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EQUIVALENCE BETWEEN RIEMANN-CHRISTOFFEL AND GAUSS-CODAZZI-MAINARDI CONDITIONS FOR A SHELL

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Abstract. We establish the equivalence between the vanishing three-dimensionnal Riemann-Christoffel curvature tensor of a shell and the two-dimensionnal Gauss-Codazzi-Mainardi compatibility conditions on its middle surface. Additionally we produce a new proof of Gauss theorema egregium and Bonnet theorem (reconstructing a surface from its two fundamental forms). This is performed in the very elegant framework of Cartan's moving frames.

Key words. Surfaces, 3D manifolds, Pfaffian systems, Frobenius integrability conditions, Riemann-Christoffel curvature tensor, moving frames, Cartan differential geometry, Tensorial calculus

1. Introduction

Let $\mathcal{D} \subset \mathbb{R}^3$ be a compact, connected, simply-connected manifold with boundary of class C^2. Let $X = (X^1, X^2, X^3)$ be a system of Cartesian coordinates and $x = (x^1, x^2, x^3)$ be a system of curvilinear coordinates in $\mathbb{R}^3$. The purpose of this paper is to revisit the integrability of the system of nonlinear partial differential equations (PDE)

$$(1) \qquad \sum_{k,l=1,..3} \frac{\partial x^k}{\partial X^i} \delta_{kl} \frac{\partial x^l}{\partial X^j} = g_{ij}(X), \quad i, j = 1, 2, 3.$$

where g is a regular, twice covariant, positive definite bilinear form, for example of class $C^2(\mathcal{D})$. In some of their previous works Vallée and Fortuné have already addressed this question in the framework of Darboux's instantaneous rotation vectors [9, 10, 5]. We consider again this question by using Cartan setting as introduced in [2] . Let us note that the interest of our approach is that it does not rely on the knowledge of the radii of curvature nor on the principal directions of the shell as in [7].

The plan of this work is as follows: in the next section, for the sake of clarity we recall some definitions and properties satisfied by the metric, in section 3 we establish the Riemann-Christoffel compatibility conditions for a three-dimensional Riemannian manifold. In section 4 with the same Frobenius approach we establish the Gauss-Codazzi-Mainardi conditions for a surface embedded in $\mathbb{R}^3$. As a by-product, Weingarten's condition on the normal at each point of the surface is therefore recovered. In section 5 we address the equivalence of Riemann-Christoffel and Gauss-Codazzi-Mainardi compatibility conditions for a shell. Finally in section 6 we state Gauss Theorema egregium and Bonnet reconstruction theorem.

2. Notations, lemmas and assumptions

Let ε and δ be respectively the Levi-Civita symbol, and the Kronecker symbols. Einstein summation convention of repeated indices and exponents is applied. In

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$\mathbb{R}^n$, the identity matrix is denoted I_n. The transpose of the matrix R is the matrix R^t. The scalar product of two vectors u and v with components u^i and v^i is denoted

$$u \cdot v = \delta_{ij} u^i v^j, \ i, j = 1, ...n.$$

Let ω and λ be two 1-forms with components ω_k and λ_k. We define their tensor product as the covariant tensor $\omega \otimes \lambda$ with components:

$$(\omega \otimes \lambda)_{ij} = \omega_i \lambda_j, \ i, j = 1, ..., n.$$

Now let us enunciate two elementary results that we will use repeatedly.

Lemma 2.1 *(i) Let g be a given positive symmetric bilinear form in $\mathbb{R}^n \times \mathbb{R}^n$, there exists n independent 1-form ω^k (the volume n-form $\omega^1 \wedge \omega^2 \wedge ... \wedge \omega^n$ does not vanish) such that g can be expanded:*

$$g = \delta_{kl} \ \omega^k \otimes \omega^l. \tag{2}$$

(ii) The form g can be defined by it covariant (g_{ij}) and contravariant (g^{kl}) components related by

$$g_{ij} g^{jk} = \delta_i^k.$$

Let us denote by (e_k) the dual vector basis associated to (ω^k), i.e. $\omega^k(e_l) = \omega_i^k e_l^i = \delta_l^k$ and g^{-1} whose components are (g^{kl}) can be expanded:

$$g^{-1} = \delta^{ij} \ e_i \otimes e_j.$$

(iii) The expansion (2) is not unique. If there exists a second expansion such that $g(x) = \delta_{kl} \ \zeta^k \otimes \zeta^l$ and if the signs of the n-forms $\omega^1 \wedge \omega^2 ... \wedge \omega^n$ and $\zeta^1 \wedge \zeta^2 ... \wedge \zeta^n$ are the same, then there exists a rotation $R \in SO(n)$ such that:

$$\zeta^k = R_l^k \omega^l.$$

In the sequel Latin indices or exponents take their value in the set $\{1, 2, 3\}$, Greek indices or exponents take their value in the set $\{1, 2\}$.
For $n = 3$ the exterior product of two 1-forms ω and λ is the 2-form $\omega \wedge \lambda$ with components

$$(\omega \wedge \lambda)_i = \delta_{ij} \ \varepsilon^{jkl} \ \omega_k \lambda_l, \quad i, j, k, l = 1, ...3.$$

Let us consider two vectorial 1-forms $\omega = (\omega^1, \omega^2, \omega^3)$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ we use the compact expression $\omega \wedge \lambda$ to represent the vectorial 2-form with components $(\omega \wedge \lambda)^i = \delta^{ij} \ \varepsilon_{jkl} \ \omega^k \wedge \lambda^l, \ \ i, j, k, l = 1, ...3$. For scalar 1-forms ω, λ we remark that $\omega \wedge \lambda = -\lambda \wedge \omega$. However, for vectorial 1-forms $\omega = (\omega^1, \omega^2, \omega^3)$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ we remark that

$$\omega \wedge \lambda = \lambda \wedge \omega.$$

For example we have $(\omega \wedge \lambda)^1 = \omega^2 \wedge \lambda^3 - \omega^3 \wedge \lambda^2 = \lambda^2 \wedge \omega^3 - \lambda^3 \wedge \omega^2 = (\lambda \wedge \omega)^1$.

Lemma 2.2 *Let R be a rotation field. There exists a vectorial 1-form $\lambda = (\lambda^k)$ such that:*

$$dR = Rj(\lambda),$$

where d represents the exterior derivative and $j(\lambda) = \begin{pmatrix} 0 & -\lambda^3 & +\lambda^2 \\ +\lambda^3 & 0 & -\lambda^1 \\ -\lambda^2 & +\lambda^1 & 0 \end{pmatrix}$.

The proof is based upon the relationship $R^t R = I_3$ which implies that $R^t dR$ is antisymmetric. For all vectorial 1-forms λ, ω a direct computation yields:

$$j(\lambda) \wedge \omega = \lambda \wedge \omega.$$

Let us remark that $j(\lambda)$ is a vectorial 1-form for the Lie algebra $so(3)$ of $SO(3)$.

3. Riemann-Christoffel compatibility condition in 3D

To facilitate the reading capital letters Ω and Λ are preferred in three-dimensional framework and we return to small letters ω, λ in two-dimensional framework.

For a given metric tensor field g, the PDE (1) can be rewritten as:

$$\delta_{kl}\ dx^k \otimes dx^l = g(X).$$

And according to Lemma 2.1, we can state that there exists a vectorial 1-form $\Omega = (\Omega^k)$ and a rotation field R such that:

$$dx = R\Omega.$$

Consequently, according to Lemma 2.2, there exists a vectorial 1-form Λ such that $dR = Rj(\Lambda)$.

We are led to consider the following Pfaff system and to study its integrability:

$$\left\{ \begin{array}{l} dx = R\Omega, \\ dR = Rj(\Lambda), \end{array} \right. \tag{3}$$

Theorem 3.1 *Frobenius integrability conditions for the Pfaff system* (3) *reads*

$$\left\{ \begin{array}{l} d\Omega + \Lambda \wedge \Omega = 0, \\ d\Lambda + \dfrac{1}{2}\Lambda \wedge \Lambda = 0. \end{array} \right. \tag{4}$$

We remark that the vectorial 2-form $d\Omega + \Lambda \wedge \Omega$ represents the torsion tensor. The vectorial 2-form $d\Lambda + \frac{1}{2}\Lambda \wedge \Lambda$ represents the Riemann-Christoffel curvature tensor, it depends only upon the vectorial 1-form Λ.

Proof of Theorem 3.1. From one hand we equate d^2x to zero and get:

$$0 = d^2x = dR \wedge \Omega + Rd\Omega = R(j(\Lambda) \wedge \Omega + d\Omega) = R(d\Omega + \Lambda \wedge \Omega),$$

and from the other hand we equate d^2R to zero and get:

$$0 = d^2R = dR \wedge j(\Lambda) + Rj(d\Lambda) = R(j(d\Lambda) + j(\Lambda) \wedge j(\Lambda)) = Rj(d\Lambda + \frac{1}{2}\Lambda \wedge \Lambda). \blacksquare$$

Lemma 3.2 *The vectorial 1-form Λ is determined from the first equation of Pfaffian system* (4)*. It depends only of the metric tensor g and can be computed from Ω and $d\Omega$ by the following formula:*

$$\mathcal{V}\Lambda^k = \delta_{ij}(d\Omega^i \wedge \Omega^k)\Omega^j - \frac{1}{2}\mathcal{T}\Omega^k, \tag{5}$$

where $\mathcal{T}$ is the 3-form $\mathcal{T} = \delta_{ij}\ \Omega^i \wedge d\Omega^j$ and the volume 3-form is $\mathcal{V} = \Omega^1 \wedge \Omega^2 \wedge \Omega^3$.

Proof of Lemma 3.2. Cramer identity [8] reads:

$$\left\{ \begin{array}{ll} \mathcal{V}\Lambda^k & = (\Omega^1 \wedge \Omega^2 \wedge \Omega^3)\Lambda^k \\ & = (\Lambda^k \wedge \Omega^2 \wedge \Omega^3)\Omega^1 + (\Omega^1 \wedge \Lambda^k \wedge \Omega^3)\Omega^2 + (\Omega^1 \wedge \Omega^2 \wedge \Lambda^k)\Omega^3 \\ & = \dfrac{1}{2}(\varepsilon_{jlm}\ \Lambda^k \wedge \Omega^j \wedge \Omega^l)\Omega^m. \end{array} \right.$$

From the relation $d\Omega = -\Lambda \wedge \Omega$ we can compute all the terms $\varepsilon_{jlm}\ \Lambda^k \wedge \Omega^j \wedge \Omega^l$:

$$\left\{ \begin{array}{ll} \varepsilon_{jlm}\ \Lambda^k \wedge \Omega^j \wedge \Omega^l & = \varepsilon_{jlm}\ (\Lambda^k \wedge \Omega^j - \Lambda^j \wedge \Omega^k) \wedge \Omega^l + \varepsilon_{jlm}\ (\Lambda^j \wedge \Omega^k) \wedge \Omega^l \\ & = -\varepsilon_{jlm}(\varepsilon^{pkj}\delta_{pr}d\Omega^r \wedge \Omega^l) - \varepsilon_{jlm}\ (\Lambda^j \wedge \Omega^l) \wedge \Omega^k \\ & = -\delta^k_m\mathcal{T} + 2\delta_{mr}(d\Omega^r \wedge \Omega^k) \end{array} \right.$$

which give the desired result. $\blacksquare$

4. Surface embedded in $\mathbb{R}^3$ and GCM conditions

Let $\mathcal{S}$ be a two-dimensional surface embedded in $\mathbb{R}^3$. To parametrize $\mathcal{S}$ we consider a system of coordinates $X = (X^1, X^2)$. The tangent plane to each point x of $\mathcal{S}$ is mapped by the two vectors $a_1 = \dfrac{\partial x}{\partial X^1}$ and $a_2 = \dfrac{\partial x}{\partial X^2}$ which are assumed to be independent. We denote by n the unit normal at this point. The two fundamental forms associated to $\mathcal{S}$ are the metric tensor denoted $a = a_{\alpha\beta}\ dX^\alpha \otimes dX^\beta$ and the curvature tensor denoted $b = b_{\alpha\beta}\ dX^\alpha \otimes dX^\beta$. The aim of this section is once more to rewrite in Cartan's framework the integrability conditions for the PDE system:

$$\text{(6)} \qquad \left\{ \begin{array}{ll} \delta_{ij}\ dx^i \otimes dx^j & = a, \\ \delta_{ij}\ dx^i \otimes dn^j & = -b. \end{array} \right.$$

The next step is therefore to replace (6) by a Pfaff system and consider its Frobenius integrability conditions.

In $\mathbb{R}^2$ the Levi-Civita symbol is still denoted ε and take the value $\varepsilon_{11} = -\varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$.

4.1. Expansion of the two fundamental forms (a, b).

Theorem 4.1 *There exist two independent 1-forms ω^1 and ω^2 and three 1-forms λ^1, λ^2 and λ^3 such that:*

$$\text{(7)} \qquad \left\{ \begin{array}{ll} a & = \delta_{\alpha\beta}\ \omega^\alpha \otimes \omega^\beta, \\ -b & = \varepsilon_{\alpha\beta}\ \omega^\alpha \otimes \lambda^\beta \end{array} \right.$$

or in other words

$$\text{(8)} \qquad \left\{ \begin{array}{ll} a & = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2, \\ -b & = \omega^1 \otimes \lambda^2 - \omega^2 \otimes \lambda^1. \end{array} \right.$$

Proof of Theorem 4.1. It is broken into 3 steps.

Step 1. According to Lemma 2.1, there exist two 1-forms ω^1 and ω^2 which allow the decomposition of the first fundamental form a.

Step 2. A direct application of Lemma 2.1 in two-dimensional framework yields the following result: *There exists a 3×2 matrix field r satisfying $r^t\ r = I_2$ such that:*

$$\text{(9)} \qquad dx^i = r^i_\alpha\ \omega^\alpha.$$

Let us denote by A and B the two columns of r. Following one of Poincaré's ideas we can construct a full rotation $R \in SO(3)$ whose columns are A, B and $A \times B$ where $\times$ denotes the vector product $(A \times B)^l = \varepsilon_{ijk}\delta^{kl}A^iB^j$. Between r and R the following matrix relation holds $r = Rk$, where $k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. According to Lemma 2.2, we denote $\lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix}$ with $\lambda^i = \lambda^i_1 dX^1 + \lambda^i_2 dX^2$ the vectorial 1-form given by:

$$\text{(10)} \qquad dR = Rj(\lambda), \qquad j(\lambda) = \begin{pmatrix} 0 & -\lambda^3 & +\lambda^2 \\ +\lambda^3 & 0 & -\lambda^1 \\ -\lambda^2 & +\lambda^1 & 0 \end{pmatrix}.$$

We are now in a position to expand the second form.

Step 3. We consider the vector $\widetilde{n} = R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. From $R^t r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ we infer that $\widetilde{n}^t r = \begin{pmatrix} 0, 0 \end{pmatrix}$. Hence vector $\widetilde{n}$ is orthogonal to A and B, it is the unit normal to the surface with the correct orientation sign. Therefore n and $\widetilde{n}$ coincide. A direct computation yields

$$dn = Rj(\lambda) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = R \begin{pmatrix} +\lambda^2 \\ -\lambda^1 \\ 0 \end{pmatrix}.$$

The expansion of the second fundamental form is obtained from $dx = r\omega$ and $r^t dn = \begin{pmatrix} +\lambda^2 \\ -\lambda^1 \end{pmatrix}$ and reads

$$-b = \omega^1 \otimes \lambda^2 - \omega^2 \otimes \lambda^1. \blacksquare$$

In the next section we will show that tensor b is symmetric. (15)
Let us emphasize what we found during Step 3.

Corollary 4.2 *Weingarten formula reads* $dn = R \begin{pmatrix} +\lambda^2 \\ -\lambda^1 \\ 0 \end{pmatrix}$ *and Frobenius integrability conduition* $d^2 n = 0$ *is always satisfied.*

4.2. Pfaff system for a surface embedded in $\mathbb{R}^3$ and Gauss-Codazzi-Mainardi conditions. Motivated by (6) let us consider the following Pfaff system associated to the surface $\mathcal{S}$.

$$\left\{ \begin{array}{l} dx = r\omega, \ r = Rk, \\ dR = Rj(\lambda). \end{array} \right. \tag{11}$$

Theorem 4.3 *Frobenius integrability of the Pfaff system yields three sets of PDE:*
(i) Gauss-Codazzi-Mainardi conditions

$$d\lambda + \frac{1}{2}\lambda \wedge \lambda = 0, \tag{12}$$

(ii) Gauss conditions

$$\left\{ \begin{array}{l} d\omega^1 - \lambda^3 \wedge \omega^2 = 0, \\ d\omega^2 + \lambda^3 \wedge \omega^1 = 0, \end{array} \right. \tag{13}$$

(iii) Symmetry of tensor b

$$\lambda^1 \wedge \omega^2 - \lambda^2 \wedge \omega^1 = 0. \tag{14}$$

Proof of Theorem 4.3. The proof is established with Frobenius integrability conditions in two steps.

Step 1. Equating d^2R to zero yields (12):

$$0 = d^2R = Rj(d\lambda) + dR \wedge j(\lambda) = R\Big(j(d\lambda) + j(\lambda) \wedge j(\lambda)\Big) = Rj\Big(d\lambda + \frac{1}{2}\lambda \wedge \lambda\Big),$$

or component-wise

$$\left\{ \begin{array}{l} d\lambda^1 + \lambda^2 \wedge \lambda^3 = 0, \\ d\lambda^2 + \lambda^3 \wedge \lambda^1 = 0, \\ d\lambda^3 + \lambda^1 \wedge \lambda^2 = 0. \end{array} \right.$$

Step 2. Equating d^2x to zero yields (13)-(14):

$$0 = d^2x = rd\omega + dr \wedge \omega = R\Big(kd\omega + (j(\lambda)\ k) \wedge \omega\Big)$$
$$= R \begin{pmatrix} d\omega^1 \\ d\omega^2 \\ 0 \end{pmatrix} + \ R \begin{pmatrix} -\lambda^3 \wedge \omega^2 \\ +\lambda^3 \wedge \omega^1 \\ +\lambda^1 \wedge \omega^2 - \lambda^2 \wedge \omega^1 \end{pmatrix}.$$

We notice that the last relation $\lambda^1 \wedge \omega^2 - \lambda^2 \wedge \omega^1 = 0$ expresses the symmetry of the second fundamental form b. More precisely we can complete relation (8) by

$$-b = \omega^1 \otimes \lambda^2 - \omega^2 \otimes \lambda^1 = \lambda^2 \otimes \omega^1 - \lambda^1 \otimes \omega^2. \tag{15}$$ ■

As usual, when working with shells, it is of interest to introduce the so-called third fundamental form $c = ba^{-1}b$ also given by its covariant components $c_{\alpha\beta} = b_{\alpha\gamma}a^{\gamma\sigma}b_{\sigma\beta}$. This form can be expanded in terms of the vectorial 1-form λ, as it has been done for the first two fundamental forms.

4.3. Expansion of the third form c.

Theorem 4.4 *The third form c can be expanded as follows:*

$$c = \delta_{\alpha\beta}\ \lambda^\alpha \otimes \lambda^\beta. \tag{16}$$

Proof. Applying Lemma 2.1 (ii) in two-dimension we can write

$$a^{-1} = \delta^{\alpha\beta}\ e_\alpha \otimes e_\beta.$$

We replace tensors a and b by their factorization in terms of cross products of ω, λ and we get

$$\begin{cases} a^{-1}b & = (\delta^{\alpha\beta}\ e_\alpha \otimes e_\beta)(\varepsilon_{\sigma\tau}\ \omega^\sigma \otimes \lambda^\tau), \\ & = \delta^{\alpha\beta}\varepsilon_{\sigma\tau}\delta^\sigma_\beta(e_\alpha \otimes \lambda^\tau) = \delta^{\alpha\beta}\varepsilon_{\beta\tau}(e_\alpha \otimes \lambda^\tau). \end{cases}$$

The symmetry of b obtained in (15) yields

$$\begin{cases} c & = -(\varepsilon_{\mu\nu}\ \lambda^\mu \otimes \omega^\nu)(\delta^{\alpha\beta}\varepsilon_{\beta\tau}(e_\alpha \otimes \lambda^\tau), \\ & = -(\varepsilon_{\mu\nu}\delta^{\alpha\beta}\varepsilon_{\beta\tau})(\delta^\nu_\alpha\ \lambda^\mu \otimes \lambda^\tau), \\ & = -(\delta^{\alpha\beta}\ \varepsilon_{\mu\alpha}e_{\beta\tau})(\lambda^\mu \otimes \lambda^\tau) = \delta_{\mu\tau}\ (\lambda^\mu \otimes \lambda^\tau) \end{cases}$$

which implies the nice factorisation of the twice covariant, bilinear and symmetric form c. ■

5. The case of a shell

We consider a surface $\mathcal{S}$ which is defined by the mapping $\varphi : (X^1, X^2) \in \mathcal{D} \longrightarrow \mathbb{R}^3$. Therefore the shell with middle surface $\mathcal{S}$ and thickness 2ε is mapped by

$$(X^1, X^2, X^3) \in \mathcal{D} \times (-\varepsilon, \varepsilon) \longrightarrow \varphi + X^3 n.$$

It is well known that the three-dimensional metric tensor G, given by its covariant expression, reads

$$G_{\alpha\beta} = g_{\alpha\beta}, \qquad G_{\alpha 3} = G_{3\alpha} = 0, \qquad G_{33} = 1, \qquad \text{with } g = a - 2X^3 b + (X^3)^2 c.$$

In the sequel we assume that X^3 is small enough so that g remains definite positive. In the next section we establish the expansion of tensor G.

5.1. Expansion of the bilinear symmetric form G.

Lemma 5.1 *There exist a vectorial 1-form $\Omega = (\Omega^1, \Omega^2, \Omega^3)$ such that the metric tensor of the shell can be expanded as:*

$$G = \delta_{ij}\ \Omega^i \otimes \Omega^j. \tag{17}$$

with

$$\Omega^1 = \omega^1 + X^3\lambda^2, \qquad \Omega^2 = \omega^2 - X^3\lambda^1, \qquad \Omega^3 = dX^3. \tag{18}$$

the two vectorial 1-forms $\omega = (\omega^1, \omega^2)$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ being those associated to the middle surface and introduced in Theorem 4.1.

Proof of Lemma 5.1. This is directly deduced from the factorisation of the three fundamental forms (7)-(16). We get the factorisation of G from the factorisation of $g = a - 2X^3 b + (X^3)^2 c$, hence

$$\left\{\begin{array}{ll} g & = \Big(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2\Big) + 2X^3\Big(\omega^1 \otimes \lambda^2 - \omega^2 \otimes \lambda^1\Big) \\ & \quad +(X^3)^2\Big(\lambda^1 \otimes \lambda^1 + \lambda^2 \otimes \lambda^2\Big), \\ & = (\omega^1 + X^3\lambda^2) \otimes (\omega^1 + X^3\lambda^2) + (\omega^2 - X^3\lambda^1) \otimes (\omega^2 - X^3\lambda^1). \end{array}\right.$$

5.2. Computation of the volume element.

Lemma 5.2 *The three-dimensional 3-form volume element $(\Omega^1 \wedge \Omega^2 \wedge \Omega^3)$ can be expanded in terms of the two-dimensional 2-form volume $(\omega^1 \wedge \omega^2)$ and in terms of the data a and b as:*

$$\Omega^1 \wedge \Omega^2 \wedge \Omega^3 = \Big(\omega^1 \wedge \omega^2 \wedge dX^3\Big)\Big(1 - X^3(b_{\alpha\gamma}a^{\gamma\alpha}) + (X^3)^2\frac{Det\ b}{Det\ a}\Big).$$

Proof of Lemma 5.2.
We use the definition of Ω given by (18) to obtain

$$\Omega^1 \wedge \Omega^2 = (\omega^1 + X^3\lambda^2) \wedge (\omega^2 - X^3\lambda^1) = (\omega^1 \wedge \omega^2) + X^3(\lambda^1 \wedge \omega^1 + \lambda^2 \wedge \omega^2) + (X^3)^2(\lambda^1 \wedge \lambda^2).$$

The proof is established in two steps.

Step 1. We express the 1-forms λ^1, λ^2 in terms of ω^1, ω^2 making use of the second fundamental form b as follows. The first components of λ^1 and λ^2 are obtained from the two relations $-b_{\alpha 1} = (\omega^1 \otimes \lambda^2 - \omega^2 \otimes \lambda^1)_{\alpha 1}$. The symmetry of b yields the computation of their second components from the relations $-b_{2\alpha} = (\lambda^2 \otimes \omega^1 - \lambda^1 \otimes \omega^2)_{2\alpha}$. Therefore we get the following three equivalent expressions by taking into account (7) $a = \delta_{\alpha\beta}\ \omega^\alpha \otimes \omega^\beta$ which implies $\omega^1_1\omega^2_2 - \omega^1_2\omega^2_1 = \sqrt{\text{Det } a}$:

$$\left\{\begin{array}{ll} (\omega^1_1\omega^2_2 - \omega^1_2\omega^2_1)\lambda^\alpha_\beta & = \varepsilon^{\tau\sigma} b_{\beta\sigma}\omega^\alpha_\tau, \\ \sqrt{\text{Det } a}\ \lambda^\alpha_\beta & = \varepsilon^{\tau\sigma} b_{\beta\sigma}\omega^\alpha_\tau, \\ (\omega^1 \wedge \omega^2)\lambda^\alpha_\beta & = (dX^1 \wedge dX^2)\varepsilon^{\tau\sigma} b_{\beta\sigma}\omega^\alpha_\tau. \end{array}\right. \tag{19}$$

Step 2. The computation of each coefficient of X^3 in the expression of $\Omega^1 \wedge \Omega^2$ yields

$$\left\{\begin{array}{ll} (\omega^1_1\omega^2_2 - \omega^1_2\omega^2_1)(\lambda^1 \wedge \omega^1 + \lambda^2 \wedge \omega^2) & = -(\text{Det } a)(b_{\alpha\gamma}a^{\gamma\alpha})(dX^1 \wedge dX^2), \\ (\omega^1_1\omega^2_2 - \omega^1_2\omega^2_1)(\lambda^1 \wedge \lambda^2) & = (\text{Det } b)(dX^1 \wedge dX^2) \end{array}\right.$$

and finally we use the definition $\omega^1 \wedge \omega^2 = (\omega^1_1\omega^2_2 - \omega^1_2\omega^2_1)(dX^1 \wedge dX^2)$. ■

5.3. Pfaff system associated to the shell. Let us denote x any point of a surface $\mathcal{S}$ while $\tilde{x} = x + X^3 n$ is a corresponding point of the shell with middle surface $\mathcal{S}$.
We recall that the Pfaff system associated to $\mathcal{S}$ is:

$$\begin{cases} dx = r\omega, \ r = Rk, \\ dR = Rj(\lambda), \end{cases}$$

and the set of PDE given by Frobenius integrability conditions is (12)-(13)-(14)

$$\begin{cases} d\omega^1 - \lambda^3 \wedge \omega^2 = 0, \\ d\omega^2 + \lambda^3 \wedge \omega^1 = 0, \\ \omega^1 \wedge \lambda^2 - \omega^2 \wedge \lambda^1 = 0, \\ d\lambda + \frac{1}{2}\lambda \wedge \lambda = 0. \end{cases}$$

The Pfaff system associated to the shell is:

$$\begin{cases} d\tilde{x} = R\Omega, \\ dR = Rj(\Lambda), \end{cases}$$

and the set of PDE given by Frobenius integrability conditions (4) reads:

$$\begin{cases} d\Omega + \Omega \wedge \Lambda = 0, \\ d\Lambda + \frac{1}{2}\Lambda \wedge \Lambda = 0. \end{cases}$$

The main result of our paper which is the equivalence announced in the Introduction amounts at showing the equivalence of compatibility conditions. This is done in the next section.

5.4. Equivalence between Riemann-Christoffel and Gauss-Codazzi -Mainardi compatibility conditions.

Theorem 5.3 or a shell *Gauss-Codazzi-Mainardi compatibility conditions are equivalent to a vanishing Riemann-Christoffel tensor.*

Proof of Theorem 5.3.
(i) We show that Gauss-Codazzi-Mainardi conditions (12) *implies that Riemann-Christoffel tensor vanishes.*
For a shell we can replace Ω by its expression (18) in terms of ω and λ and get:

$$\begin{cases} d\Omega^1 & = d(\omega^1 + X^3\lambda^2) = d\omega^1 + X^3 d\lambda^2 + dX^3 \wedge \lambda^2 \\ & = \lambda^3 \wedge \omega^2 - X^3(\lambda^3 \wedge \lambda^1) + dX^3 \wedge \lambda^2, \\ & = \lambda^3 \wedge (\omega^2 - X^3\lambda^1) + \Omega^3 \wedge \lambda^2 = \lambda^3 \wedge \Omega^2 + \Omega^3 \wedge \lambda^2, \\ & = -(\Omega \wedge \lambda)^1, \end{cases}$$

and similarly $d\Omega^2 + (\Omega \wedge \lambda)^2 = 0$ and $d\Omega^3 = d^2 X^3 = 0$. According to (14) we deduce:

$$(\Omega\wedge\lambda)^3 = \Omega^1\wedge\lambda^2 - \Omega^2\wedge\lambda^1 = (\omega^1 + X^3\lambda^2)\wedge\lambda^2 - (\omega^2 - X^3\lambda^1)\wedge\lambda^1 = \omega^1\wedge\lambda^2 - \omega^2\wedge\lambda^1 = 0.$$

In other words $d\Omega + \Omega \wedge \lambda = 0$. For the shell we have $d\Omega + \Omega \wedge \Lambda = 0$, which implies $\Omega \wedge (\Lambda - \lambda) = 0$. Since ω^1 and ω^2 is a basis, the three forms Ω^1, Ω^2 and Ω^3 is a basis and Λ coincides with λ. Therefore we get the result.
(ii) Conversely we show that Riemann-Christoffel compatibility conditions $d\Lambda + \frac{1}{2}\Lambda \wedge \Lambda = 0$ imply Gauss-Codazzi-Mainardi conditions.
This part is obtained in 3 steps: in Step 1 we show that the vectorial 1-form Λ is independent of dX^3, in Step 2 we show that $\Lambda^1 = \lambda^1, \Lambda^2 = \lambda^2$ and that Gauss condition is satisfied, finally in Step 3 we show that $\Lambda^3 = \lambda^3$ and that Gauss-Codazzi-Mainardi conditions are satisfied

Step 1. From the first equality of (4) that relies Ω and Λ we get $0 = (d\Omega + \Omega \wedge \Lambda)^3 = \Omega^1 \wedge \Lambda^2 - \Omega^2 \wedge \Lambda^1$. Since the third components of Ω^1 and Ω^2 vanish then, the third components of (Λ^1, Λ^2) also vanish. The second equality of (4), i.e., $0 = d\Lambda^1 + \Lambda^2 \wedge \Lambda^3$ and $0 = d\Lambda^2 + \Lambda^3 \wedge \Lambda^1$ imply that the third component of Λ^3 also vanishes (otherwise we would have obtained $\Lambda^1 = \Lambda^2 = 0$).

Step 2. We return to the expression of $d\Omega$:

$$\begin{cases} 0 & = d\Omega^1 + (\Omega \wedge \Lambda)^1 = d\Omega^1 + \Omega^2 \wedge \Lambda^3 - \Omega^3 \wedge \Lambda^2, \\ & = \Big(d\omega^1 + X^3 d\lambda^2 + dX^3 \wedge \lambda^2\Big) + \Big((\omega^2 - X^3\lambda^1) \wedge \Lambda^3 - dX^3 \wedge \Lambda^2\Big), \\ & = (d\omega^1 + \omega^2 \wedge \Lambda^3) + X^3(d\lambda^2 - \lambda^1 \wedge \Lambda^3) - dX^3 \wedge (\Lambda^2 - \lambda^2). \end{cases}$$

We make use of the integrability condition (13) $d\omega^1 + \omega^2 \wedge \lambda^3 = 0, d\omega^2 - \omega^1 \wedge \lambda^3 = 0$ to simplify the previous relation which now reads:

$$\begin{cases} 0 & = \omega^2 \wedge (\Lambda^3 - \lambda^3) + X^3(d\lambda^2 - \lambda^1 \wedge \Lambda^3) - dX^3 \wedge (\Lambda^2 - \lambda^2), \\ & = \Omega^2 \wedge (\Lambda^3 - \lambda^3) + X^3(d\lambda^2 - \lambda^1 \wedge \lambda^3) - dX^3 \wedge (\Lambda^2 - \lambda^2). \end{cases}$$

Equating to zero the coefficients of $dX^3 \wedge dX^1$ and $dX^3 \wedge dX^2$ we get $\Lambda^2 = \lambda^2$ and similarly $\Lambda^1 = \lambda^1$, which in turns yields

$$\Omega^1 \wedge \Lambda^2 - \Omega^2 \wedge \Lambda^1 = 0.$$

In other words $\omega^1 \wedge \lambda^2 - \omega^2 \wedge \lambda^1 = 0$ and Gauss-Codazzi-Mainardi conditions are satisfied.

Step 3. From Step 2, Riemann-Christoffel's compatibility conditions combined with the second relation in (18) give $0 = \Omega^2 \wedge (\Lambda^3 - \lambda^3) + X^3\lambda^1 \wedge (\Lambda^3 - \lambda^3) = \omega^2 \wedge (\Lambda^3 - \lambda^3)$ and similarly $0 = \omega^1 \wedge (\Lambda^3 - \lambda^3)$. Since $\Omega^1, \Omega^2, \Omega^3$ is a basis we deduce that ω^1, ω^2 is also a basis, hence $\Lambda^3 = \lambda^3$. ■

6. Gauss theorema egregium

The total curvature of a surface is related to the two fundamental forms a and b, it is given by the formula:

$$K = \frac{\text{Det } b}{\text{Det } a}.$$

Let us enunciate and prove Gauss theorema egregium.

Theorem 6.1 *The total curvature only depends upon the first fundamental form.*

The proof is now very simple. We recall that b can de decomposed as $-b = \omega^1 \otimes \lambda^2 - \omega^2 \otimes \lambda^1$ and that, according to lemma 3.2, λ depends only upon ω and $d\omega$. Hence K only depends upon a. ■

7. Bonnet reconstruction theorem

As we showed in section 4, a surface $\mathcal{S}$ is associated to Pfaff system (11) which must satisfy the integrability conditions (13)-(12)-(14). Then the surface $\mathcal{S}$ can be completely recovered in a very easy way as detailed by the following Theorem.

Theorem 7.1 *Let the two fundamental forms a and b of $\mathcal{S}$ be given. Then $\mathcal{S}$ can be completely recovered up to a fixed rigid displacement.*

Proof of Theorem 7.1. It is broken into 5 steps.

Step 1. Since tensor a is definite positive we can select two independent 1-forms ω^1 and ω^2 such that $a = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$ and the volume element $\omega^1 \wedge \omega^2$ does not vanish. This determination is not unique.

Step 2. The second fundamental form b yields the computation of λ^1 and λ^2 as proved in (19).

Step 3. We compute the two 2-forms $d\omega^1$ and $d\omega^2$. By using Cramer identity as we did in the proof of Lemma 3.2 and the two Gauss conditions (13) we obtain the 1-form λ^3:

$$(\omega_1^1\omega_2^2 - \omega_2^1\omega_1^2)\lambda^3 = (\lambda^3 \wedge \omega^2)_1\omega^1 + (\omega^1 \wedge \lambda^3)_1\omega^2 = (d\omega^1)_1\omega^1 + (d\omega^2)_1\omega^2.$$

Step 4. We solve the PDE $dR = Rj(\lambda)$ and get R up to a fixed rotation $\widehat{R}$ on the left. Next, we get the partial rotations $r = Rk$ and $\widehat{r}r = (\widehat{R}R)k$.
Step 5. We solve the PDE $dx = r\omega$, up to a fixed translation $\widehat{t}$. Since the unknown x does not appear in the right-hand side the integration turns out to be very simple. Finally the solution is obtained up to a rigid displacement $\widehat{x} = \widehat{r}x + \widehat{t}$. ■

Let us notice that no tedious use of the Christoffel symbols have been necessary to establish the previous computations. The interest of our approach is once more obvious.

8. Conclusion and outlook

In this paper we showed how to describe surfaces embedded in $\mathbb{R}^3$ and 3D manifolds by Pfaff systems. The introduction of Cartan's differential framework proved to be very fruitfull in order to simplify the proofs, in particular the equivalence of 3D and 2D compatibility conditions for shells. We also gave a new method to recover a surface from its fundamental forms. In a forthcoming paper [6] we still apply this approach to establish the equilibrium equations of a shell.

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