrightarrow U_{ad}, \quad P_{[a,b]}(q(t,x)) = max(a, min(b, q(t,x))),$$

then the optimality condition (5) can be simplified further as

$$u(t,x) = P_{[a,b]} \left(\frac{-1}{\alpha} B^* p(t,x) \right).$$

Here, B^* is the adjoint of operator B. We also note that the projection operator $P_{[a,b]}$ satisfies the following regularity properties

$$\left\| \nabla (P_{[a,b]}(v))(t) \right\|_{L^\infty(\Omega)} \leq \|\nabla v(t)\|_{L^\infty(\Omega)}, \quad \forall v \in L^2(W^{1,\infty}), \tag{6}$$

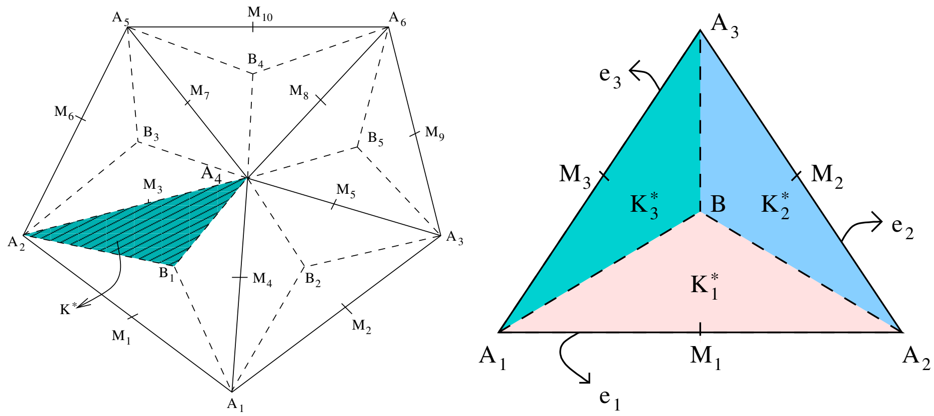
for almost all $t \in I$. We stress that with some extra regularity on the solution operator $\mathcal{G}$ the local solution $u \in U_{ad}$ also satisfies the following second order sufficient optimality condition. This assumption seems to be legitimate, for details we refer to [32] (see also [9]).

There exists a positive constant C such that

$$j''(u)(\tilde{u}, \tilde{u}) \geq C \|\tilde{u}\|^2_{L^2(L^2)}, \quad \forall \tilde{u} \in U. \tag{7}$$

2. The discrete formulation

In this section, as mentioned earlier, we will use *optimize-then-discretize* approach to approximate the continuous optimal system directly by applying the piecewise linear discontinuous finite volume schemes with three different control discretization (variational discretization, piecewise linear and constant discretization) techniques. We first introduce discontinuous finite volume methods for spatial discretization of the optimal control problem.



(a) The dual partition of a triangulation. (b) A triangular partition and its dual.

FIGURE 1.

2.1. Discontinuous finite volume scheme. Let $\mathcal{T}_h$ denotes a regular, quasi-uniform triangulation of $\bar{\Omega}$ into closed triangles K, where we define the discretization parameter h by setting $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K is the diameter of the triangle K. Moreover, we denote the set of all interior and boundary edges in $\mathcal{T}_h$ by $\mathcal{E}_h$ and $\mathcal{E}_h^{\Gamma}$, respectively. The dual partition $\mathcal{T}_h^*$ of the primal partition $\mathcal{T}_h$ is constructed the following way. We split each triangle $K \in \mathcal{T}_h$ into three sub-triangles $(K_i^*)_{i=1}^3$ by joining the barycenter B of the triangle K to its vertices as shown in the Figure 1, for more details we refer to [35]. Let $\mathcal{T}_h^*$ consists of all these dual elements/control volumes K_i^* generated by the barycentric subdivison. As depicted in Figure 1 the control volumes have support in the triangle in which they belong whereas for conforming FVE methods, the dual elements may have support in the neighboring triangles, for details see [6]. The advantages of DFV methods (in terms of small support of the control volume and other aspects of the computational issues) over the other numerical methods are clearly mentioned in [5, 35]. Now, let us denote by $P_r(K)$ or $P_r(K^*)$ the space of polynomials of degree less than or equal to r defined on the element K or K^*, respectively. Then the finite dimensional trial and test spaces associated with $\mathcal{T}_h$ and $\mathcal{T}_h^*$ are given respectively, by

$$V_h = \{v_h \in L^2(\Omega) : v_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\} \quad \text{and}$$
$$V_h^* = \{v_h \in L^2(\Omega) : v_h|_{K^*} \in P_0(K^*) \quad \forall K^* \in \mathcal{T}_h^*\}.$$

Let $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega)$. The connection between trial and test spaces is characterized by the transfer operator $\gamma : V(h) \longrightarrow V_h^*$ defined as

$$\gamma v|_{K^*} = \frac{1}{h_e}\int_e v|_{K^*} ds, \quad \forall K^* \in \mathcal{T}_h^*, \tag{8}$$

where h_e represents the length of the edge e. Some useful results satisfied by the map γ are collected in the following Lemma (for a proof, see [3, 18, 35])

Lemma 2.1. *Let γ be a transfer operator defined as in* (8). *Then*

(1) *γ satisfies the self-adjoint property with respect to the L^2-inner product, i.e.*

$$(v_h, \gamma q_h)_{0,\Omega} = (q_h, \gamma v_h)_{0,\Omega}, \quad \forall v_h, q_h \in V_h.$$

(2) *For* $v_h \in V_h$ *if* $|||v_h|||_0^2 := (v_h, \gamma v_h)$, *then the norms* $|||\cdot|||_0$ *and* $\|\cdot\|_{0,\Omega}$ *are equivalent.*

(3) γ *is stable with respect to the norm* $\|\cdot\|_{0,\Omega}$, *i.e.*

$$\|\gamma v_h\|_{0,\Omega} = \|v_h\|_{0,\Omega}, \quad \forall v_h \in V_h.$$

(4) *For all* $v \in V(h)$ *and* $K \in \mathcal{T}_h$, *we have*

$$\int_e (v - \gamma v) ds = 0; \quad \int_K (v - \gamma v) dx = 0; \quad \|v - \gamma v\|_{0,K} \leq C h_K \|v\|_{1,K}.$$

Now, let e be an interior edge shared by two elements K_1 and K_2 in $\mathcal{T}_h$, and let $\mathbf{n_1}$ and $\mathbf{n_2}$ denote unit normal vectors on e pointing outward to K_1 and K_2, respectively. Then the average $\langle\cdot\rangle$ and jump $[\![\cdot]\!]$ on e for a generic scalar q and a generic vector $\mathbf{r}$ are defined respectively by

$$\langle q \rangle = \frac{1}{2}(q_1 + q_2), \ [\![q]\!] = q_1\mathbf{n_1} + q_2\mathbf{n_2} \text{ and } \langle \mathbf{r} \rangle = \frac{1}{2}(\mathbf{r_1} + \mathbf{r_2}), \ [\![\mathbf{r}]\!] = \mathbf{r_1} \cdot \mathbf{n_1} + \mathbf{r_2} \cdot \mathbf{n_2}.$$

Here, $q_i = (q|_{\partial K_i})$, $\mathbf{r}_i = (\mathbf{r}|_{\partial K_i})$. For $e \in \mathcal{E}_h^\Gamma$ with outward normal $\mathbf{n}$ we take $\langle q \rangle = q$, $[\![q]\!] = q\mathbf{n}$, $\langle \mathbf{r} \rangle = \mathbf{r}$ and $[\![\mathbf{r}]\!] = \mathbf{r} \cdot \mathbf{n}$.

Testing (2) against γv_h, integrating by parts over control volumes, applying Gauss divergence methods and following the arguments used in [18, 19], we end up with the following formulation: Find $y_h(t, \cdot) \in V_h$ with $0 < t \leq T$ such that

$$(\partial_{tt} y_h, \gamma v_h) + A_h(y_h, v_h) + (\phi(y_h), \gamma v_h) = (Bu_h + f, \gamma v_h), \quad \forall v_h \in V_h,$$
$$y_h(0, x) = g_h(x), \quad \partial_t y_h(0, x) = w_h(x), \quad x \in \Omega,$$

where, g_h and w_h are certain approximations of $g(x)$ and $w(x)$ to be defined later and the bilinear form $A_h(\cdot, \cdot) : V_h \times V_h \longrightarrow \mathbb{R}$ is defined as (see [18])

$$\begin{aligned} A_h(v_h, q_h) \quad = \quad & -\sum_{K \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{A_{j+1}BA_j} (\mathcal{A}\nabla v_h \cdot \mathbf{n}) \gamma q_h ds + \theta \sum_{e \in \mathcal{E}_h} \int_e [\![\gamma v_h]\!] \cdot \langle \mathcal{A}\nabla q_h \rangle ds \\ & - \sum_{e \in \mathcal{E}_h} \int_e [\![\gamma q_h]\!] \cdot \langle \mathcal{A}\nabla v_h \rangle ds + \sum_{e \in \mathcal{E}_h} \int_e \frac{\alpha_d}{h_e^\beta} [\![v_h]\!] \cdot [\![q_h]\!] ds, \quad \forall v_h, q_h \in V_h, \end{aligned}$$

where, $A_4 = A_1$, (see Figure 1(b)), and α_d and β are penalty parameters independent of h. In general, $\theta \in \{-1, 0, 1\}$ correspond to SIPG, IIPG and NIPG methods, respectively, in the context of DG methods. We note that the different values of β are required for achieving the optimal rate of convergence in the L^2-norm for $\theta \neq -1$, for more details kindly see [18]. Now, we introduce the following natural mesh-dependent norm on space $V(h)$ which is naturally associated with the bilinear form $A_h(\cdot, \cdot)$:

$$|||v_h|||_h^2 := \sum_{K \in \mathcal{T}_h} |v_h|_{1,K}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-\beta} \|[\![v_h]\!]\|_{0,e}^2.$$

We have the following discrete Poincaré-Friedrichs type inequality (see [35])

$$\|v_h\| \leq C |||v_h|||_h, \qquad \forall v_h \in V_h. \tag{9}$$

In addition, we have the following results.

Lemma 2.2. *The bilinear form* $A_h(\cdot, \cdot)$ *possess the following properties:*

(1) $A_h(\cdot,\cdot)$ *is bounded and coercive with respect to* $\|\!|\cdot|\!\|_h$*, i.e. there exists* $C > 0$ *such that*

$$|A_h(v_h, q_h)| \le C \,|\!|\!| v_h |\!|\!|_h \,|\!|\!| q_h |\!|\!|_h\,, \quad \forall v_h, q_h \in V_h,$$

$$A_h(v_h, v_h) \ge C \,|\!|\!| v_h |\!|\!|_h^2\,, \qquad \forall v_h \in V_h.$$

The proof can be obtained with the help of lemma 2.1, for details kindly see [18].

(2) *For all* $v_h, q_h \in V_h$*, the following relation holds true*

$$|A_h(v_h, q_h) - A_h(q_h, v_h)| \le Ch \,|\!|\!| v_h |\!|\!|_h \,|\!|\!| q_h |\!|\!|_h\,.$$

For proof details cf. [3, 35].

(3) *Let* $\epsilon_a(v_h, q_h) := a_h(v_h, q_h) - A_h(v_h, q_h)$*. Then we have the following estimate*

$$|\epsilon_a(v_h, q_h)| \le Ch \,|\!|\!| v_h |\!|\!|_h \,|\!|\!| q_h |\!|\!|_h\,, \quad \forall\, v_h, q_h \in V_h,$$

where, the bilinear form $a_h(\cdot,\cdot)$ *is defined by*

$$a_h(v_h, q_h) = \sum_{K\in\mathcal{T}_h} \int_K \mathcal{A}\nabla v_h \cdot \nabla q_h \; dx + \theta \sum_{e\in\mathcal{E}_h} \int_e [\![v_h]\!] \cdot \langle \mathcal{A}\nabla q_h\rangle ds$$
$$- \sum_{e\in\mathcal{E}_h} \int_e [\![q_h]\!] \cdot \langle \mathcal{A}\nabla v_h\rangle ds + \sum_{e\in\mathcal{E}_h} \int_e \frac{\alpha_d}{h_e^{\beta}} [\![v_h]\!] \cdot [\![q_h]\!] ds.$$

For a proof we refer to Lemma 3.2 *of* [4].

2.2. Discretization of control. In this subsection, we describe briefly three different discretization techniques for control variable. Let U_h denote a finite dimensional subspace of $L^2(L^2)$, then the discrete admissible set for control is given by $U_{h,ad} = U_h \cap U_{ad}$.

Variational discretization. Introduced by Hinze in [14], this approach does not use explicit discretization of control variable. For this case, one simply chooses $U_h = L^2(L^2)$ and therefore, the discrete admissible space $U_{h,ad}$ coincides with U_{ad}. The discretization error with this approach will be analyzed in Section 3.2.

Piecewise linear discretization. Here, we use the similar space as for the approximation of the state and costate variables, i.e, piecewise linear subspace on triangulation. That is,

$$U_h = \{u_h(t,\cdot) \in L^2(L^2) : u_h(t,\cdot)|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h,\, t \in I\}.$$

Piecewise constant discretization. Another possibility for control discretization is to use elementwise constant functions. In this case, the discrete control space is defined by

$$U_h = \{u_h(t,\cdot) \in L^2(L^2) : u_h(t,\cdot)|_K \in P_0(K) \quad \forall K \in \mathcal{T}_h,\, t \in I\}.$$

The error estimates using piecewise linear and constant control discretization will be derived in Section 3.3. On applying DFV scheme to discretize the state and costate equations directly, the semidiscrete formulation of semilinear hyperbolic optimal control problem is given by :

Find $(y_h(t,\cdot), p_h(t,\cdot), u_h(t,\cdot)) \in V_h \times V_h \times U_{h,ad}$ with $0 < t \le T$ such that

$$(\partial_{tt} y_h, \gamma v_h) + A_h(y_h, v_h) + (\phi(y_h), \gamma v_h) = (Bu_h + f, \gamma v_h), \quad \forall v_h \in V_h, \tag{10}$$
$$y_h(0,x) = g_h(x), \quad \partial_t y_h(0,x) = w_h(x), \quad x \in \Omega,$$

$$(\partial_{tt} p_h, \gamma q_h) + A_h(p_h, q_h) + (\phi'(y_h) p_h, \gamma q_h) = (y_h - y_d, \gamma q_h), \quad \forall q_h \in V_h, \tag{11}$$

$$p_h(T,x) = 0, \quad \partial_t p_h(T,x) = 0, \quad x \in \Omega,$$

$$(\alpha u_h + B^* p_h, \tilde{u}_h - u_h)_{L^2(L^2)} \geq 0, \quad \forall \tilde{u}_h \in U_{h,ad}. \tag{12}$$

3. Error estimates

In this section, we provide *a priori* error estimates for the optimal control problem, in context of fixed local reference solution of the problem (3) which fulfills the first and second order optimality conditions. We will derive the estimates for three different control discretization approaches as mentioned earlier in Section 2.2.

3.1. Preliminaries. For a given arbitrary $\tilde{u} \in L^2(L^2)$ and $\tilde{y} = y(\tilde{u}) \in L^2(H_0^1)$, let $y_h(\tilde{u})$ and $p_h(\tilde{y})$ be the solutions of auxiliary equations

(13)

$$(\partial_{tt} y_h(\tilde{u}), \gamma v_h) + A_h(y_h(\tilde{u}), v_h) + (\phi(y_h(\tilde{u})), \gamma v_h) = (B\tilde{u} + f, \gamma v_h), \ \forall v_h \in V_h,$$
$$y_h(\tilde{u})(0,x) = g_h(x), \quad \partial_t y_h(\tilde{u})(0,x) = w_h(x), \quad x \in \Omega,$$

and

(14)

$$(\partial_{tt} p_h(\tilde{y}), \gamma q_h) + A_h(p_h(\tilde{y}), q_h) + (\phi'(\tilde{y}) p_h(\tilde{y}), \gamma q_h) = (\tilde{y} - y_d, \gamma q_h), \ \forall q_h \in V_h,$$
$$p_h(\tilde{y})(T,x) = 0, \quad \partial_t p_h(\tilde{y})(T,x) = 0, \quad x \in \Omega,$$

respectively. To avoid ambiguity, we will be using the following notations: $y_h = y_h(u_h)$, $p_h = p_h(y_h)$ and $p_h(\tilde{u}) = p_h(y_h(\tilde{u}))$. Proceeding with the similar steps involved in proving Lemma 4.1 of [28], we have the following estimates for $\tilde{u} = u$ and $\tilde{y} = y$.

Lemma 3.1. *There exists a positive constant C independent of h such that the following assertions hold*

$$\|y_h(u) - y_h\|_{L^\infty(V(h))} \leq C \|u - u_h\|_{L^2(L^2)},$$
$$\|p_h(y) - p_h\|_{L^\infty(V(h))} \leq C \|y - y_h\|_{L^2(L^2)}.$$

Proof. On subtracting (10) from (13), we have the following relation for $v_h \in V_h$

$$\begin{aligned}&(\partial_{tt} y_h(u) - \partial_{tt} y_h, \gamma v_h) + A_h(y_h(u) - y_h, v_h) + (\phi(y_h(u)) - \phi(y_h), \gamma v_h)\\ =&(B(u - u_h), \gamma v_h).\end{aligned}$$

Denoting $y_h(u) - y_h = \vartheta$ and choosing $v_h = \partial_t \vartheta$ in the above equation, we get

$$(\partial_{tt}\vartheta, \gamma\partial_t\vartheta) + A_h(\vartheta, \partial_t\vartheta) + (\phi(y_h(u)) - \phi(y_h), \gamma\partial_t\vartheta) = (B(u - u_h), \gamma\partial_t\vartheta).$$

Using the self-adjoint property of γ from Lemma 2.1 and rearranging the terms we can easily obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[(\partial_t\vartheta, \gamma\partial_t\vartheta) + A_h(\vartheta, \vartheta)\right] =&(B(u - u_h), \gamma\partial_t\vartheta) + \frac{1}{2}[A_h(\partial_t\vartheta, \vartheta) - A_h(\vartheta, \partial_t\vartheta)]\\ &- (\phi(y_h(u)) - \phi(y_h), \gamma\partial_t\vartheta).\end{aligned}$$

Integrating above equation from 0 to t and taking into account that $\vartheta(0,x) = 0$ and $\partial_t\vartheta(0,x) = 0$

$$|\!|\!|\partial_t\vartheta|\!|\!|_0^2 + A_h(\vartheta, \vartheta) = 2\int_0^t (B(u - u_h), \gamma\partial_t\vartheta) d\tau + \int_0^t [A_h(\partial_t\vartheta, \vartheta) - A_h(\vartheta, \partial_t\vartheta)] d\tau$$

$$+\,2\int_0^t (\phi(y_h)-\phi(y_h(u)),\gamma\partial_t\vartheta)d\tau.$$

The equivalence of the norms $|||\cdot|||_0$ and $\|\cdot\|_{0,\Omega}$ and the ellipticity of $A_h(\cdot,\cdot)$ gives

$$\|\partial_t\vartheta\|_{0,\Omega}^2+|||\vartheta|||_h^2 \le C\int_0^t (B(u-u_h),\gamma\partial_t\vartheta)d\tau + C\int_0^t [A_h(\partial_t\vartheta,\vartheta)-A_h(\vartheta,\partial_t\vartheta)]d\tau$$

$$+\,C\int_0^t (\phi(y_h)-\phi(y_h(u)),\gamma\partial_t\vartheta)d\tau. \tag{15}$$

Since B is a continuous linear operator, the first term of (15) can be bounded as

$$\begin{aligned}\int_0^t (B(u-u_h),\gamma\partial_t\vartheta)d\tau &\le \int_0^t C\,\|u-u_h\|_{0,\Omega}\,\|\partial_t\vartheta\|_{0,\Omega}\,d\tau \\ &\le C\int_0^t \|u-u_h\|_{0,\Omega}^2\,d\tau + C\int_0^t \|\partial_t\vartheta\|_{0,\Omega}^2\,d\tau.\end{aligned} \tag{16}$$

Applying the estimate of result (2) as mentioned in Lemma 2.2 and standard inverse estimate we obtain

(17)

$$\begin{aligned}\int_0^t |A_h(\partial_t\vartheta,\vartheta)-A_h(\vartheta,\partial_t\vartheta)|d\tau &\le \int_0^t Ch\,|||\vartheta|||_h\,|||\partial_t\vartheta|||_h\,d\tau \le \int_0^t C\,|||\vartheta|||_h\,\|\partial_t\vartheta\|_{0,\Omega}\,d\tau \\ &\le C\int_0^t |||\vartheta|||_h^2\,d\tau + C\int_0^t \|\partial_t\vartheta\|_{0,\Omega}^2\,d\tau.\end{aligned}$$

From the Lipschitz continuity of nonlinear term $\phi(\cdot)$, the property (3) of γ in Lemma 2.1 and the inequality (9), we can get

(18)

$$\begin{aligned}\int_0^t (\phi(y_h)-\phi(y_h(u)),\gamma\partial_t\vartheta)d\tau &\le \int_0^t C\,\|\vartheta\|_{0,\Omega}\,\|\gamma\partial_t\vartheta\|_{0,\Omega}\,d\tau \le C\int_0^t |||\vartheta|||_h\,\|\partial_t\vartheta\|_{0,\Omega}\,d\tau \\ &\le C\int_0^t |||\vartheta|||_h^2\,d\tau + C\int_0^t \|\partial_t\vartheta\|_{0,\Omega}^2\,d\tau.\end{aligned}$$

Inserting the estimates of (16), (17) and (18) in (15) we find that

$$\|\partial_t\vartheta\|_{0,\Omega}^2+|||\vartheta|||_h^2 \le C\int_0^t \|u-u_h\|_{0,\Omega}^2\,d\tau + C\int_0^t \left(\|\partial_t\vartheta\|_{0,\Omega}^2+|||\vartheta|||_h^2\right)d\tau. \tag{19}$$

The application of Gronwall's Lemma in (19) implies

$$\|\partial_t\vartheta\|_{0,\Omega}^2+|||\vartheta|||_h^2 \le C\int_0^T \|u-u_h\|_{0,\Omega}^2\,d\tau = C\,\|u-u_h\|_{L^2(L^2)}^2$$

which further leads to the first required result

$$\|y_h(u) - y_h\|_{L^\infty(V(h))} \leq C \|u - u_h\|_{L^2(L^2)} .$$

For the second result, we proceed similarly by subtracting (11) from (14), denoting $p_h(y) - p_h = \eta$ and choosing $q_h = \partial_t \eta$ to get

$$(\partial_{tt}\eta, \gamma\partial_t\eta) + A_h(\eta, \partial_t\eta) + (\phi'(y)p_h(y) - \phi'(y_h)p_h, \gamma\partial_t\eta) = (y - y_h, \gamma\partial_t\eta).$$

Following the similar arguments used previously, we can obtain the relation

$$\begin{aligned}\|\partial_t\eta\|_{0,\Omega}^2 + \|\eta\|_h^2 \leq & C\int_0^t (y - y_h, \partial_t\eta)d\tau + C\int_0^t [A_h(\partial_t\eta, \eta) - A_h(\eta, \partial_t\eta)]d\tau \\ & - C\int_0^t (\phi'(y)p_h(y) - \phi'(y_h)p_h, \gamma\partial_t\eta)d\tau.\end{aligned}$$

Using the result (2) of Lemma 2.2, inverse estimate and monotonicity of nonlinear term we can readily obtain

$$\|\partial_t\eta\|_{0,\Omega}^2 + \|\eta\|_h^2 \leq C\int_0^t \|y - y_h\|_{0,\Omega}^2 \, d\tau + C\int_0^t \left(\|\partial_t\eta\|_{0,\Omega}^2 + \|\eta\|_h^2\right) d\tau,$$

which on application of Gronwall's Lemma yields

$$\|p_h(y) - p_h\|_{L^\infty(V(h))} \leq C \|y - y_h\|_{L^2(L^2)} .$$

□

For our forthcoming analysis, we would need the following assertion which can be easily proved by using the similar arguments used in the proof of Lemma 2.1 in [17]. Therefore, we provide a sketch of the proof.

Lemma 3.2. *There exists a positive constant C independent of h such that the following relation holds:*

$$\|\partial_{tt}(y_h(u) - y_h)\|_{L^\infty(L^2)} \leq C \|\partial_t(u - u_h)\|_{L^2(L^2)} ,$$

Proof. Differentiating (2) with respect to t and multiplying by γv_h, we can obtain the following relation by employing discrete state equation for y_h and $y_h(u)$

$$\begin{aligned}&(\partial_{ttt}y_h(u) - \partial_{ttt}y_h, \gamma v_h) + A_h(\partial_t(y_h(u) - y_h), v_h) \\ &+ (\phi'(y_h(u))\partial_t y_h(u) - \phi'(y_h)\partial_t y_h, \gamma v_h) \\ =&(B\partial_t(u - u_h), \gamma v_h).\end{aligned}$$

Denoting $y_h(u) - y_h = \mu$ and choosing $v_h = \partial_{tt}\mu$ in the above equation, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[(\partial_{tt}\mu, \gamma\partial_{tt}\mu) + A_h(\partial_t\mu, \partial_t\mu)\right] \\ =&(B\partial_t(u - u_h), \partial_{tt}\mu) + \frac{1}{2}[A_h(\partial_{tt}\mu, \partial_t\mu) - A_h(\partial_t\mu, \partial_{tt}\mu)] \\ &- (\phi'(y_h(u))\partial_t y_h(u) - \phi'(y_h)\partial_t y_h, \gamma\partial_{tt}\mu).\end{aligned}$$

Integrating from 0 to t and using the estimate (2) of Lemma 2.2 and monotonicity of nonlinear term, we can obtain

$$\|\partial_{tt}\mu\|_{0,\Omega}^2 + |||\partial_t\mu|||_h^2 \le C\int_0^t \|\partial_t(u-u_h)\|_{0,\Omega}^2\,d\tau + C\int_0^t \left(\|\partial_{tt}\mu\|_{0,\Omega}^2 + |||\partial_t\mu|||_h^2\right)d\tau.$$

Using Gronwall's Lemma we find that

$$\|\partial_{tt}(y_h(u)-y_h)\|_{L^\infty(L^2)} \le C\,\|\partial_t(u-u_h)\|_{L^2(L^2)}\,.$$

□

Now, let us define the Ritz projection operator $R_h : H_0^1(\Omega) \to V_h$ by

$$A_h(R_h y, \chi_h) = A_h(y,\chi_h), \quad \forall \chi_h \in V_h.$$

With the help of the Ritz projection defined above and the analogous steps involved in the proof of Theorem 4.1 of [17], for a given $\tilde{u}$ one can easily obtain the following estimates.

Theorem 3.3. *Let us assume $g_h(x) = R_h g(x)$ and $w_h(x) = R_h w(x)$. Then there exists a positive constant C independent of h such that*

$$\|y(\tilde{u}) - y_h(\tilde{u})\|_{L^2(L^2)} \le Ch^2,\ \|p(\tilde{y}) - p_h(\tilde{y})\|_{L^2(L^2)} \le Ch^2,$$
$$\|p(\tilde{u}) - p_h(\tilde{u})\|_{L^2(L^2)} \le Ch^2.$$

In particular, for $\tilde{u} = u_h$ we have

$$\|p(u_h) - p_h(u_h)\|_{L^2(L^2)} \le Ch^2. \tag{20}$$

3.2. Error analysis with variational discretization approach. Since in this approach the control space is not discretized explicitly, we choose $U_h = L^2(L^2)$ and thus $U_{h,ad} = U_{ad}$. Using the coercivity of j'' and proceeding in the same manner as in the proof of [22, Theorem 3.3], we have the following theorem dealing with the error estimates for control, state and costate variables.

Theorem 3.4. *Let u be a fixed local control of the problem* (3) *with associated state y and costate p and let (u_h, y_h, p_h) be their DFV approximations with variational discretization approach, then the following results hold true*

$$\|u-u_h\|_{L^2(L^2)} \le Ch^2,\ \|y-y_h\|_{L^2(L^2)} \le Ch^2,\ \|p-p_h\|_{L^2(L^2)} \le Ch^2.$$

Proof. Employing the discrete and continuous variational inequalities with variational discretization approach, we can obtain

$$(\alpha u_h + B^* p_h, u-u_h)_{L^2(L^2)} \ge 0 \ge (\alpha u + B^* p, u-u_h)_{L^2(L^2)}. \tag{21}$$

Applying second order sufficient condition (7) for $u - u_h \in U$, we have

$$C\,\|u-u_h\|_{L^2(L^2)}^2 \le j'(u)(u-u_h) - j'(u_h)(u-u_h)$$
$$= (\alpha u + B^* p, u-u_h)_{L^2(L^2)} - (\alpha u_h + B^* p(u_h), u-u_h)_{L^2(L^2)}.$$

We use (21) in the above relation to get

$$C\,\|u-u_h\|_{L^2(L^2)}^2 \le (\alpha u_h + B^* p_h, u-u_h)_{L^2(L^2)} - (\alpha u_h + B^* p(u_h), u-u_h)_{L^2(L^2)},$$

which on application of (20) gives the required estimate for control with variational discretization

$$\|u-u_h\|_{L^2(L^2)} \le C\,\|p(u_h) - p_h\|_{L^2(L^2)} \le Ch^2. \tag{22}$$

Now, to estimate the error for state and costate we split $y - y_h = y - y_h(u) + y_h(u) - y_h$ and $p - p_h = p - p_h(y) + p_h(y) - p_h$. Then we apply triangle inequality, estimate (22) alongwith the results of Lemma 3.1 and Theorem 3.3 to obtain

$$\|y - y_h\|_{L^2(L^2)} \leq Ch^2 \quad \text{and} \quad \|p - p_h\|_{L^2(L^2)} \leq Ch^2.$$

□

3.3. Error analysis for control with explicit discretization. In this subsection, we will analyze the error estimates with two different discretization (piecewise linear and piecewise constant) techniques for control in space. We first lay out some assumptions on the structure of the active sets. Let $\tilde{u}_h(t,x)$ be a function in discrete admissible set $U_{h,ad}$ defined on an arbitrary triangle $K \in \mathcal{T}_h$ for $0 < t \leq T$ by

$$\tilde{u}_h(t,x) = \begin{cases} a & \text{if} \min_{x \in K} u(t,x) = a, \\ b & \text{if} \max_{x \in K} u(t,x) = b, \\ \tilde{I}_h u & \text{else}. \end{cases} \tag{23}$$

Here, $\tilde{I}_h u$ represents Lagrange interpolate of u. To avoid confusion, the mesh size h is chosen sufficiently small so that $\min_{x \in K} u(t,x) = a$ and $\max_{x \in K} u(t,x) = b$ cannot happen simultaneously in the same element K. Now, the elements $K \in \mathcal{T}_h$ are grouped into three sets $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2 \cup \mathcal{T}_h^3$ with $\mathcal{T}_h^i \cap \mathcal{T}_h^j = \emptyset$ for $i \neq j$ according to the value of $u(t,x)$ on K. These sets are defined as follows:

$$\begin{aligned} \mathcal{T}_h^1 &= \{K \in \mathcal{T}_h : u(t,x) = a \quad \textit{or} \quad u(t,x) = b \quad \forall x \in K\}, \\ \mathcal{T}_h^2 &= \{K \in \mathcal{T}_h : a < u(t,x) < b \quad \forall x \in K\}, \\ \mathcal{T}_h^3 &= \mathcal{T}_h \setminus (\mathcal{T}_h^1 \cup \mathcal{T}_h^2). \end{aligned}$$

As shown in [10, Lemma 2.1], with the help of definition (23) it follows that for any $\tilde{u}_h \in U_{h,ad}$ we have

$$(\alpha u + B^* p, \tilde{u} - \tilde{u}_h)_{L^2(L^2)} \geq 0 \quad \forall \tilde{u} \in U_{ad}. \tag{24}$$

For our subsequent analysis, we will exploit the following assumption
Assumption 1. There exists a positive constant C independent of h such that

$$\sum_{K \in \mathcal{T}_h^3} |K| \leq Ch. \tag{25}$$

Now we state the following assertion which will be used in deriving the error estimates for control with piecewise linear discretization. The proof can be found in [30].

Lemma 3.5. *Let u be a local control of the optimization problem* (3)*. Then, under the assumption* (25)*, the following estimate holds, provided $p \in L^2(W^{1,\infty})$:*

$$|(\alpha u + B^* p, \tilde{u}_h - u)_{L^2(L^2)}| \leq \frac{C}{\alpha} h^3 \|\nabla p\|^2_{L^2(L^\infty)}, \quad \forall \tilde{u}_h \in U_{h,ad}.$$

To this end, we would like to mention that the proof of Theorem 3.6 is analogous to the proof of [22, Theorem 3.5]; however, for the sake of completeness, we only give main ideas of the proof. Here, again the key idea is to make use of the coercivity of j'' and Lemma 3.5.

Theorem 3.6. *Let u be a local optimal control of the problem* (3) *and u_h be the solution of the discrete problem* (10)-(12) *with piecewise linear control discretization technique, then we have the following convergence result.*

$$\|u - u_h\|_{L^2(L^2)} \leq Ch^{3/2}.$$

Proof. From the continuous and discrete variational inequality, we have the relation

$$(\alpha u_h + B^* p_h, \tilde{u}_h - u_h)_{L^2(L^2)} \geq 0 \geq (\alpha u + B^* p, u - u_h)_{L^2(L^2)}. \tag{26}$$

On application of condition (7) for $u - u_h \in U$, we get

$$C \|u - u_h\|^2_{L^2(L^2)} \leq (\alpha u + B^* p, u - u_h)_{L^2(L^2)} - (\alpha u_h + B^* p(u_h), u - u_h)_{L^2(L^2)}.$$

Now using the result (26) in the above relation we can obtain

$$\begin{aligned} C \|u - u_h\|^2_{L^2(L^2)} \leq &(\alpha u_h + B^* p_h, \tilde{u}_h - u_h)_{L^2(L^2)} - (\alpha u_h + B^* p(u_h), u - u_h)_{L^2(L^2)} \\ \leq &(B^*(p_h - p(u_h)), u - u_h)_{L^2(L^2)} \\ &+ (\alpha(u - u_h) + B^*(p - p_h), u - \tilde{u}_h)_{L^2(L^2)} \\ &+ (\alpha u + B^* p, \tilde{u}_h - u)_{L^2(L^2)}, \end{aligned}$$

and therefore, by applying Cauchy-Schwarz inequality, using (20), property (9) and results of Lemma 3.1 and Theorem 3.3, we get

$$\begin{aligned} \|u - u_h\|^2_{L^2(L^2)} \leq &Ch^2 \|u - u_h\|_{L^2(L^2)} + \|u - u_h\|_{L^2(L^2)} \|u - \tilde{u}_h\|_{L^2(L^2)} \\ &+ |(\alpha u + B^* p, \tilde{u}_h - u)_{L^2(L^2)}|. \end{aligned} \tag{27}$$

To estimate the term $\|u - \tilde{u}_h\|_{L^2(L^2)}$ we use the definition 23 on the sets $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$ alongwith the projection property (6) and assumption 25. For details, we refer to [22, Theorem 3.5].

$$\|u - \tilde{u}_h\|_{L^2(L^2)} \leq \frac{C}{\alpha} \left(h^2 \left\|\nabla^2 p\right\|_{L^2(L^2)} + h^{3/2} \|\nabla p\|_{L^2(L^\infty)} \right). \tag{28}$$

Using the above estimate (28) in (27), applying Young's inequality and Lemma 3.5 we can obtain $\|u - u_h\|_{L^2(L^2)} \leq Ch^{3/2}$. □

Next, we will establish the error estimate for the control variable u in the L^2-norm, i.e., $\|u - u_h\|_{L^2(L^2)}$ when the control is discretized by piecewise constant polynomials in space. In order to accomplish this purpose, we follow the analysis of [10] and again utilize the coercivity of j''. We start by introducing an L^2-projection operator $\Pi_0 : U \longrightarrow U_h$ with the following approximation property: There exists a positive constant C independent of h such that

$$\|\tilde{u} - \Pi_0 \tilde{u}\|_{0,K} \leq Ch \|\tilde{u}\|_{1,K}. \tag{29}$$

Now, the error estimate is formulated as follows and the main tools of the proof are borrowed from the [22, Theorem 3.6]. Therefore, we present the outline of the proof.

Theorem 3.7. *Let u be a local optimal control of the problem* (3) *and u_h be the solution of the discrete problem* (10)-(12) *with piecewise constant control discretization technique, then the following convergence result holds.*

$$\|u - u_h\|_{L^2(L^2)} \leq Ch.$$

Proof. Due to the fact that $\Pi_0 U_{ad} \subset U_{h,ad}$, the continuous and discrete variational inequalities imply the relation

$$(\alpha u_h + B^* p_h, \Pi_0 u - u_h)_{L^2(L^2)} \geq 0 \geq (\alpha u + B^* p, u - u_h)_{L^2(L^2)}. \tag{30}$$

Applying second order sufficient condition (7) for $u - u_h \in U$, we get

$$C \|u - u_h\|^2_{L^2(L^2)} \leq (\alpha u + B^* p, u - u_h)_{L^2(L^2)} - (\alpha u_h + B^* p(u_h), u - u_h)_{L^2(L^2)}.$$

Utilizing (30) in the above relation we can obtain

$$\begin{aligned} C \|u - u_h\|^2_{L^2(L^2)} \leq& (\alpha u_h + B^* p_h, \Pi_0 u - u_h)_{L^2(L^2)} - (\alpha u_h + B^* p(u_h), u - u_h)_{L^2(L^2)} &(31)\\ \leq& (B^*(p_h - p(u_h)), u - u_h)_{L^2(L^2)} + (\alpha u_h + B^* p_h, \Pi_0 u - u)_{L^2(L^2)} &(32)\\ =& J_1 + J_2. \end{aligned}$$

To bound the first term, we use (20) and continuity of operator B to get

$$J_1 \leq C \|p(u_h) - p_h\|_{L^2(L^2)} \|u - u_h\|_{L^2(L^2)} \leq Ch^2 \|u - u_h\|_{L^2(L^2)}.$$

Using orthogonal property of projection Π_0 and (29) to bound the second term, we arrive at

$$\begin{aligned} J_2 &= (B^* p_h - \Pi_0(B^* p_h), \Pi_0 u - u)_{L^2(L^2)} \\ &\leq \|B^* p_h - \Pi_0(B^* p_h)\|_{L^2(L^2)} \|\Pi_0 u - u\|_{L^2(L^2)} \\ &\leq Ch^2 \|p_h\|_{L^2(V(h))} \|u\|_{L^2(H^1)}. \end{aligned}$$

Now, we need to show the uniform boundedness of p_h. Testing the discrete state equation (10) for $v_h = \partial_t y_h$, employing the coercivity of $A_h(\cdot,\cdot)$, estimate (2) of Lemma 2.2, properties of nonlinear term and applying Gronwall's inequality, we can obtain the bound

$$\|y_h\|_{L^2(V(h))} \leq C \left(\|u_h\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} \right). \tag{33}$$

Similarly, from the discrete costate equation, we can obtain

$$\|p_h\|_{L^2(V(h))} \leq C \left(\|y_h\|_{L^2(L^2)} + \|y_d\|_{L^2(L^2)} \right). \tag{34}$$

The uniform boundedness of p_h can be achieved from (33) and (34) by utilizing property (9) and the fact that $U_{h,ad}$ is uniformly bounded. Finally, plugging the bounds of the terms of (31), we can obtain the desired estimate. ☐

3.4. Error analysis for state and costate with explicit control discretization. As seen before for variational discretization approach one can obtain optimal convergence order for state and costate error without much difficulty. But if we follow analogously with the piecewise constant or linear control discretization techniques then we end up with suboptimal order of convergence. Using a more detailed analysis we can overcome this difficulty and obtain optimal convergence of $\mathcal{O}(h^2)$ for these two different schemes. For both choices of the space U_h (piecewise constant and linear) as described in Section 2.2 the following results hold.

Theorem 3.8. *Let u be a fixed local control of the problem* (3) *with associated state y and costate p and let (u_h, y_h, p_h) be their DFV approximations, then we have*

$$\|y - y_h\|_{L^2(L^2)} \leq Ch^2, \quad \|p - p_h\|_{L^2(L^2)} \leq Ch^2.$$

Proof. Splitting the error $y - y_h = y - y_h(u) + y_h(u) - y_h(\Pi_h u) + y_h(\Pi_h u) - y_h$ and applying triangle inequality we can write

$$\|y - y_h\|_{L^2(L^2)} \leq \|y - y_h(u)\|_{L^2(L^2)} + \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} + \|y_h(\Pi_h u) - y_h\|_{L^2(L^2)}. \tag{35}$$

Here, Π_h denotes the L^2 projection operator onto U_h. Now, let us assume that $\tilde{p}_h(t,\cdot) \in V_h$, $(0 < t \leq T)$ be the solution of auxiliary discrete dual equation

$$(\xi, \partial_{tt}\tilde{p}_h) + a_h(\tilde{p}_h, \xi) = (\xi, y_h(u) - y_h(\Pi_h u)) - (\xi, \hat{\phi}\tilde{p}_h), \quad \forall \xi \in V_h \tag{36}$$
$$\tilde{p}_h(T,x) = 0, \quad \partial_t \tilde{p}_h(T,x) = 0$$

with

$$\hat{\phi}(t,x) = \begin{cases} \frac{\phi(y_h(u)) - \phi(y_h(\Pi_h u))}{y_h(u) - y_h(\Pi_h u)}, & \text{if} \quad y_h(u) \neq y_h(\Pi_h u)) \\ 0, & \text{else.} \end{cases}$$

We note that $\left\|\hat{\phi}\right\|_{L^\infty(L^\infty)} \leq c$ for a $c > 0$ due to boundedness of U_{ad}. Choosing $\xi = \partial_t \tilde{p}_h$ in (36), using coercivity of $a_h(\cdot,\cdot)$ and applying Gronwall's Lemma, it is easy to check that the following result holds

$$\|\tilde{p}_h\|_{L^\infty(V(h))} \leq C \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}. \tag{37}$$

Testing (36) against $\xi = y_h(u) - y_h(\Pi_h u)$, we find that

$$(y_h(u) - y_h(\Pi_h u), \partial_{tt}\tilde{p}_h) + a_h(\tilde{p}_h, y_h(u) - y_h(\Pi_h u)) \tag{38}$$
$$= \|y_h(u) - y_h(\Pi_h u))\|_{0,\Omega}^2 - (\phi(y_h(u)) - \phi(y_h(\Pi_h u)), \tilde{p}_h).$$

Employing the discrete state equation for $y_h(u)$ and $y_h(\Pi_h u)$, we have

$$(\partial_{tt}(y_h(u) - y_h(\Pi_h u)), \gamma\tilde{p}_h) + A_h(y_h(u) - y_h(\Pi_h u), \tilde{p}_h) \tag{39}$$
$$=(B(u - \Pi_h u), \gamma\tilde{p}_h) - (\phi(y_h(u)) - \phi(y_h(\Pi_h u)), \gamma\tilde{p}_h).$$

Subtracting (39) from (38), integrating from 0 to T and rearranging the terms we can obtain

$$\begin{aligned} &\|y_h(u) - y_h(\Pi_h u))\|_{L^2(L^2)}^2 \\ &= \int_0^T (B(u - \Pi_h u), \gamma\tilde{p}_h) d\tau + \int_0^T (\partial_{tt}(y_h(u) - y_h(\Pi_h u)), \tilde{p}_h - \gamma\tilde{p}_h) d\tau \\ &+ \int_0^T \epsilon_a(y_h(u) - y_h(\Pi_h u), \tilde{p}_h) d\tau + \int_0^T (\phi(y_h(u)) - \phi(y_h(\Pi_h u)), \tilde{p}_h - \gamma\tilde{p}_h) d\tau \\ &= S_1 + S_2 + S_3 + S_4. \end{aligned} \tag{40}$$

Using the property of projection Π_h, results of Lemma 2.1 and (37) we can get

$$\begin{aligned} S_1 &= \int_0^T \left[(B(u - \Pi_h u), \gamma\tilde{p}_h - \tilde{p}_h) + (u - \Pi_h u, B^*\tilde{p}_h - \Pi_h B^*\tilde{p}_h) \right] d\tau \\ &\leq Ch \|u - \Pi_h u\|_{L^2(L^2)} \|\tilde{p}_h\|_{L^2(V(h))} \\ &\leq Ch \|u - \Pi_h u\|_{L^2(L^2)} \|\tilde{p}_h\|_{L^\infty(V(h))} \\ &\leq Ch \|u - \Pi_h u\|_{L^2(L^2)} \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}. \end{aligned}$$

Following similar steps as in the proof of Lemma 3.1 and Lemma 3.2, it is easy to establish

$$\|y_h(u) - y_h(\Pi_h u)\|_{L^\infty(V(h))} \leq \|u - \Pi_h u\|_{L^2(L^2)}, \tag{41}$$

$$\|\partial_{tt}(y_h(u) - y_h(\Pi_h u))\|_{L^\infty(L^2)} \leq \|\partial_t(u - \Pi_h u)\|_{L^2(L^2)}. \tag{42}$$

Using the approximation property of γ as mentioned in Lemma 2.1, above result (42) and result (37) readily gives

$$\begin{aligned} S_2 &\leq Ch \|\partial_{tt}(y_h(u) - y_h(\Pi_h u))\|_{L^2(L^2))} \|\tilde{p}_h\|_{L^2(V(h))} \\ &\leq Ch \|\partial_{tt}(y_h(u) - y_h(\Pi_h u))\|_{L^\infty(L^2)} \|\tilde{p}_h\|_{L^\infty(V(h))} \\ &\leq Ch \|\partial_t(u - \Pi_h u)\|_{L^2(L^2)} \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}. \end{aligned}$$

From the estimate of $\epsilon_a(\cdot,\cdot)$ in Lemma 2.2 and using (37), one can obtain

$$\begin{aligned} S_3 &\leq Ch \|y_h(u) - y_h(\Pi_h u)\|_{L^2(V(h))} \|\tilde{p}_h\|_{L^2(V(h))} \\ &\leq Ch \|y_h(u) - y_h(\Pi_h u)\|_{L^\infty(V(h))} \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} \\ &\leq Ch \|u - \Pi_h u\|_{L^2(L^2)} \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}. \end{aligned}$$

To bound S_4 we use Lipschitz continuity of nonlinear term $\phi(\cdot)$, estimate of γ and (37)

$$\begin{aligned} S_4 &\leq Ch \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} \|\tilde{p}_h\|_{L^2(V(h))} \\ &\leq Ch \|y_h(u) - y_h(\Pi_h u)\|_{L^\infty(V(h))} \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} \\ &\leq Ch \|u - \Pi_h u\|_{L^2(L^2)} \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}. \end{aligned}$$

Finally substituting the estimates for S_1, S_2, S_3 and S_4 in (40) we find that

$$\|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} \leq Ch \left(\|u - \Pi_h u\|_{L^2(L^2)} + \|\partial_t(u - \Pi_h u)\|_{L^2(L^2)} \right). \tag{43}$$

For the third term in (35), using (9) and proceeding with similar steps in the proof of Lemma 3.1 we can obtain

$$\|y_h(\Pi_h u) - y_h\|_{L^2(L^2)} \leq \|y_h(\Pi_h u) - y_h\|_{L^\infty(V(h))} \leq C \|\Pi_h u - u_h\|_{L^2(L^2)}. \tag{44}$$

Applying condition (7) for $\Pi_h u - u_h \in U_{h,ad} \subset U$, we find that

$$\begin{aligned} &C \|\Pi_h u - u_h\|^2_{L^2(L^2)} \\ &\quad\leq (\alpha \Pi_h u + B^* p(\Pi_h u), \Pi_h u - u_h)_{L^2(L^2)} - (\alpha u_h + B^* p(u_h), \Pi_h u - u_h)_{L^2(L^2)} \\ &\quad\leq \alpha \|\Pi_h u - u_h\|^2_{L^2(L^2)} - (B^* p(u_h) - B^* p(\Pi_h u), \Pi_h u - u_h)_{L^2(L^2)}. \end{aligned} \tag{45}$$

The discrete variational inequality and projection property of Π_h alongwith result (24) implies the following relation

$$\begin{aligned} &\alpha \|\Pi_h u - u_h\|^2_{L^2(L^2)} = \alpha(u - u_h, \Pi_h u - u_h)_{L^2(L^2)} \\ &\leq (B^* p_h - B^* p, \Pi_h u - u_h)_{L^2(L^2)} \\ &= (B^* p_h - B^* p(u_h), \Pi_h u - u_h)_{L^2(L^2)} + (B^* p(u_h) - B^* p(\Pi_h u), \Pi_h u - u_h)_{L^2(L^2)} \\ &\quad + (B^* p(\Pi_h u) - B^* p, \Pi_h u - u_h)_{L^2(L^2)}. \end{aligned}$$

Therefore, we can rewrite the above relation as

$$\begin{aligned} &\alpha \|\Pi_h u - u_h\|^2_{L^2(L^2)} - (B^* p(u_h) - B^* p(\Pi_h u), \Pi_h u - u_h)_{L^2(L^2)} \\ &\quad\leq (B^* p_h - B^* p(u_h), \Pi_h u - u_h)_{L^2(L^2)} + (B^* p(\Pi_h u) - B^* p, \Pi_h u - u_h)_{L^2(L^2)}. \end{aligned} \tag{46}$$

To bound the first term of (46) we use (20) and continuity of operator B to obtain the estimate

$$(B^*p_h - B^*p(u_h), \Pi_h u - u_h)_{L^2(L^2)} \leq Ch^2 \left\|\Pi_h u - u_h\right\|_{L^2(L^2)}. \tag{47}$$

Decomposing the second term of (46) as

$$\begin{aligned}(B^*p(\Pi_h u) - B^*p, \Pi_h u - u_h)_{L^2(L^2)} =& (B^*p(\Pi_h u) - B^*p_h(\Pi_h u), \Pi_h u - u_h)_{L^2(L^2)} \\ &+ (B^*p_h(\Pi_h u) - B^*p_h(u), \Pi_h u - u_h)_{L^2(L^2)} \\ &+ (B^*p_h(u) - B^*p, \Pi_h u - u_h)_{L^2(L^2)}.\end{aligned}$$

Applying the results of Lemma 3.3 in the above relation we can obtain the bound (48)

$$\begin{aligned}&(B^*p(\Pi_h u) - B^*p, \Pi_h u - u_h)_{L^2(L^2)} \\ &\leq Ch^2 \left\|\Pi_h u - u_h\right\|_{L^2(L^2)} + \left\|p_h(y_h(\Pi_h u)) - p_h(y_h(u)\right\|_{L^2(L^2)} \left\|\Pi_h u - u_h\right\|_{L^2(L^2)} \\ &\leq Ch^2 \left\|\Pi_h u - u_h\right\|_{L^2(L^2)} + \left\|y_h(\Pi_h u) - y_h(u)\right\|_{L^2(L^2)} \left\|\Pi_h u - u_h\right\|_{L^2(L^2)} \\ &\leq Ch^2 \left\|\Pi_h u - u_h\right\|_{L^2(L^2)} \\ &+ Ch \left(\left\|u - \Pi_h u\right\|_{L^2(L^2)} + \left\|\partial_t(u - \Pi_h u)\right\|_{L^2(L^2)}\right) \left\|\Pi_h u - u_h\right\|_{L^2(L^2)},\end{aligned}$$

where, the last inequality follows from the proof of Lemma 3.1 and estimate (43). Now, we use the estimates of (46), (47) and (48) in (45) and insert it in (44) to get

$$\left\|y_h(\Pi_h u) - y_h\right\|_{L^2(L^2)} \leq Ch \left(\left\|u - \Pi_h u\right\|_{L^2(L^2)} + \left\|\partial_t(u - \Pi_h u)\right\|_{L^2(L^2)}\right). \tag{49}$$

Putting the bounds of (43) and (49) in (40), using the estimates of Lemma 3.3 and approximation properties of $\Pi_h u$, the optimal order of convergence for state with piecewise constant or piecewise linear discretization of control can be obtained, i.e.,

$$\left\|y - y_h\right\|_{L^2(L^2)} = \mathcal{O}(h^2). \tag{50}$$

On utilizing the estimates of Lemma 3.1, Lemma 3.3 and above result (50) it is easy to derive

$$\begin{aligned}\left\|p - p_h\right\|_{L^2(L^2)} &\leq \left\|p - p_h(y)\right\|_{L^2(L^2)} + \left\|p_h(y) - p_h\right\|_{L^\infty(V(h))} \\ &\leq \left\|p - p_h(y)\right\|_{L^2(L^2)} + \left\|y - y_h\right\|_{L^2(L^2)} = \mathcal{O}(h^2).\end{aligned}$$

□

4. Numerical Experiments

In this section, we present two numerical examples to illustrate the performance of the proposed scheme applied to distributed semilinear hyperbolic optimal control problem. In order to validate the theoretical error estimates derived for control, state and costate variables, we consider the following example.

Example 4.1.

$$\min_{u \in U_{ad}} J(y,u) := \frac{1}{2}\int_0^1 \left\|(y(t,x) - y_d(t,x))\right\|_{0,\Omega}^2 dt + \frac{1}{2}\int_0^1 \left\|u(t,x)\right\|_{0,\Omega}^2 dt,$$

subject to

$$\begin{aligned}&\partial_{tt}y - \Delta y + y^3 = u + f, \quad \textit{in} \quad (0,1] \times \Omega, \\ &y(t,x) = 0, \quad \textit{on} \quad (0,1] \times \partial\Omega,\end{aligned}$$

$$y(0,x) = \sin(\pi x_1)\sin(\pi x_2),\ \partial_t y(0,x) = \sin(\pi x_1)\sin(\pi x_2),\ \ in\ \Omega.$$

Here, the space domain $\Omega = \{x = (x_1, x_2) : 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$*, the source term* f *and the desired state* y_d *are of the form*

$$\begin{aligned} f(t,x) =& (1+2\pi^2)e^t \sin(\pi x_1)\sin(\pi x_2) + e^{3t}\sin(\pi x_1)^3\sin(\pi x_2)^3 - u(t,x),\\ y_d(t,x) =& \sin(\pi x_1)\sin(\pi x_2)(e^t + 2 + 2\pi^2(t-1)^2) + 3e^{2t}(t-1)^2\sin(\pi x_1)^3\sin(\pi x_2)^3. \end{aligned}$$

To assess the experimental convergence, we would require the exact solution of the above mentioned control problem. Therefore, with the choice of the source term f *and the desired state* y_d*, the exact state* y *and the costate* p *is given in the following manner*

$$y(t,x) = e^t \sin(\pi x_1)\sin(\pi x_2), \quad p(t,x) = -(t-1)^2\sin(\pi x_1)\sin(\pi x_2).$$

Moreover, the control variable is defined as: $u(t,x) = max(0, min(1, -p(t,x)))$.

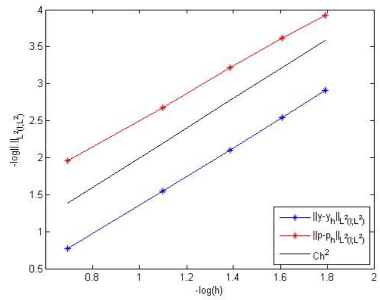
To compute the convergence of the approximate solutions we consider the partition $0 = t_0 < t_1 < ...t_M = T$ of the time interval $(0,T]$ with step size $k = \frac{T}{M}$. Let $\partial_{tt} y_h^i = \dfrac{y_h^{i+1} - 2y_h^i + y_h^{i-1}}{k^2}$ where $y_h^i = y_h(t_i, x)$, then we have the following scheme: Find $(y_h^i, p_h^i, u_h^i) \in V_h \times V_h \times U_{h,ad}$ such that for all $v_h, q_h \in V_h$

$$\begin{aligned} (\partial_{tt} y_h^i, \gamma v_h) + A_h(y_h^i, v_h) + (\phi(y_h^i), \gamma v_h) &= (Bu_h^i + f^i, \gamma v_h),\ i = 0, 1, ..., M;\\ y_h^0(x) = g_h(x), \quad \partial_t y_h^0(x) &= w_h(x), \quad x \in \Omega,\\ (\partial_{tt} p_h^i, \gamma q_h) + A_h(p_h^i, q_h) + (\phi'(y_h^i)p_h^i, \gamma q_h) &= (y_h^i - y_d^i, \gamma q_h),\ i = M, ..., 1, 0;\\ p_h^M(x) = 0, \quad \partial_t p_h^M(x) &= 0, \quad x \in \Omega,\\ (\alpha u_h^i + B^* p_h^i, \tilde{u}_h - u_h^i) &\ge 0 \quad \forall \tilde{u}_h \in U_{h,ad},\ \ i = 0, 1, ..., M. \end{aligned}$$

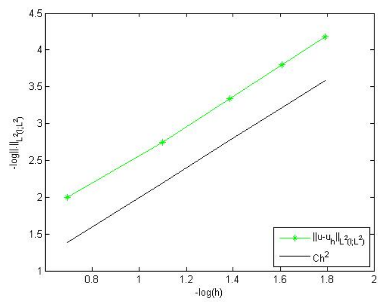
The convergence of the approximate solutions measured by the errors for optimal state, costate and control variables and corresponding observed rates are denoted by

$$\begin{aligned} e_h(y) &:= \|y - y_h\|_{L^2(L^2)}, e_h(p) := \|p - p_h\|_{L^2(L^2)}, e_h(u) := \|u - u_h\|_{L^2(L^2)},\\ r_h(y) &:= \frac{\log(e_h(y)/\hat{e}_h(y))}{\log(h/\hat{h})}, r_h(p) := \frac{\log(e_h(p)/\hat{e}_h(p))}{\log(h/\hat{h})}, r_h(u) := \frac{\log(e_h(u)/\hat{e}_h(u))}{\log(h/\hat{h})}. \end{aligned}$$

Here, e and $\hat{e}$ represent computed errors on two consecutive meshes of length h and $\hat{h}$, respectively.



(a) Convergence of state and costate.



(b) Convergence of control.

FIGURE 2. Example 4.1: The order of convergence of the errors of DFV discretization of the state, costate and control variables with variational discretization approach computed with $\theta = -1$, $\beta = 1$, $\alpha_d = 10$ and time step size $k = 0.01$.

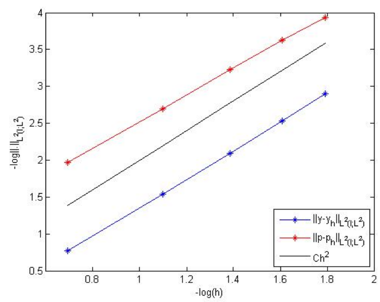
TABLE 1. Example 4.1: The development of the errors with spatial triangulation and fixed time step size $k = 0.01$ for state, adjoint state and control variables using variational discretization method for $\theta = -1$, $\beta = 1$ and $\alpha_d = 10$.

h	$e_h(y)$	$r_h(y)$	$e_h(p)$	$r_h(p)$	$e_h(u)$	$r_h(u)$
0.5000000	0.4609884	-	0.1413975	-	0.1361673	-
0.3333333	0.2128814	1.9055585	0.0688366	1.7753422	0.0640524	1.8600432
0.2500000	0.1223231	1.9259781	0.0401296	1.8757560	0.0355903	2.0426318
0.2000000	0.0787966	1.9709106	0.0268189	1.8060407	0.0224223	2.0704921
0.1666667	0.0546035	2.0116705	0.0197060	1.6903316	0.0154010	2.0602029

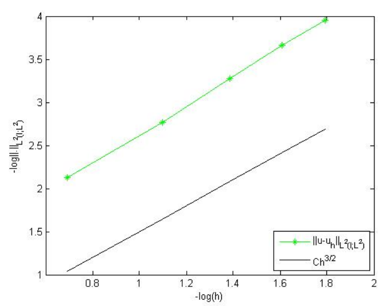
The results concerning the errors of approximation for the optimal state, costate and control variables using variational discretization approach with time step length $k = 0.01$ on a sequence of uniformly refined meshes are reported in Table 4. The corresponding convergence orders are shown in Figure 2 which match with the estimates of Theorem 3.4.

For piecewise linear discretization of control the numerical errors and convergence rates for a fixed time step $k = 0.01$ are shown in Table 4 and Figure 3, respectively which is in agreement with the theoretical results.

Finally, using the piecewise constant discretization technique for control the errors in the state, costate and control variables for refinement of space discretization and fixed time step $k = 0.01$ is listed in Table 4 and the corresponding convergence orders are shown in Figure 4, which are consistent with the results proved in the previous section.



(a) Convergence of state and costate.



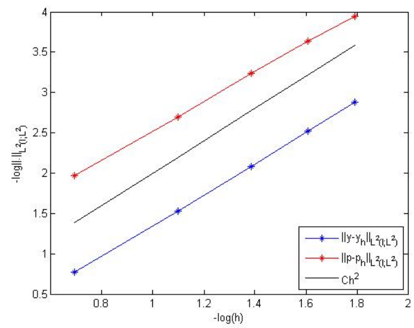
(b) Convergence of control.

FIGURE 3. Example 4.1: The order of convergence of the errors of DFV discretization of the state, adjoint state and control variables with piecewise linear discretization of control for $\theta = -1$, $\beta = 1$, $\alpha_d = 10$ and time step size $k = 0.01$.

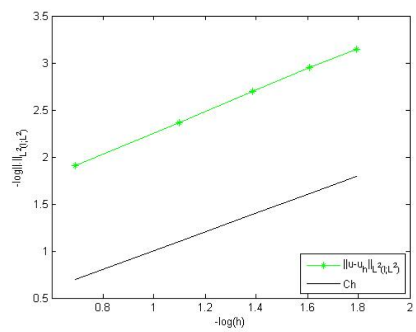
To illustrate the performance of the proposed numerical scheme we consider another example corresponding to (1)-(2) in which exact solutions are not available.

TABLE 2. Example 4.1: The development of the errors with spatial triangulation and fixed time step size $k = 0.01$ for state, adjoint state and control variables using piecewise linear discretization of control for $\theta = -1$, $\beta = 1$ and $\alpha_d = 10$.

h	$e_h(y)$	$r_h(y)$	$e_h(p)$	$r_h(p)$	$e_h(u)$	$r_h(u)$
0.5000000	0.4630198	-	0.1398337	-	0.1192073	-
0.3333333	0.2152126	1.8895423	0.0678123	1.7848868	0.0626460	1.5867297
0.2500000	0.1236614	1.9260119	0.0396571	1.8648169	0.0379108	1.7459024
0.2000000	0.0796403	1.9719417	0.0265457	1.7988475	0.0257728	1.7294516
0.1666667	0.0550646	2.0239735	0.0195618	1.6744457	0.0191732	1.6224268



(a) Convergence of state and costate.



(b) Convergence of control.

FIGURE 4. Example 4.1: The convergence order of the errors of DFV approximations of the state, costate and control variables using piecewise constant discretization of control which are computed for $\theta = -1$, $\beta = 1$, $\alpha_d = 10$ and time step $k = 0.01$.

Example 4.2. *The problem represents the optimal oscillations of a membrane which is fixed on the boundary. The domain consists of unit square and the final time* $T = 1$. *The displacement and velocity at time zero are given by the initial data* $g(x) = w(x) = (x_1^2 - x_1)(x_2^2 - x_2)$. *The applied body force* f *and the target function* y_d *are*

$$\begin{aligned} f &= e^t[(x_1^2 - x_1)(x_2^2 - x_2) - 2(x_1^2 - x_1 + x_2^2 - x_2)] + sin(e^t(x_1^2 - x_1)(x_2^2 - x_2)) \\ &\quad -max(0, min(0.8, 2(t-1)^2(x_1^2 - x_1)(x_2^2 - x_2))) \quad and \\ y_d &= (e^t + 2)(x_1^2 - x_1)(x_2^2 - x_2) \\ &\quad +(t-1)^2[(x_1^2 - x_1)(x_2^2 - x_2)cos(e^t(x_1^2 - x_1)(x_2^2 - x_2)) \\ &\quad -2(x_1^2 - x_1 + x_2^2 - x_2)], \end{aligned}$$

respectively. The nonlinear term is $\phi(y) = sin(y)$, *the control bounds are* $a = 0$, $b = 0.8$ *and the regularization parameter is* $\alpha = 0.5$.

The computed optimal control acting as a force on the membrane and the corresponding displacement at final time T with mesh size $h = 0.1$ and time step length $k = 0.01$ are depicted in Figure 5. In addition, the effect of control cost on the the minimum values of objective functional is listed in Table 4.

We close this section by making the following remarks.

Remark 4.1. *In order to approximate the time derivative, one may appeal to any finite difference schemes such as: Euler forward/backward schemes, central difference schemes etc., and the present analysis can be extended to fully discrete (considering the time discretization also) scheme in the usual manner without putting much efforts. Therefore, we refrain from doing so, as the main purpose of the paper is to analyze the performance of discontinuous finite volume methods which are used in the space discretization.*

Remark 4.2. *For our numerical experiments, we have considered* $\theta = -1$ *(SIPG). However, we have observed similar rate of convergence for the other two cases* $\theta = 1$ *(NIPG) and* $\theta = 0$ *(IIPG).*

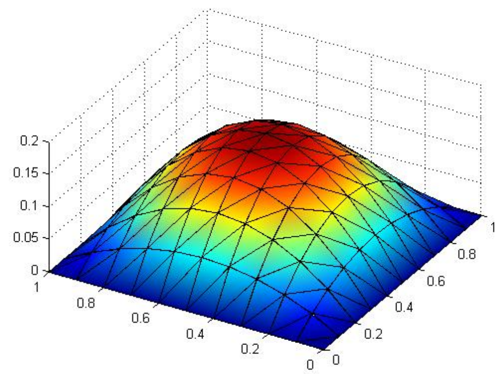
TABLE 3. Example 4.1: Computational errors with time step $k = 0.01$ for state, costate and control variables using piecewise constant discretization of control for $\theta = -1$, $\beta = 1$ and $\alpha_d = 10$.

h	$e_h(y)$	$r_h(y)$	$e_h(p)$	$r_h(p)$	$e_h(u)$	$r_h(u)$
0.5000000	0.4636564	-	0.1393451	-	0.1486513	-
0.3333333	0.2162951	1.8805558	0.0673734	1.7922693	0.0943047	1.1223461
0.2500000	0.1248519	1.9101482	0.0392757	1.8758371	0.0673654	1.1693404
0.2000000	0.0806279	1.9596488	0.0262638	1.8033765	0.0523758	1.1279186
0.1666667	0.0559691	2.0022050	0.0193140	1.6858232	0.0429525	1.0879131

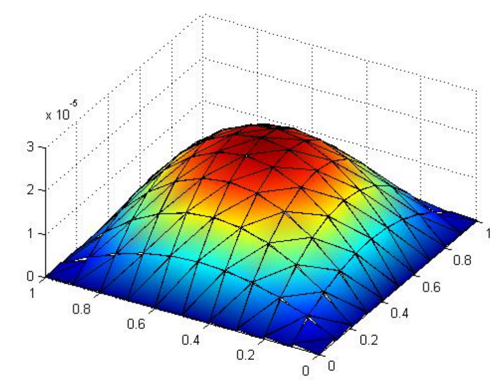
5. Concluding Remarks

In this paper, discontinuous finite volume schemes are introduced to approximate a distributed optimal control problem governed by semilinear hyperbolic equations. *A priori* error estimates in $\|\cdot\|_{L^2(L^2)}$-norm using three different control discretization techniques (variational discretization, piecewise linear and constant discretization) have been derived. We stress that the derivation of optimal error estimate of $\mathcal{O}(h^2)$ for state and costate variables with variational discretization of control, is straightforward and is achieved by decomposing the errors and using the estimate of control. However, using the similar arguments for the case of piecewise linear and piecewise constant discretization of control leads to suboptimal order of convergence for state and costate variables. In order to obtain the optimal error estimates, we have used the duality arguments.

We would also like to mention that in the case of variational discretization, we are able to derive $\mathcal{O}(h^2)$ convergence rate for control by following the standard arguments. But, we could obtain only $\mathcal{O}(h^{3/2})$ and $\mathcal{O}(h)$ convergence order for control when piecewise linear and constant discretization techniques are used, respectively. Hence, theoretically this approach have advantages over the others. However, there would be some computational difficulties with this scheme (variational discretization), and this can be explained as follows. Since here the control set is not discretized explicitly but discretized by a projection of the discrete costate variables, we observe that the discrete control does not belong to the finite dimensional space associated with mesh and hence one would need to handle nonstandard numerical algorithm and require some advanced tools in order to set the stopping criteria.



(a) The computed optimal state.



(b) The computed optimal control.

FIGURE 5. Example 4.2: The DFV approximation of optimal control and associated state with piecewise linear discretization of control for $\theta = -1$, $\beta = 1$ and $\alpha_d = 10$.

TABLE 4. Example 4.2: The values of objective functional for different regularization parameter for the DFV approximations of the semilinear hyperbolic optimal control problem.

α	0.1	0.01	0.001	0.0001	0.00001
$J(y_h, u_h)$	0.0709148	0.0703449	0.0682066	0.0677848	0.0677252

Acknowledgement. Authors thank Prof. Raju K. George (IIST, Trivandrum) for his valuable suggestion during the preparation of this manuscript and the referees for their constructive comments and suggestions, which led to a number of improvements to this manuscript. SK acknowledges the research support by the Department of Science and Technology, Government of India, through the National Programme on Differential Equations: Theory, Computation and Applications.

References

[1] H.T. Banks, S.L. Keeling and R.J. Silcox, Optimal control techniques for active noise suppression, in: Decision and Control , vol. 3, Austin, Texas, 1988, pp. 2006-2011, proceedings of the 27th Conference on Decision and Control.

[2] M. Braack, Optimal control in fluid mechanics by finite elements with symmetric stabilization, SIAM J. Control Optim. (2009), 48:672–687.

[3] C. Bi and J. Geng, Discontinuous finite volume element method for parabolic problems, Numer. Methods Part. Diff. Eq. (2010), 26:367–383.

[4] C. Bi and M. Liu, A Discontinuous finite volume element method for second-order elliptic problems, Numer. Methods Part. Diff. Eq. (2012), 28:425–440.

[5] R. Bürger, S. Kumar, R. Ruiz-Baier, Discontinuous finite volume element discretization for coupled flow-transport problems arising in models of sedimentation, J. Comput. Phys. (2015), 299:446–471.

[6] Z. Cai, On the finite volume element method, Numer. Math. (1991), 58:713–735.

[7] C. Clason, B. Kaltenbacher, and S. Veljović, Boundary optimal control of the Westervelt and the Kuznetsov equation, J. Math. Anal. Appl. (2009), 356:738–751.

[8] E. Casas and F. Tröltzsch, Error estimates for the finite-element approximation of a semilinear elliptic control problem, Control and Cybernetics (2002), 31:695–712.

[9] E. Casas and F. Tröltzsch, Second order optimality conditions and their role in PDE control, Jahresber. Dtsch. Math. Ver. (2015), 117:3–44.

[10] E. Casas and F. Tröltzsch, Error estimates for linear-quadratic elliptic control problems, IFIP: Anal. and Optimization of Diff. Systems (2003), 121:89–100.

[11] K. Chrysafinos, Convergence of discontinuous Galerkin approximations of an optimal control problem associated to semilinear parabolic pde's, ESIAM: M2AN (2009), electronically publishes, DOI10.1051/ m2an/ 2009046.

[12] S.H. Chou, D.Y. Kwak and Q. Li, L^p error estimates and superconvergence for covolume or finite volume element methods, Numer. Meth. Part. Diff. Eqns. (2003), 19:463–486.

[13] R.E. Ewing, T. Lin and Y. Lin, On the accuracy of the finite volume element method based on piecewise linear polynomials, SIAM J. Numer. Anal. (2002), 39:1865–1888.

[14] M. Hinze, A variational discretization concept in control constrained optimization: the linear-quadratic case , Computational Optimization and Applications (2005), 30:45–61.

[15] A. Kröner, Adaptive finite element methods for optimal control of second order hyperbolic equations, Comput. Methods Appl. Math. (2011), 11:214–240.

[16] A. Kröner, K. Kunisch and B. Vexler, Semismooth Newton methods for optimal control of the wave equation with control constraints, SIAM J. Control Optim. (2011), 49:830–858.

[17] S. Kumar, N. Nataraj and A.K. Pani, Finite volume element method for second order hyperbolic equations, Int. J. Numer. Anal. Model. (2008), 5:132–151.

[18] S. Kumar, N. Nataraj and A.K. Pani, Discontinuous Galerkin finite volume element methods for second order linear elliptic problems, Numer. Meth. Part. Diff. Eqns. (2009), 25:1402–1424.

[19] S. Kumar, A mixed and discontinuous Galerkin finite volume element method for incompressible miscible displacement problems in porous media, Numer. Meth. Part. Diff. Eqns. (2012), 28:1354–1381.

[20] S. Kumar and R. Ruiz-Baier, Equal order discontinuous finite volume element methods for the Stokes problem, J. Sci. Comput. (2015), 65:956–978.

[21] R. Sandilya and S. Kumar, Convergence analysis of discontinuous finite volume methods for elliptic optimal control problems, Int. J. Comput. Methods (2015), 13:1640012–20.

[22] R. Sandilya and S. Kumar, On discontinuous finite volume approximations for semilinear parabolic optimal control problems, Int. J. Numer. Anal. Model. (2016), 13:545–568.

[23] S. Kumar, R. Ruiz-Baier and R. Sandilya, Error estimates for a discontinuous finite volume discretization of the Brinkman optimal control problem, APNUM, Submitted.

[24] Y. Lin, J. Liu and M. Yang, Finite volume element methods: An overview on recent developments, Int. J. Numer. Anal. Model. B (2013), 4:14–24.

[25] J.L. Lions, Optimal control of systems governed by partial differential equations, Springer Verlag, Berlin(1971).

[26] X. Luo, Y. Chen and Y. Huang, Some error estimates of finite volume element approximation for elliptic optimal control problems, Int. J. Num. Anal. Model. (2013), 10:697–711.

[27] X. Luo, Y. Chen, Y. Huang and T. Hou, Some error estimates of finite volume element approximation for parabolic optimal control problems, Optim. Control Appl. Meth. (2014), 35:145–165.
[28] X. Luo, Y. Chen and Y. Huang, A priori error estimates of finite volume element method for hyperbolic optimal control problems, Sci China Math (2013), 56:901–914.
[29] D. Meidner and B. Vexler, A priori error estimates for space-time finite element discretization of parabolic optimal control problems part I: problems without control constraints, SIAM J. Control and Optimization. (2008), 47:1150–1177.
[30] D. Meidner and B. Vexler, A priori error estimates for space-time finite element discretization of parabolic optimal control problems part II: problems with control constraints, SIAM J. Control and Optimization (2008), 47:1301–1329.
[31] B.S. Mordukhovich and J.P. Raymond, Dirichlet boundary control of hyperbolic equations in the presence of state constraints, Appl. Math. Optim. (2004), 49: 145-157.
[32] I. Neitzel and B. Vexler, A priori error estimates for space-time finite element discretization of semilinear parabolic optimal control problems, Numer. Math. (2012), 120:345–386.
[33] P. Nestler, Optimales Design einer Zylinderschale - eine Problemstellung der optimalen Steuerung in der linearen Elastizitätstheorie, Ph. D. thesis, Institut für Angewandte Mathematik, Universität Greifswald, 2010.
[34] F. Tröltzsch, Optimal control of partial differential equations: theory, methods and applications, American Math. Society, book series Graduate Studies in Mathematics (2010).
[35] X. Ye, A new discontinuous finite volume method for elliptic problems, SIAM J. Numer. Anal. (2004), 42:1062–1072.

Department of Mathematics, Indian Institute of Space Science and Technology, Thiruvananthapuram 695 547, Kerala , India
E-mail: `ruchisandilya.12@iist.ac.in`

Department of Mathematics, Indian Institute of Space Science and Technology, Thiruvananthapuram 695 547, Kerala , India
E-mail: `sarvesh@iist.ac.in`