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A SEMI DISCRETE MODEL FOR MORTGAGE VALUATION AND ITS COMPUTATION BY AN ADAPTIVE FINITE ELEMENT METHOD

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Abstract. In traditional models for valuation of mortgages with a stochastic interest rate, one parabolic equation starting from the maturity is assumed to govern the whole life of a mortgage. Following the valuation of zero-coupon bond, a new model is proposed, where an initial value problem is restarted after a mortgage payment each month. In addition, the low and high limits on the interest rate are incorporated into the initial-boundary value problems, so that the partial differential equation remains regular and the solution better approximates the real value. We show the existence and uniqueness of the solution and the free boundary (which determines early prepayment). A finite element method is introduced with a convergence analysis. Numerical tests are presented and the results are interpreted for guiding mortgage practice.

Key words. Finite element method, parabolic equation, free boundary problem, mortgage valuation.

1. Model

We consider the standard fixed rate mortgage, where the loan borrower pays an equal amount of money to the lender for the duration of the contract. Typically the borrower has a choice of early settlement, for example, by refinance if a much lower fixed-rate mortgage is available, or by the fund from his or her other investment where the return is too low. The free boundary computed by a mortgage model would help such a borrower to determine if and when to pay off a loan. On the other hand, the mortgage valuation would help financial institutions to assess their loan equity, for example, in issuing mortgage backed security or bond.

The mortgage securities constitute one of the world's largest fixed income market. An adequate and efficient model for pricing mortgage contracts is not only useful for bankers and home owners to make financial decisions, but also critical for the sustainable development of mortgage market. For this reason, there exist considerable literature dealing with mortgage valuation and relevant topics. Most researchers have studied the problem from theoretical option pricing viewpoint, where the mortgage contracts are treated as an American style financial option [9, 1, 4]. A survey in this regard can be found in [10]. Since financial option valuation rarely assumes closed form solutions, efforts have been made to solve such problems numerically. For instance, a binomial iteration scheme is proposed in [12]. A projected successive over-relaxation iterative method is applied in [14]. A Monte-Carlo simulation method is tried in [8]. One notices (see [17, 7, 16], for instance) that usual numerical techniques such as binomial method typically provide poor accuracy and stability, which are mainly attributed to the difficulty in handling free boundary conditions, in addition to low convergence rate. In this manuscript, we propose a new model simplifying the free boundary setting while reflecting the real market practice.

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Assuming the interest rate follows the CIR [3] brownian motion, a mortgage valuation $V(x,t)$ either grows at the loan rate or decays at the higher market interest rate, cf. [11],

$$\begin{aligned}(1)\qquad &\max\Big\{\frac{\partial V}{\partial t}-\frac{\sigma^2}{2}x\frac{\partial^2 V}{\partial x^2}-k(\theta-x)\frac{\partial V}{\partial x}+xV-m,\\ &\qquad V-\frac{m}{c}(1-e^{-ct})\Big\}=0,\quad -\infty<x<\infty,\ 0\le t\le T,\\ &\qquad V(x,0)=0.\end{aligned}$$

(Details and notations are provided next section.) This model is not well posed (may have multiple solutions) and is not computable. Assume there is only one free boundary ($h(t)$ one point separating PDE and ODE regions) and assume the two pieces of solution of (1) join smoothly, then the model is equivalent to an over-constrained system of differential equations, as commonly done in the American option models and in the zero-coupon bond models: Find $h(t)$ and $V(x,t)$ such that

$$(2)\qquad \begin{aligned}\frac{\partial V}{\partial t}&=\frac{\sigma^2}{2}x\frac{\partial^2 V}{\partial x^2}+k(\theta-x)\frac{\partial V}{\partial x}-xV+m, && h(t)<x<\infty,\ 0<t<T,\\ V(h(t),t)&=\frac{m}{c}(1-e^{-ct}), && 0<t<T,\\ \frac{\partial}{\partial x}V(h(t),t)&=0, && 0<t<T,\\ V(x,0)&=0, && x\ge c,\\ h(0)&=c.\end{aligned}$$

Like the solutions to the American option problems, or to the zero-coupon bond problems, the mortgage valuation model (2) is not well posed either. By the free-boundary condition $V_x(h(t),t)=0$, the PDE provides a solution $V(x,t)>V(h(t),t)$ for some $x>h(t)$. That is, a second free-boundary $\tilde h(t)(>h(t))$ would be created at which $V(\tilde h(t),t)=(m/c)(1-e^{-ct})$ and $V_x(\tilde h(t),t)<0$. This violates the refinance principle.

As both models are not well posed, we propose a new model where the free boundary is converted to an initial condition of one parabolic PDE: in $(x,t)\in(c_{\min},c_{\max})\times(0,\frac{1}{12})$

$$(3)\qquad \frac{\partial V^{(n)}}{\partial t}-\frac{\sigma^2}{2}x\frac{\partial^2 V^{(n)}}{\partial x^2}-k(\theta-x)\frac{\partial V^{(n)}}{\partial x}+xV^{(n)}=0,$$

for $n=1,2,\ldots$, with initial and boundary conditions

$$\begin{aligned}V^{(n)}(x,0)&=me^{-\frac{\max\{c,x\}}{12}}+\min\Big\{V^{(n-1)}(x,\frac{1}{12}),V^{(n-1)}(c_{\min},0)e^{-\frac{c}{12}}\Big\},\\ V^{(n)}(c_{\min},t)&=V^{(n)}(c_{\min},0)e^{-c_{\min}t},\\ V^{(n)}(c_{\max},t)&=V^{(n)}(c_{\max},0)e^{-c_{\max}t},\\ V^{(0)}(x,0)&=0.\end{aligned}$$

(Details are given in the next section.) That is, we limit the time of refinance to the time of monthly mortgage payment. This way we avoid mathematical problems in the other two models and provide a practical and computable model. A significance of the new model is its avoidance of numerical computation of the exponential growth of the old model (1), by entering the exponential growth term as an exact initial condition. We will show the uniqueness and well-posedness of the model (3) in section 3. In section 4, we present a finite element method for the parabolic problem. We show the stability and convergence for the discrete approximation. In section 5, we present some numerical examples and interpret the numerical results for guiding mortgage practice for loaners and borrowers. In an appendix, section 6, we present the proof of some theorems in sections 3 and 4.

2. A model based on the monthly payment and refinance

Let c be the fixed mortgage rate written in a contract, t the time to maturity of the contract, and $M(t)$ the loan balance that the borrower owes to the lender. $M(t)$ is determined by the ordinary differential equation

$$\frac{dM(t)}{dt} = m - cM(t), \tag{4}$$

where m is the continuous payment equivalent to the monthly payment in the contract, depending on c. When the contract expires, we have $M(0) = 0$. So the above ODE has a unique solution

$$M(t) = \frac{m}{c}(1 - e^{-ct}). \tag{5}$$

Since the borrower always has the choice of prepayment, the market value of the contract is always bounded above by $M(t)$. Assume that market interest rate follows the CIR [3] model, which says

$$dx = k(\theta - x)dt + \sigma\sqrt{x}dB, \tag{6}$$

where x is the market interest rate, k is the speed of adjustment, θ is the mean of interest rate, σ is the volatility, and B is the standardized Wiener process for interest rate. By Ito's lemma, the value $V(x,t)$ of mortgage loan, at time t and the corresponding interest rate x at t, is modeled by the partial differential equation [9, 11, 14, 17]

$$\frac{\partial V}{\partial t} - \frac{\sigma^2}{2}x\frac{\partial^2 V}{\partial x^2} - k(\theta - x)\frac{\partial V}{\partial x} + xV - m = 0, \tag{7}$$

where m is, again, the continuous payment and the time t is measured from the future expiry of the mortgage contract. However, as assumed above, when the interest rate x (at time t) is lower enough than the fixed mortgage rate c, the borrower would terminate the mortgage contract by refinancing, i.e., obtaining a new loan from the same or different bank to pay off the existing loan. So the value of the mortgage cannot be calculated based on the observed low interest rate. This leads to a free boundary problem that (7) holds for $x > h(t)$ for some unknown function $h(t)$, while the mortgage value remains $M(t)$:

$$V(x,t) = M(t) = \frac{m}{c}(1 - e^{-ct}) \quad \forall x \le h(t). \tag{8}$$

Nevertheless, to stablize the problem, we do not allow refinance all the time, but only at some discrete times, for example, once a month. This is, however, what happens in practice that the refinance takes place mostly at a monthly payment due time. Thus, we propose a new model where the free boundary condition (8) holds at some discrete times only. That is, it is converted in to an initial condition to a parabolic partial differential equation.

We propose a new model for the mortgage evaluation: for $n = 1, 2, 3, ...$, find $V^n(x,t)$ such that

$$\frac{\partial V^{(n)}}{\partial t} - \frac{\sigma^2}{2} x \frac{\partial^2 V^{(n)}}{\partial x^2} - k(\theta - x)\frac{\partial V^{(n)}}{\partial x} + xV^{(n)} = 0,$$

$$c_{\min} < x < c_{\max}, \quad 0 < t < \frac{1}{12}, \tag{9}$$

with the initial condition

$$V^{(n)}(x,0) = me^{-\frac{\max\{c,x\}}{12}} + \begin{cases} 0, & n = 0, \\ \min\left\{V^{(n-1)}(x, \frac{1}{12}), V^{(n-1)}(c_{\min}, 0)e^{-\frac{c}{12}}\right\}, & n > 0, \end{cases}$$

$$c_{\min} < x < c_{\max}, \tag{10}$$

and the boundary conditions

$$V^{(n)}(c_{\min}, t) = V^{(n)}(c_{\min}, 0)e^{-c_{\min} t},$$

$$V^{(n)}(c_{\max}, t) = V^{(n)}(c_{\max}, 0)e^{-c_{\max} t}, \qquad 0 < t < \frac{1}{12}. \tag{11}$$

Remark 2.1. *In the new model, an initial value problem is restarted after each mortgage payment. We call it semi discrete model where the time range of the partial differential equation (7) is separated into discrete segments, by the time of monthly payment. It seems that in real-life simulation people always use a discrete payment* m *(*[4, 9]*) instead of a continuous payment. Here in* (10) *we convert the continuous payment* m *to an equivalent monthly payment. In (9),* n *is the number of months from the mortgage expiration. So when* $t = 1/12$ *(one month of time), a new payment is made and the mortgage value is jumped from* $V^{n-1}(x, 1/12)$ *to* $V^n(x, 0)$*.*

Remark 2.2. *In the new model, the free boundary is no longer sought continuously for all time* t*, but only for the time of each payment. This means, early pay-off occurs only at a monthly payment due date. Of course, the model remains the same mathematically if pay-off is limited to the end of each day, or the end of each hour. But we avoid the challenge of continuous free-boundary in mathematics and in computation. The free boundary* $x = h(t)$ *at the end of each month is defined by the contract rate* c *and the unique (to be shown in the manuscript) intersection point* $x = h_n$*:*

$$h(\frac{n}{12}) = \min\{c, h_n\}, \tag{12}$$

where h_n *is determined by, cf. (10),*

$$V^{(n)}(h_n, \frac{1}{12}) = V^{(n)}(c_{\min}, 0)e^{-\frac{c}{12}}. \tag{13}$$

The reason for using the minimum value in (12) is that the value a mortgage can never be higher than its principal value.

Remark 2.3. (9) *is a typical model for zero-coupon bond. With discrete coupon payments, a jump condition such as (10) should be added to the initial condition, as pointed out by* [17] *on Page 272. Further, the early-settlement (not partial pre-payment) option makes mortgage similar to convertible bond. Nevertheless, a convertible bond can be much higher than its face value while a mortgage loan cannot be higher than its face value — the principal. Such a difference makes the bond value function differentiable at free-boundary while that of mortgage in* (2) *may not be differentiable. If the interest rate is introduced to convertible bond computation,* *in addition to the security price, the non-smoothness problem would arise at the high interest rate portion of boundary where the bond discount and the underlying security are both very low (i.e., one puts money in bank, not in the bond.) This is then similar to our mortgage model. However, to our knowledge, there is no study on such a model yet. This is understandable from practical point of view where, due to short life of convert bond, the interest rate plays much less significant role in convertible bond.*

Remark 2.4. *Another change in the new model is a limit of range for the Brownian motion $x(t)$ in (6), i.e., the x range in* (2) *is no longer from* 0 *to* ∞*, but from $c_{\min}$ to $c_{\max}$. Here we propose a range such as*

$$c_{\min} = \frac{c}{40}, \quad c_{\max} = 40c, \tag{14}$$

i.e., from 0.25% *to* 400% *if the contract interest rate is* 10%. *This limit would lead to a more realistic valuation. On the other hand, the singular points $x = 0$ and $x = \infty$ for the parabolic equation (7) are avoided. This is also crucial for numerical computation. At a very high interest rate or a very low interest rate, the short term valuation V is closely determined by the local rate within the month. We therefore propose boundary conditions at $x = c_{\min}$ and $x = c_{\max}$ as in (11).*

We note that the lower bound in (14) would be implied by more sophisticated interest model by Hull-White (cf. [17]*) than CIR model (6). For the upper limit in (14), the Hull-White model can be further extended so that the random interest rate x has an upper limit as well. This would complicate the analysis, but not the computation if truncated boundary conditions are defined similarly. We note further that the upper interest limit in (14) would not affect the solution much, as indicated by the numerical computation late in the manuscript.*

Remark 2.5. *Again, we use a discrete (monthly, to be specific) mortgage payment. In the traditional finance computation, m is a constant denoting a continuous payment equivalent to the monthly payment. The equivalence is based on a fixed interest rate, in this case, c, the mortgage rate at signing the contract. But the future interest rate x determined by a Brownian motion (6) can be both much higher and much lower than c. Furthermore, the value of a monthly payment m is discounted according to the actual interest rate within the month, cf. (10). This adjustment would make the solution smoother and more accurate in short term. From practical point of view, this method values a mortgage one month ahead so that the mortgage is terminated one month after the last mortgage payment.*

Remark 2.6. *We should point out that this mortgage valuation is done by an investor when comparing investments in mortgage market or in saving, i.e., buying a package of mortgage loans/mortgage-backed bonds or securities, or saving the fund in a return-guaranteed bank account. For example, when $\theta =$ 9%, if the current interest rate is 3.3%, a 30-year home mortgage loan at a rate higher than 6% would be less than its principal, but more than its principal if the loan rate is less than 6%, shown in Figure 1. This computation is conservative as we assume that a borrower could refinance at the right time when the market interest rate is lower than the contract rate. But as we point out, all banks would offer mortgages a point or more higher than the market rate at any time, computed by models similar to ours. That is, there is a gap between the saving rate and mortgage rate. If we modify the model further accordingly, i.e., allowing a mortgage value slightly higher than its principal, then the break-even point would be lower in the above example. That is, a*

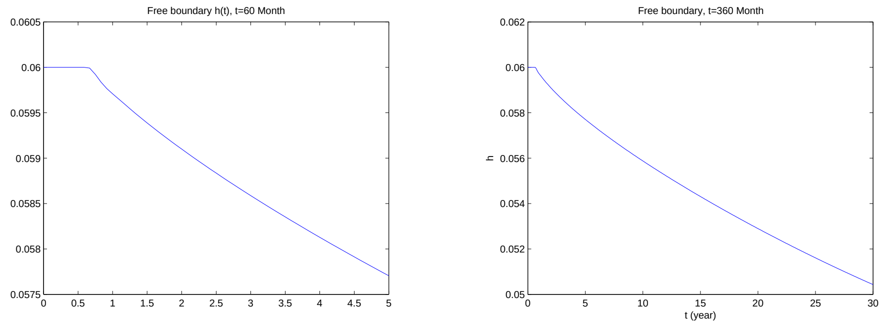

FIGURE 1. The computed free boundary $x = h(t)$, cf. (40), (41) and (42).

30-year home mortgage loan of, say, 5.5% *would make money when the market rate is* 3.3% *(see more discussion in Section 5.) To be exact, this is a nonlinear problem that the mortgage valuation depends on the free-boundary, while the free-boundary depends on the refinance rate, and the refinance rate depends back on the mortgage valuation.*

Remark 2.7. *A final remark is that the model (9) does not include mortgage defaults, i.e., if a house-backed mortgage is higher than the value of the house and the borrower returns the house instead of the mortgage. It is standard to extend (9) to two random variables, interest rate and house price, as in* [11]. *This would slightly decrease the mortgage value* $V(x,t)$. *The computation method proposed here, an implicit in time with space-adaptive finite element method, would remain the same, but the analysis would require somewhat more. We leave it to a further study.*

3. Existence and uniqueness

We will show the existence and uniqueness of solution to the initial value problem (9)–(11). We first introduce two changes of variables to make the bi-linear form coercive and to derive an equivalent homogeneous boundary condition. We then prove the existence of a unique solution to the problem in a finite dimensional subspace of the H^1 Sobolev space. The weak limit of finite dimensional solutions is shown to be a strong solution by a stability result. Finally, the regularity of the solution is provided.

To cancel the boundary value of $V(x,t)$, let

$$W(x,t) = V^{(n)}(x,t) - w_0(x,t), \tag{15}$$

where

$$w_0(x,t) = \frac{(c_{\max} - x)V^{(n)}(c_{\min}, 0) + (x - c_{\min})V^{(n)}(c_{\max}, 0)}{x_{\max} - x_{\min}} e^{-xt}.$$

$W(x,t)$ is the solution of the following initial value problem with homogeneous boundary conditions:

$$\frac{\partial W}{\partial t} - \frac{\sigma^2}{2} x \frac{\partial^2 W}{\partial x^2} - k(\theta - x)\frac{\partial W}{\partial x} + xW = w_1(x,t) \quad \forall (x,t) \in \Omega \times \Theta,$$
$$W(x,0) = g(x) \quad \forall x \in \Omega,$$
$$W(c_{\min}, t) = W(c_{\max}, t) = 0 \quad \forall t \in \Theta,$$

where $\Omega = (c_{\min}, c_{\max})$, $\Theta = (0, 1/12)$, and

$$w_1(x,t) = \frac{\partial w_0}{\partial t} - \frac{\sigma^2}{2} x \frac{\partial^2 w_0}{\partial x^2} - k(\theta - x)\frac{\partial w_0}{\partial x} + x w_0,$$
$$g(x) = V^{(n)}(x,0) - w_0(x,0).$$

In order to define a coercive bi-linear form for the variation problem, let

$$v(x,t) = e^{-\lambda_0 t} W(x,t), \tag{16}$$

where

$$\lambda_0 = 2\left(\frac{(kc_{\max} + \sigma^2/2)^2}{c_{\min}\sigma^2} - c_{\min}\right).$$

It leads to the following initial value problem for $v(x,t)$:

$$\frac{\partial v}{\partial t} - \frac{\sigma^2}{2} x \frac{\partial^2 v}{\partial x^2} - k(\theta - x)\frac{\partial v}{\partial x} + (\lambda_0 + x)v = w_1 e^{\lambda_0 t} \quad \forall (x,t) \in \Omega \times \Theta, \tag{17}$$

$$v(x,0) = g(x) \quad \forall x \in \Omega, \tag{18}$$

$$v(c_{\min}, t) = v(c_{\max}, t) = 0 \quad \forall t \in \Theta, \tag{19}$$

Multiplying (17) by a test function $w(x) \in H_0^1(\Omega)$, after an integration by parts, we obtain the following weak variation problem: Find $v \in L^2(\Theta, H_0^1(\Omega))$ such that

$$(\dot{v}, w) + a(v, w) = (w_1 e^{\lambda_0 t}, w) \quad \forall w \in H_0^1(\Omega), \tag{20}$$

$$v(x,0) = g(x), \tag{21}$$

where the bi-linear form

$$a(v,w) = \frac{\sigma^2}{2}(xv', w') + ((k(x-\theta) + \frac{\sigma^2}{2})v', w) + ((\lambda_0 + x)v, w). \tag{22}$$

Here $\dot{v}$ and v' represent the derivatives in time and in space respectively, and the inner product denotes the L^2 inner product:

$$(v,w) = \int_\Omega vw \, dx.$$

Lemma 3.1. *(Proved in Section 6.) The bi-linear form $a(v,w)$ is continuous and coercive, i.e.,*

$$a(v,w) \le C_1 \|v\|_{H^1} \|w\|_{H^1} \quad \forall v, w \in H_0^1(\Omega), \tag{23}$$

$$a(v,v) \ge C_2 \|v\|_{H^1}^2 \qquad \forall v \in H_0^1(\Omega). \tag{24}$$

where C_1 and C_2 are independent of the functions in the estimation.

Let $\{\phi_i(x),\ i = 1, 2, ...\}$ be the orthonormal basis of $L^2(\Omega)$ which is also orthogonal in the separable Hilbert space $H_0^1(\Omega)$. To be specific, ϕ_i are normalized eigenfunctions of the Laplace operator in L^2 inner product, which form a basis for $H_0^1(\Omega)$. Let H_n be subspaces of $H_0^1(\Omega)$ defined by

$$H_n = \left\{ v(t) = \sum_{i=1}^{n} \nu_i(t)\phi_i(x) \right\} \subset H_0^1(\Omega).$$

We first solve the following approximation problem: Determine $v_n \in H^1(\Theta, H_0^1(\Omega))$, such that, for $i = 1, 2, ..., n$,

$$(\dot{v}_n, \phi_i) + a(v_n, \phi_i) = (w_1 e^{\lambda_0 t}, \phi_i), \quad \text{a.e. } \ t \in \Theta, \tag{25}$$

$$v_n(0) = \sum_{j=1}^{n} g_j \phi_j,$$

where $g_j = (g(x), \phi_j)$.

Lemma 3.2. *(Proved in Section 6.) The linear system of n ordinary differential equations (25) has a unique solution v_n for each n such that*

$$v_n \in H^1(\Theta, H_0^1(\Omega)), \quad \dot{v}_n \in L^2(\Theta, H_0^1(\Omega)).$$

Lemma 3.3. *(Proved in Section 6.) Let v_n be the unique solution of (25). It holds that*

$$\|v_n(t)\|_{L^2}^2 + C_2 \int_0^t \|v_n(s)\|_{H^1}^2 ds \le C_3 \|V^{(n)}(0)\|_{H^1}^2. \tag{26}$$

Here C_2 is defined in Lemma 3.1, and C_3 is to be specified below.

Lemma 3.4. *(Proved in Section 6.) Let v_n be the unique solution of (25). It holds that*

$$\int_\Theta \|\dot{v}_n(s)\|_{H^{-1}}^2 ds \le C_5 \|V^{(n)}(0)\|_{H^1}^2. \tag{27}$$

where C_5 is to be specified below.

Since we have two Hilbert spaces $L^2(\Theta, H_0^1(\Omega))$ and $L^2(\Theta, H^{-1}(\Omega))$, by Lemmas 3.3 and 3.4, we have a subsequence v_{n_i} which converges weakly in the first space, and a sub-subsequence $v_{n_{i_j}}$ so that $\dot{v}_{n_{i_j}}$ converges weakly in the second space. For simplicity, we denote the sub-subsequence by v_n too:

$$v_n \to v \quad \text{weakly in } \ L^2(\Theta, H_0^1(\Omega)), \tag{28}$$

$$\dot{v}_n \to \dot{v} \quad \text{weakly in } \ L^2(\Theta, H^{-1}(\Omega)). \tag{29}$$

Theorem 3.1. *(Proved in Section 6.) Let v be defined in (28). Then v is the unique solution to (20), and furthermore, (26) and (27) hold for v too, with v_n replaced with v.*

As v is the unique solution to (20), the original initial-boundary value problem (9)–(11) has a unique solution too, cf (15) and (16),

$$V^{(n)} = e^{\lambda_0 t} v(x,t) + w_0(x,t).$$

We conclude this section by showing there is a unique free boundary point $x = h_n$ in (12)–(13).

Theorem 3.2. *For $n = 1, 2, \ldots$, there is a unique $h_n \in \Omega$ such that (13) holds.*

Proof. It is standard to prove the regularity of v in $L^2(\Theta, H^2(\Omega))$ and $L^\infty(\Theta, H_0^1(\Omega))$, via the regularity results for elliptic equations, cf. [6]. Further we have interior regularity that $v(x,t)$ and $V^{(n)}(x,t)$ are $C^1(\Omega \times \Theta)$ functions. By continuity of $V^{(n)}(x, 1/12)$ and its boundary conditions, we have that

$$V^{(n)}(c_{\min}, \frac{1}{12}) < V^{(n)}(c_{\min}, 0) e^{-c/12} < V^{(n)}(c_{\max}, \frac{1}{12})$$

and that the horizontal line of graph $V = V^{(n)}(c_{\min}, 0)e^{-c/12}$ crosses the graph of $V = V^{(n)}(x, 1/12)$ at least once, cf. Figure 2(a).

Now, if (13) has more than one solution, i.e., the horizontal line crosses the graph of $V^{(n)}(x, 1/12)$ more than once, by the boundary conditions, we would have an odd number of solutions, see Figure 2(b). Similar to the proof of the maximal principle, we will show a contradictory in this case. In fact, we will show $V^{(n)}(x,t)$ is a decreasing function in x for each t. Let

$$\delta(t) = \max_{x', x'' \in \Omega} \{ V^{(n)}(x', t) - V^{(n)}(x'', t) \mid x' \le x'' \}, \quad t \in \Theta.$$

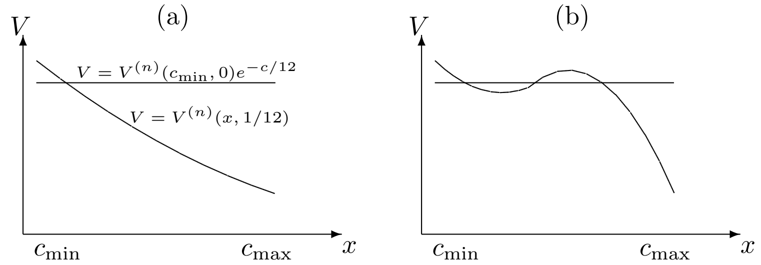

FIGURE 2. The uniqueness of solution to the free boundary problem (13).

By the initial condition (10) $\delta(0) = 0$. By the assumption of multiple solutions in (13), see Figure 2(b), $\delta(1/12) > 0$. As $\delta(t)$ is a continuous function, we have an $0 < t_0 \leq 1/12$ such that

$$\delta(t_0) = \max_{0<t\leq 1/12}\{\delta(t)\}. \tag{30}$$

Since $V^{(n)}(x, t_0)$ is continuous, there are x_0' and x_0'' such that

$$x_0' < x_0'' \quad \text{and} \quad \delta(t_0) = V^{(n)}(x_0', t_0) - V^{(n)}(x_0'', t_0).$$

Let us assume $x_0' \neq c_{\min}$ and $x_0'' \neq c_{\max}$ to avoid some technical details. As $V^{(n)}$ is C^1 inside the region $\Omega \times \Theta$, we have

$$\frac{dV^{(n)}}{dx}(x_0', t_0) = \frac{dV^{(n)}}{dx}(x_0'', t_0) = 0, \quad \frac{d^2V^{(n)}}{dx^2}(x_0', t_0) > 0, \quad \frac{d^2V^{(n)}}{dx^2}(x_0'', t_0) < 0.$$

By the partial differential equation (9),

$$\begin{aligned}
\frac{dV^{(n)}}{dt}(x_0', t_0) &= \frac{\sigma^2}{2}x_0'\frac{d^2V^{(n)}}{dx^2}(x_0', t_0) - x_0'V^{(n)}(x_0', t_0) \\
&> -x_0'V^{(n)}(x_0', t_0) = (x_0'' - x_0')\delta(t_0) - x_0''V^{(n)}(x_0'', t_0) \\
&> (x_0'' - x_0')\delta(t_0) + \frac{\sigma^2}{2}x_0''\frac{d^2V^{(n)}}{dx^2}(x_0', t_0) - x_0'V^{(n)}(x_0', t_0) \\
&= (x_0'' - x_0')\delta(t_0) + \frac{dV^{(n)}}{dt}(x_0'', t_0).
\end{aligned}$$

Then there is a small $\delta_0 > 0$ such that the above inequality holds for all $t_0 - \delta_0 < t < t_0$. By the mean value theorem,

$$\begin{aligned}
V^{(n)}(x_0', t_0 - \delta_0) - V^{(n)}(x_0'', t_0 - \delta_0) >& (x_0'' - x_0')\delta(t_0)\delta_0 + V^{(n)}(x_0', t_0) - V^{(n)}(x_0'', t_0) \\
=& (x_0'' - x_0')\delta(t_0)\delta_0 + \delta(t_0) > \delta(t_0),
\end{aligned}$$

which contradicts to (30). ∎

4. The finite element method

In this section, we define a finite element method for solving the mortgage valuation problem, where in the space we use piecewise linear elements and in the time direction we use the Crank-Nicholson method.

Let Ω_{δ_x} be a uniform grid on $\Omega = (c_{\min}, c_{\max})$:

$$\Omega_{\delta_x} = \{[x_i, x_{i+1}] \mid x_{i+1} - x_i = \delta_x,\ x_0 = c_{\min},\ x_N = c_{\max}\}.$$

For better accuracy, we can use graded grids as well, as shown in Figure 3, where $x_{i+1} - x_i \simeq \delta_x\sqrt{x_i}$. Or, in addition, we can use adaptive grids where the grid

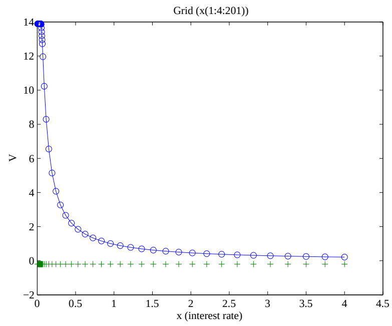

FIGURE 3. A graded grid, and the finite element solution $V^{(n)}(x, 0)$.

size can be much smaller near the free boundary point $x = h_n$ for each variational problem (20), see the numerical computation next section. Let the $C^0 - P_1$ finite element spaces be

$$X_{\delta_x} = \{v(x) \in C^0(\Omega) \mid v(x)|_E \in P_1(x), \quad E \in \Omega_{\delta_x}\} \subset H_0^1(\Omega).$$

Let $\{\psi_i\}$ be the nodal basis functions for X_{δ_x}, where the piecewise linear function ψ_i has nodal value 1 at $x = x_i$ and 0 at the rest nodes x_j. The semi discrete approximation to (20) reads: Find $v_{\delta_x} = \sum_i v_i(t)\psi_i(x)$ such that

$$(\dot{v}_{\delta_x}, \psi_j) + a(v_{\delta_x}, \psi_j) = (w_1 e^{\lambda_0 t}, \psi_j), \quad 1 \leq j \leq (N-1), \tag{31}$$

$$v_{\delta_x}(0) = \sum_{i=1}^{N-1} g_i \psi_i(x), \tag{32}$$

where $\sum_i g_i \psi(x) \in X_{\delta_x}$ is the nodal interpolation of $g(x)$ on the grid Ω_{δ_x}. We first prove the convergence of this semi-discretization method.

Theorem 4.1. *(Proved in Section 6.) The semi discrete problem (31)–(32) has a unique solution $v_{\delta_x}(t)$, which approximates the exact solution v of (20) in the optimal order:*

$$\|v(t) - v_{\delta_x}(t)\|_{L^2} \leq C\delta_x^2 \Big(\|V^{(n)}(x,0)\|_{H^2(c_{\min}, h_{n-1})} + \|V^{(n)}(x,0)\|_{H^2(h_{n-1}, c_{\max})} + \|v(t)\|_{H^2} + \int_0^t \|\dot{v}(s)\|_{H^2} ds \Big). \tag{33}$$

Further estimation on (33) requires regularity results on the smooth solution. To avoid technical details, we refer to Theorem 2 on Page 55 of [15] for the following result:

$$\|v(t) - v_{\delta_x}(t)\|_{L^2} \leq C\delta_x^2 t^{-1} \|v(0)\|\|_{L^2} \quad \forall t \in \Theta.$$

Applying this result to our problem, the following estimate would be derived immediately.

Corollary 4.1. *The unique solution to the semi discrete problem (31)–(32) approximates the exact solution v of (20) in the optimal order:*

$$\|v(\frac{1}{12}) - v_{\delta_x}(\frac{1}{12})\|_{L^2} \leq C\delta_x^2 \|V^{(n)}(x,0)\|_{H^1(\Omega)}.$$

∎

We next discretize the problem fully, both in the space and in the time domain. The time domain is uniformly subdivided:

$$0 = t_0 < t_1 = \delta_t < ... < t_M = M\delta_t = \frac{1}{12}.$$

We then apply the Crank-Nicholson method in the time discretization. The fully discretized finite element problem for (20) reads: For $i = 1, 2, ..., M$, find $v^i_{\delta_x} \in X_{\delta_x}$ such that

$$(D_i v^i_{\delta_x}, w_{\delta_x}) + a(A_i v^i_{\delta_x}, w_{\delta_x}) = (A_i(w_1 e^{\lambda_0 t}), w_{\delta_x}) \quad \forall w_{\delta_x} \in X_{\delta_x}, \tag{34}$$

$$v^0_{\delta_x} = I_{\delta_x} g(x), \tag{35}$$

where D_i and A_i are a finite difference operator and an averaging operator respectively

$$D_i v^i_{\delta_x} = \frac{v^{i+1}_{\delta_x} - v^i_{\delta_x}}{\delta_t}, \qquad A_i w = \frac{w(x, t_{i+1}) + w(x, t_i)}{2}.$$

Here we have an implicit scheme that $v^{i+1}_{\delta_x}$ is defined by a linear system of equations

$$(A + \frac{1}{2}\delta_t B)v^{i+1} = (A - \frac{1}{2}\delta_t B)v^i + kw^i,$$

where the entries of matrices and vectors are defined by

$$\begin{aligned} A_{ij} &= (\psi_i(x), \psi_j(x)), \\ B_{ij} &= a(\psi_i(x), \psi_j(x)), \\ w^i_j &= (A_i(w_1 e^{\lambda_0 t}), \psi_j). \end{aligned}$$

By the coercive condition (24), the matrix $(A + \frac{1}{2}\delta_t B)$ is invertible and the linear system (34) has a unique solution. we then prove the following convergence theorem for the fully discretized problem.

Theorem 4.2. *(Proved in Section 6.) The full-discrete problem (34)–(35) has a unique solution $v^M_{\delta_x}$, which approximates the exact solution v of (20) in the optimal order:*

$$\begin{aligned} \|v(\frac{1}{12}) - v^M_{\delta_x}\|_{L^2} \le C\delta_x^2 \Big(&\|V^{(n)}(x,0)\|_{H^2(c_{\min}, h_{n-1})} \\ &+ \|V^{(n)}(x,0)\|_{H^2(h_{n-1}, c_{\max})} + \|v(\frac{1}{12})\|_{H^2} + \int_\Theta \|v_t\|_{H^2} ds \Big) \\ &+ C\delta_t \int_\Theta (\|v_{ttt}\|_{L^2} + \|v_{tt}\|_{H^2}) ds. \end{aligned} \tag{36}$$

As we use the exact (but artificial) boundary conditions at $c_{\min}$ and $c_{\max}$, the numerical error would not grow exponentially. The exponential growth of the mortgage payment was computed without numerical error as the payment enters the initial conditions exactly. We then have the following global error estimate.

Corollary 4.2. *The global solution of full-discrete problem (34)–(35), with the initial condition* (10) *that $v^0_{\delta t,(n)} = v^M_{\delta t,(n-1)}$, or $V^{(n)}(0)$, for $n = 1, 2, ..., 30 \times 12$, approximates the exact solution $V^{(n)}$ of (20) in the optimal order:*

$$\begin{aligned} \|V^{(n)}(\frac{1}{12}) - v^M_{\delta_x,(n)}\|_{L^2} \le \sum_{i=1}^{n} C\delta_x^2 \Big(&\|V^{(i)}(0)\|_{H^2(c_{\min}, h_{n-1})} + \|V^{(i)}(0)\|_{H^2(h_{n-1}, c_{\max})} \\ &+ \int_\Theta \|V^{(i)}_t\|_{H^2} ds \Big) + C\delta_t \int_\Theta (\|V^{(i)}_{ttt}\|_{L^2} + \|V^{(i)}_{tt}\|_{H^2}) ds. \end{aligned}$$

Proof. As we enter the exact payment into the initial condition for the next level partial differential equation, the PDE is no longer of exponential growth type. Thus, the numerical error would not grow exponentially. By (26), the true solution for the initial condition perturbed, and homogeneous boundary conditioned problem is zero. Thus,

$$\begin{aligned}
&\|V^{(n)}(\frac{1}{12}) - v^M_{\delta_x,(n)}\|_{L^2} \\
&\le C\|V^{(n)}(0) - v^M_{\delta_x,(n-1)}\|_{L^2} + C\delta_x^2\Big(\|V^{(n)}(0)\|_{H^2(c_{\min},h_{n-1})} \\
&\quad + \|V^{(n)}(0)\|_{H^2(h_{n-1},c_{\max})} + \|V^{(n)}(\frac{1}{12})\|_{H^2} + \int_\Theta \|V_t^{(n)}\|_{H^2} ds\Big) \\
&\quad + C\delta_t \int_\Theta (\|V_{ttt}^{(n)}\|_{L^2} + \|V_{tt}^{(n)}\|_{H^2}) ds \\
&\le \sum_{i=1}^n C\delta_x^2\Big(\|V^{(i)}(0)\|_{H^2(c_{\min},h_{n-1})} + \|V^{(i)}(0)\|_{H^2(h_{n-1},c_{\max})} \\
&\quad + \int_\Theta \|V_t^{(i)}\|_{H^2} ds\Big) + C\delta_t \int_\Theta (\|V_{ttt}^{(i)}\|_{L^2} + \|V_{tt}^{(i)}\|_{H^2}) ds.
\end{aligned}$$

∎

5. Numerical Computation

We apply the finite element method (34) directly to the original problem (9)–(11), where the treatment for inhomogeneous boundary conditions is standard, i.e., extending the boundary condition into the space domain by setting the interior nodal values 0. First, the time domain Θ is subdivided uniformly by

$$\delta_t = \frac{1}{12N_t}, \quad N_t = 300. \tag{37}$$

At each time level, i.e., after each monthly payment, we remesh the space grid by letting

$$\delta_x = \frac{h_{n-1} - c_{\min}}{N_1 - N_0} \tag{38}$$

and

$$x_i = \begin{cases} c_{\min} + i\delta_x, & i = 0, 1, ..., N_1, \\ c_{\min} + N_1\delta_x + \frac{(i-N_1)(i-N_1+1)}{2}\delta_x', & i - N_1 = 1, 2, ...N_2, \end{cases} \tag{39}$$

where

$$\delta_x' = \max\{0, \frac{2(c_{\max} - x_{N_1})}{N_2(N_2+1)}\}.$$

We note that we enforce one grid point to match exactly the nonsmooth point of initial condition $V^{(n)}(x, 0)$ at $x = h_{n-1}$, where we have a jump in the space derivative. This would improve the numerical order by 1/2, as in typical interface problems solved by the finite element method, indicated by both the numerical results and the analysis (33). One grid is shown in Figure 3, where we plot only one grid point out of every four points. In addition, we choose, for example,

$$N_0 = 20, \; N_1 = 100, \; N_2 = 200, \tag{40}$$

so that we have small elements near $x = c_{\min}$ and also near the free boundary point $x = h_n$. For the former, we have small diffusion when x is close to 0. For the latter, we use smaller elements near the free boundary point where the solution changes

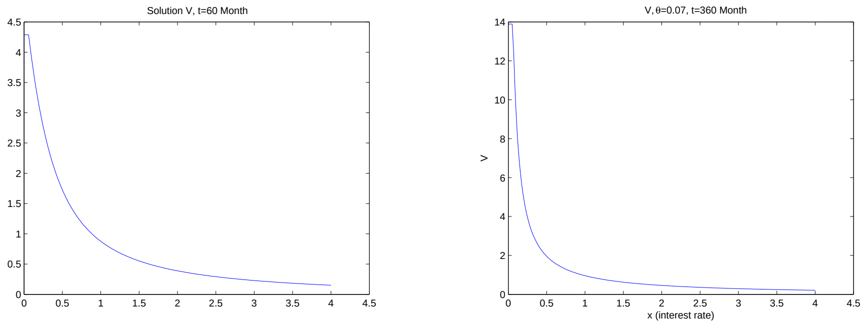

FIGURE 4. The solution $V_{(1)}^{(n)}$ on spacial grid (42).

rapidly (cf. Figure 3). For the portion of graded grid in (39), we make the discrete diffusion on each element roughly the same:

$$x_i \frac{d^2}{dx^2} \simeq x_i(x_{i+1} - x_i)^{-2} \sim \text{ constant.}$$

In our numerical tests, the constants in (9) are from [16] (except $c_{\min}$ and $c_{\max}$), where the authors used maximum likelihood method to derive the parameter values using historical data of treasury bills.

$$\sigma = 0.01,\ k = 0.1,\ \theta = 0.07,\ c = 0.06,\ m = \frac{1}{12},\ c_{\min} = 0.005,\ c_{\max} = 4. \tag{41}$$

We first show that the numerical results are very accurate, that is, the numerical errors can be ignored, even for relatively large grid size. This is because we enter the payment exactly in initial conditions so that we have a diffusive problem in between. As long as the numerical scheme is stable, the numerical error would be reduced rapidly as the time level increases. First, for the following two sets of spacial grids, cf. (37) and (39),

$$\delta_{(1)}:\ N_1 = 100,\ N_2 = 200, N_t = 300, \tag{42}$$

the solution $V^{(n)}$ at the end of 5 years and the end of 30 years, is computed, and plotted in Figure 4.

We vary the grid size of δt to see the error of numerical free boundary $h(t)$ at the end of 5 years. The results are listed in Table 1. Also the graph of $h(t)$ is shown in Figure 1. For this nonlinear problem, the error is nearly independent of grid size δ_t. But the free boundary depends strongly on space discretization. In Table 2, we computed the free boundary with various spacial grid sizes. By extrapolation, we list the order of convergence, roughly one, for this nonlinear problem. This is the reason we use an adaptive grid (39) which is very fine near the free boundary.

TABLE 1. The computed free boundary for $N_1 = 80$ and $N_2 = 160$ in (39), on adaptive grids (42).

N_t in (37)	$h^{(60)}(1/12)$
4	$5.8021e-002$
16	$5.8022e-002$
64	$5.8022e-002$
256	$5.8022e-002$

TABLE 2. The computed free boundary for $N_t = 160$ in (37), on adaptive grids (42).

N_1 in (39)	N_2 in (39)	$h^{(60)}(1/12)$	$O(\delta_x^m)$
40	40	0.060000	
80	80	0.058022	
160	160	0.057334	1.5236
320	320	0.057097	1.5375
640	640	0.056998	1.2594
1280	1280	0.056951	1.0748

We test the convergence order of the finite element method. We vary the spacial grid size. We use the next level solution ($N_1 = N_2 = 1280$) as the exact solution for finding numerical errors, in Table 3. Different from the standard case that the finite element solutions do not converge at the second order, the L^2 error converges at a little more than first order, see column 3 of Table 3. This is caused by the first order approximation of the free boundary, see Table 2. On the other side, the order of convergence of the H^1 error remains at the optimal order, one. Next, we vary the time step size to test the order of convergence in Table 4, where we use the discrete solution of $N_t = 1024$ as the exact solution. It turns out the error is purely the computer error. This is because the exact initial conditions are presented and the diffusion equation each time level damps out the discrete error quickly.

TABLE 3. The L^2 and H^1 convergence of $V^{(60)}(1/12)$, for various N_i and $N_t = 160$ in (37).

$N_1 = N_2$ in (39)	$\Vert V^{(60)} - \tilde{V}^{(60)}\Vert_{L^2}$	$O(\delta_x^m)$	$\vert V^{(60)} - \tilde{V}^{(60)}\vert_{H^1}$	$O(\delta_x^m)$
40	$3.1584e-001$		$2.3796e+000$	
80	$1.1151e-001$	1.5020	$1.2862e+000$	0.88760
160	$4.6313e-002$	1.2677	$7.0819e-001$	0.86091
320	$1.8900e-002$	1.2930	$3.6772e-001$	0.94553
640	$6.1605e-003$	1.6173	$1.4522e-001$	1.34037

TABLE 4. The L^2 and H^1 convergence of $V^{(60)}(1/12)$, for various N_t and $N_1 = N_2 = 160$ in (39).

N_t in (37)	$\Vert V^{(60)} - \tilde{V}^{(60)}\Vert_{L^2}$	$\vert V^{(60)} - \tilde{V}^{(60)}\vert_{H^1}$
4	$3.7889e-006$	$1.3306e-004$
8	$1.9668e-007$	$4.5296e-005$
16	$6.3501e-007$	$2.8339e-005$
32	$8.1042e-007$	$2.6249e-005$
64	$8.5462e-007$	$2.5941e-005$
128	$8.6569e-007$	$2.5878e-005$
256	$8.6846e-007$	$2.5863e-005$

Next, we vary the computational space domain by changing the constants in (41):

$$\begin{cases} \delta_{(1)}: & c_{\min} = 0.001,\ c_{\max} = 10, \\ \delta_{(2)}: & c_{\min} = 0.005,\ c_{\max} = 4. \end{cases} \tag{43}$$

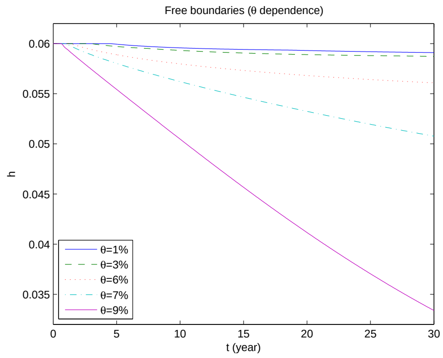

FIGURE 5. The computed free boundary $x = h(t)$, cf (41).

We compare the computed $V^{(n)}_{\delta_{(i)}}$ and free boundary $h_{\delta_{(i)}}(t)$. The maximal differences and relative differences between $V^{(n)}_{\delta_{(i)}}$ are listed in Table 5. But the maximal error for the free boundary $x = h(t)$ occurs in the first few months, near the turning point (from concave down to concave up) of its graph, cf. Figure 1.

TABLE 5. The difference of numerical solutions and relative error, on computational domains defined by (43).

$\left\|V^{(360)}_{\delta_{(1)}} - V^{(360)}_{\delta_{(2)}}\right\|_{L^\infty}$	$\left\|V^{(360)}_{\delta_{(1)}} - V^{(360)}_{\delta_{(2)}}\right\|_{L^\infty} \Big/ \left\|V^{(360)}_{\delta_{(1)}}\right\|_{L^\infty}$
0.0069	0.05%
$\left\|h_{360,\delta_{(1)}} - h_{360,\delta_{(2)}}\right\|$	$\left\|h_{360,\delta_{(1)}} - h_{360,\delta_{(2)}}\right\| / \left\|h_{360,\delta_{(1)}}\right\|$
0.0001	0.24%
$\max_n \left\|h_{n,\delta_{(1)}} - h_{n,\delta_{(2)}}\right\|$	$\max_n \left\|h_{n,\delta_{(1)}} - h_{n,\delta_{(2)}}\right\| / \max_n \left\|h_{n,\delta_{(1)}}\right\|$
0.0005	1.1%

Finally, we examine the dependence of the solution to (9) on the expected future interest rate θ, numerically. This function provides the practical guideline in mortgage industry. We keep the other constants in (41) except θ. The computed mortgage value for a 30-year loan almost does not depend on θ at all, as shown in Figure 6. This is because of the diffusive nature of the parabolic problem (9). But the free boundary is determined mainly by the expected interest rate θ. In Figure 5, we plotted $h(t)$ for various θ. From the free boundary diagrams, we can tell the equilibrium points. For example, when $\theta = 9\%$, if the current interest rate is 3.3%, a bank should offer 30-year home mortgage loans at a rate no less than 6% (i.e., 2.7% higher than market interest rate) in order to break even, comparing to depositing the fund into a saving account. But one only needs to raise market interest from 4.5% to 6% for a 15-year mortgage if the long term interest rate is 9%. On the other hand, if the long term interest rate is 6%, then one can propose 30-year mortgages at 6%, only 0.3% higher than the rate when the contract is signed. We list the computed mortgage rates in Table 6.

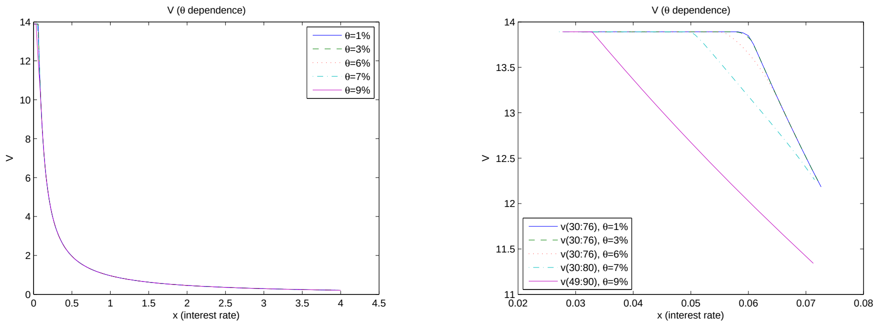

FIGURE 6. The computed $V^{(360)}(x, 0)$, and a zoom-in graph, cf (41).

TABLE 6. The computed mortgage rate for (9), for various long-term interest rates and mortgage terms.

Long-term interest rate θ	Mortgage rate c	above market rate
9%	6%, 30-year	+2.7%
9%	6%, 15-year	+1.5%
6%	6%, 30-year	+0.3%

However, this computation is conservative as we assume that a borrower would refinance immediately when the market interest rate is lower than the contract rate. But as we point out, all banks would offer mortgages a point or more higher than the market rate at any time. That is, there is a gap between the rates for savings and for loans. To be more realistic, we may modify the initial condition (10) to

$$V^{(n)}(x, 0) = me^{-\frac{\max\{c,x\}}{12}} + \min\left\{V^{(n-1)}(x, \frac{1}{12}), V^{(n-1)}(c_{\min}, 0)e^{-\frac{c'}{12}}\right\}$$

for some constant $c' < c$ such as $c' = 0.9c$, which prices in the refinancing cost and the gap between the average bank interest rate and the average mortgage rate.

6. Appendix

This section provides details of the proofs to the lemmas and theorems contained in Sections 3 and 4 where they were referred to in these two Sections without proof.

Proof. (For Lemma 3.1.) By (22), we have that

$$\begin{aligned} a(v, w) \leq & \frac{\sigma^2}{2} c_{\max} |v|_{H^1} |w|_{H^1} + \left(kc_{\max} + \frac{\sigma^2}{2}\right) |v|_{H^1} \|w\|_{L^2} + (\lambda_0 + c_{\max}) \|v\|_{L^2} \|w\|_{L^2} \\ \leq & C_1 \|v\|_{H^1} \|w\|_{H^1}, \end{aligned}$$

where C_1 is chosen to be the maximum of the three constant coefficients, multiplied by the Poincaré constant which bounds the L^2 norm of H_0^1 functions by the semi-H^1 norm. In our application, C_1 would be $(\lambda_0 + c_{\max})$ which is much bigger than the other two constants. For coercivity, we have

$$\begin{aligned}a(v,v) \geq & \frac{\sigma^2}{2} c_{\min} |v|_{H^1}^2 - \left(kc_{\max} + \frac{\sigma^2}{2}\right) |v|_{H^1} \|v\|_{L^2} + (\lambda_0 + c_{\min}) \|v\|_{L^2}^2 \\ \geq & \frac{\sigma^2}{2} c_{\min} |v|_{H^1}^2 - \left(kc_{\max} + \frac{\sigma^2}{2}\right) \left(\frac{\gamma_0}{2} |v|_{H^1}^2 + \frac{1}{\gamma_0 2} \|v\|_{L^2}^2\right) \\ & + (\lambda_0 + c_{\min}) \|v\|_{L^2}^2 \\ = & \frac{\sigma^2}{4} c_{\min} |v|_{H^1}^2 + \frac{\lambda_0}{2} \|v\|_{L^2}^2 \geq C_2 \|v\|_{H^1}^2,\end{aligned}$$

where we choose

$$\gamma_0 = \frac{c_{\min} \sigma^2}{2kc_{\max} + \sigma^2}.$$

Again, C_2 is the minimum of the two constants, which would be $\sigma^2 c_{\min}/4$ in general, divided by one plus the Poincaré constant. ∎

Proof. (For Lemma 3.2.) Under the double orthogonal basis $\{\phi_i\}$, the linear ODE system (25) can be written in the matrix form:

$$\dot{\mathbf{V}}_n = -A_n \mathbf{V}_n + \mathbf{F}_n$$

with initial conditions

$$\nu_j(0) = g_j.$$

Here the vectors and matrix are

$$\mathbf{V}_n = \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \vdots \\ \nu_n(t) \end{pmatrix}, \quad \mathbf{F}_n = e^{\lambda_0 t} \begin{pmatrix} (w_1, \phi_1) \\ (w_1, \phi_2) \\ \vdots \\ (w_1, \phi_n) \end{pmatrix}, \quad A_n = \begin{pmatrix} a(\phi_1, \phi_1) & \cdots & a(\phi_n, \phi_1) \\ \vdots & & \\ a(\phi_1, \phi_n) & \cdots & a(\phi_n, \phi_n) \end{pmatrix}.$$

By Lemma 3.1, A_n is invertible with all positive eigenvalues. Converting the system to a block-diagonal one by the eigen-system of A_n, we obtain unique solutions $\nu_j(t) \in H^1(\Theta, R)$. The regularities are derived by the expression of the solutions, as we have variable-separated solutions. ∎

Proof. (For Lemma 3.3.) Combining the n ODEs of (25), it follows that

$$(\dot{v}_n, v_n) + a(v_n, v_n) = (w_1 e^{\lambda_0 t}, v_n).$$

Noting that

$$\begin{aligned}(\dot{v}_n, v_n) &= \frac{1}{2} \frac{d}{dt} \|v_n\|_{L^2}^2, \\ a(v_n, v_n) &\geq C_2 \|v_n\|_{H^1}^2, \\ (w_1 e^{\lambda_0 t}, v_n) &\leq \|w_1 e^{\lambda_0 t}\|_{L^2} \leq \frac{1}{2C_2} \|w_1 e^{\lambda_0 t}\|_{L^2}^2 + \frac{C_2}{2} \|v_n\|_{H^1}^2,\end{aligned}$$

we obtain

$$\frac{d}{dt} \|v_n\|_{L^2}^2 + C_2 \|v_n\|_{H^1}^2 \leq \frac{1}{C_2} \|w_1 e^{\lambda_0 t}\|_{L^2}^2.$$

By the initial condition, it follows that

$$\|v_n(t)\|_{L^2}^2 + C_2 \int_0^t \|v_n(s)\|_{H^1}^2 ds \leq \|g\|_{L^2}^2 + \frac{1}{C_2} \int_0^t \|w_1 e^{\lambda_0 t}\|_{L^2}^2.$$

By the definition of g, w_1 and λ_0, we can bound the terms in the right hand side:

$$\|g\|_{L^2} \le C_4 \|V^{(n)}(0)\|_{H^1}, \tag{44}$$

$$\|w_1 e^{\lambda_0 t}\|_{L^2} \le C_4 \|V^{(n)}(0)\|_{H^1}, \tag{45}$$

where C_4 depends on the constants in (9) and the C is from the following Sobolev embedding theorem,

$$\|w\|_{L^\infty} \le C\|w\|_{H^1} \quad \forall w \in H^1(\Omega).$$

Therefore

$$\|v_n(t)\|_{L^2}^2 + C_2 \int_0^t \|v_n(s)\|_{H^1}^2 ds \le C_3 \|V^{(n)}(0)\|_{H^1}^2.$$

Here $C_3 = C_4^2(1 + 1/(12C_2))$. ∎

Proof. (For Lemma 3.4.) We consider $\dot{v}_n$ as a linear functional on the dual space of $H_0^1(\Omega)$. Let $w \in H_0^1(\Omega)$ and $w = w_n \oplus z_n$, where $w_n \in H_n$ and $z_n \in H_n^\perp$ under H^1–inner product. By (25), we get

$$\begin{aligned}
(\dot{v}_n(s), w) &= (\dot{v}_n(s), w_n) = -a(v_n(s), w_n) + (w_1 e^{\lambda_0 s}, w_n),\\
|(\dot{v}_n(s), w)| &\le C_1 \|v_n(s)\|_{H^1} \|w_n\|_{H^1} + \|w_1 e^{\lambda_0 s}\|_{L^2} \|w_n\|_{L^2}\\
&\le (C_1 \|v_n(s)\|_{H^1} + \|w_1 e^{\lambda_0 s}\|_{L^2}) \|w\|_{H^1}.
\end{aligned}$$

Here we used the bounds $\|w_n\|_{L^2} \le \|w_n\|_{H^1}$ and $\|w_n\|_{H^1} \le \|w\|_{H^1}$. Therefore, by (45) and Lemma 3.3,

$$\begin{aligned}
\|\dot{v}_n(s)\|_{H^{-1}} &\le C_1 \|v_n(s)\|_{H^1} + \|w_1 e^{\lambda_0 s}\|_{L^2},\\
\int_\Theta \|\dot{v}_n(s)\|_{H^{-1}}^2 ds &\le \int_\Theta (2C_1^2 \|v_n(s)\|_{H^1}^2 + 2C_4^2 \|V^{(n)}(0)\|_{H^1}) ds\\
&\le (2C_1^2 \frac{C_3}{C_2} + \frac{C_4^2}{6}) \|V^{(n)}(0)\|_{H^1}.
\end{aligned}$$

Therefore the lemma holds with $C_5 = 2C_1^2 C_3/C_2 + C_4^2/6$. ∎

Proof. (For Theorem 3.1.) By (28) and (29),

$$\begin{aligned}
\int_\Theta (v_n(t), w)_{H^1} dt &\to \int_\Theta (v(t), w)_{H^1} dt \quad \forall w \in L^2(\Theta, H_0^1(\Omega)),\\
\int_\Theta (\dot{v}_n(t), w)_{H^1} dt &\to \int_\Theta (\dot{v}(t), w) dt \quad \forall w \in L^2(\Theta, H^{-1}(\Omega)).
\end{aligned}$$

Let

$$w = \sum_{i=1}^\infty w_i(t)\phi_i, \quad w_N = \sum_{i=1}^N w_i(t)\phi_i.$$

Let $n \ge N$. By (25),

$$\int_\Theta [(\dot{v}_n(t), w_N(t)) + a(v_n(t), w_N(t))] dt = \int_\Theta (w_1 e^{\lambda_0 t}, w_N(t)) dt.$$

Taking the weak limit by letting $n \to \infty$ while N is fixed,

$$\int_\Theta [(\dot{v}(t), w_N(t)) + a(v(t), w_N(t))] dt = \int_\Theta (w_1 e^{\lambda_0 t}, w_N(t)) dt.$$

Let $N \to \infty$, because $w_N \to w$ weakly too,

$$\int_\Theta [(\dot{v}(t), w) + a(v(t), w)]dt = \int_\Theta (w_1 e^{\lambda_0 t}, w)dt.$$

Because w is arbitrary, we get the equality pointwise, i.e.,

$$(\dot{v}(t), w) + a(v(t), w) = (w_1 e^{\lambda_0 t}, w) \quad \text{a.e. } t \in \Theta.$$

Hence (20) holds. It is then standard to check the initial condition for $v(x,t)$, via an integration by parts. Now, if there are two solutions to (20), v_a and v_b, by the linearity and (24),

$$\begin{aligned}(\dot{v}_a - \dot{v}_b, w) &= -a(v_a - v_b, w),\\ \frac{1}{2}\frac{d}{dt}\|v_a - v_b\|_{L^2}^2 &= -a(v_a - v_b, v_a - v_b),\\ \|v_a(t) - v_b(t)\|_{L^2}^2 &\le -C_2 \int_0^t \|v_a - v_b\|_{H_1}^2 ds \le 0.\end{aligned}$$

Hence $v_a = v_b$. Finally, by taking limits, the bounds (26) and (27) hold for v too. ∎

Proof. (For Theorem 4.1.) We first define the $a(\cdot,\cdot)$-projection operator $P_1 : v \mapsto v_1$,

$$a(v_1(t), w) = a(v(t), w) \quad \forall w \in X_{\delta_x}. \tag{46}$$

By Céa's Lemma, cf. [2], and the coerciveness of the bi-linear form $a(\cdot,\cdot)$ provided in (24), it follows that

$$\|v_1(t) - v(t)\|_{H^1} \le C\delta_x \|v(t)\|_{H^2}.$$

Further, by Nitsch's trick, i.e., a duality argument, cf. [2, 15], we have

$$\|v_1(t) - v(t)\|_{L^2} \le C\delta_x^2 \|v(t)\|_{H^2}^2. \tag{47}$$

Now we introduce this projection into the estimate:

$$\|v(t) - v_{\delta_x}(t)\|_{L^2} \le \|v(t) - v_1(t)\|_{L^2} + \|v_1(t) - v_{\delta_x}(t)\|_{L^2}.$$

Noting that $v_1(t) - v_{\delta_x}(t) \in X_{\delta_x}$, by (20) and (31), it follows that

$$\begin{aligned}(\frac{d}{dt}(v - v_{\delta_x}), v_1 - v_{\delta_x}) + a(v - v_{\delta_x}, v_1 - v_{\delta_x}) &= 0,\\ (\frac{d}{dt}(v_1 - v_{\delta_x}), v_1 - v_{\delta_x}) + a(v_1 - v_{\delta_x}, v_1 - v_{\delta_x}) &= (\frac{d}{dt}(v_1 - v), v_1 - v_{\delta_x}),\end{aligned}$$

and that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_1 - v_{\delta_x}\|_{L^2}^2 &= -a(v_1 - v_{\delta_x}, v_1 - v_{\delta_x}) + (\frac{d}{dt}(v_1 - v), v_1 - v_{\delta_x})\\ &\le -C_2\|v_1 - v_{\delta_x}\|_{H^1}^2 + \|\frac{d}{dt}(v_1 - v)\|_{L^2}\|v_1 - v_{\delta_x}\|_{L^2}\\ &\le \|\frac{d}{dt}(v_1 - v)\|_{L^2}\|v_1 - v_{\delta_x}\|_{L^2},\\ \|v_1(t) - v_{\delta_x}(t)\|_{L^2} &\le \|v_1(0) - v_{\delta_x}(0)\|_{L^2} + \int_0^t \|P_1\dot{v} - \dot{v}\|_{L^2} ds\\ &\le C\delta_x^2(\|V^{(n)}(x,0)\|_{H^2(c_{\min}, h_{n-1})}\\ &\quad + \|V^{(n)}(x,0)\|_{H^2(h_{n-1}, c_{\max})} + \int_0^t \|\frac{d}{dt}v\|_{H^2} ds).\end{aligned}$$

Combined with the estimate (47), the theorem is proven. ∎

Proof. (For Theorem 4.2.) As in Theorem 4.1, we introduce the P_1 projection:

$$v_{\delta x}^{i+1} - v(x, t_{i+1}) = (v_{\delta x}^{i+1} - P_1 v(x, t_{i+1})) + (P_1 v(x, t_{i+1}) - v(x, t_{i+1})),$$

where the second term is estimated by Theorem 4.1, and the first term is to be shortened by a notation

$$\theta^i = v_{\delta x}^i(x) - P_1 v(x, t_i).$$

Subtracting (34) from the continuous equation at $t = t_{i+1/2} = t_i + \delta_t/2$, we have

$$\begin{aligned}(D_i\theta^i, w_{\delta_x}) + a(A_i\theta^i, w_{\delta_x}) &= (P_1 D_i v(t) - D_i v(t), w_{\delta_x}) \\ &+ (D_i v(t) - v(t_{i+1/2}), w_{\delta_x}) + a(v(t_{i+1/2}) - A_i v(t), w_{\delta_x}),\end{aligned} \tag{48}$$

where P_1 is defined in (46). Let T be the inverse operator of $a(\cdot,\cdot)$ with respect to $(\cdot,\cdot)$, i.e., $a(Tv, w) = (v, w)$ for all $w \in H_0^1(\Omega)$. We can rewrite (48) as

$$(D_i\theta^i, w_{\delta_x}) + a(A_i\theta^i, w_{\delta_x}) = (\tilde{w}, w_{\delta_x}), \tag{49}$$

where

$$\begin{aligned}\tilde{w} &= \tilde{w}_1 + \tilde{w}_2 + \tilde{w}_3, \\ \tilde{w}_1 &= P_1 D_i v(t), - D_i v(t), \\ \tilde{w}_2 &= D_i v(t) - v(t_{i+1/2}), \\ \tilde{w}_3 &= T(v(t_{i+1/2}) - A_i v(t)).\end{aligned}$$

Letting $w_{\delta_x} = A_i\theta^i$ in (49), we obtain

$$\begin{gathered}(D_i\theta^i, A_i\theta^i) \le \|\tilde{w}\|_{L^2}\|A_i\theta^i\|_{L^2}, \\ \|\theta^{i+1}\|_{L^2}^2 - \|\theta^i\|_{L^2}^2 \le \delta_t\|\tilde{w}\|_{L^2}(\|\theta^{i+1}\|_{L^2} + \|\theta^i\|_{L^2}), \\ \|\theta^{i+1}\|_{L^2} \le \|\theta^i\|_{L^2} + \delta_t\|\tilde{w}\|_{L^2}.\end{gathered}$$

Hence

$$\|\theta^{i+1}\|_{L^2} \le \|\theta^0\|_{L^2} + \delta_t \sum_{j=1}^{i+1}(\|\tilde{w}_1\|_{L^2} + \|\tilde{w}_2\|_{L^2} + \|\tilde{w}_3\|_{L^2}).$$

We estimate each of the three terms.

$$\|\tilde{w}_1\|_{L^2} = \|(P_1 - I)D_i v(t_i)\|_{L^2} \le C\delta_x^2\delta_t^{-1}\int_{t_i}^{t_{i+1}} \|\dot{v}\|_{L^2} ds.$$

Next,

$$\begin{aligned}\|\tilde{w}_2\|_{L^2} &= \|D_i v(t_i) - \dot{v}(t_{i+1/2})\|_{L^2} \\ &= \frac{1}{2}\delta_t^{-1}\left\|\int_{t_i}^{t_{i+1/2}}(s - t_i)^2 v_{ttt} ds + \int_{t_{i+1/2}}^{t_{i+1}}(s - t_{i+1})^2 v_{ttt} ds\right\|_{L^2} \\ &\le C\delta_t\int_{t_i}^{t_{i+1}}\|v_{ttt}\|_{L^2}.\end{aligned}$$

Lastly, $\tilde{w}_3$ is the error of finite difference operators on the continuous solution, and can be treated standardly as in [15],

$$\begin{aligned}\|\tilde{w}_3\|_{L^2} &= \|T(v(t_{i+1/2}) - A_i v(t))\|_{L^2} \le C\|v(t_{i+1/2}) - A_i v(t)\|_{H^2} \\ &\le \delta_t\int_{t_i}^{t_{i+1}}\|\dot{v}\|_{H^2} ds.\end{aligned}$$

∎

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