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CONVERGENCE OF A FINITE DIFFERENCE SCHEME FOR 3D FLOW OF A COMPRESSIBLE VISCOUS MICROPOLAR HEAT-CONDUCTING FLUID WITH SPHERICAL SYMMETRY

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Abstract. We consider the nonstationary 3D flow of a compressible viscous heat-conducting micropolar fluid in the domain to be a subset of $\mathbf{R^3}$, bounded with two concentric spheres. In the thermodynamical sense the fluid is perfect and polytropic. The homogeneous boundary conditions for velocity, microrotation, heat flux and spherical symmetry of the initial data are proposed. Due to the assumption of spherical symmetry, the problem can be considered as one–dimensional problem in Lagrangian description on the domain that is a segment. We define the approximate equations system by using the finite difference method and construct the sequence of approximate solutions for our problem. By analyzing the properties of these approximate solutions we prove their convergence to the generalized solution of our problem globally in time and establish the convergence of the defined numerical scheme, which is the main result of the paper. The practical application of the proposed numerical scheme is performed on the chosen test example.

Key words. micropolar fluid flow; spherical symmetry; finite difference approximations; strong and weak convergence.

1. Introduction

The theory of micropolar fluids, established by Eringen [13], provides a mathematical foundation for studying the model of a fluid, which takes into account the interactions between the micromotion effects of fluid particles and the macromotion. Eringen's theory has provided a good model to study a fluid flow in which the influence of the microstructure on the flow itself is not negligible. The examples of such fluids are polymeric suspensions, biological fluids, liquid cristals, biofluidics, muddy fluids, etc. The application of these fluids are in blood flow, lubrication theory, flow in capillares and microchannels, etc. Due to a number of practical applications (see [14, 19]), in recent years, the model has become an important area of interest for theoretical and applied mathematicians [6, 16, 26, 29, 5], as well as, for the engineers [17, 18, 27]. The majority of work considering the micropolar fluids refers to the incompressible flow. The theory and known results from mathematical point of view are very well systematized in the book of Lukaszewicz [19], but there are still many open problems. The compressible flow of the micropolar fluid has begun to be intensively studied in the last few years [6, 16, 29, 5].

In this paper we focus on the compressible flow of the isotropic, viscous and heat conducting micropolar fluid, which is in the thermodynamical sense perfect and polytropic. We consider the fluid in the domain $\Omega = \{\mathbf{x} \in \mathrm{R}^3 |\ a < |\mathbf{x}| < b\}$, where $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $b > a > 0$ and assume that the initial data are spherically symmetric. The homogeneous boundary conditions for velocity, microrotation and heat flux are proposed. Taking into account the spherical symmetry of our problem, we get one-dimensional problem, which is analysed here in the Lagrangian description.

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The model for this type of flow was first considered by Mujaković in [20] where she developed a one-dimensional case. In the same work, the local existence and uniqueness of the solution, which is called generalized, for the model with the homogeneous boundary conditions are proved. Mujaković in [21] and in the references cited therein proved the local and global existence for the same one-dimensional model with the nonhomogeneous boundary conditions for velocity and microrotation, as well as the stabilization and the regularity of the solution. In [22] the Cauchy problem for this one-dimensional problem was also considered.

The described model of compressible micropolar fluid in the three-dimensional case was first considered in [9]. Assuming the homogeneous boundary conditions and spherical symmetry, the existence and uniqueness of the generalized solution locally in time was proved by using the Faedo-Galerkin method. Applying this result, with the help of the extension principle, the existence of the generalized solution globally in time was proved in [11].

The main goal of this paper is to apply the finite difference method to the described micropolar fluid flow problem and to prove its convergence. The method is based on the defined finite difference approximate equations system and its convergence is established by analysing the properties of the sequence of approximate solutions. We prove that the limit of this sequence is the solution to our problem with the same properties as the solutions obtained in [9] and [11]. In this way, the global existence of generalized solution is established again, but now by using the finite difference method. Let us mention that this proof is technically more demanding that the already obtained proof in which the Faedo-Galerkin method, together with the extension principle, was used [9, 11], but it has the methodological advantages as follows. First, we do not need the local existence theorem for the proof of the global existence. Second, the Faedo-Galerkin method is limited to the problems with the smooth enough initial functions and to the problems with homogeneous boundary conditions, while the method from this work could be extended to other classes of initial functions, for example, to the initial functions with discontinuities (see [3]) and to the problems with non-homogeneous boundary conditions. Thus, on the one hand, the paper could be of interest to theoretical mathematicians working in the area of compressible flows, since it offers a possible approach to the proof of the existence of global solutions to similar types of problems. On the other hand, the paper provides the convergent numerical scheme, which could be of interest to the applied mathematicians and engineers when performing numerical simulations for the considered type of problems.

The procedure used in this work is similar to the procedure applied in [25], where we proved the existence of the solution globally in time for the model that governs one-dimensional flow. However, in comparison with [25], the mathematical model used in this work is more complex because it contains function $r(x,t)$ that represents the Eulerian coordinate, which refers to the Lagrangian coordinate x in time t. As a consequence, we need to derive more a priori estimates for approximate solution, including the estimates for approximations of the function r, for which we have to show the strong convergence.

In our research, there is an important influence of the results from [4], where the existence of the solution globally in time for the three-dimensional spherically symmetric model of the classical fluid flow (with microrotation equals zero) is proved by using the finite difference method. We follow some ideas from [3] also.

The paper is organized as follows. In the second section we introduce the mathematical formulation of our problem. In the third section we derive the finite difference approximate equations system and in the fourth section present the main result. In Sections 5-9, we prove uniform a priori estimates for the approximate solutions. The proof of convergence of a sequence of approximate solutions to a solution of our problem is given in the tenth section. Finally, in order to validate the proposed finite difference method, the numerical experiment is presented in the last section.

2. Statement of the mathematical model

In this work we consider the three-dimensional flow of the compressible viscous and heat-conducting micropolar fluid, which is thermodynamically perfect and polytropic. In the Eulerian description the starting domain is $\{\mathbf{x} \mid \mathbf{x} \in \mathrm{R}^3, a < |\mathbf{x}| < b\}$, where $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $b > a > 0$. The homogeneous boundary conditions for velocity, microrotation, heat flux and spherical symmetry of the initial data are proposed. This spherically symmetric problem is transformed in [9] to the one-dimensional problem in Lagrangian coordinates in the domain $\langle 0, 1\rangle$, and we investigate it in this work. Let ρ, v, w and θ denote, respectively, the mass density, velocity, microrotation velocity and temperature in the Lagrangian description. The motion of the fluid under consideration is described by the following system of four equations ([9]):

$$\frac{\partial \rho}{\partial t} = -\frac{1}{L}\rho^2 \frac{\partial}{\partial x}\left(r^2 v\right), \tag{1}$$

$$\frac{\partial v}{\partial t} = -\frac{R}{L}r^2\frac{\partial}{\partial x}(\rho\theta) + \frac{\lambda + 2\mu}{L^2}r^2\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r^2 v\right)\right), \tag{2}$$

$$\rho\frac{\partial \omega}{\partial t} = -\frac{4\mu_r}{j_I}\omega + \frac{c_0 + 2c_d}{j_I L^2}r^2\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r^2\omega\right)\right), \tag{3}$$

$$\begin{aligned}\rho\frac{\partial \theta}{\partial t} = {} & \frac{K}{c_v L^2}\rho\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial \theta}{\partial x}\right) - \frac{R}{c_v L}\rho^2\theta\frac{\partial}{\partial x}\left(r^2 v\right) + \frac{\lambda+2\mu}{c_v L^2}\left(\rho\frac{\partial}{\partial x}\left(r^2 v\right)\right)^2 \\ & -\frac{4\mu}{c_v L}\rho\frac{\partial}{\partial x}\left(r v^2\right) + \frac{c_0+2c_d}{c_v L^2}\left(\rho\frac{\partial}{\partial x}\left(r^2\omega\right)\right)^2 - \frac{4c_d}{c_v L}\rho\frac{\partial}{\partial x}\left(r\omega^2\right) + \frac{4\mu_r}{c_v}\omega^2,\end{aligned} \tag{4}$$

where

$$L = \int_a^b s^2\rho_0(s)\, ds, \tag{5}$$

(a and b are radii of the starting domain)

$$\mu \geq 0,\ 3\lambda + 2\mu \geq 0,\ \mu_r \geq 0, \tag{6}$$

$$c_d \geq 0,\ 3c_0 + 2c_d \geq 0,\ K > 0, \tag{7}$$

$$c_v > 0,\ R > 0,\ j_I \geq 0. \tag{8}$$

The system is considered in the domain $Q_T = \langle 0,1\rangle \times \langle 0, T\rangle$, where $T > 0$ is arbitrary. Equations (1)-(4) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment and energy. We take the following non-homogeneous initial conditions:

$$\rho(x,0) = \rho_0(x),\ v(x,0) = v_0(x),\ \omega(x,0) = \omega_0(x),\ \theta(x,0) = \theta_0(x) \tag{9}$$

and homogeneous boundary conditions:

$$v(0,t) = v(1,t) = 0,\ \omega(0,t) = \omega(1,t) = 0, \tag{10}$$

$$\partial_x \theta(0,t) = \partial_x \theta(1,t) = 0, \tag{11}$$

for $x \in \langle 0,1\rangle$ and $t \in \langle 0,T\rangle$. Here ρ_0, v_0, ω_0 and θ_0 are given functions. The function r is defined by

$$r(x,t) = r_0(x) + \int_0^t v(x,\tau)\,d\tau, \quad (x,t) \in Q_T, \tag{12}$$

where

$$r_0(x) = \left(a^3 + 3L\int_0^x \frac{1}{\rho_0(y)}\,dy\right)^{1/3}, \quad x \in \langle 0,1\rangle \tag{13}$$

(a is radius of a smaller boundary sphere). We assume that there exists a constant $m \in \mathrm{R}^+$ such that

$$\rho_0(x) \geq m,\ \theta_0(x) \geq m, \quad x \in \langle 0,1\rangle. \tag{14}$$

Let the initial data (9) have the following properties of smoothness

$$\rho_0, \theta_0 \in H^1(\langle 0,1\rangle) \quad \text{and} \quad v_0, \omega_0 \in H_0^1(\langle 0,1\rangle). \tag{15}$$

From (13) we conclude that the function r_0 belongs to $H^2(\langle 0,1\rangle)$. Because of the embedding $H^k(\langle 0,1\rangle)$ into $C^{k-1}([0,1])$, it is easy to check that there exists $M \in \mathrm{R}^+$ such that

$$\rho_0(x), |v_0(x)|, |\omega_0(x)|, \theta_0(x) \leq M, \quad x \in [0,1]. \tag{16}$$

The function r_0 has the properties:

$$r_0 \in C^1([0,1]), \tag{17}$$

$$0 < a \leq r_0(x) \leq M; 0 < a_1 \leq r_0'(x) \leq M_1, \quad x \in [0,1], \tag{18}$$

where $a_1 = LM^{-3}$, $M_1 = L(ma^2)^{-1}$. Under the stated assumptions (14)-(15) in the previous papers [9, 11, 24] is proven that the problem (1)-(11) has unique solution $(\rho, v, \omega, \theta)$ in the domain Q_T, for every $T > 0$, with the following properties:

$$\rho \in L^\infty(0,T; H^1(\langle 0,1\rangle)) \cap H^1(Q_T), \tag{19}$$

$$v, \omega, \theta \in L^\infty(0,T; H^1(\langle 0,1\rangle)) \cap H^1(Q_T) \cap L^2(0,T; H^2(\langle 0,1\rangle)), \tag{20}$$

$$\rho > 0, \quad \theta > 0 \quad \text{on} \quad \bar{Q}_T. \tag{21}$$

For the function r it holds

$$r \in L^\infty(0,T; H^2(\langle 0,1\rangle)) \cap H^2(Q_T) \cap C(\bar{Q}_T), \tag{22}$$

$$r \geq a \quad \text{in} \quad \bar{Q}_T. \tag{23}$$

These results where obtained by using the Faedo-Galerkin method for a local existence theorem ([9]) and the principle of extension for a global existence theorem ([11]). From embedding and interpolation theorem (e.g. see [7, 8]) we can conclude that from (19) and (20) it follows:

$$\rho \in L^\infty(0,T; C([0,1]) \cap C([0,T]; L^2(\langle 0,1\rangle)), \tag{24}$$

$$v, \omega, \theta \in L^2(0,T; C^1([0,1])) \cap C([0,T]; H^1(\langle 0,1\rangle)), \tag{25}$$

$$v, \omega, \theta \in C(\bar{Q}_T). \tag{26}$$

We refer to the solution with the properties (19)-(21) as to the generalized solution. In this work we prove the existence of the generalized solution of the system (1)-(11) globally in time with the properties (19)-(21) by using the finite difference method. We use the similar procedure as in [25].

3. Finite-difference spatial discretization and approximate solutions

In this section we introduce a space discrete difference scheme in order to obtain an approximate system for the equation system (1)-(4) and (9)-(11). We construct semi-discrete finite difference approximate solutions on a uniform staggered grid. In making a discrete scheme we use some ideas from [4], [3] and [25].

Let h be an increment in x such that $Nh = 1$ for $N \in \mathbf{Z}^+$. The staggered grid points are denoted with $x_k = kh$, $k \in \{0, 1, \dots, N\}$ and $x_j = jh$, $j \in \left\{\frac{1}{2}, \dots, N - \frac{1}{2}\right\}$. For each integer N, we construct the following time dependent functions

$$\rho_j(t), \theta_j(t), \quad j = \tfrac{1}{2}, \dots, N - \tfrac{1}{2}, \tag{27}$$

$$v_k(t), \omega_k(t), \quad k = 0, 1, \dots, N, \tag{28}$$

that form a discrete approximation to the solution at defined grid points

$$\rho(x_j, t), \theta(x_j, t), \quad j = \tfrac{1}{2}, \dots, N - \tfrac{1}{2},$$
$$v(x_k, t), \omega(x_k, t), \quad k = 0, 1, \dots, N.$$

We define the operator δ with

$$\delta g_l = \frac{g_{l+\frac{1}{2}} - g_{l-\frac{1}{2}}}{h}, \tag{29}$$

where $l = j$ or $l = k$. For $k \in \{1, \dots, N\}$ and $j \in \{\frac{1}{2}, \dots, N - \frac{1}{2}\}$, the functions ρ_k, θ_k and v_j, ω_j we define by

$$\rho_k = \rho_{k-\frac{1}{2}}, \quad \theta_k = \theta_{k-\frac{1}{2}} \quad \text{and} \quad v_j = v_{j+\frac{1}{2}}, \quad \omega_j = \omega_{j+\frac{1}{2}}. \tag{30}$$

In accordance with the given initial conditions (9) we introduce the initial conditions as

(31)

$$(\rho_j, \theta_j)(0) = \left(\frac{1}{h} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \rho_0(x)dx, \ \frac{1}{h} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \theta_0(x)dx \right), \ j \in \left\{\tfrac{1}{2}, \dots, N - \tfrac{1}{2}\right\},$$

(32)

$$(v_k, \omega_k)(0) = \left(\frac{1}{h} \int_{(k-1)h}^{kh} v_0(x)dx, \ \frac{1}{h} \int_{(k-1)h}^{kh} \omega_0(x)dx \right), \ k \in \{1, \dots, N-1\},$$

and

$$v_0(0) = v_N(0) = 0, \quad \omega_0(0) = \omega_N(0) = 0, \quad \delta\theta_0(0) = \delta\theta_N(0) = 0. \tag{33}$$

Because of (13) we take

$$r_k(0) = \left(a^3 + 3L \sum_{j=\frac{1}{2}}^{k-\frac{1}{2}} \frac{h}{\rho_j(0)} \right)^{1/3}, \quad k = 1, \dots, N, \tag{34}$$

$$r_0(0) = a. \tag{35}$$

Now, the functions $\rho_j(t), v_k(t), \omega_k(t), \theta_j(t)$, $j=\frac{1}{2},\dots,N-\frac{1}{2}$, $k=1,\dots,N-1$, and $r_k(t)$, $k=0,\dots,N$ are determined by using an appropriate spatial discretization of the equation system (1)-(4) and (12):

$$\dot{\rho}_j = -\frac{1}{L}\rho_j^2\delta(r^2v)_j, \quad j=\tfrac{1}{2},\dots,N-\tfrac{1}{2}, \tag{36}$$

$$\dot{v}_k = -\frac{R}{L}\, r_k^2\,\delta(\rho\theta)_k + \frac{\lambda+2\mu}{L^2}\, r_k^2\,\delta(\rho\delta(r^2v))_k, \quad k=1,\dots,N-1, \tag{37}$$

$$\rho_k\dot{\omega}_k = -\frac{4\mu_r}{j_I}\omega_k + \frac{c_0+2c_d}{j_IL^2}\, r_k^2\,\rho_k\delta(\rho\delta(r^2\omega))_k, \quad k=1,\dots,N-1, \tag{38}$$

$$\begin{aligned}\rho_j\dot{\theta}_j &= \frac{K}{c_vL^2}\rho_j\delta(r^4\rho\delta\theta)_j - \frac{R}{c_vL}\rho_j^2\theta_j\delta(r^2v)_j - \frac{4\mu}{c_vL}\rho_j\delta(rv^2)_j - \frac{4c_d}{c_vL}\rho_j\delta(r\omega^2)_j + \\ &+ \frac{\lambda+2\mu}{c_vL^2}\big(\rho_j(\delta(r^2v)_j)\big)^2 + \frac{c_0+2c_d}{c_vL^2}\big(\rho_j(\delta(r^2\omega)_j)\big)^2 + \frac{4\mu_r}{c_v}\omega_j^2,\ j=\tfrac{1}{2},\dots,N-\tfrac{1}{2},\end{aligned} \tag{39}$$

$$\dot{r}_k = v_k, \quad k=0,\dots,N. \tag{40}$$

In accordance with the boundary conditions (10)-(11) we take

$$v_0(t)=v_N(t)=0,\ \omega_0(t)=\omega_N(t)=0, \tag{41}$$

$$\delta\theta_0(t)=\delta\theta_N(t)=0. \tag{42}$$

We consider equations (36)-(40) as a system of ordinary differential equations with initial conditions (31)-(35). Using (14), (16) and (18) we can easily see that from (31), (32) and (34) follows

$$m \le \rho_j(0), \theta_j(0) \le M, \quad j=\tfrac{1}{2},\dots,N-\tfrac{1}{2}, \tag{43}$$

$$\frac{1}{M} \le \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{h}{\rho_j(0)} \le \frac{1}{m}, \tag{44}$$

$$a \le r_0(0) \le r_1(0) \le \dots \le r_N(0) \le C, \tag{45}$$

where $C\in\mathbf{R}^+$. Taking into account the properties (15) from (31)-(35) we obtain that for initial data $(\rho_j,\theta_j)(0)$, $j=\frac{1}{2},\dots,N-\frac{1}{2}$, $(v_k,\omega_k,r_k)(0)$, $k=0,\dots,N$, the following estimates, which shall be used through the article, are valid:

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \big|(\rho_j,\theta_j)\big|^2(0)h \le C, \quad \sum_{k=0}^{N} \big|(v_k,\omega_k,r_k)\big|^2(0)h \le C, \tag{46}$$

$$\sum_{k=1}^{N-1} \big|(\delta\rho_k,\delta\theta_k)\big|^2(0)h \le C, \quad \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \big|(\delta v_j,\delta\omega_j,\delta r_j)\big|^2(0)h \le C, \tag{47}$$

and

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \big|(\rho_j,\theta_j)\big|^4(0)h \le C, \quad \sum_{k=1}^{N} \big|(v_k,\omega_k,r_k)\big|^4(0)h \le C. \tag{48}$$

Even more, using

$$\delta r_j(0) = \frac{3L}{\rho_j(0)\left(r^2_{j+\frac{1}{2}}(0) + r_{j+\frac{1}{2}}(0) r_{j-\frac{1}{2}}(0) + r^2_{j-\frac{1}{2}}(0)\right)} \tag{49}$$

and (43), (45), (47) we easily conclude that

$$\left|\delta r_j(0)\right| \leq C, \quad \text{for} \quad j = \frac{1}{2}, \dots, N - \frac{1}{2}, \tag{50}$$

and

$$\sum_{k=1}^{N-1} \left(\delta^2 r_k\right)^2 (0)\, h \leq C. \tag{51}$$

Notice that the system (36)-(40) with (41)-(42) contains $5N+5$ equations for $5N+5$ required functions.

From the basic theory of differential equations and the local existence theorem, it is known that there exists a smooth solution of the Cauchy problem (36)-(42), with the initial conditions (31)-(35), locally on some time interval $[0, T\rangle$, $T > 0$ ([1, 28]). Because of the positivity of the initial conditions (see (43) and (45)) and smoothness of the solution on the considered interval, we can choose such T so that

$$0 < \rho_j(t), \theta_j(t) < \infty, \quad j = \tfrac{1}{2}, \dots, N - \tfrac{1}{2} \tag{52}$$

$$0 < r_k(t) < \infty, \quad |v_k(t)|, |\omega_k(t)| < \infty, \quad k = 0, \dots, N \tag{53}$$

for $t \in [0, T\rangle$. Let $[0, T_{max}\rangle$ be the maximal time interval on which the smooth solution satisfying (52) and (53) exists. Our first goal, as in [25] and [4], is to show that the solution is globally defined on $[0, \infty\rangle$, i.e., that $T_{max} = \infty$. We will achieve this by showing, for fixed $h > 0$, the boundedness of the functions ρ_j, θ_j, v_k, ω_k, r_k uniformly by j and k, as well as the lower boundedness of ρ_j, θ_j and r_k away from zero (see Sections 6 and 9) uniformly by j and k also. From that, we conclude that the solution $(\rho_j, v_k, \omega_k, r_k, \theta_j)$, $j = \frac{1}{2}, \dots, N - \frac{1}{2}$, $k = 0, \dots, N$, can be defined globally in time.

Now, using the solution of the Cauchy problem (36)-(40), we construct, for $t \geq 0$ the following approximate functions. For each fixed $h = \frac{1}{N}$, $x \in [kh, (k+1)h]$, $k = 0, \dots, N-1$, we define

$$v^N(x,t) = v_k(t) + \frac{1}{h}(x - kh)(v_{k+1}(t) - v_k(t)), \tag{54}$$

$$\omega^N(x,t) = \omega_k(t) + \frac{1}{h}(x - kh)(\omega_{k+1}(t) - \omega_k(t)), \tag{55}$$

$$r^N(x,t) = r_k(t) + \frac{1}{h}(x - kh)(r_{k+1}(t) - r_k(t)), \tag{56}$$

and similarly for $x \in [jh, (j+1)h]$, $j = \frac{1}{2}, \dots, N - \frac{3}{2}$, we define

$$\rho^{N-\frac{1}{2}}(x,t) = \rho_j(t) + \frac{1}{h}(x - jh)(\rho_{j+1}(t) - \rho_j(t)), \tag{57}$$

$$\theta^{N-\frac{1}{2}}(x,t) = \theta_j(t) + \frac{1}{h}(x - jh)(\theta_{j+1}(t) - \theta_j(t)). \tag{58}$$

For $x \in [0, \frac{1}{2}h]$ we take

$$\rho^{N-\frac{1}{2}}(x,t) = \rho_{\frac{1}{2}}(t), \quad \theta^{N-\frac{1}{2}}(x,t) = \theta_{\frac{1}{2}}(t),$$

and for $x \in [1 - \frac{1}{2}h, 1]$

$$\rho^{N-\frac{1}{2}}(x,t) = \rho_{N-\frac{1}{2}}(t), \quad \theta^{N-\frac{1}{2}}(x,t) = \theta_{N-\frac{1}{2}}(t).$$

We also introduce the corresponding step functions:

(59) $$(v_h, \omega_h, r_h)(x,t) = (v_k, \omega_k, r_k)(t),\ \ x \in \langle kh, (k+1)h],\ k = 0, \dots, N-1,$$

(60) $$(\rho_{h-\frac{1}{2}}, \theta_{h-\frac{1}{2}})(x,t) = (\rho_j, \theta_j)(t),\ \ x \in \langle jh, (j+1)h],\ j = \tfrac{1}{2}, \dots, N - \tfrac{3}{2},$$

(61) $$(\rho_{h-\frac{1}{2}}, \theta_{h-\frac{1}{2}})(x,t) = (\rho_{\frac{1}{2}}, \theta_{\frac{1}{2}})(t),\ \ x \in [0, \tfrac{1}{2}h],$$

(62) $$(\rho_{h-\frac{1}{2}}, \theta_{h-\frac{1}{2}})(x,t) = (\rho_{N-\frac{1}{2}}, \theta_{N-\frac{1}{2}})(t),\ \ x \in \langle 1 - \tfrac{1}{2}h, 1].$$

In this section the semidiscrete finite difference scheme resulting with the system of ordinary differential equations is defined. In what follows the main research will relate to the analysis of the convergence of the obtained schemes.

4. The main result

The aim of this paper is to prove the following statements.

Theorem 4.1. *Suppose that the initial data* $(\rho_0, v_0, \omega_0, \theta_0)$ *satisfy the conditions (14)-(15). Then, for the sequences of the approximate solutions* $\{(\rho^{N-\frac{1}{2}}, v^N, \omega^N, r^N, \theta^{N-\frac{1}{2}})\}$ *and* $\{(\rho_{h-\frac{1}{2}}, v_h, \omega_h, r_h, \theta_{h-\frac{1}{2}})\}$ *in the domain* Q_T *(for any* $T \in \mathrm{R}^+$*), as* $N \to \infty$ *(or* $h \to 0$*), we have:*

(63) $$(\rho^{N-\frac{1}{2}}, v^N, \omega^N, r^N, \theta^{N-\frac{1}{2}}) \to (\rho, v, \omega, r, \theta) \quad \textit{strongly in}\ \ (C(\overline{Q}_T))^5,$$

(64) $$\textit{* weakly in}\ \ (L^\infty(0,T; H^1(\langle 0,1\rangle)))^5,$$

(65) $$\textit{weakly in}\ \ (H^1(Q_T))^5,$$

(66) $$(v^N, \omega^N, r^N, \theta^{N-\frac{1}{2}}) \to (v, \omega, r, \theta) \quad \textit{weakly in}\ \ (L^2(0,T; H^2(\langle 0,1\rangle)))^4,$$

(67) $$(\rho_{h-\frac{1}{2}}, v_h, \omega_h, r_h, \theta_{h-\frac{1}{2}}) \to (\rho, v, \omega, r, \theta) \quad \textit{strongly in}\ \ (L^\infty(0,T; L^2(\langle 0,1\rangle)))^5,$$

(68) $$r^N \to r \quad \textit{* weakly in}\ \ (L^\infty(0,T; H^2(\langle 0,1\rangle))).$$

The function $(\rho, v, \omega, r, \theta)$ *satisfies equations (1)-(4) a.e. in* Q_T*, conditions (9)-(11), (13) in the sense of traces and* ρ*,* θ *and* r *have the properties*

(69) $$\inf_{Q_T} \rho > 0, \quad \inf_{Q_T} \theta > 0, \quad \inf_{Q_T} r > 0,$$

(70) $$r(x,t) = r_0(x) + \int_0^t v(x,\tau)\, d\tau, \quad (x,t) \in Q_T.$$

Remark. Theorem 4.1 will follow as a consequence of the various results stated and proved in the upcoming sections. The limit in Theorem 4.1 shall be obtained as the limit of some subsequences (see Section 10). It will be proved that this limit function $(\rho, v, \omega, r, \theta)$ solves our problem (1)-(15) and satisfies properties (19)-(23). Therefore, it coincides with the function $(\rho, v, \omega, r, \theta)$ introduced in Section 2, so it is the generalized solution of our problem (1)-(15). It is proved in [10] that such a generalized solution is unique, thus, the analysed strong and weak convergences of the subsequences imply the convergence not only of the subsequences, but of the whole sequences $\{(\rho^{N-\frac{1}{2}}, v^N, \omega^N, r^N, \theta^{N-\frac{1}{2}})\}$ and $\{(\rho_{h-\frac{1}{2}}, v_h, \omega_h, r_h, \theta_{h-\frac{1}{2}})\}$ as $N \to \infty$ (or $h \to 0$). This property means that a sequence of numerical solutions obtained by applying the proposed numerical method converge to the solution of the considered problem.

The proof of the Theorem 4.1 is performed through several stages. It is essentially based on a careful examination of a priori estimates and a limit procedure. We first study, for each N, the approximate problem (36)-(42), (31)-(35) and derive the a priori estimates for its solution independent of N (or h) by utilizing a technique of articles [4, 3, 25]. This is done in Sections 5-9. Using the obtained a priori estimates and results of weak and strong compactness [7, 8], we extract in Section 10, the subsequences of approximate solutions, which, when N tends to infinity (or $h \to 0$), have the limit in the strong or weak sense on the domain $Q_T = \langle 0,1\rangle \times \langle 0,T\rangle$, where $T > 0$ is arbitrary. Finally, we show that this limit is the solution to our problem.

In this article we use some ideas from [2] and as mentioned above, we use the same procedure as in [25]. Some of our considerations are very similar or identical to those in [4, 3, 25]. In these cases we omit proofs or details of proofs, making references to corresponding pages of the articles [4, 3, 25].

The proof of our theorem is a direct consequence of the results that we obtain in the following sections.

5. Basic relations and estimates. Global construction of the difference scheme

Throughout this paper we denote by $C > 0$ or $C_i > 0$ ($i = 1, 2, \ldots$) generic constants independent of N (i.e. h), having possibly different values at different places.

In this section we first make some key relations and estimates for $(\rho_j, v_k, \omega_k, r_k, \theta_j)(t)$, $j = \frac{1}{2}, \ldots, N-\frac{1}{2}$, $k = 0, \ldots, N$ in the domain $[0, T_{max}\rangle$, where T_{max} is introduced in Section 3. Then we show that the Cauchy problem (36)-(42), (31)-(35) has a solution in the domain $[0, T\rangle$, where $T > 0$ is arbitrary.

In the following two lemmas we have the estimates which are formed in the same manner as in [25]. In the first lemma we made some basic estimates, that shall be used through the whole article. The second lemma is crucial for proving that the solutions of the system (36)-(42) are globally defined.

Lemma 5.1. *For $t \in [0, T_{max}\rangle$, it holds*

$$\frac{1}{M} \leq \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{h}{\rho_j(t)} \leq \frac{1}{m}, \tag{71}$$

$$r_k^3(t) = a^3 + 3L \sum_{j=\frac{1}{2}}^{k-\frac{1}{2}} \frac{h}{\rho_j(t)}, \quad k = 1, \ldots, N \tag{72}$$

$$a = r_0(t) < r_1(t) \leq \ldots \leq r_N(t) \leq C, \tag{73}$$

$$(\delta r^3)_j(t) = \frac{3L}{\rho_j(t)}, \quad j = \tfrac{1}{2}, \ldots, N-\tfrac{1}{2}, \tag{74}$$

where $C \in \mathrm{R}^+$ is independent of T_{max}.

Proof. From (36) and (33) we obtain

$$\frac{d}{dt}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{h}{\rho_j(t)} \right) = \frac{1}{L} \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \delta(r^2 v)_j h = 0. \tag{75}$$

Integrating over $[0,t]$, $t < T_{max}$, and using (44) we immediately get (71). Notice that, for $k = 1, \ldots, N$, from (40) and (36) follows

$$\dot{r}_k(t) = r_k^{-2} \sum_{j=\frac{1}{2}}^{k-\frac{1}{2}} \delta(r^2 v)_j h = r_k^{-2} L \sum_{j=\frac{1}{2}}^{k-\frac{1}{2}} \frac{d}{dt}\Big(\frac{1}{\rho_j}\Big) h,$$

i.e.,

$$\frac{d}{dt}(r_k^3) = 3L \frac{d}{dt}\Big(\sum_{j=\frac{1}{2}}^{k-\frac{1}{2}} \frac{1}{\rho_j} h\Big).$$

Using (34), (35), (40), (71) and integrating over $[0,t]$ we easily conclude that (72), (73) and (74) are fulfilled. □

Lemma 5.2. *There exists $C > 0$ such that, for $t \in [0, T_{max}\rangle$, the following inequality holds*

$$\begin{aligned} &\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Phi(\frac{1}{\rho_j})h + \sum_{k=1}^{N-1} v_k^2 h + \sum_{k=1}^{N-1} \omega_k^2 h + \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Phi\left(\theta_j\right) h + \\ &+ \int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{\rho_j}{\theta_j} (\delta(r^2 v)_j) h \, d\tau + \int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{\rho_j}{\theta_j} (\delta(r^2 \omega)_j)^2 h \, d\tau + \\ &+ \int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{\omega_{j+\frac{1}{2}}^2}{\rho_j \theta_j} h \, d\tau + \int_0^t \sum_{k=1}^{N-1} \frac{r_k^4 \rho_k (\delta\theta_k)^2}{\theta_{k-\frac{1}{2}} \theta_{k+\frac{1}{2}}} h \, d\tau \le C, \end{aligned} \tag{76}$$

where C depends only of the initial data, and $\Phi(x) = x - 1 - \ln x$ is a nonnegative convex function.

Proof. In a similar manner as in [14, Lemma 5.1], multiplying (36), (37), (38) and (39), respectively, by $R(\rho_j - 1)\rho_j^{-2} h$, $v_k h$, $j_I \rho_k^{-1} \omega_k h$ and $c_v \rho_j^{-1}(1 - \theta_j^{-1}) h$, summing over $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$ and $k = 1, \ldots, N-1$, and adding the obtained equations, we get

$$\begin{aligned} &\frac{d}{dt}\left[R \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Phi(\frac{1}{\rho_j})h + \frac{1}{2}\sum_{k=1}^{N-1} v_k^2 h + \frac{j_I}{2} \sum_{k=1}^{N-1} \omega_k^2 h + c_v \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Phi\left(\theta_j\right) h \right] + \\ &+ \tfrac{\lambda+2\mu}{L^2} \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{\rho_j}{\theta_j} (\delta(r^2 v)_j)^2 h + \tfrac{c_0 + 2c_d}{L^2} \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{\rho_j}{\theta_j} (\delta(r^2\omega)_j)^2 h + 4\mu_r \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{\omega_j^2}{\rho_j \theta_j} h + \\ &+ \tfrac{K}{L^2} \sum_{k=1}^{N-1} \frac{r_k^4 \rho_k (\delta\theta_k)^2}{\theta_{k-\frac{1}{2}}\theta_{k+\frac{1}{2}}} h = \frac{4\mu}{L} \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{1}{\theta_j} \delta(rv^2)_j h + \frac{4c_d}{L} \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{1}{\theta_j} \delta(r\omega^2)_j h. \end{aligned} \tag{77}$$

Applying (74) on the right-hand side of (77) we obtain

$$\tfrac{4\mu}{L} \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{1}{\theta_j} \delta(rv^2)_j h = -\tfrac{4\mu}{3L^2} \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{\rho_j r_{j+\frac{1}{2}}^3 r_{j-\frac{1}{2}}^3}{\theta_j} \big(\delta\big(\frac{v}{r}\big)_j\big)^2 h + \tfrac{4\mu}{3L^2} \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \frac{\rho_j}{\theta_j} \left(\delta(r^2 v)_j\right)^2 h \tag{78}$$

(79)
$$\tfrac{4c_d}{L}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{1}{\theta_j}\delta(r\omega^2)_j h=-\tfrac{4c_d}{3L^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\rho_j r^3_{j+\frac{1}{2}}r^3_{j-\frac{1}{2}}}{\theta_j}\big(\delta\big(\frac{\omega}{r}\big)_j\big)^2 h+\tfrac{4c_d}{3L^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\rho_j}{\theta_j}\left(\delta(r^2\omega)_j\right)^2 h$$

Inserting (78) and (79) into (77), integrating over $[0,t]$, $t<T_{max}$ and using estimates (43) and (46) we conclude that (76) is valid. □

Now in the same way as in [25] and [4], this result verifies the existence of solution to the Cauchy problem (36)-(42), (31)-(35) for all time, that is $T_{max}=\infty$. Indeed, for fixed $h>0$, the estimate

$$\Phi\big(\frac{1}{\rho_j(t)}\big)+v_k(t)^2+\omega_k(t)^2+\Phi\big(\theta_j(t)\big)\le\frac{C}{h},$$

together with (73), implies the global bounds of the functions $(\rho_j,v_k,\omega_k,r_k,\theta_j)$:

$$0<\Phi_-^{-1}\big(\frac{C}{h}\big)\le\rho_j^{-1}(t),\,\theta_j(t)\le\Phi_+^{-1}\big(\frac{C}{h}\big)<\infty,$$
$$a\le r_k(t)\le C,$$
$$|v_k(t)|<\frac{C}{\sqrt{h}},\quad |\omega_k(t)|<\frac{C}{\sqrt{h}}$$

for all $j\in\{\frac{1}{2},\dots,N-\frac{1}{2}\}$ and $k\in\{0,\dots,N\}$. Here $\Phi_\pm^{-1}$ denote the two branches of the inverse function of Φ defined on $\langle 0,1]$ and $[1,\infty\rangle$, respectively. In this case, $(\rho_j,v_k,\omega_k,r_k,\theta_j)$ can be locally extended beyond the maximal time interval $[0,T_{max}\rangle$, that is a contradiction unless $T_{max}=\infty$.

Hence, we have our construction of the difference scheme $(\rho_j,v_k,\omega_k,r_k,\theta_j)(t)$ and the corresponding approximate solutions

$$(\rho^{N-\frac{1}{2}},v^N,\omega^N,r^N,\theta^{N-\frac{1}{2}})(x,t)\qquad\text{and}\qquad(\rho_{h-\frac{1}{2}},v_h,\omega_h,r_h,\theta_{h-\frac{1}{2}})(x,t)$$

defined on $[0,T\rangle$, where $T>0$ is arbitrary.

We shall derive some basic estimates, which we need to prove that the sequences of approximate solutions obtained by the proposed scheme, converge in the spaces introduced in Theorem 4.1.

It is important to note that hereafter all estimates will not depend on N (or h) and T.

Lemma 5.3. *There exists $C>0$ such that, for $t\in[0,T]$, it holds*

$$\sum_{k=1}^{N-1}\omega_k^2 h+\int_0^t\sum_{k=1}^{N-1}\frac{\omega_k^2}{\rho_k}h\,d\tau+\int_0^t\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\rho_j(\delta(r^2\omega)_j)^2 h\,d\tau\le C. \tag{80}$$

Proof. We multiply (38) by $\rho_k^{-1}\omega_k h$ and sum over k. After integration over $[0,t]$ and using (46), we get immediately (80). □

Lemma 5.4. *There exists $C>0$ such that, for $t\in[0,T]$, the estimate*

$$\sum_{k=1}^{N-1}v_k^2 h+\sum_{k=1}^{N-1}\omega_k^2 h+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\theta_j h\le C \tag{81}$$

is satisfied.

Proof. Multiplying (37), (38) and (39), respectively, by $v_k h$, $j_I \rho_k^{-1} \omega_k h$ and $c_v \rho_j^{-1} h$ and summing over j and k, we get, in the same way as in [14, Lemma 5.3], estimate (81). □

The estimates from Lemmas 5.1-5.4 ensure that the sequences $\{r^N\}$ and $\{r_h\}$ are bounded in $C(\bar{Q}_T)$ and $L^\infty(Q_T)$, respectively. Moreover, the sequences $\{(v^N, \omega^N)\}$ and $\{(v_h, \omega_h)\}$ are bounded in $L^\infty(0,T; L^2(\langle 0,1\rangle))$. The sequences $\{\theta^{N-\frac{1}{2}}\}$ and $\{\theta_{h-\frac{1}{2}}\}$ with positive terms are bounded in $L^\infty(0,T; L^1(\langle 0,1\rangle))$.

6. Boundedness of the density

From the results obtained in the next lemma we shall conclude that the sequences $\{\rho^{N-\frac{1}{2}}\}$ and $\{\rho_{h-\frac{1}{2}}\}$ are bounded in $C(\bar{Q}_T)$ and $L^\infty(Q_T)$, respectively. As a consequence, we get also the boundedness of the sequence $\{\partial_x r^N\}$ in $L^\infty(Q_T)$.

First, from (76), as in [25, 4], we have

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Phi\big(\frac{1}{\rho_j}\big) + \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Phi(\theta_j) \le \frac{C}{h} = CN,$$

which implies that, for $t \in [0,T]$, there exists at least one $a_t \in \{\frac{1}{2}, \dots, N-\frac{1}{2}\}$ such that

$$C^{-1} \le \rho_{a_t}(t) \le C, \quad C^{-1} \le \theta_{a_t}(t) \le C, \tag{82}$$

where $C \in \mathrm{R}^+$. This fact we use in the following lemmas.

Lemma 6.1. *There exist constants $C_1, C_2 \in \mathrm{R}^+$ such that, for all $t \in [0,T]$*

$$C_1 \le \rho_j(t) \le C_2, \quad j \in \{\frac{1}{2}, \dots, N-\frac{1}{2}\}. \tag{83}$$

Proof. The procedure is similar to that in [14, Lemma 6.1]. Inserting (36) into (37) we get

$$\frac{L}{\lambda+2\mu} r_k^{-2} \dot{v}_k = -\frac{R}{\lambda+2\mu}\delta(\rho\theta)_k - \frac{d}{dt}(\delta \ln\rho)_k, \quad k=1,\dots,N-1.$$

After multiplying by h and summarizing, it takes the form

$$\begin{aligned} &-\frac{L}{\lambda+2\mu}\sum_{m=a_t+\frac{1}{2}}^{k}\frac{d}{dt}\left(\frac{v_m}{r_m^2}\right)h - \frac{2L}{\lambda+2\mu}\sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m^2}{r_m^3}h = \\ &=\frac{R}{\lambda+2\mu}(\rho_{k+\frac{1}{2}}(t)\theta_{k+\frac{1}{2}}(t) - \rho_{a_t}(t)\theta_{a_t}(t)) + \frac{d}{dt}(\ln\rho_{k+\frac{1}{2}}(t) - \ln\rho_{a_t}(t)). \end{aligned} \tag{84}$$

(we have used a convention notation $\sum_{m=x}^{y} = -\sum_{m=y}^{x}$ for the case $y < x$). Integrating (84) over $[0,t]$ we obtain

$$\begin{aligned} &\frac{L}{\lambda+2\mu}\left(\sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m(0)}{r_m^2(0)}h - \sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m(t)}{r_m^2(t)}h - 2\int_0^t\sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m^2(\tau)}{r_m^3(\tau)}h\,d\tau\right) = \\ &=\frac{R}{\lambda+2\mu}\left(\int_0^t \rho_{k+\frac{1}{2}}(\tau)\theta_{k+\frac{1}{2}}(\tau)\,d\tau - \int_0^t \rho_{a_t}(\tau)\theta_{a_t}(\tau)\,d\tau\right) + \ln\frac{\rho_{k+\frac{1}{2}}(t)\rho_{a_t}(0)}{\rho_{k+\frac{1}{2}}(0)\rho_{a_t}(t)}. \end{aligned} \tag{85}$$

Now, we define the discrete Kazhikov-Shelukhin type of functions by

$$Y(t) = \exp\left\{\frac{R}{\lambda+2\mu}\int_0^t \rho_{a_t}(\tau)\theta_{a_t}(\tau)d\tau\right\}, \tag{86}$$

(87)
$$B_k(t) = \exp\left\{\tfrac{L}{\lambda+2\mu}\left(\sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m(0)}{r_m^2(0)}h - \sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m(t)}{r_m^2(t)}h - 2\int_0^t\sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m^2(\tau)}{r_m^3(\tau)}h\,d\tau\right)\right\}$$

Inserting (86) and (87) into (85) and multiplying by $\frac{R}{\lambda+2\mu}\,\rho_{k+\frac{1}{2}}(t)\theta_{k+\frac{1}{2}}(t)$ we get, for $k = 1,\ldots,N-1$, that

$$\tfrac{R}{\lambda+2\mu}B_k(t)Y(t)\frac{\rho_{k+\frac{1}{2}}(0)\rho_{a_t}(t)\theta_{k+\frac{1}{2}}(t)}{\rho_{a_t}(0)} = \frac{d}{dt}\left\{\tfrac{R}{\lambda+2\mu}\int_0^t\rho_{k+\frac{1}{2}}(\tau)\theta_{k+\frac{1}{2}}(\tau)d\tau\right\}.$$

Integrating over $[0,t]$ and taking into account (85)–(87), we obtain

(88)
$$\tfrac{R}{\lambda+2\mu}\frac{\rho_{k+\frac{1}{2}}(0)}{\rho_{a_t}(0)}\int_0^t B_k(\tau)Y(\tau)\rho_{a_t}(\tau)\theta_{k+\frac{1}{2}}(\tau)d\tau = \frac{\rho_{a_t}(t)\rho_{k+\frac{1}{2}}(0)}{\rho_{k+\frac{1}{2}}(t)\rho_{a_t}(0)}B_k(t)Y(t) - 1.$$

Equality (88) which contains $\rho_{\frac{1}{2}}$ $(a_t > \frac{1}{2})$ we obtain analogously. Notice that, because of (81) and (73) we easily conclude that

$$\Big| - \sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m(0)}{r_m^2(0)}h - \sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m(t)}{r_m^2(t)}h - 2\int_0^t\sum_{m=a_t+\frac{1}{2}}^{k}\frac{v_m^2(\tau)}{r_m^3(\tau)}h\,d\tau\Big| \le$$

$$\le C\left(\left(\sum_{k=1}^{N-1}v_k^2(0)h\right)^{1/2} + \left(\sum_{k=1}^{N-1}v_k^2(t)h\right)^{1/2} + \int_0^t\sum_{k=1}^{N-1}v_k^2(\tau)h\,d\tau\right) \le C, \tag{89}$$

and from (87) follows that there exists $C \in \mathrm{R}^+$ such that

$$C^{-1} \le B_k(t) \le C, \tag{90}$$

for all $t \in [0,T]$ and $k = 1,\ldots,N-1$. It is obvious that $Y(t) > 1$, $t \in [0,T]$.

With the help of (90), (82) and (43), from (88) we obtain the inequality

$$\frac{Y(t)}{\rho_{k+\frac{1}{2}}(t)} \le C + \int_0^t\theta_{k+\frac{1}{2}}(\tau)Y(\tau)d\tau.$$

Multiplying the above inequality by h, summing up for $k = 0,\ldots,N-1$, and using estimates (71) and (81), we obtain

$$Y(t) \le C\left(1 + \int_0^t Y(\tau)d\tau\right),$$

from which, after applying the Gronwall inequality as in [14, Lemma 6.1] and [4, Proposition 5.1], we get that

$$1 \le Y(t) \le C \tag{91}$$

for all $t \in [0,T]$.

Now, using (85)-(87) and (90)-(91) we obtain

$$\begin{aligned}\rho_{k+\frac{1}{2}}(t) &\le \rho_{k+\frac{1}{2}}(t)\exp\left\{\tfrac{R}{\lambda+2\mu}\int_0^t\rho_{k+\frac{1}{2}}(\tau)\theta_{k+\frac{1}{2}}(\tau)d\tau\right\} = \\ &= \frac{\rho_{k+\frac{1}{2}}(0)\rho_{a_t}(t)}{\rho_{a_t}(0)}B_k(t)Y(t) \le C,\end{aligned} \tag{92}$$

for each $k = 0,\ldots,N-1$ and $t \in [0,T]$. Notice that from (88) follows the inequality

$$\frac{1}{\rho_{k+\frac{1}{2}}(t)} \le C\left(1 + \int_0^t\theta_{k+\frac{1}{2}}(\tau)d\tau\right). \tag{93}$$

Taking into account that for a_t, introduced by (82), we have

(94)
$$\theta_{k+\frac{1}{2}}(t) = \Big(\sqrt{\theta_{a_t}(t)} + \sum_{r=a_t+\frac{1}{2}}^{k} \delta(\sqrt{\theta})_r h\Big)^2 \leq C\Bigg(1 + \sum_{r=a_t+\frac{1}{2}}^{k} \frac{(\delta\theta)_r^2 \rho_{r-\frac{1}{2}} h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}} \sum_{r=a_t+\frac{1}{2}}^{k} \frac{\theta_{r+\frac{1}{2}}}{\rho_{r-\frac{1}{2}}} h\Bigg).$$

Using (81) and inserting (94) into (93) we get immediately

(95)
$$\max_{0\leq k\leq N-1} \frac{1}{\rho_{k+\frac{1}{2}}(t)} \leq C\left(1 + \int_0^t \max_{0\leq k\leq N-1} \frac{1}{\rho_{k+\frac{1}{2}}(\tau)} \sum_{k=1}^{N-1} \frac{(\delta\theta_k)^2 \rho_{k-\frac{1}{2}}}{\theta_{k-\frac{1}{2}}\theta_{k+\frac{1}{2}}} h\, d\tau\right).$$

Applying (73), (76) and the Gronwall inequality from (95) we conclude that

$$\frac{1}{\rho_{k+\frac{1}{2}}(t)} \leq C, \tag{96}$$

for each $k = 0, \ldots, N-1$ and $t \in [0, T]$. The estimates (96) and (92) constitute the proof of (83). □

Using (83) and (73) from (74) we easily get that there exists $C \in \mathrm{R}^+$ such that

$$C^{-1} \leq \delta r_j(t) \leq C \tag{97}$$

for $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$ and $t \in [0, T]$.

7. Boundedness of the energy density and its consequences

To be able to derive estimates for the derivatives of the approximate solutions $(v^N, \omega^N, \theta^{N-\frac{1}{2}})$ and to prove the boundedness of this sequence in appropriate vector spaces, we will make first the estimate of the energy density defined as

$$W_k(t) = \frac{1}{2} v_k^2(t) + \frac{j_I}{2}\omega_k^2(t) + c_v \theta_{k-\frac{1}{2}}(t), \tag{98}$$

for $k = 1, \ldots, N$ and $t \in [0, T]$. It is easy to see that $W_k(t) > 0$, for each k. Multiply equations (37), (38) and (39), respectively, by $v_k W_k h$, $j_I \rho_k^{-1} \omega_k W_k h$ and $c_v \rho_j^{-1} W_{j+\frac{1}{2}} h$ $(j = k - \frac{1}{2})$ and sum up the resulting equality for $k = 1, \ldots, N$. Then, in the similar way as in [14, Section 7], we obtain the following inequality

$$\begin{aligned}
&\sum_{k=1}^{N} W_k^2 h + \int_0^t \sum_{k=1}^{N} (\delta W_{k-\frac{1}{2}})^2 h\, d\tau + \int_0^t \sum_{k=1}^{N-1} (\delta\theta_k)^2 h\, d\tau \leq \\
&\leq C_1 \left(1 + \int_0^t \left(\sum_{k=1}^{N} v_k^4 h + \sum_{k=1}^{N} \omega_k^4 h + \sum_{k=1}^{N} (\delta v_k)^2 (v_k^2 + v_{k-1}^2) h + \right.\right. \\
&\qquad \left.\left. \sum_{k=1}^{N} (\delta\omega_k)^2 (\omega_k^2 + \omega_{k-1}^2) h + \sum_{k=1}^{N} \theta_{k-\frac{1}{2}}^2 (v_k^2 + v_{k-1}^2) h\right) d\tau\right),
\end{aligned}$$

which is crucial for obtaining the results of the following Lemmas 7.1-7.4. As a consequence of these results, we get the boundedness of the sequence $\{(v^N, \omega^N, \theta^{N-\frac{1}{2}})\}$ in $L^2(0, T; H^1(\langle 0, 1\rangle)) \cap L^2(0, T; C([0, 1]))$.

Lemma 7.1. *For $t \in [0,T]$, the inequality*

$$\sum_{k=1}^{N} W_k^2 h + \sum_{k=1}^{N} v_k^4 h + \sum_{k=1}^{N} \omega_k^4 h + \int_0^t \sum_{k=1}^{N} \omega_k^4 h d\tau + \int_0^t \sum_{k=1}^{N} (\delta W_{k-\frac{1}{2}})^2 h \; d\tau + \tag{99}$$

$$+ \int_0^t \sum_{k=1}^{N-1} (\delta\theta_k)^2 h \; d\tau \leq C\bigg(1 + \int_0^t \Big(\sum_{k=1}^{N} v_k^4 h + \sum_{k=1}^{N} \omega_k^4 h + \sum_{k=1}^{N} \theta_{k-\frac{1}{2}}^2 (v_k^2 + v_{k-1}^2) h\Big) \, d\tau\bigg).$$

is satisfied.

Proof. Multiplying (37) and (38), respectively, by $\frac{L^2}{\lambda+2\mu} v_k^3 h$ and $\frac{j_I L^2}{c_0+2c_d}\rho_k^{-1}\omega_k^3 h$, summing over k, we get

$$\frac{L^2}{4(\lambda+2\mu)} \frac{d}{dt} \sum_{k=1}^{N} v_k^4 h = \tag{100}$$

$$= \frac{LR}{\lambda+2\mu} \sum_{k=1}^{N} \rho_{k-\frac{1}{2}} \theta_{k-\frac{1}{2}} \delta(r^2 v^3)_{k-\frac{1}{2}} h - \sum_{k=1}^{N} \rho_{k-\frac{1}{2}} \delta(r^2 v)_{k-\frac{1}{2}} \delta(r^2 v^3)_{k-\frac{1}{2}} h$$

$$\frac{j_I L^2}{4(c_0+2c_d)} \frac{d}{dt} \sum_{k=1}^{N} \omega_k^4 h + \frac{4\mu_r L^2}{c_0+2c_d} \sum_{k=1}^{N} \frac{\omega_k^4}{\rho_k} h = - \sum_{k=1}^{N} \rho_{k-\frac{1}{2}} \delta(r^2\omega)_{k-\frac{1}{2}} \delta(r^2\omega^3)_{k-\frac{1}{2}} h. \tag{101}$$

Using (97) and (73) we obtain the following inequalities

$$\frac{LR}{\lambda+2\mu} \sum_{k=1}^{N} \rho_{k-\frac{1}{2}} \theta_{k-\frac{1}{2}} \delta(r^2 v^3)_{k-\frac{1}{2}} h \leq \tag{102}$$

$$\leq \epsilon r_{k-1}^4 (\delta v_{k-\frac{1}{2}})^2 (v_k^2 + v_{k-1}^2) + C\big(\theta_{k-\frac{1}{2}}^2 (v_k^2 + v_{k-1}^2) + v_k^4\big),$$

$$-\rho_{k-\frac{1}{2}} \delta(r^2 v)_{k-\frac{1}{2}} \delta(r^2 v^3)_{k-\frac{1}{2}} h \leq -\frac{1}{4} r_{k-1}^4 (\delta v_{k-\frac{1}{2}})^2 (v_k^2 + v_{k-1}^2) + C(v_k^4 + v_{k-1}^4). \tag{103}$$

Notice that (103) satisfies the function ω_k also.

Inserting (102) and (103) into (100) and (101), integrating over $[0,t]$ we conclude (for sufficiently small $\epsilon > 0$) that

$$\sum_{k=1}^{N} v_k^4 h + \int_0^t \sum_{k=1}^{N} (\delta v_{k-\frac{1}{2}})^2 (v_k^2 + v_{k-1}^2) h \, d\tau \leq \tag{104}$$

$$\leq C_2 \bigg(1 + \int_0^t \sum_{k=1}^{N} v_k^4 h \, d\tau + \int_0^t \sum_{k=1}^{N} \theta_{k-\frac{1}{2}}^2 (v_k^2 + v_{k-1}^2) h \, d\tau\bigg),$$

$$\sum_{k=1}^{N} \omega_k^4 h + \int_0^t \sum_{k=1}^{N} \frac{\omega_k^4}{\rho_k} h d\tau + \int_0^t \sum_{k=1}^{N} (\delta\omega_{k-\frac{1}{2}})^2 (\omega_k^2 + \omega_{k-1}^2) h \, d\tau \leq \tag{105}$$

$$\leq C_3 \bigg(1 + \int_0^t \sum_{k=1}^{N} \omega_k^4 h \, d\tau\bigg).$$

Now, we multiply (104) and (105) by C_1 (determined in (98)), so that after adding the obtained inequalities to (98), the parts consisting of $(\delta\omega_{k-\frac{1}{2}})^2(\omega_k^2 + \omega_{k-1}^2)$ and $(\delta v_{k-\frac{1}{2}})^2 (v_k^2 + v_{k-1}^2)$ cancel each other out. So, using (83), we get (99). $\square$

Lemma 7.2. *There exists $C \in \mathrm{R}^+$, such that*

$$\sum_{k=1}^{N}(\theta^2_{k-\frac{1}{2}} + v_k^4 + \omega_k^4)h \le C. \tag{106}$$

Proof. Inserting (94) into (99), using estimates (83), (81), the Young inequality and the property $\theta^2_{k-\frac{1}{2}} \le W_k^2$, from (99) we obtain

(107)
$$\sum_{k=1}^{N}(\theta^2_{k-\frac{1}{2}}+ v_k^4 + \omega_k^4)\, h \le C\bigg(1 +\int_0^t\Big(1 +\sum_{r=1}^{N-1} \frac{(\delta\theta_r)^2\rho_{r-\frac{1}{2}}h}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\Big) \sum_{k=1}^{N}(\theta^2_{k-\frac{1}{2}}+ v_k^4 + \omega_k^4)\, h\, d\tau\bigg)$$

Applying the Gronwall inequality, estimates (73) and (76), from (107) we get immediately (106) for any $t \in [0, T]$. $\square$

Taking into account (94), (83), (81) and the Young inequality, we have

$$\int_0^t \sum_{k=1}^{N} \theta^2_{k-\frac{1}{2}}(v_k^2 + v_{k-1}^2)\, h\, d\tau \le C \int_0^t \Big(1 + \sum_{r=1}^{N-1} \frac{(\delta\theta_r)^2\rho_{r-\frac{1}{2}}}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\, h \sum_{r=1}^{N-1} \frac{\theta_{r+\frac{1}{2}}}{\rho_{r-\frac{1}{2}}}\, h\Big)\, d\tau,$$

$$\sum_{k=1}^{N}\theta_{k-\frac{1}{2}}(v_k^2 + v_{k-1}^2)\, h \le C\int_0^t\Big(1 + \sum_{r=1}^{N-1} \frac{(\delta\theta_r)^2\rho_{r-\frac{1}{2}}}{\theta_{r-\frac{1}{2}}\theta_{r+\frac{1}{2}}}\, h\Big) \sum_{k=1}^{N}(\theta^2_{k-\frac{1}{2}}+ v_k^4 + v_{k-1}^4)\, h\, d\tau,$$

and then, with the help of (76) and (106), we easily conclude that

$$\int_0^t \sum_{k=1}^{N} \theta^2_{k-\frac{1}{2}}(v_k^2 + v_{k-1}^2)\, h\, d\tau \le C. \tag{108}$$

Using (108) and (106) from (104) and (105) we get the estimates

$$\int_0^t \sum_{k=1}^{N}(\delta v_{k-\frac{1}{2}})^2(v_k^2 + v_{k-1}^2)\, h\, d\tau \le C, \tag{109}$$

$$\int_0^t \sum_{k=1}^{N}(\delta \omega_{k-\frac{1}{2}})^2(\omega_k^2 + \omega_{k-1}^2)\, h\, d\tau \le C, \tag{110}$$

and from (99) follows

$$\int_0^t \sum_{k=1}^{N-1}(\delta\theta_k)^2\, h\, d\tau \le C \tag{111}$$

for all $t \in [0, T]$.

Lemma 7.3. *There exists $C \in \mathrm{R}^+$ such that, for all $t \in [0, T]$, it holds*

$$\int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h d\tau \le C, \tag{112}$$

$$\int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta \omega_j)^2 h d\tau \le C. \tag{113}$$

Proof. Multiplying (37) by $v_k h$ and summing over $k = 1, \ldots, N-1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{k=1}^{N-1} v_k^2 h + \frac{\lambda+2\mu}{L^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j\big(\delta(r^2v)_j\big)^2 h = \frac{R}{L}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j\theta_j\delta(r^2v)_j h. \tag{114}$$

Integrating over $[0,t]$ and using (83), the Young inequality and (46) for v_k, from (114) we get

$$\begin{aligned}&\sum_{k=1}^{N-1} v_k^2 h + C_1\int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \big(\delta(r^2v)_j\big)^2 h\,d\tau \leq\\ &\leq \frac{C_1}{2}\int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \big(\delta(r^2v)_j\big)^2 h\,d\tau + C\Big(1+\int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \theta_j^2 h\,d\tau\Big)\end{aligned}$$

and with the help of (106), it follows

$$\int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \big(\delta(r^2v)_j\big)^2 h\,d\tau \leq C. \tag{115}$$

Applying the inequality

$$|\delta v_j|^2 \leq C|\delta v_j\, r_{j-\frac{1}{2}}^2|^2 = C|\delta(r^2v)_j - \delta(r^2)_j\, v_{j+\frac{1}{2}}|^2 \leq C\left(\big(\delta(r^2v)_j\big)^2 + (v_{j+\frac{1}{2}})^2\right)$$

and (81), from (115) we get (112). Multiplying (38) by $\rho_k^{-1}\omega_k h$ and summing over $k = 1, \ldots, N-1$ we get (113) analogous to (112). $\square$

Lemma 7.4. *There exists $C \in \mathrm{R}^+$ such that, for all $t \in [0,T]$ the estimates*

$$\int_0^t \max_{0\leq k\leq N} |v_k|\,d\tau \leq C, \tag{116}$$

$$\int_0^t \max_{0\leq k\leq N} |\omega_k|\,d\tau \leq C, \tag{117}$$

$$\int_0^t \max_{\frac{1}{2}\leq j\leq N-\frac{1}{2}} \theta_j^2\,d\tau \leq C, \qquad \int_0^t \max_{\frac{1}{2}\leq j\leq N-\frac{1}{2}} \theta_j\,d\tau \leq C. \tag{118}$$

are satisfied.

Proof. (116)-(118) are proven in [14, pp. 15-16]. $\square$

From

$$\begin{aligned}\int_0^t \max_{0\leq k\leq N} v_k^2\,d\tau &\leq C\Big(1+\int_0^t \max_{0\leq k\leq N} v_k^3\,h\,d\tau\Big) \leq\\ &\leq C\Big(1+\int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} |\delta v_j|(v_{j+\frac{1}{2}}^2+v_{j-\frac{1}{2}}^2)h\,d\tau\Big) \leq\\ &\leq C\Big(1+\int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h\,d\tau + \int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (v_{j+\frac{1}{2}}^4+v_{j-\frac{1}{2}}^4)\,h\,d\tau\Big),\end{aligned} \tag{119}$$

using (112) and (106) we get

$$\int_0^t \max_{0\le k\le N} v_k^2\, d\tau \le C. \tag{120}$$

In the same way we obtain

$$\int_0^t \max_{0\le k\le N} \omega_k^2\, d\tau \le C. \tag{121}$$

The estimates (81), (106), (111)-(113), (116)-(118), and (120)-(121) enable us to conclude that the sequence $\{(v^N, \omega^N, \theta^{N-\frac{1}{2}})\}$ is bounded in $L^2(0,T;H^1(\langle 0,1\rangle)) \cap L^2(0,T;C([0,1]))$.

8. Further bounds for the density, velocity and microrotation velocity

We proceed with the further bounds for the functions ρ_j, v_k, ω_k, r_k and θ_k, needed for proving the main theorem. The results from Lemmas 8.1-8.4 will ensure the boundedness of the sequence $\{\rho^{N-\frac{1}{2}}\}$ in $L^\infty(0,T;H^1(\langle 0,1\rangle)) \cap H^1(Q_T)$ and of the sequence $\{(v^N, \omega^N, r^N)\}$ in $L^2(0,T;H^2(\langle 0,1\rangle)) \cap C(\bar{Q}_T) \cap H^1(Q_T)$.

Lemma 8.1. *There exists a constant $C \in R^+$ such that, for all $t \in [0,T]$,*

$$\sum_{k=1}^{N-1} (\delta\rho_k)^2(t) h \le C \tag{122}$$

is true.

Proof. This estimate we derive by the same method as in [14, Lemma 8.1]. From (88) we have

$$\rho_{k+\frac{1}{2}}(t) = F_{k+\frac{1}{2}}(t) \cdot G^{-1}_{k+\frac{1}{2}}(t) \tag{123}$$

where

$$F_{k+\frac{1}{2}}(t) = \frac{\rho_{k+\frac{1}{2}}(0)\rho_{a_t}(t)}{\rho_{a_t}(0)} B_k(t)Y(t), \tag{124}$$

$$G_{k+\frac{1}{2}}(t) = 1 + \frac{R}{\lambda+2\mu}\frac{\rho_{k+\frac{1}{2}}(0)}{\rho_{a_t}(0)}\int_0^t B_k(\tau)Y(\tau)\rho_{a_t}(\tau)\theta_{k+\frac{1}{2}}(\tau)d\tau \tag{125}$$

(B_k and Y are defined by (87) and (86)). For estimating

$$\delta\rho_k = \frac{\delta F_k\, G_{k-\frac{1}{2}} - F_{k-\frac{1}{2}}\,\delta G_k}{G_{k+\frac{1}{2}}\, G_{k-\frac{1}{2}}} \tag{126}$$

we use the fact that there exists $C \in \mathrm{R}^+$ such that

$$C^{-1} \le G_{k+\frac{1}{2}}(t) \le C, \qquad C^{-1} \le F_{k+\frac{1}{2}}(t) \le C \tag{127}$$

for all $t \in [0,T]$ and $k = 0, \dots, N-1$. By the Taylor development we obtain

$$\delta B_{k-\frac{1}{2}}(t) = \frac{B_k(t)}{h}\left(1 - \exp\left\{\tfrac{L}{\lambda+2\mu}\Big(\frac{v_k(t)}{r_k^2(t)} - \frac{v_k(0)}{r_k^2(0)} + 2\int_0^t \frac{v_k^2(\tau)}{r_k^3(\tau)}\,d\tau\Big)\,h\right\}\right)$$
$$= -B_k(t)\Big(\tfrac{L}{\lambda+2\mu}\Big(\frac{v_k(t)}{r_k^2(t)} - \frac{v_k(0)}{r_k^2(0)} + 2\int_0^t \frac{v_k^2(\tau)}{r_k^3(\tau)}\,d\tau\Big) +$$
$$+ \big(\tfrac{L}{\lambda+2\mu}\big)^2 \tfrac{h}{2}\Big(\frac{v_k(t)}{r_k^2(t)} - \frac{v_k(0)}{r_k^2(0)} + 2\int_0^t \frac{v_k^2(\tau)}{r_k^3(\tau)}\,d\tau\Big)^2 \cdot$$
$$\exp\left\{\nu\Big(\frac{v_k(t)}{r_k^2(t)} - \frac{v_k(0)}{r_k^2(0)} + 2\int_0^t \frac{v_k^2(\tau)}{r_k^3(\tau)}\,d\tau\Big)\,h\right\}\Big),$$

for some $0 < \nu < 1$ and using (90), (73) and (120) we conclude that

$$|\delta B_{k-\frac{1}{2}}(t)| \le C\left(|v_k(t)| + |v_k(0)| + 1\right). \tag{128}$$

For $\delta F_k(t)$ and $\delta G_k(t)$ we obtain the following inequalities

$$|\delta F_k| \le C\left(|\delta\rho_k(0)| + |\delta B_{k-\frac{1}{2}}(t)|\right) \le C\left(|\delta\rho_k(0)| + |v_k(t)| + |v_k(0)| + 1\right), \tag{129}$$

$$|\delta G_k| \le C\left(|\delta\rho_k(0)| + \int_0^t (|v_k(\tau)| + |v_k(0)| + 1)\theta_{k+\frac{1}{2}}(\tau)\,d\tau + \int_0^t |\delta\theta_k(\tau)|\,d\tau\right). \tag{130}$$

Inserting (127) and (129)-(130) into (126) we easily get

$$\sum_{k=1}^{N-1}(\delta\rho_k(t))^2 h \le C\left(\sum_{k=1}^{N-1}(\delta\rho_k(0))^2 h + \sum_{k=1}^{N-1} v_k^2(t)h + \sum_{k=1}^{N-1} v_k^2(0)h + \sum_{k=1}^{N-1} h + \right.$$
$$\left. + \int_0^t \max_{0\le k\le N-1}\theta_{k+\frac{1}{2}}^2 \sum_{k=1}^{N-1}(v_k^2(0) + v_k^2(\tau))h d\tau + \int_0^t \sum_{k=1}^{N-1}(\delta\theta_k)^2 h d\tau\right). \tag{131}$$

Taking into account (47), (81), (111) and (118) from (130) we obtain (122). □

The next lemma and the comment after it are the auxiliary results, which help us to obtain the estimates for the derivative of the function (v^N, ω^N, r^N).

Lemma 8.2. *([14, Lemma 8.3]) For all $t \in [0,T]$ and $j \in \{\frac{1}{2}, \dots, N-\frac{1}{2}\}$ the functions δv_j and $\delta\omega_j$ satisfy the following inequalities:*

$$|\delta v_j| \le C\Big(\sum_{r=\frac{1}{2}}^{N-\frac{3}{2}}(\delta^2 v_{r+\frac{1}{2}})^2 h\Big)^{1/4}\Big(\sum_{r=\frac{1}{2}}^{N-\frac{1}{2}}(\delta v_r)^2 h\Big)^{1/4}, \tag{132}$$

$$|\delta\omega_j| \le C\Big(\sum_{r=\frac{1}{2}}^{N-\frac{3}{2}}(\delta^2 \omega_{r+\frac{1}{2}})^2 h\Big)^{1/4}\Big(\sum_{r=\frac{1}{2}}^{N-\frac{1}{2}}(\delta \omega_r)^2 h\Big)^{1/4}. \tag{133}$$

Notice that from (74) we get

$$\delta^2 r_k = -3L\frac{\delta\rho_k(r_{k+1}^2 + r_{k+1}r_k + r_k^2) + \rho_{k-\frac{1}{2}}(\delta r_{k+\frac{1}{2}} + \delta r_{k-\frac{1}{2}})(r_{k+1} + r_k + r_{k-1})}{\rho_{k+\frac{1}{2}}\rho_{k-\frac{1}{2}}(r_{k+1}^2 + r_{k+1}r_k + r_k^2)(r_k^2 + r_k r_{k-1} + r_{k-1}^2)}$$

and using (73), (83) and (97) follows the inequality

$$|\delta^2 r_k| \le C\left(|\delta\rho_k| + 1\right), \quad k = 1, \dots, N-1, \tag{134}$$

which will be used in future estimates.

Lemma 8.3. *For $t \in [0,T]$ we have*

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h + \int_0^t \sum_{k=1}^{N-1} (\delta^2 v_k)^2 h d\tau \leq C, \tag{135}$$

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta \omega_j)^2 h + \int_0^t \sum_{k=1}^{N-1} (\delta^2 \omega_k)^2 h d\tau \leq C, \tag{136}$$

$$|v_k(t)| \leq C, \quad |\omega_k(t)| \leq C, \qquad k \in 1, \ldots, N-1. \tag{137}$$

Proof. Multiplying (37) by $\delta^2 v_k h$ and summing up for $k = 1, \ldots, N-1$ we get

(138)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h + \tfrac{\lambda+2\mu}{L^2} \sum_{k=1}^{N-1} \rho_{k+\frac{1}{2}} r_k^4 (\delta^2 v_k)^2 h = -\tfrac{\lambda+2\mu}{L^2} \Bigg(\sum_{k=1}^{N-1} r_k^4 \delta\rho_k \delta^2 v_k \delta v_{k-\frac{1}{2}} h + \\
&+ \sum_{k=1}^{N-1} r_k^2 \rho_{k-\frac{1}{2}} \delta r_{k-\frac{1}{2}}^2 \delta^2 v_k \delta v_{k-\frac{1}{2}} h + \sum_{k=1}^{N-1} r_k^2 \delta\rho_k \delta r_{k+\frac{1}{2}}^2 v_{k+1} \delta^2 v_k h + \\
&+ \sum_{k=1}^{N-1} r_k^2 \rho_{k-\frac{1}{2}} \delta^2 r_k \delta^2 v_k v_{k+1} h + \sum_{k=1}^{N-1} r_k^2 \rho_{k-\frac{1}{2}} \delta r_{k-\frac{1}{2}}^2 \delta v_{k+\frac{1}{2}} \delta^2 v_k h \Bigg) + \\
&+ \frac{R}{L} \left(\sum_{k=1}^{N-1} r_k^2 \delta\rho_k \theta_{k+\frac{1}{2}} \delta^2 v_k h + \sum_{k=1}^{N-1} r_k^2 \rho_{k-\frac{1}{2}} \delta\theta_k \delta^2 v_k h \right).
\end{aligned}$$

With the help of (73), (132), (122), (83), (97), (134), (81), using the Hölder inequality and the Young inequality with a parameter $\epsilon > 0$, for the terms on the right-hand side of (138) we find the following estimates. For instance,

$$\begin{aligned}
&\Big| -\frac{\lambda+2\mu}{L^2} \sum_{k=1}^{N-1} r_k^4 \delta\rho_k \delta^2 v_k \delta v_{k-\frac{1}{2}} h \Big| \leq \\
&\leq C \Big(\sum_{k=1}^{N-1} (\delta^2 v_k)^2 h \Big)^{1/4} \Big(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h \Big)^{1/4} \sum_{k=1}^{N-1} |\delta\rho_k| \, |\delta^2 v_k| h \leq \\
&\leq C \Big(\sum_{k=1}^{N-1} (\delta^2 v_k)^2 h \Big)^{3/4} \Big(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h \Big)^{1/4} \Big(\sum_{k=1}^{N-1} (\delta\rho_k)^2 h \Big)^{1/2} \leq \\
&\leq \epsilon \sum_{k=1}^{N-1} (\delta^2 v_k)^2 h + C_\epsilon \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h.
\end{aligned} \tag{139}$$

In an analogous way one obtains inequalities:

(140)
$$\Big| -\tfrac{\lambda+2\mu}{L^2}\sum_{k=1}^{N-1} r_k^2\rho_{k-\frac12}\delta r_{k-\frac12}^2\delta^2 v_k\delta v_{k-\frac12}h\Big| \le \epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2h + C_\epsilon\sum_{k=1}^{N-1}(\delta v_{k-\frac12})^2h,$$

(141)
$$\Big| -\tfrac{\lambda+2\mu}{L^2}\sum_{k=1}^{N-1} r_k^2\delta\rho_k\delta r_{k+\frac12}^2 v_{k+1}\delta^2 v_k h\Big| \le \epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2h + C_\epsilon\max_{1\le k\le N-1} v_k^2,$$

(142)
$$\Big| -\tfrac{\lambda+2\mu}{L^2}\sum_{k=1}^{N-1} r_k^2\rho_{k-\frac12}\delta^2 r_k\delta^2 v_k v_{k+1}h\Big| \le 2\epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2h + C_\epsilon\max_{1\le k\le N-1} v_k^2 + C,$$

(143)
$$\Big| -\tfrac{\lambda+2\mu}{L^2}\sum_{k=1}^{N-1} r_k^2\rho_{k-\frac12}\delta r_{k-\frac12}^2\delta v_{k+\frac12}\delta^2 v_k h\Big| \le \epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2h + C_\epsilon\sum_{k=1}^{N-1}(\delta v_{k+\frac12})^2h,$$

$$\Big|\tfrac{R}{L}\sum_{k=1}^{N-1} r_k^2\delta\rho_k\theta_{k+\frac12}\delta^2 v_k h\Big| \le \epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2h + C_\epsilon\max_{\frac12\le k\le N-\frac12}\theta_j^2, \tag{144}$$

$$\Big|\tfrac{R}{L}\sum_{k=1}^{N-1} r_k^2\rho_{k-\frac12}\delta\theta_k\delta^2 v_k h\Big| \le \epsilon\sum_{k=1}^{N-1}(\delta^2 v_k)^2h + C_\epsilon\sum_{k=1}^{N-1}(\delta\theta_k)^2, \tag{145}$$

Inserting (139)-(145) into (138), integrating over $[0,t]$ and using (111), (118), (120), (73) and (82) (for sufficiently small $\epsilon>0$) we get (135).

Multiplying (38) by $\rho_k^{-1}\delta^2\omega_k h$ and applying the same procedure as for (135) we obtain (136).

From the inequality

$$|v_k(t)| \le \sum_{j=\frac12}^{N-\frac12}|\delta v_j|h \le \Big(\sum_{j=\frac12}^{N-\frac12}(\delta v_j)^2h\Big)^{\frac12}$$

that satisfies the function ω_k, $k=1,\dots,N-1$, also, follows (137). $\square$

Notice that because of (134) and (122) we easily get

$$\max_{0\le k\le N}|\dot r_k(t)| \le C, \qquad \sum_{k=1}^{N-1}(\delta^2 r_k)^2h \le C, \tag{146}$$

and from (132) and (133) follows

$$\int_0^t \max_{\frac12\le j\le N-\frac12}(\delta v_j)^2 d\tau \le C, \qquad \int_0^t \max_{\frac12\le j\le N-\frac12}(\delta\omega_j)^2 d\tau \le C \tag{147}$$

immediately. Taking into account (147), (137) and

$$|\delta(r^2v)|_j^2 \le C\left(|v_{j+\frac12}|^2 + |\delta v_j|^2\right)$$

we conclude that

$$\int_0^t \max_{\frac12\le j\le N-\frac12}\left(\delta(r^2v)_j\right)^2 d\tau \le C. \tag{148}$$

Analogously, we get

$$\int_0^t \max_{\frac{1}{2}\le j\le N-\frac{1}{2}} \big(\delta(r^2\omega)_j\big)^2\, d\tau \le C, \tag{149}$$

$$\int_0^t \sum_{k=1}^{N-1} \big(\delta^2(r^2v)_k\big)^2\, h\, d\tau \le C, \quad \int_0^t \sum_{k=1}^{N-1} \big(\delta^2(r^2\omega)_k\big)^2\, h\, d\tau \le C. \tag{150}$$

Lemma 8.4. *There exists $C \in \mathrm{R}^+$ such that*

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\dot\rho_j)^2 h \le C, \tag{151}$$

$$\int_0^t \sum_{k=1}^{N-1} (\dot v_k)^2 h d\tau \le C, \tag{152}$$

$$\int_0^t \sum_{k=1}^{N-1} (\dot\omega_k)^2 h d\tau \le C, \tag{153}$$

for all $t \in [0,T]$.

Proof. Applying (82), (73) and (97), from (36)-(38) we get following inequalities

$$\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\dot\rho_j)^2 h \le C\Bigg(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} v_{j+\frac{1}{2}}^2 h + \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta v_j)^2 h\Bigg),$$

$$\begin{aligned}\sum_{k=1}^{N-1} (\dot v_k)^2 h \le C\Big(&\max_{\frac{1}{2}\le j\le N-\frac{1}{2}} \theta_j^2 \sum_{k=1}^{N-1} (\delta\rho_k)^2 h + \sum_{k=1}^{N-1} (\delta\theta_k)^2 h + \\ &+ \max_{\frac{1}{2}\le j\le N-\frac{1}{2}} (\delta(r^2v)_j)^2 \sum_{k=1}^{N-1} (\delta\rho_k)^2 h + \sum_{k=1}^{N-1} (\delta^2(r^2v)_k)^2 h\Big),\end{aligned}$$

$$\sum_{k=1}^{N-1} (\dot\omega_k)^2 h \le C\Big(\sum_{k=1}^{N-1} \omega_k^2 h + \max_{\frac{1}{2}\le j\le N-\frac{1}{2}} (\delta(r^2\omega)_j)^2 \sum_{k=1}^{N-1} (\delta\rho_k)^2 h + \sum_{k=1}^{N-1} (\delta^2(r^2\omega)_k)^2 h\Big).$$

Integrating over $[0,t]$ and using estimates (122), (135)-(137), (148)-(150), (118) and (111) we obtain that (151)-(153) are satisfied. □

9. Further bounds for the temperature

Now, we prove in Lemma 9.1 the boundedness from bellow of the function θ_j uniformly by $j = \frac{1}{2}, \ldots, N-\frac{1}{2}$. This result ensures the boundedness from below of the sequences $\{\theta^{N-\frac{1}{2}}\}$ and $\{\theta_{h-\frac{1}{2}}\}$. In the proof we apply the same procedure as in [25] and [4].

Lemma 9.1. *There exists $C \in \mathrm{R}^+$ such that, for $j = \frac{1}{2}, \ldots, N-\frac{1}{2}$, and all $t \in [0,T]$, it holds*

$$\theta_j(t) \ge C. \tag{154}$$

Proof. We use the equality

$$-\frac{1}{\theta_j^2}\delta(r^4\rho\delta\theta)_j = \delta\big(r^4\rho\delta(\frac{1}{\theta})\big)_j - \left(r^4_{j+\frac{1}{2}}\rho_{j+\frac{1}{2}}\frac{(\delta\theta_{j+\frac{1}{2}})^2}{\theta_{j+1}\theta_j^2} + r^4_{j-\frac{1}{2}}\rho_{j-\frac{1}{2}}\frac{(\delta\theta_{j-\frac{1}{2}})^2}{\theta_{j-1}\theta_j^2}\right) \tag{155}$$

and the following inequalities

$$\begin{aligned}&\frac{\lambda+2\mu}{c_vL^2}\rho_j(\delta(r^2v)_j)^2 - \frac{4\mu}{c_vL}\delta(rv^2)_j = \\ &= \frac{1}{c_vL^2}\left(\big(\lambda+\frac{2\mu}{3}\big)\rho_j(\delta(r^2v)_j)^2 + \frac{4\mu}{3}\rho_j r_{j+\frac{1}{2}}r_{j-\frac{1}{2}}(\delta(rv)_j)^2\right) \geq 0,\end{aligned} \tag{156}$$

$$\begin{aligned}&\frac{c_0+2c_d}{c_vL^2}\rho_j(\delta(r^2\omega)_j)^2 - \frac{4c_d}{c_vL}\delta(r\omega^2)_j = \\ &= \frac{1}{c_vL^2}\left(\big(c_0+\frac{2c_d}{3}\big)\rho_j(\delta(r^2\omega)_j)^2 + \frac{4c_d}{3}\rho_j r_{j+\frac{1}{2}}r_{j-\frac{1}{2}}(\delta(r\omega)_j)^2\right) \geq 0,\end{aligned} \tag{157}$$

which are obtained by property (74). After multiplying (39) by $\rho_j^{-1}\theta_j^{-2}$ and using (155)-(157) we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{\theta_j}\right) + \frac{K}{c_vL^2}\left(r^4_{j+\frac{1}{2}}\rho_{j+\frac{1}{2}}\frac{(\delta\theta_{j+\frac{1}{2}})^2}{\theta_{j+1}\theta_j^2} + r^4_{j-\frac{1}{2}}\rho_{j-\frac{1}{2}}\frac{(\delta\theta_{j-\frac{1}{2}})^2}{\theta_{j-1}\theta_j^2}\right) = \\ &= -\frac{1}{c_vL^2\theta_j^2}\left(\big(\lambda+\frac{2\mu}{3}\big)\rho_j(\delta(r^2v)_j)^2 + \frac{4\mu}{3}\rho_j r_{j+\frac{1}{2}}r_{j-\frac{1}{2}}(\delta(rv)_j)^2\right) - \\ &\quad - \frac{1}{c_vL^2\theta_j^2}\left(\big(c_0+\frac{2c_d}{3}\big)\rho_j(\delta(r^2\omega)_j)^2 + \frac{4c_d}{3}\rho_j r_{j+\frac{1}{2}}r_{j-\frac{1}{2}}(\delta(r\omega)_j)^2\right) - \\ &\quad - \frac{4\mu_r}{c_v}\frac{\omega_j^2}{\rho_j\theta_j} + \frac{R}{c_vL}\frac{\rho_j}{\theta_j}\delta(r^2v)_j + \frac{K}{c_vL^2}\delta(r^4\rho\delta(\frac{1}{\theta}))_j \leq \\ &\leq \frac{R}{c_vL}\frac{\rho_j}{\theta_j}\delta(r^2v)_j + \frac{K}{c_vL^2}\delta(r^4\rho\delta(\frac{1}{\theta}))_j.\end{aligned} \tag{158}$$

Multiplying (158) by $2q(\frac{1}{\theta_j})^{2q-1}h$, $q\in \mathrm{N}\backslash\{1\}$, and summing up for $j=\frac{1}{2},\ldots,N-\frac{1}{2}$, we get for $\Psi_j = \frac{1}{\theta_j}$, the following inequality

$$\frac{d}{dt}\left(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Psi_j^{2q}h\right) \leq 2qC_1\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\rho_j\Psi_j^{2q}\delta(r^2v)_jh + 2qC_2\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\delta(r^4\rho\,\delta\Psi)_j\Psi_j^{2q-1}h. \tag{159}$$

Taking into account that

$$\begin{aligned}&2qC_2\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\delta(r^4\rho\,\delta\Psi)_j\Psi_j^{2q-1}h = \\ &= -2qC_2\sum_{k=1}^{N-1}r_k^4\rho_k(\delta\Psi_k)^2(\Psi^{2q-2}_{k+\frac{1}{2}} + \Psi^{2q-3}_{k+\frac{1}{2}}\cdot\Psi_{k-\frac{1}{2}} + \ldots + \Psi^{2q-2}_{k-\frac{1}{2}})h \leq 0,\end{aligned}$$

from (159) we conclude that

$$\frac{d}{dt}\Big(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Psi_j^{2q}h\Big) \leq 2qC_1\max_{\frac{1}{2}\leq j\leq N-\frac{1}{2}}|\delta(r^2v)_j|\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\Psi_j^{2q}h \tag{160}$$

and obtain the following differential inequality

$$D'(t) \le C_1 \max_{\frac{1}{2}\le j\le N-\frac{1}{2}} |\delta(r^2 v)_j|\, D(t), \tag{161}$$

where the function $D(t)$ is defined by

$$D(t) = \Big(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \Psi_j^{2q} h\Big)^{\frac{1}{2q}}. \tag{162}$$

With the help of (148), from (161) it follows

$$D(t) \le D(0) \exp\Big\{ C_1 \int_0^t \max_{\frac{1}{2}\le j\le N-\frac{1}{2}} |\delta(r^2 v)_j|\, d\tau \Big\} \le CD(0),$$

for $t \in [0,T]$, i.e., for $\Psi = (\Psi_{\frac{1}{2}}, \ldots, \Psi_{N-\frac{1}{2}})$, we have

$$\|\Psi(t)\|_{L^{2q}} \le CD(0), \tag{163}$$

where $D(0) \le \frac{1}{m}$. If $q \to \infty$ in (163) we get

$$\|\Psi(t)\|_{L^\infty} \le \frac{C}{m},$$

which implies

$$\theta_j(t) \ge \Big(\frac{m}{C}\Big)$$

for $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$ and all $t \in [0,T]$. □

In what follows we make the estimates for the functions $\delta\theta_k$ and $\delta^2\theta_j$. Notice that due to $\delta\theta_0 = 0$ we have inequalities

$$(\delta\theta)_k^2 \le \Big(\sum_{r=1}^{N} (\delta^2\theta_{r-\frac{1}{2}})^2 h\Big)^{\frac{1}{2}} \Big(\sum_{r=1}^{N-1} (\delta\theta_r)^2 h\Big)^{\frac{1}{2}}, \tag{164}$$

$$(\delta\theta)_k^2 \le \Big(\sum_{r=1}^{N} (\delta^2\theta_{r-\frac{1}{2}})^2 h\Big)^{\frac{1}{2}} \tag{165}$$

for each $k = 1, \ldots, N-1$.

By using the result of Lemma 9.1 and of the next two Lemmas 9.2-9.3, we shall conclude that the sequence $\{\theta^{N-\frac{1}{2}}\}$ is bounded in $L^2(0,T;H^2(\langle 0,1\rangle)) \cap C(\bar{Q}_T) \cap H^1(Q_T)$.

Lemma 9.2. *There exists $C \in \mathrm{R}^+$ such that, for all $t \in [0,T]$, it holds*

$$\sum_{k=1}^{N-1} (\delta\theta_k)^2 h + \int_0^t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} (\delta^2\theta_j)^2 h d\tau \le C, \tag{166}$$

$$|\theta_j(t)| \le C, \quad j = \tfrac{1}{2}, \ldots, N - \tfrac{1}{2}. \tag{167}$$

Proof. Multiplying (39) by $\rho_j^{-1}\delta^2\theta_j h$ and summing up for $j=\frac{1}{2},\dots,N-\frac{1}{2}$ we obtain

(168)

$$\frac{1}{2}\frac{d}{dt}\sum_{k=1}^{N-1}(\delta\theta_k)^2h+\tfrac{K}{c_vL^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}r^4_{j-\frac{1}{2}}\rho_{j-\frac{1}{2}}(\delta^2\theta_j)^2h=-\tfrac{K}{c_vL^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\delta(r^4\rho)_j\delta\theta_{j+\frac{1}{2}}\delta^2\theta_jh+$$

$$+\tfrac{R}{c_vL}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\rho_j\theta_j\delta(r^2v)_j\delta^2\theta_jh-\tfrac{\lambda+2\mu}{c_vL^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\rho_j(\delta(r^2v)_j)^2\delta^2\theta_jh+\tfrac{4\mu}{c_vL}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\delta(rv^2)_j\delta^2\theta_jh-$$

$$-\tfrac{c_0+2c_d}{c_vL^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\rho_j(\delta(r^2\omega)_j)^2\delta^2\theta_jh+\tfrac{4c_d}{c_vL}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\delta(r\omega^2)_j\delta^2\theta_jh-\tfrac{4\mu_r}{c_v}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\frac{\omega_j^2}{\rho_j}\delta^2\theta_jh.$$

To make the estimates for the terms on the right-hand side of (168) we use the following inequalities

$$|\delta(r^4\rho)_j|\le C(1+|\delta\rho_j|),\tag{169}$$

$$|\delta(r^2v)_j|\le C(1+|\delta v_j|),\tag{170}$$

$$|\delta(rv^2)_j|\le C(1+|\delta v_j|^2)\tag{171}$$

(the functions $r^2\omega$, and $r\omega^2$ satisfy (170)–(171), also) and then, with the help of (164), (122), (135), (106), (136), (137) and using the Young inequality with a parameter $\epsilon>0$ we find the following estimates. For instance,

$$\begin{aligned}(172)\quad &\Big|-\frac{K}{c_vL^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\delta(r^4\rho)_j\delta\theta_{j+\frac{1}{2}}\delta^2\theta_jh\Big|\le\\ &\le C\Bigg(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}|\delta\theta_{j+\frac{1}{2}}||\delta^2\theta_j|h+\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}|\delta\rho_j||\delta\theta_{j+\frac{1}{2}}||\delta^2\theta_j|h\Bigg)\le\\ &\le\epsilon\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta^2\theta_j)^2h+C\sum_{k=1}^{N-1}(\delta\theta_k)^2h+C\Big(\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta^2\theta_j)^2h\Big)^{3/4}\Big(\sum_{k=1}^{N-1}(\delta\theta_k)^2h\Big)^{1/2}\le\\ &\le 2\epsilon\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta^2\theta_j)^2h+C\sum_{k=1}^{N-1}(\delta\theta_k)^2h.\end{aligned}$$

In an analogous way one obtains the inequalities

$$\Big|\tfrac{R}{c_vL}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\rho_j\theta_j\delta(r^2v)_j\delta^2\theta_j\,h\Big|\le2\epsilon\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta^2\theta_j)^2h+C\max_{\frac{1}{2}\le j\le N-\frac{1}{2}}\theta_j^2,\tag{173}$$

(174)

$$\Big|-\tfrac{\lambda+2\mu}{c_vL^2}\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}\rho_j(\delta(r^2v)_j)^2\delta^2\theta_jh\Big|\le2\epsilon\sum_{j=\frac{1}{2}}^{N-\frac{1}{2}}(\delta^2\theta_j)^2h+C\Big(1+\max_{\frac{1}{2}\le j\le N-\frac{1}{2}}(\delta v_j)^2\Big),$$

$$\left|\tfrac{4\mu}{c_v L}\sum_{j=\frac12}^{N-\frac12}\delta(rv^2)_j\delta^2\theta_j h\right|\le 2\epsilon\sum_{j=\frac12}^{N-\frac12}(\delta^2\theta_j)^2h+C, \tag{175}$$

$$\left|-\tfrac{c_0+2c_d}{c_v L^2}\sum_{j=\frac12}^{N-\frac12}\rho_j(\delta(r^2\omega)_j)^2\delta^2\theta_j h\right|\le 2\epsilon\sum_{j=\frac12}^{N-\frac12}(\delta^2\theta_j)^2h+C, \tag{176}$$

$$\left|\tfrac{4c_d}{c_v L}\sum_{j=\frac12}^{N-\frac12}\delta(r\omega^2)_j\delta^2\theta_j h\right|\le 2\epsilon\sum_{j=\frac12}^{N-\frac12}(\delta^2\theta_j)^2h+C, \tag{177}$$

$$\left|-\tfrac{4\mu_r}{c_v}\sum_{j=\frac12}^{N-\frac12}\frac{\omega_j^2}{\rho_j}\delta^2\theta_j h\right|\le 2\epsilon\sum_{j=\frac12}^{N-\frac12}(\delta\theta_j)^2h+C. \tag{178}$$

Inserting (172)-(178) into (168), integrating over $[0,t]$ and using (83), (73), (147) and (118) (for sufficiently small $\epsilon>0$), we get (166).

Notice that the function θ_j, $j=\frac12,\dots,N-\frac12$ satisfies the property

$$|\theta_j(t)|\le\left(\sum_{k=1}^{N-1}(\delta\theta_k)^2h\right)^{\frac12}+\theta_{a_t}$$

where θ_{a_t} is introduced by (82). Applying (166) we get (167) immediately. $\square$

It remains to prove the following estimate.

Lemma 9.3. *There exists $C\in\mathrm{R}^+$ such that, for any $t\in[0,T]$, we have*

$$\int_0^t\sum_{j=\frac12}^{N-\frac12}(\dot\theta_j(t))^2\,h\,d\tau\le C. \tag{179}$$

Proof. In the proof we apply inequalities (169)-(171) and the inequality

$$\delta(r^4\rho\delta\theta)_j\le C\left(|\delta\theta_{j+\frac12}|+|\delta\rho_{j-\frac12}||\delta\theta_{j+\frac12}|+|\delta^2\theta_j|\right), \tag{180}$$

also. Multiplying (39) by ρ_j^{-1}, squaring, summing up for $j=\frac12,\dots,N-\frac12$ and using (73), (83), (165) we find that

$$\begin{aligned}\sum_{j=\frac12}^{N-\frac12}(\dot\theta_j)^2h\le C\Bigg(&\sum_{j=\frac12}^{N-\frac12}(\delta\theta_j)^2h+\sum_{j=\frac12}^{N-\frac12}(\delta^2\theta_j)^2h\sum_{k=1}^{N-1}(\delta\rho_k)^2h+\sum_{j=\frac12}^{N-\frac12}(\delta^2\theta_j)^2h+\\&+\max_{\frac12\le j\le N-\frac12}\theta_j^2\sum_{j=\frac12}^{N-\frac12}(\delta v_j)^2h+\sum_{j=\frac12}^{N-\frac12}\theta_j^2h+\max_{\frac12\le j\le N-\frac12}(\delta v_j)^2\sum_{j=\frac12}^{N-\frac12}(\delta v_j)^2h+\\&+\max_{\frac12\le j\le N-\frac12}(\delta\omega_j)^2\sum_{j=\frac12}^{N-\frac12}(\delta\omega_j)^2h+\max_{\frac12\le j\le N-\frac12}\omega_j^2\sum_{j=\frac12}^{N-\frac12}\omega_j^2h\Bigg).\end{aligned} \tag{181}$$

Taking into account estimates (106), (101), (118), (122), (135), (136), (147) and (167) and integrating over $[0,T]$, from (181) we get (179). $\square$

10. Convergence of approximate solutions to a solution of (1)-(4), (9)-(12)

In this section we show the compactness of sequences of approximate solutions $(\rho^{N-\frac{1}{2}}, v^N, \omega^N, r^N, \theta^{N-\frac{1}{2}})$ and $(\rho_{h-\frac{1}{2}}, v_h, \omega_h, r_h, \theta_{h-\frac{1}{2}})$, which are defined by (54)-(62) and their convergence to a solution $(\rho, v, \omega, r, \theta)$ of (1)-(4), (9)-(12). The results that follow are similar to those in [25]. Therefore, we omit the proofs of some statements.

With the help of (83), (122) and (150) we conclude that there exists $C \in \mathrm{R}^+$ (independent of N), such that

$$|\rho^{N-\frac{1}{2}}(x,t)| + \int_0^1 (\partial_x \rho^{N-\frac{1}{2}})^2(x,t)dx + \int_0^1 (\partial_t \rho^{N-\frac{1}{2}})^2(x,t)dx \leq C, \tag{182}$$

which implies the following statements.

Lemma 10.1. *([14, Lemma 9.1]) There exists a function*

$$\rho \in C(\overline{Q}_T) \cap H^1(Q_T) \cap L^\infty(0,T;H^1(\langle 0,1\rangle)) \tag{183}$$

and a subsequence of $\{\rho^{N-\frac{1}{2}}\}$ (for simplicity denoted again as $\{\rho^{N-\frac{1}{2}}\}$), such that

$$\rho^{N-\frac{1}{2}} \longrightarrow \rho \qquad \text{strongly in } C(\overline{Q}_T), \tag{184}$$

$$* \text{ weakly in } L^\infty(0,T;H^1(\langle 0,1\rangle)), \tag{185}$$

$$\text{weakly in } H^1(Q_T) \tag{186}$$

(when $N \to \infty$ or $h \to 0$). There exists a subsequence of $\{\rho_{h-\frac{1}{2}}\}$ (still denoted $\{\rho_{h-\frac{1}{2}}\}$) such that

$$\rho_{h-\frac{1}{2}} \longrightarrow \rho \quad \text{strongly in } L^\infty(0,T;L^2(\langle 0,1\rangle)). \tag{187}$$

The function ρ satisfies the condition

$$C_1 \leq \rho(x,t) \leq C_2 \text{ for } (x,t) \in \overline{Q}_T, \tag{188}$$

where $C_1, C_2 \in \mathrm{R}^+$.

Taking into account estimates (135), (137), (152) for the function v^N, (136), (137), (153) for the function ω^N and estimates (166), (167) and (179) for the function $\theta^{N-\frac{1}{2}}$ and (73), (97) and (146) for the function r^N, we conclude again there exists $C \in \mathrm{R}^+$ (independent of N), such that

$$|v^N(x,t)| + \int_0^1 (\partial_x v^N)^2(x,t)dx + \int_0^T\int_0^1 \big[(\partial_t v^N)^2 + (\partial_{xx} v^N)^2\big](x,t)dxd\tau \leq C, \tag{189}$$

$$|\omega^N(x,t)| + \int_0^1 (\partial_x \omega^N)^2(x,t)dx + \int_0^T\int_0^1 \big[(\partial_t \omega^N)^2 + (\partial_{xx} \omega^N)^2\big](x,t)dxd\tau \leq C, \tag{190}$$

$$|\theta^{N-\frac{1}{2}}(x,t)| + \int_0^1 (\partial_x \theta^{N-\frac{1}{2}})^2(x,t)dx + \int_0^T\int_0^1 \Big[(\partial_t \theta^{N-\frac{1}{2}})^2 + (\partial_{xx} \theta^{N-\frac{1}{2}})^2\Big](x,t)dxd\tau \leq C, \tag{191}$$

$$|r^N(x,t)| + \int_0^1 (\partial_x r^N)^2(x,t)dx + \int_0^T\int_0^1 \big[(\partial_t r^N)^2 + (\partial_{xx} r^N)^2\big](x,t)dxd\tau \leq C. \tag{192}$$

Because of (74) we have that

$$\partial_t\partial_x(r^3)^N = \partial_t\left(\frac{3L}{\rho_j(t)}\right) \quad \text{on} \quad \left[(j-\tfrac{1}{2})h,(j+\tfrac{1}{2})h\right]$$

and so we get

$$\int_0^1 \left(\partial_t\partial_x(r^3)^N\right)^2 dx \le C. \tag{193}$$

Now, we have the following statements.

Lemma 10.2. *There exist functions*

$$v,\omega,r,\theta \in C(\overline{Q}_T)\cap H^1(Q_T)\cap L^\infty(0,T;H^1(\langle 0,1\rangle))\cap L^2(0,T;H^2(\langle 0,1\rangle)) \tag{194}$$

$$r \in L^\infty(0,T;H^2(\langle 0,1\rangle)) \tag{195}$$

and a subsequence of $\{(v^N,\omega^N,r^N,\theta^{N-\frac{1}{2}})\}$ *(denoted again as* $\{(v^N,\omega^N,r^N,\theta^{N-\frac{1}{2}})\}$*) such that*

$$(v^N,\omega^N,r^N,\theta^{N-\frac{1}{2}}) \longrightarrow (v,\omega,r,\theta) \ \textit{strongly in} \ (C(\overline{Q}_T))^4, \tag{196}$$

$$* \ \textit{weakly in} \ (L^\infty(0,T;H^1(\langle 0,1\rangle)))^4, \tag{197}$$

$$\textit{weakly in} \ (L^2(0,T;H^2(\langle 0,1\rangle)))^4, \tag{198}$$

$$\textit{weakly in} \ (H^1(Q_T))^4, \tag{199}$$

$$r^N \longrightarrow r \ * \ \textit{weakly in} \ (L^\infty(0,T;H^2(\langle 0,1\rangle))), \tag{200}$$

$$\partial_t\partial_x(r^3)^N \longrightarrow \partial_t\partial_x r^3 \ * \ \textit{weakly in} \ (L^\infty(0,T;L^2(\langle 0,1\rangle))), \tag{201}$$

(when $N\to\infty$ *or* $h\to 0$*). There exists a subsequence of* $\{(v_h,\omega_h,r_h,\theta_{h-\frac{1}{2}})\}$ *(still denoted* $\{(v_h,\omega_h,r_h,\theta_{h-\frac{1}{2}})\}$*) such that*

$$(v_h,\omega_h,\theta_{h-\frac{1}{2}}) \longrightarrow (v,\omega,\theta) \ \textit{strongly in} \ (L^\infty(0,T;L^2(\langle 0,1\rangle)))^3 \tag{202}$$

$$r_h \longrightarrow r \ \textit{strongly in} \ L^\infty(0,T;L^2(\langle 0,1\rangle)) \tag{203}$$

There exists $C\in \mathrm{R}^+$ *such that*

$$\theta(x,t)\ge C, \tag{204}$$

$$r(x,t)\ge C, \tag{205}$$

for $(x,t)\in\overline{Q}_T$.

Proof. Conclusions (196)-(201) follow immediately from (189)-(193). Notice that from

$$\int_0^1 (r^N-r_h)^2(x,t)dx \le h^2\int_0^1(\partial_x r^N)^2(x,t)\,dx \le Ch^2 \tag{206}$$

we get a strong convergence (203). The properties (202) and (204) are given in [15, Lemma 9.2]. Using estimate (73) we can easily conclude that (205) is correct. □

Notice, as in [25], that the sequences $\{\rho_{h-\frac{1}{2}}(x,0)\}$, $\{v_h(x,0)\}$, $\{\omega_h(x,0)\}$, $\{r_h(x,0)\}$, $\{\theta_{h-\frac{1}{2}}(x,0)\}$ strongly converge, respectively, to ρ_0, v_0, ω_0, r_0 and θ_0 in $L^2(\langle 0,1\rangle)$ (ρ_0, v_0, ω_0, r_0 and θ_0 are introduced by (9) and (13)).

Since $v_h\to v$ strongly converges as $h\to 0$, from (40) we get that

$$r(x,t) = r_0(x)+\int_0^t v(x,\tau)\,d\tau \quad \text{on} \quad \bar{Q}_T. \tag{207}$$

Lemma 10.3. *The functions ρ, v, ω, r, θ defined by Lemmas 10.1 and 10.2 satisfy equations (1)-(4) a.e. in Q_T.*

Proof. Equation (36) can be written in the from

$$\partial_t \rho_{h-\frac{1}{2}}(x,t) = -\tfrac{1}{3L}\rho^2_{h-\frac{1}{2}}(x,t)\partial_{tx}(r^3)^N(x,t). \tag{208}$$

For any test function $\varphi \in \mathcal{D}(Q_T)$ from (208) we obtain

$$\int_0^T\!\!\int_0^1 \rho_{h-\frac{1}{2}}(x,\tau)\partial_t\varphi(x,\tau)\,dx\,d\tau - \tfrac{1}{3L}\int_0^T\!\!\int_0^1 \rho^2_{h-\frac{1}{2}}(x,\tau)\partial_{tx}(r^3)^N(x,\tau)\varphi(x,\tau)\,dx\,d\tau = 0. \tag{209}$$

Using the convergence $\rho_{h-\frac{1}{2}}(x,t) \to \rho(x,t)$ strongly and $\partial_{tx}(r^3)^N(x,t) \to \partial_{tx}(r^3)(x,t)$ weakly, from (209) we easily get

$$\int_0^T\!\!\int_0^1 \partial_t\rho(x,\tau)\varphi(x,\tau)\,dx\,d\tau + \tfrac{1}{3L}\int_0^T\!\!\int_0^1 \rho^2(x,\tau)\partial_{tx}r^3(x,\tau)\varphi(x,\tau)\,dx\,d\tau = 0.$$

for all $\varphi \in \mathcal{D}(Q_T)$, so (1) is satisfied.

Now, as in [25], we choose $N = \frac{1}{h}$ large enough so that the support of the test function φ is away enough from the boundaries, that is $\operatorname{supp}\varphi \subset (h, 1-h)\times(0,T) = (\frac{1}{N}, 1-\frac{1}{N})\times(0,T)$. Define

$$\varphi_k(t) = \varphi_h(x,t) \equiv \varphi(kh,t), \;\; kh \le x < (k+1)h, \tag{210}$$

$$\varphi_j(t) = \varphi_{h-\frac{1}{2}}(x,t) \equiv \varphi(jh,t), \;\; jh \le x < (j+1)h. \tag{211}$$

We can see that

$$\varphi_k(t) = 0, \;\text{ for }\; k = 0, 1, N-1, N, \tag{212}$$

$$\varphi_j(t) = 0, \;\text{ for }\; j = \tfrac{1}{2}, N-\tfrac{1}{2}. \tag{213}$$

First, equation (37) we write in the form

$$\frac{\partial}{\partial t}\left(\frac{v_k}{r_k^2}\right) + \frac{2v_k^2}{r_k^3} = -\frac{R}{L}\,\delta(\rho\theta)_k + \frac{\lambda+2\mu}{L^2}\,\delta\big(\rho\delta(r^2v)\big)_k \tag{214}$$

and then, after multiplying by $\varphi_k h$, summing up for $k = 1,\ldots,N-1$ and integrating over $[0,T]$ we get

$$\int_0^T \sum_{k=1}^{N-1} \partial_t\left(\frac{v_k}{r_k^2}\right)\varphi_k h\,d\tau = -\int_0^T \sum_{k=1}^{N-1}\frac{2v_k^2}{r_k^3}\varphi_k h\,d\tau + \frac{R}{L}\int_0^T \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j\theta_j\delta\varphi_j h\,d\tau -$$
$$-\frac{\lambda+2\mu}{L^2}\int_0^T \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \rho_j\delta(r^2v)_j\varphi_j h\,d\tau. \tag{215}$$

Since $\varphi_h \to \varphi$, $\partial_t\varphi_h \to \partial_t\varphi$, $\delta\varphi^N \to \partial_x\varphi$ strongly converge, as $h \to 0$, we can write (215) as follows

$$-\int_0^T\!\!\int_0^1 \left(\frac{1}{r^2}\right)_h v_h\partial_t\varphi\,dx\,d\tau = \mathcal{O}(h) - 2\int_0^T\!\!\int_0^1 \left(\frac{1}{r^3}\right)_h v_h^2\varphi\,dx\,d\tau +$$
$$+\frac{R}{L}\int_0^T\!\!\int_0^1 \rho_{h-\frac{1}{2}}\theta_{h-\frac{1}{2}}\partial_x\varphi^N\,dx\,d\tau - \frac{\lambda+2\mu}{L^2}\int_0^T\!\!\int_0^1 \rho_{h-\frac{1}{2}}\partial_{tx}(r^3)^N\partial_x\varphi^N\,dx\,d\tau, \tag{216}$$

where $\mathcal{O}(h) \to 0$ as $h \to 0$. Taking into account that $\left(\left(\frac{1}{r^3}\right)_h, \left(\frac{1}{r^2}\right)_h, v_h, \rho_{h-\frac{1}{2}}, \theta_{h-\frac{1}{2}}\right)$ $\to \left(\frac{1}{r^3}, \frac{1}{r^2}, v, \rho, \theta\right)$ strongly converge and $\partial_{tx}(r^3)^N \to \partial_{tx}(r^3)$ weakly converges, from (216) we conclude that, as $h \to 0$

$$-\int_0^T\int_0^1\left(\partial_t\left(\frac{1}{r^2}v\right) + \frac{2}{r^3}v^2 + \frac{R}{L}\partial_x(\rho\theta) - \frac{\lambda+2\mu}{L^2}\partial_x\left(\rho\partial_{tx}r^3\right)\right)\varphi\, dx\, d\tau = 0$$

for all $\varphi \in \mathcal{D}(Q_T)$ and (2) is satisfied. In the same way as for the equation (37), from (38) we get

$$\int_0^T \sum_{k=1}^{N-1} \partial_t\left(\frac{\omega_k}{r_k^2}\right)\varphi_k h\, d\tau + 2\int_0^T \sum_{k=1}^{N-1} \frac{\omega_k v_k}{r_k^3}\varphi_k h\, d\tau =$$
$$= -\frac{4\mu_r}{j_I}\int_0^T \sum_{k=1}^{N-1} \frac{\omega_k}{\rho_k r_k^2}\varphi_k h\, d\tau + \frac{c_0+2c_d}{j_I L^2}\int_0^T \sum_{k=1}^{N-1} \delta\left(\rho\delta(r^2\omega)\right)_k\varphi_k h\, d\tau,$$

i.e.

(217)
$$-\int_0^T\int_0^1\left(\frac{1}{r^2}\right)_h\omega_h\partial_t\varphi\, dx\, d\tau + 2\int_0^T\int_0^1\left(\frac{1}{r^3}\right)_h\omega_h v_h\partial_t\varphi\, dx\, d\tau = \mathcal{O}(h) -$$
$$-\frac{4\mu_r}{j_I}\int_0^T\int_0^1\left(\frac{1}{\rho}\right)_h\left(\frac{1}{r^2}\right)_h\omega_h\,\varphi\, dx\, d\tau - \frac{c_0+2c_d}{j_I L^2}\int_0^T\int_0^1\rho_h\partial_x(r^2\omega)^N\partial_x\varphi^N\, dx\, d\tau.$$

Since $\partial_x(r^2\omega)^N \to \partial_x(r^2\omega)$ weakly converges in $L^2(Q_T)$, as in (215), as $h \to 0$, we obtain

$$\int_0^T\int_0^1\left(\partial_t\left(\frac{1}{r^2}\omega\right) + \frac{2}{r^3}v\omega + \frac{4\mu_r}{j_I}\frac{1}{\rho r^2}\omega - \frac{c_0+2c_d}{j_I L^2}\partial_x\left(\rho\partial_x(r^2\omega)\right)\right)\varphi\, dx\, d\tau = 0$$

for all $\varphi \in \mathcal{D}(Q_T)$ and (3) is satisfied.

Similarly, multiplying (37)-(39), respectively, by $v_k\varphi_k h$, $j_I\rho_k^{-1}\omega_k\varphi_k h$ and $c_v\rho_j^{-1}\varphi_j h$, summing up for $k = 1, \ldots, N-1$ and $j = \frac{1}{2}, \ldots, N-\frac{1}{2}$ and integrating over $[0, T]$ we get, after summing the obtained equations, that

(218)
$$-\int_0^T\int_0^1\left(\tfrac{1}{2}v_h^2 + \tfrac{j_I}{2}\omega_h^2 + c_v\theta_{h-\frac{1}{2}}\right)\partial_t\varphi\, dx\, d\tau + \tfrac{\lambda+2\mu}{L^2}\int_0^T\int_0^1 r_h^2 v_h\rho_{h-\frac{1}{2}}\partial_x(r^2v)^N\,\partial_x\varphi\, dx\, d\tau$$
$$-\tfrac{R}{L}\int_0^T\int_0^1\rho_{h-\frac{1}{2}}\theta_{h-\frac{1}{2}}r_h^2 v_h\partial_x\varphi\, dx\, d\tau + \tfrac{c_0+2c_d}{L^2}\int_0^T\int_0^1\rho_{h-\frac{1}{2}}\partial_x(r^2\omega)^N\, r_h^2\omega_h\partial_x\varphi\, dx\, d\tau +$$
$$+\tfrac{K}{L^2}\int_0^T\int_0^1 r_h^4\rho_{h-\frac{1}{2}}\partial_x\theta^N\partial_x\varphi\, dx\, d\tau - \tfrac{4\mu}{L}\int_0^T\int_0^1 r_h v_h^2\partial_x\varphi\, dx\, d\tau -$$
$$-\tfrac{4c_d}{L}\int_0^T\int_0^1 r_h\omega_h^2\partial_x\varphi\, dx\, d\tau = \mathcal{O}(h).$$

Using the aforementioned strong and weak convergence, from (218) we obtain

(219)

$$\int_0^T\int_0^1 (v\partial_t v + j_I\omega\partial_t\omega + c_v\partial_t\theta)\,\varphi\,dx\,d\tau + \tfrac{\lambda+2\mu}{L^2}\int_0^T\int_0^1 r^2 v\rho\partial_x(r^2 v)\,\partial_x\varphi\,dx\,d\tau -$$
$$- \tfrac{R}{L}\int_0^T\int_0^1 \rho\theta r^2 v\partial_x\varphi\,dx\,d\tau + \tfrac{c_0+2c_d}{L^2}\int_0^T\int_0^1 \rho\partial_x(r^2\omega)\,r^2\omega\partial_x\varphi\,dx\,d\tau +$$
$$+ \tfrac{K}{L^2}\int_0^T\int_0^1 r^4\rho\partial_x\theta\partial_x\varphi\,dx\,d\tau - \tfrac{4\mu}{L}\int_0^T\int_0^1 r v^2\partial_x\varphi\,dx\,d\tau - \tfrac{4c_d}{L}\int_0^T\int_0^1 r\omega^2\partial_x\varphi\,dx\,d\tau = 0.$$

Now, we multiply the already proven equations (2) and (3), respectively, by $-v\varphi$ and $-j_I\rho^{-1}\omega\varphi$, integrate over $[0,1]\times[0,T]$ and add up to (219). So we get that (4) is satisfied. □

Lemma 10.4. *([14, Lemma 9.4]) The functions ρ, v, ω and θ satisfy the following conditions*

$$\rho(x,0)=\rho_0(x),\ \ v(x,0)=v_0(x),\ \ \omega(x,0)=\omega_0(x),\ \ \theta(x,0)=\theta_0(x), \tag{220}$$

$$v(0,t)=v(1,t)=0,\ \ \omega(0,t)=\omega(1,t)=0,\ \ \partial_x\theta(0,t)=\partial_x\theta(1,t)=0, \tag{221}$$

for $x\in(0,1)$ and $t\in(0,T)$.

Final proof of Theorem 4.1. In Lemmas 10.1-10.2 we prove that there exists some subsequence of approximate solutions, which converge in a strong or a weak sense in spaces proposed by Theorem 4.1. In Lemmas 10.3-10.4 we prove that the limit of this subsequence satisfy equations (1)-(4) and the given initial and boundary conditions (9)-(11). As a consequence of the convergence in the appropriate vector spaces, we conclude that this limit is actually the generalized solution of our problem with properties (19)-(23). Due to the fact that such a generalized solution is unique [10], it follows that not only the subsequence converge, but the whole sequences $\{(\rho^{N-\frac12}, v^N, \omega^N, r^N, \theta^{N-\frac12})\}$ and $\{(\rho_{h-\frac12}, v_h, \omega_h, r_h, \theta_{h-\frac12})\}$ converge to this limit, as $N\to\infty$ (or $h\to 0$), which completes the proof of the Theorem 4.1.

At the end we give a brief summary of the complete process of proving the main theorem. As mentioned earlier, the lemmas in the article are grouped such that the proof of the Theorem 4.1 is carried through several stages. At the first stage, in Section 3, we formed the sequences of approximate solutions. At the second stage, by using results of lemmas from Sections 5-9, we show that these sequences belong to the same vector spaces as we expect that the solution of our problem belongs to. The analysis of the weak and the strong convergence of the sequences of the approximate solution is performed at the third stage in Lemmas 10.1-10.2. Finally, the fourth stage consists of the analysis of the obtained limit in Lemmas 10.3-10.4. We proved that this limit is indeed the generalized solution of our problem.

11. Numerical experiment

In order to validate the proposed numerical method we apply it on the chosen test example.

The system (36)-(40) is ordinary differential equations system in time variable and has the form $\dot{\mathbf{u}}(t)=\mathbf{F}(\mathbf{u}(t))$, where $\mathbf{u}=(\rho_j, v_k, \omega_k, \theta_j, r_k)$, $j=\frac12,\ldots,N-\frac12$,

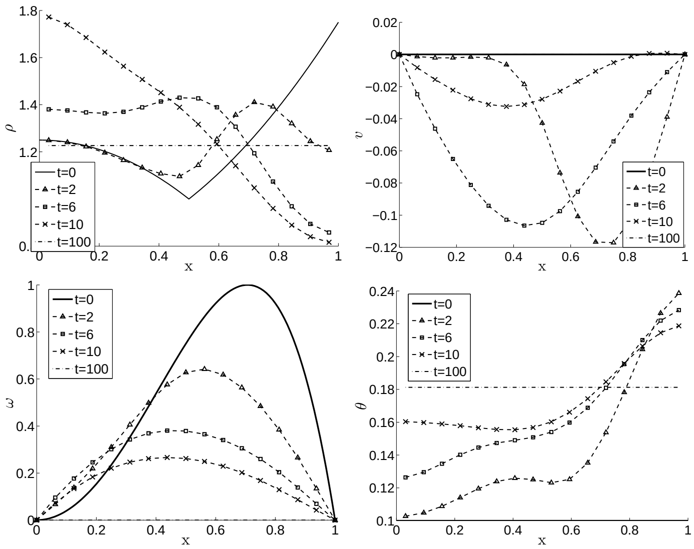

FIGURE 1. Numerical results obtained at different time moments.

$k = 1, \ldots, N-1$. For solving it numerically, the second order strong stability preserving Runge-Kutta method [15] is used:

$$\begin{aligned}\mathbf{u}^{(1)} &= \mathbf{u}^n + \Delta t \mathbf{F}(\mathbf{u}^n)\\ \mathbf{u}^{n+1} &= \frac{1}{2}\mathbf{u}^n + \frac{1}{2}\mathbf{u}^{(1)} + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{u}^{(1)}).\end{aligned}$$

Here $\mathbf{u}^n$ denotes the numerical solution at time moment $t^n = n\Delta t$ for the chosen time step Δt. For stability reasons of the numerical method, we choose $\Delta t = \mathcal{O}(h^2)$. In this way the positivity of the density and the temperature are preserved.

We take the following initial conditions: $\rho_0(x) = \left|x^2 - \frac{1}{4}\right| + 1$, $v_0(x) = 0$, $\omega_0(x) = 4(x^2 - x^4)$, $\theta_0(x) = 0.1$ and parameters: $\mu = \mu_r = K = c_0 = c_d = 0.01$, $R = c_v = 1$, $j_I = 1$, $a = 1$, $L = 1$. Numerical parameters are set to $N = 16$, $t = 10^{-3}$.

In Figure 1 we present the numerical results obtained with the proposed finite difference method at different time moments. In shown figures, it is nicely visible that for larger t the stabilization of the solution arises, which is in accordance with the result that was proved in [12] and which states that the solution of our problem converges to the stationary constant solution of the form $(\rho^*, 0, 0, \theta^*)$ in the space $(H_1(\langle 0,1\rangle))^4$ (when $t \to \infty$), where

$$\rho^* = \left(\int_0^1 \frac{1}{\rho_0(x)}dx\right)^{-1},\ \theta^* = \frac{1}{c_v}\int_0^1 \left(\frac{1}{2}|v_0(x)|^2 + \frac{j_I}{2}|\omega_0(x)|^2 + c_v|\theta_0(x)|\right)dx.$$

At the time moment $t = 100$ the discrepancy of the numerical results from the stationary constant solution is on the level of the round of error. Thus, we can conclude that we successfully applied the proposed numerical method.

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